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DISPERSION RELATIONS FOR A MAGNETOACTIVE FINITE TEMPERATURE PLASMA

TECHNICAL REPORT NO. 95

By:

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September, 1966

ELECTRON PHYSICS LABORATORY

DEPARTMENT OF ELECTRICAL ENGINEERING
THE UNIVERSITY OF MICHIGAN, ANN ARBOR

CONTRACT WITH:

OFFICE OF SPACE SCIENCE AND APPLICATIONS, NATIONAL AERONAUTICS AND SPACE ADMINISTRATION, WASHINGTON, D. C. RESEARCH GRANT NO. NsG 696.

THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN

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Project 06621

RESEARCH GRANT NO. NsG 696
OFFICE OF SPACE SCIENCE AND APPLICATIONS
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546

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ABSTRACT

The dispersion relation for a finite temperature magnetoactive plasma is derived in a form particularly suitable for the study of the effects of transverse static electric and magnetic fields upon the coupling between the transverse and longitudinal modes. The derivation is based on the coupled Boltzmann-Vlasov-Maxwell equations under the one-dimensional small-signal assumptions.

The time-varying parts of the particle distribution functions for a two-component plasma are divided into three parts; namely, those associated respectively with the right-hand and left-hand circularly polarized transverse waves and that associated with the longitudinal mode.

The mode coupling equation, which relates the dynamic electric fields of these modes, is derived in terms of the time-independent part of the distribution function for two cases: (a) longitudinal propagation in the presence of a transverse static electric field, and (b) oblique propagation in the absence of static electric field.

If the time-independent portions of the distribution functions are taken to be Maxwellian it is shown that in the low-temperature limit the dispersion relationship reduces to the familiar expression for the cold plasma. Possible applications of the derived dispersion relationship are briefly discussed.

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DISPERSION RELATIONS FOR A MAGNETOACTIVE FINITE TEMPERATURE PLASMA

I. INTRODUCTION

Wave phenomena in plasmas have been studied by many authors[1-6] under a variety of assumptions and, in general, coupling between transverse and longitudinal modes is neglected. The longitudinal and transverse oscillations in plasmas are strictly uncoupled only in the case of a nonrelativistic plasma and in the absence of any external magnetic fields and temperature or density gradients. The presence of an external magnetic field[7] or inhomogeneities in plasma density[8,9] and/or temperature result in the coupling of the longitudinal and transverse modes.

It is also a well-known fact that in the absence of a transverse magnetostatic field there exist two purely transverse and two purely longitudinal waves. The existence of a transverse magnetostatic field introduces a coupling between the transverse and longitudinal motion of the particles. Thus there appear mixed modes having both transverse and longitudinal components. This fact has been demonstrated theoretically; for example, by Denisse and Delcroix[2] for a uniform, unbounded two-component plasma based on a macroscopic description which uses Maxwell's equations together with the continuity equation and the equation of momentum conservation. They assume that the thermal velocity is negligible compared to the phase velocity of the wave and, of course, develop a linear theory.

It is the purpose of the present report to derive the dispersion relationship for a magnetoactive finite temperature plasma in a form which is suitable for the study of the coupling of transverse and longitudinal modes due to the presence of transverse static electric and magnetic fields. The derivation uses Maxwell's equations together with the Boltzmann-Vlasov equation and the effect of particle thermal motions is taken into account.

II. MATHEMATICAL FORMULATION

Consider a two-component plasma in which collision effects are assumed to be negligible. The electron distribution function $f(\vec{r},\vec{v},t)$ and the ion distribution function $F(\vec{r},\vec{v},t)$ for this plasma are governed by the Boltzmann-Vlasov equation:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B})\cdot\nabla_v f = 0$$

and

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla f + \frac{e}{M}(\vec{E} + \vec{v} \times \vec{B})\cdot\nabla_v F = 0 \quad , \tag{1}$$

where m and M are the electron and ion mass respectively and e is the electronic charge taken as a positive quantity. The electromagnetic fields in the plasma are governed by Maxwell's equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ,$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad ,$$

$$\nabla\cdot\vec{D} = \rho \quad ,$$

$$\nabla\cdot\vec{B} = 0 \quad , \tag{2}$$

where the electric displacement vector $\vec{D}$ and the magnetic flux density $\vec{B}$ are, respectively, related to the electric field intensity $\vec{E}$ and the magnetic field intensity $\vec{H}$ in the following manner:

$$\vec{D} = \epsilon_o \vec{E} \quad \text{and} \quad \vec{B} = \mu_o \vec{H} \ , \tag{3}$$

where ϵ_o and μ_o denote the permittivity and the permeability of the vacuum. The convection current density $\vec{J}$ and the charge density ρ may be written in terms of the distribution functions as

$$\vec{J} = e\int \vec{v}(F-f)d^3v \quad \text{and} \quad \rho = e\int (F-f)d^3v \ . \tag{4}$$

Consider that all quantities of interest are composed of a time-independent part denoted by the subscript 0 and a time-dependent part denoted by the subscript 1:

$$\vec{B} = \vec{B}_o(\vec{r}) + \vec{B}_1(\vec{r},t) \ ,$$

$$\vec{E} = \vec{E}_o(\vec{r}) + \vec{E}_1(\vec{r},t) \ ,$$

$$\vec{J} = \vec{J}_o(\vec{r}) + \vec{J}_1(\vec{r},t) \ ,$$

$$\rho = \rho_o(\vec{r}) + \rho_1(\vec{r},t) \ ,$$

$$f = f_o(\vec{r},\vec{v}) + f_1(\vec{r},\vec{v},t) \ ,$$

$$F = F_o(\vec{r},\vec{v}) + F_1(\vec{r},\vec{v},t) \ . \tag{5}$$

Upon substituting Eqs. 5 into Eqs. 1, 2 and 4, the following time-independent set of differential equations (Eqs. 6) and the time-dependent set of equations (Eqs. 7) are obtained:

$$\vec{v}\cdot\nabla f_o - \frac{e}{m}(\vec{E}_o + \vec{v} \times \vec{B}_o)\cdot\nabla_v f_o = 0 \ , \tag{6a}$$

$$\vec{v}\cdot\nabla F_o + \frac{e}{M}(\vec{E}_o + \vec{v} \times \vec{B}_o)\cdot\nabla_v F_o = 0 \ , \tag{6b}$$

$$\nabla \times \vec{E}_o = 0 \quad , \tag{6c}$$

$$\nabla \times \vec{H}_o = \vec{J}_o \quad , \tag{6d}$$

$$\nabla \cdot \vec{E}_o = \frac{\rho_o}{\epsilon_o} \quad , \tag{6e}$$

$$\nabla \cdot \vec{B}_o = 0 \quad , \tag{6f}$$

$$\vec{J}_o = e \int \vec{v}(F_o - f_o) d^3v \quad , \tag{6g}$$

$$\rho_o = e \int (F_o - f_o) d^3v \tag{6h}$$

and

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 - \frac{e}{m} (\vec{E}_o + \vec{v} \times \vec{B}_o) \cdot \nabla_v f_1 - \frac{e}{m} (\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \nabla_v f_o$$

$$= \frac{e}{m} [\vec{E}_1 + (\vec{v} \times \vec{B}_1)] \cdot \nabla_v f_1 \quad , \tag{7a}$$

$$\frac{\partial F_1}{\partial t} + \vec{v} \cdot \nabla F_1 + \frac{e}{M} (\vec{E}_o + \vec{v} \times \vec{B}_o) \cdot \nabla_v F_1 + \frac{e}{M} (\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \nabla_v F_o$$

$$= \frac{-e}{M} [\vec{E}_1 + (\vec{v} \times \vec{B}_1)] \cdot \nabla_v F_1 \quad , \tag{7b}$$

$$\nabla \times \vec{E}_1 = - \frac{\partial \vec{B}_1}{\partial t} \quad , \tag{7c}$$

$$\nabla \times \vec{H}_1 = \vec{J}_1 + \frac{\partial \vec{D}_1}{\partial t} \quad , \tag{7d}$$

$$\nabla \cdot \vec{E}_1 = \frac{\rho_1}{\epsilon_o} \quad , \tag{7e}$$

$$\nabla \cdot \vec{B}_1 = 0 \quad , \tag{7f}$$

$$\vec{J}_1 = e\int \vec{v}(F_1-f_1)d^3v \quad , \tag{7g}$$

$$\rho_1 = e\int (F_1-f_1)d^3v \quad . \tag{7h}$$

In the present report the following assumptions are made and a rectangular coordinate system is employed:

1. Small amplitude conditions are satisfied so that the terms involving the products of time-dependent quantities are regarded as negligible.

2. All quantities vary only with one spatial variable, z.

3. All time-dependent quantities in the system have the $e^{j(\omega t-kz)}$ time and distance dependence.

By Assumption No. 2 Eqs. 6c and 6e imply that

$$E_{ox} = \text{constant} \quad , \quad E_{oy} = \text{constant} \quad \text{and} \quad \frac{\partial E_{oz}}{\partial z} = \frac{\rho_o(z)}{\epsilon_o} \quad , \tag{8}$$

and Eqs. 6d and 6f yield

$$\frac{\partial B_{ox}}{\partial z} = \mu_o J_{oy} \quad , \quad \frac{\partial B_{oy}}{\partial z} = -\mu_o J_{ox} \quad \text{and} \quad B_{oz} = \text{constant} \quad . \tag{9}$$

Under the above-mentioned assumptions, Eqs. 7a and 7b become, respectively,

$$j(\omega-kv_z)f_1 - \frac{e}{m}\left[(E_{ox} + v_yB_{oz} - v_zB_{oy})\frac{\partial f_1}{\partial v_x} + (E_{oy} + v_zB_{ox} - v_xB_{oz})\frac{\partial f_1}{\partial v_y}\right.$$

$$\left. + (E_{oz} + v_xB_{oy} - v_yB_{ox})\frac{\partial f_1}{\partial v_z}\right] = \frac{e}{m}\left[(E_{1x} + v_yB_{1z} - v_zB_{1y})\frac{\partial f_o}{\partial v_x}\right.$$

$$\left. + (E_{1y} + v_zB_{1x} - v_xB_{1z})\frac{\partial f_o}{\partial v_y} + (E_{1z} + v_xB_{1y} - v_yB_{1x})\frac{\partial f_o}{\partial v_z}\right] \tag{10}$$

and

$$j(\omega - kv_z)F_1 + \frac{e}{M}\left[(E_{ox} + v_y B_{oz} - v_z B_{oy})\frac{\partial F_1}{\partial v_x} + (E_{oy} + v_z B_{ox} - v_x B_{oz})\frac{\partial F_1}{\partial v_y}\right.$$

$$\left. + (E_{oz} + v_x B_{oy} - v_y B_{ox})\frac{\partial F_1}{\partial v_z}\right] = \frac{-e}{M}\left[(E_{1x} + v_y B_{1z} - v_z B_{1y})\frac{\partial F_o}{\partial v_x}\right.$$

$$\left. + (E_{1y} + v_z B_{1x} - v_x B_{1z})\frac{\partial F_o}{\partial v_y} + (E_{1z} + v_x B_{1y} - v_y B_{1x})\frac{\partial F_o}{\partial v_z}\right] . \qquad (11)$$

Equations 7c and 7f give

$$B_{1x} = -\frac{k}{\omega} E_{1y} \ , \quad B_{1y} = \frac{k}{\omega} E_{1x} \quad \text{and} \quad \frac{\partial B_{1z}}{\partial z} = 0 \ , \qquad (12)$$

which implies that

$$E_{1x}B_{1x} + E_{1y}B_{1y} = 0 \ , \qquad (13)$$

which in turn suggests that the transverse time-varying electric field is perpendicular to the magnetic field. On the other hand Eqs. 7c and 7d can be combined to give

$$\nabla^2\vec{E}_1 + \frac{\omega^2}{c^2}\vec{E}_1 - \nabla(\nabla\cdot\vec{E}_1) = j\omega\mu_o\vec{J}_1 \ , \qquad (14)$$

where $c = 1/\sqrt{\mu_o\epsilon_o}$ is the speed of light in vacuum. Equation 14 can be written in its component form as

$$\frac{\partial^2 E_{1x}}{\partial z^2} + \frac{\omega^2}{c^2}E_{1x} = j\omega\mu_o J_{1x} \ , \qquad (15a)$$

$$\frac{\partial^2 E_{1y}}{\partial z^2} + \frac{\omega^2}{c^2}E_{1y} = j\omega\mu_o J_{1y} \qquad (15b)$$

and

$$\frac{\omega^2}{c^2}E_{1z} = j\omega\mu_o J_{1z} \ . \qquad (15c)$$

Now consider a transformation of velocity coordinates as given by

$$v_x = v_\perp \cos\varphi \ , \quad v_y = v_\perp \sin\varphi \quad \text{and} \quad v_z = v_z \ , \tag{16}$$

and, for convenience of discussion, define the quantities $\vec{\omega}_c$ and $\vec{a}$ as

$$\vec{\omega}_c \triangleq \left(\frac{e}{m}\vec{B}_o\right) \quad \text{and} \quad \vec{a} \triangleq \left(\frac{e}{m}\vec{E}_o\right) . \tag{17}$$

Then Eq. 10 can be transformed into the following, using Eq. 12;

$$\begin{aligned} &\left[j(\omega - kv_z) + \omega_z \frac{\partial}{\partial\varphi}\right] f_1 \\ &\quad - \left[a_-\left(\frac{\partial f_1}{\partial v_\perp} + j\frac{1}{v_\perp}\frac{\partial f_1}{\partial\varphi}\right) + \frac{v_z}{v_\perp}\omega_-\frac{\partial f_1}{\partial\varphi} + j\omega_- D(f_1)\right] e^{j\varphi} \\ &\quad - \left[a_+\left(\frac{\partial f_1}{\partial v_\perp} - j\frac{1}{v_\perp}\frac{\partial f_1}{\partial\varphi}\right) + \frac{v_z}{v_\perp}\omega_+\frac{\partial f_1}{\partial\varphi} - j\omega_+ D(f_1)\right] e^{-j\varphi} - a_z\frac{\partial f_1}{\partial v_z} \\ &= \frac{e}{m}M_-(f_o)E_-\, e^{j\varphi} + \frac{e}{m}M_+(f_o)E_+\, e^{-j\varphi} + \frac{e}{m}E_{1z}\frac{\partial f_o}{\partial v_z} - \frac{e}{m}B_{1z}\frac{\partial f_o}{\partial\varphi} \ , \end{aligned} \tag{18}$$

where

$$E_\pm = \frac{1}{2}(E_{1x} \pm jE_{1y}) \ , \tag{19a}$$

$$B_\pm = \frac{1}{2}(B_{1x} \pm jB_{1y}) \ , \tag{19b}$$

$$\omega_\pm = \frac{1}{2}(\omega_x \pm j\omega_y) \ , \tag{19c}$$

$$a_\pm = \frac{1}{2}(a_x \pm ja_y) \ , \tag{19d}$$

$$\omega_{cx} \triangleq \omega_x \;, \quad \omega_{cy} \triangleq \omega_y \;, \quad \omega_{cz} \triangleq \omega_z \;, \tag{19e}$$

$$M_+(f_o) = \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_o}{\partial v_\perp} + \frac{j}{v_\perp}\frac{\partial f_o}{\partial \varphi}\right) + \frac{fv_\perp}{\omega}\frac{\partial f_o}{\partial v_z}\right] \;, \tag{19f}$$

$$M_-(f_o) = \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_o}{\partial v_\perp} - \frac{j}{v_\perp}\frac{\partial f_o}{\partial \varphi}\right) + \frac{kv_\perp}{\omega}\frac{\partial f_o}{\partial v_z}\right] \;, \tag{19g}$$

and the differential operator D is defined as

$$D \triangleq \left(v_\perp \frac{\partial}{\partial v_z} - v_z \frac{\partial}{\partial v_\perp}\right) .$$

It should be noted that E_- and E_+ appearing in Eq. 18 correspond to the electric fields of the left-hand and right-hand circularly polarized waves respectively. Furthermore, from Eq. 12, B_{1z} is a constant, and for the present one-dimensional analysis, from Eq. 7c, it must be zero.

III. DERIVATION OF DISPERSION RELATIONSHIPS

Consider the time-dependent electron distribution function f_1 as consisting of three parts as indicated below:

$$f_1(z,t,v_\perp,v_z,\varphi) = f_+(z,t,v_\perp,v_z)e^{-j\varphi} + f_-(z,t,v_\perp,v_z)e^{+j\varphi} + g(z,t,v_\perp,v_z) \;, \tag{20}$$

where the first, second and third terms of Eq. 20 can be regarded as the distribution of these electrons associated with the right-hand circularly polarized, left-hand circularly polarized and longitudinal waves, respectively. Since Eq. 18 must be valid for an arbitrary value of φ, the substitution of Eq. 20 into Eq. 18 yields the following system of equations:

$$a_-\left(\frac{\partial f_-}{\partial v_\perp} - \frac{1}{v_\perp} f_-\right) + j\omega_- \left[D(f_-) + \frac{v_z}{v_\perp} f_-\right] = 0 \quad , \tag{21a}$$

$$a_+\left(\frac{\partial f_+}{\partial v_\perp} - \frac{1}{v_\perp} f_+\right) - j\omega_+ \left[D(f_+) + \frac{v_z}{v_\perp} f_+\right] = 0 \quad , \tag{21b}$$

$$j(\omega - kv_z + \omega_z)f_- - a_z \frac{\partial f_-}{\partial v_z} - j\omega_- D(g) - a_- \frac{\partial g}{\partial v_\perp} = \frac{e}{m} M_-(f_o)E_- \quad , \tag{21c}$$

$$j(\omega - kv_z - \omega_z)f_+ - a_z \frac{\partial f_+}{\partial v_z} + j\omega_+ D(g) - a_+ \frac{\partial g}{\partial v_\perp} = \frac{e}{m} M_+(f_o)E_+ \tag{21d}$$

and

$$j(\omega - kv_z)g - a_z \frac{\partial g}{\partial v_z} - a_-\left(\frac{\partial f_+}{\partial v_\perp} + \frac{1}{v_\perp} f_+\right) - j\omega_- \left[D(f_+) - \frac{v_z}{v_\perp} f_+\right]$$

$$- a_+\left(\frac{\partial f_-}{\partial v_\perp} + \frac{1}{v_\perp} f_-\right) + j\omega_+ \left[D(f_-) - \frac{v_z}{v_\perp} f_-\right] = \frac{e}{m} \frac{\partial f_o}{\partial v_z} E_{1z} \quad . \tag{21e}$$

It is of interest to note that when the transverse static electric and magnetic fields are absent, i.e., $a_+ = a_- = \omega_+ = \omega_- = 0$, the system of equations (Eqs. 21) reduces to the following set of equations:

$$j(\omega - kv_z + \omega_z)f_- - a_z \frac{\partial f_-}{\partial v_z} = \frac{e}{m} M_-(f_o)E_- \quad ,$$

$$j(\omega - kv_z - \omega_z)f_+ - a_z \frac{\partial f_+}{\partial v_z} = \frac{e}{m} M_+(f_o)E_+ \quad ,$$

$$j(\omega - kv_z)g - a_z \frac{\partial g}{\partial v_z} = \frac{e}{m} \frac{\partial f_o}{\partial v_z} E_{1z} \quad , \tag{22}$$

which clearly suggests that no coupling between the transverse and longitudinal modes can take place. However, it is obvious that the presence of either electric or magnetic transverse static fields will lead to coupling between the modes.

In the present analysis two cases are considered:

Case 1. Absence of static transverse magnetic field and longitudinal electric field: $(\omega_+ = \omega_- = 0,\ a_z = 0)$.

Case 2. Absence of electrostatic field: $(a_+ = a_- = a_z = 0)$.

For these cases, it is possible to solve Eqs. 25 for f_-, f_+ and g explicitly in terms of E_-, E_+, and E_{1z} which can be expressed as follows (see Appendix A for details):

$$
\begin{aligned}
f_- &= k_{11}E_- + k_{12}E_+ + k_{13}E_{1z} \ , \\
f_+ &= k_{21}E_- + k_{22}E_+ + k_{23}E_{1z} \ , \\
g &= k_{31}E_- + k_{32}E_+ + k_{33}E_{1z} \ ,
\end{aligned}
\tag{23}
$$

where for Case 1

$$
k_{11} = \frac{\frac{e}{m} M_-(f_o)}{j(b+\omega_z)} \ , \quad k_{12} = 0 \ , \quad k_{13} = \frac{-\frac{e}{m} a_- \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b+\omega_z)} \ ,
$$

$$
k_{21} = 0 \ , \quad k_{22} = \frac{\frac{e}{m} M_+(f_o)}{j(b-\omega_z)} \ , \quad k_{23} = \frac{-\frac{e}{m} a_+ \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b-\omega_z)} \ ,
$$

$$
k_{31} = \frac{-2\frac{e}{m}\frac{a_+}{v_\perp} M_-(f_o)}{b(b+\omega_z)} \ , \quad k_{32} = \frac{-2\frac{e}{m}\frac{a_-}{v_\perp} M_+(f_o)}{b(b-\omega_z)} \ ,
$$

$$
k_{33} = \frac{\frac{e}{m}\frac{\partial f_o}{\partial v_z}}{jb} + \frac{j4a_-a_+\frac{e}{m}\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{v_\perp b(b^2-\omega_z^2)} \ , \tag{24}
$$

with $b \triangleq (\omega - kv_z)$, and for Case 2

$$k_{11} = \frac{e}{m}\frac{M_o(f_o)}{\delta}\left[\frac{2\omega_-\omega_+\omega}{b} - b(b-\omega_z)\right] ,$$

$$k_{12} = \frac{e}{m}\frac{M_o(f_o)}{\delta}\left(\frac{2\omega_-^2\omega}{b}\right) ,$$

$$k_{13} = \frac{-e}{m}\frac{1}{\delta}\left[D\left(\frac{\partial f_o}{\partial v_z}\right) + \frac{kv_\perp}{b}\frac{\partial f_o}{\partial v_z}\right][\omega_-(b-\omega_z)] ,$$

$$k_{21} = \frac{e}{m}\frac{M_o(f_o)}{\delta}\left(\frac{2\omega_+^2\omega}{b}\right) ,$$

$$k_{22} = \frac{e}{m}\frac{M_o(f_o)}{\delta}\left[\frac{2\omega_-\omega_+\omega}{b} - b(b+\omega_z)\right] ,$$

$$k_{23} = \frac{e}{m}\frac{1}{\delta}\left[D\left(\frac{\partial f_o}{\partial v_z}\right) + \frac{kv_\perp}{b}\frac{\partial f_o}{\partial v_z}\right][\omega_+(b+\omega_z)] ,$$

$$k_{31} = \frac{-e}{m}\frac{M_o(f_o)}{\delta}\left[2\omega_+\left(\frac{v_z}{v_\perp}\right)(b-\omega_z)\right] ,$$

$$k_{32} = \frac{e}{m}\frac{M_o(f_o)}{\delta}\left[2\omega_-\left(\frac{v_z}{v_\perp}\right)(b+\omega_z)\right] ,$$

$$k_{33} = \frac{e}{m}\frac{1}{\delta}\left(\frac{\partial f_o}{\partial v_z}\right)(\omega_z^2-b^2+4\omega_+\omega_-) - \frac{e}{m}\frac{1}{\delta}D\left(\frac{\partial f_o}{\partial v_z}\right)\left[4\left(\frac{v_z}{v_\perp}\right)\omega_+\omega_-\right] ,$$

$$M_o(f_o) \triangleq \left[\frac{\partial f_o}{\partial v_\perp} + \frac{k}{\omega}D(f_o)\right] , \qquad (25)$$

with

$$\delta \triangleq [jb(\omega_z^2-b^2) + j4\omega_+\omega_-\omega] .$$

Similarly by writing the ion distribution function as

$$F_1(z,t,v_\perp,v_z,\varphi) = F_-(z,t,v_\perp,v_z)e^{j\varphi} + F_+(z,t,v_\perp,v_z)e^{-j\varphi} + G(z,t,v_\perp,v_z) , \tag{26}$$

and in view of the fact that Eq. 11 has exactly the same form as Eq. 10, the substitution of Eq. 26 into Eq. 11 results in a system of equations governing F_-, F_+ and G, similar to the system (Eqs. 21). By defining $\vec{\Omega}$ and $\vec{A}$ as

$$\vec{\Omega} \triangleq \left(-\frac{e}{M}\vec{B}_o\right) \text{ and } \vec{A} \triangleq \left(\frac{-e}{M}\vec{E}_o\right) , \tag{27}$$

F_-, F_+ and G can be expressed as

$$\begin{aligned} F_- &= K_{11}E_- + K_{12}E_+ + K_{13}E_{1z} , \\ F_+ &= K_{21}E_- + K_{22}E_+ + K_{23}E_{1z} , \\ G &= K_{31}E_- + K_{32}E_+ + K_{33}E_{1z} , \end{aligned} \tag{28}$$

where for Case 1

$$K_{11} = \frac{-\frac{e}{M}M_-(F_o)}{j(b+\Omega_z)} , \quad K_{12} = 0 , \quad K_{13} = \frac{\frac{e}{M}A_-\frac{\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b+\Omega_z)} ,$$

$$K_{21} = 0 , \quad K_{22} = \frac{-\frac{e}{M}M_+(F_o)}{j(b-\Omega_z)} , \quad K_{23} = \frac{\frac{e}{M}A_+\frac{\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b-\Omega_z)} ,$$

$$K_{31} = \frac{2\frac{e}{M}\frac{A_+}{v_\perp}M_-(F_o)}{b(b+\Omega_z)} , \quad K_{32} = \frac{2\frac{e}{M}\frac{A_-}{v_\perp}M_+(F_o)}{b(b-\Omega_z)} ,$$

$$K_{33} = \frac{-\frac{e}{M}\frac{\partial F_o}{\partial v_z}}{jb} - j\frac{4A_+A_-}{v_\perp}\frac{\frac{e}{M}\frac{\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b^2-\Omega_z^2)} , \tag{29}$$

and for Case 2

$$K_{11} = -\frac{e}{M}\frac{M_o(F_o)}{\Delta}\left[\frac{2\Omega_+\Omega_-\omega}{b} - b(b-\Omega_z)\right] ,$$

$$K_{12} = -\frac{e}{M}\frac{M_o(F_o)}{\Delta}\left(\frac{2\Omega_-^2\omega}{b}\right) ,$$

$$K_{13} = \frac{e}{M}\frac{1}{\Delta}\left[D\left(\frac{\partial F_o}{\partial v_z}\right) + \frac{kv_\perp}{b}\frac{\partial F_o}{\partial v_z}\right][\Omega_-(b-\Omega_z)] ,$$

$$K_{21} = -\frac{e}{M}\frac{M_o(F_o)}{\Delta}\left(\frac{2\Omega_+^2\omega}{b}\right) ,$$

$$K_{22} = -\frac{e}{M}\frac{M_o(F_o)}{\Delta}\left[\frac{2\Omega_+\Omega_-\omega}{b} - b(b+\Omega_z)\right] ,$$

$$K_{23} = -\frac{e}{M}\frac{1}{\Delta}\left[D\left(\frac{\partial F_o}{\partial v_z}\right) + \frac{kv_\perp}{b}\frac{\partial F_o}{\partial v_z}\right][\Omega_+(b+\Omega_z)] ,$$

$$K_{31} = \frac{e}{M}\frac{M_o(F_o)}{\Delta}\left[2\Omega_+\left(\frac{v_z}{v_\perp}\right)(b-\Omega_z)\right] ,$$

$$K_{32} = \frac{-e}{M}\frac{M_o(F_o)}{\Delta}\left[2\Omega_-\left(\frac{v_z}{v_\perp}\right)(b+\Omega_z)\right] ,$$

$$K_{33} = \frac{-e}{M}\frac{1}{\Delta}\frac{\partial F_o}{\partial v_z}(\Omega_z^2 - b^2 + 4\Omega_+\Omega_-) + \frac{e}{M}\frac{1}{\Delta}D\left(\frac{\partial F_o}{\partial v_z}\right)\left[4\left(\frac{v_z}{v_\perp}\right)\Omega_+\Omega_-\right] ,$$

$$M_o(F_o) = \left[\frac{\partial F_o}{\partial v_\perp} + \frac{k}{\omega}D(f_o)\right] , \qquad (30)$$

where $\Delta \triangleq [jb(\Omega_z^2-b^2) + j4\Omega_+\Omega_-\omega]$,

$$\Omega_\pm \triangleq \frac{1}{2}(\Omega_x \pm j\Omega_y), \text{ and}$$

$$A_\pm \triangleq \frac{1}{2}(A_x \pm jA_y).$$

Since the time-dependent distribution function is now explicitly expressed in terms of the time-varying electric field, the convection current density $\vec{J}_1$ and the space-charge density ρ_1 can be expressed in terms of the electric field with the aid of Eqs. 7g and 7h, respectively. On the other hand, the electric field is related to the current density by Eqs. 15. Consequently the electric field can be written as

$$2\left(\frac{\omega^2}{c^2} - k^2\right)E_\pm = j\omega\mu_o e \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} e^{\pm j\varphi}(F_1 - f_1)v_\perp^2 d\varphi dv_\perp dv_z \qquad (31a)$$

and

$$E_{1z} = \frac{je}{\omega\epsilon_o}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}(F_1 - f_1)v_\perp v_z d\varphi dv_\perp dv_z \quad . \qquad (31b)$$

Upon substituting F_1 and f_1 given by Eqs. 26 and 20 respectively into Eqs. 31 the following set of equations is obtained:

$$E_- = R_{11}E_- + R_{12}E_+ + R_{13}E_{1z} \quad ,$$

$$E_+ = R_{21}E_- + R_{22}E_+ + R_{23}E_{1z} \quad ,$$

$$E_{1z} = R_{31}E_- + R_{32}E_+ + R_{33}E_{1z} \quad , \qquad (32)$$

where

$$R_{p,q} = P(S_{p,q}) \;;\; p = 1, 2 \;;\; q = 1, 2, 3$$

$$= Q(S_{p,q}) \;;\; p = 3 \;;\; q = 1, 2, 3 \quad , \qquad (33)$$

in which the integration operators P and Q are defined as

$$P(S) \equiv \frac{j\left(\frac{\omega e}{\epsilon_o}\right)}{2(\omega^2-c^2k^2)} \int_{-\infty}^{\infty}\int_{0}^{\infty} S(v_\perp,v_z)v_\perp^2 dv_\perp dv_z \quad ,$$

$$Q(S) \equiv \frac{je}{\omega\epsilon_o} \int_{-\infty}^{\infty}\int_{0}^{\infty} S(v_\perp,v_z)v_\perp v_z dv_\perp dv_z \qquad (34)$$

and

$$S_{11} = \int_0^{2\pi} \left[(K_{11}-k_{11}) + (K_{21}-k_{21})e^{-j2\varphi} + (K_{31}-k_{31})e^{-j\varphi}\right] d\varphi \quad ,$$

$$S_{12} = \int_0^{2\pi} \left[(K_{12}-k_{12}) + (K_{22}-k_{22})e^{-j2\varphi} + (K_{32}-k_{32})e^{-j\varphi}\right] d\varphi \quad ,$$

$$S_{13} = \int_0^{2\pi} \left[(K_{13}-k_{13}) + (K_{23}-k_{23})e^{-j2\varphi} + (K_{33}-k_{33})e^{-j\varphi}\right] d\varphi \quad ,$$

$$S_{21} = \int_0^{2\pi} \left[(K_{11}-k_{11})e^{j2\varphi} + (K_{21}-k_{21}) + (K_{31}-k_{31})e^{j\varphi}\right] d\varphi \quad ,$$

$$S_{22} = \int_0^{2\pi} \left[(K_{12}-k_{12})e^{j2\varphi} + (K_{22}-k_{22}) + (K_{32}-k_{32})e^{j\varphi}\right] d\varphi \quad ,$$

$$S_{23} = \int_0^{2\pi} \left[(K_{13}-k_{13})e^{j2\varphi} + (K_{23}-k_{23}) + (K_{33}-k_{33})e^{j\varphi}\right] d\varphi \quad ,$$

$$S_{31} = \int_0^{2\pi} \left[(K_{11}-k_{11})e^{j\varphi} + (K_{21}-k_{21})e^{-j\varphi} + (K_{31}-k_{31})\right] d\varphi \quad ,$$

$$S_{32} = \int_0^{2\pi} \left[(K_{12}-k_{12})e^{j\varphi} + (K_{22}-k_{22})e^{-j\varphi} + (K_{32}-k_{32})\right] d\varphi \quad ,$$

$$S_{33} = \int_0^{2\pi} \left[(K_{13}-k_{13})e^{j\varphi} + (K_{23}-k_{23})e^{-j\varphi} + (K_{33}-k_{33})\right] d\varphi \quad . \qquad (35)$$

Therefore the dispersion relationship for the system under consideration is given, from Eqs. 32, as

$$d(\omega,k) = \begin{vmatrix} (R_{11}-1) & R_{12} & R_{13} \\ R_{21} & (R_{22}-1) & R_{23} \\ R_{31} & R_{32} & (R_{33}-1) \end{vmatrix} = 0 \ . \qquad (36)$$

It should be observed that once the time-independent distribution functions f_o and F_o are known, the parameters $k_{p,q}$ and $K_{p,q}$ are specified so that the $R_{p,q}$ integrals can be evaluated. Then a detailed study of dispersion relation (36) can be made to obtain the propagation characteristic of waves in the system.

Before considering the time-independent distribution functions, it is of interest to observe that for Case 2 the parameters $K_{p,q}$ and $k_{p,q}$ are independent of φ, as shown in Appendix A, and Eqs. 35 are reduced to

$$S_{p,q} = 2\pi(K_{p,q}-k_{p,q}) \ ; \ p = 1, 2, 3; \ q = 1, 2, 3 \ . \qquad (37)$$

Furthermore, if $B_{ox} = B_{oy} = 0$, i.e., $\omega_{\pm} = \Omega_{\pm} = 0$, then $S_{p,q} = 0$ for $p \neq q$, which implies that $R_{p,q} = 0$ for $p \neq q$. In other words, the off-diagonal elements of the determinant in dispersion relationship (36) vanish, so that Eq. 36 gives

$$(W_{11}-1)\ (W_{22}-1)\ (W_{33}-1) = 0 \ , \qquad (38)$$

where $W_{11} = R_{11}$, $W_{22} = R_{22}$ and $W_{33} = R_{33}$ for the case $\omega_{\pm} = \Omega_{\pm} = 0$. Equation 38 implies that

$$W_{11} = 1 \ , \ W_{22} = 1 \ \text{and} \ W_{33} = 1 \ , \qquad (39)$$

which represent the dispersion relationships for the left-hand and right-hand circularly polarized modes, and the longitudinal mode respectively:

$$1 + \frac{\pi\left(\frac{\omega\epsilon}{\epsilon_0}\right)}{(\omega^2-c^2k^2)} \int_{-\infty}^{\infty}\int_{0}^{\infty} \left[\frac{\frac{e}{M} M_0(F_0)}{b+\Omega_z} + \frac{\frac{e}{m} M_0(f_0)}{b+\omega_z} \right] v_\perp^2 dv_\perp dv_z = 0 \ , \qquad (40a)$$

$$1 + \frac{\pi\left(\frac{\omega\epsilon}{\epsilon_0}\right)}{(\omega^2-c^2k^2)} \int_{-\infty}^{\infty}\int_{0}^{\infty} \left[\frac{\frac{e}{M} M_0(F_0)}{b-\Omega_z} + \frac{\frac{e}{m} M_0(f_0)}{b-\omega_z} \right] v_\perp^2 dv_\perp dv_z = 0 \qquad (40b)$$

and

$$1 + \frac{2\pi e}{\omega\epsilon_0} \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{1}{b}\left(\frac{e}{M}\frac{\partial F_0}{\partial v_z} + \frac{e}{m}\frac{\partial f_0}{\partial v_z}\right) v_\perp v_z dv_\perp dv_z = 0 \ , \qquad (40c)$$

in which Eqs. 40a and 40b are the same as those given by Montgomery and Tidman[4].

IV. TIME-INDEPENDENT DISTRIBUTION FUNCTIONS

The time-independent distribution functions f_0 and F_0 must satisfy Eqs. 6a and 6b respectively. It is not difficult to show that the solution of Eq. 6a has the form

$$f_0(\vec{r},\vec{v}) = \overline{f_0}(w) \ , \qquad (41)$$

where $w = (1/2)m|\vec{v}|^2 - e\Phi(\vec{r})$, in which the electric scalar potential $\Phi(\vec{r})$ is related to the electrostatic field $\vec{E}_0(\vec{r})$ by

$$\vec{E}_0 = -\nabla\Phi \ . \qquad (42)$$

Similarly the solution of Eq. 6b has the form

$$F_0(\vec{r},\vec{v}) = \overline{F_0}(W) \ ,$$

where

$$W = \frac{1}{2} M|\vec{v}|^2 + e\Phi(r) \quad . \tag{43}$$

It should be noted that the electrostatic field $\vec{E}_o$, appearing in Eqs. 6a and 6b, in general consists of two parts; $\vec{E}_o = \vec{E}_s + \vec{E}_a$, where $\vec{E}_a$ is the externally applied static electric field and $\vec{E}_s$ is the space-charge field which must also satisfy Eq. 6e.

For a one-dimensional analysis in a Maxwellian plasma f_o and F_o can be written as

$$f_o = n_o \left(\frac{\alpha_e}{\pi}\right)^{3/2} \exp(-\alpha_e w_e) \tag{44}$$

and

$$F_o = N_o \left(\frac{\alpha_i}{\pi}\right)^{3/2} \exp(-\alpha_i w_i) \quad , \tag{45}$$

where

$$w_e \triangleq (v_\perp^2+v_z^2) - \frac{2e}{m} \Phi(z) \quad ,$$

$$w_i \triangleq (v_\perp^2+v_z^2) + \frac{2e}{M} \Phi(z) \quad ,$$

$$\alpha_e \triangleq \frac{m}{2KT_e} \quad , \quad \alpha_i \triangleq \frac{M}{2KT_i} \quad , \tag{46}$$

in which K is the Boltzmann constant, n_o and T_e are the concentration and the temperature of the electron respectively, and N_o and T_i are the concentration and the temperature of the ion respectively. In view of the fact that both f_o and F_o are expressed as even functions of v_x, v_y and v_z in Eqs. 46, Eq. 6g gives $J_{ox} = J_{oy} = J_{oz} = 0$. Then from Eqs. 9, the magnetostatic field must be constant, i.e., B_{ox}, B_{oy} and B_{oz} are all independent of z.

On the other hand, Eq. 6h gives

$$\rho_o(z) = eN_o\, e^{-\frac{e\Phi(z)}{KT_i}} - en_o\, e^{\frac{e\Phi(z)}{KT_e}} \quad . \tag{47}$$

If electrostatic fields E_{ox} and E_{oy} are constant, then E_{oz} can be determined from Eqs. 8 and 47. For the one-dimensional analysis under consideration the x- and y-components of the space-charge field E_s are absent. For the two cases under consideration, the assumption $a_z = 0$ implies that $E_{sz} = 0$ which will be the case if $(\partial\Phi/\partial z) = 0$. In other words, Φ is independent of z, which is equivalent to requiring that f_o and F_o be independent of z and the plasma under consideration be homogeneous. If the space-charge potential $\Phi(z)$ is set equal to zero, Eq. 47 suggests that $\rho_o = e(N_o - n_o)$ and since E_{os} must be zero, ρ_o is zero. Consequently $N_o = n_o$ when the condition of electrical neutrality is met.

It is of interest to note that for a homogeneous plasma pervaded by a uniform static electric field $\vec{E}_a$ and magnetic field $\vec{B}_o$, Eq. 6a becomes

$$(\vec{E}_a + \vec{v} \times \vec{B}_o)\cdot\nabla_v f_o = 0 \quad , \tag{48}$$

and f_o can be given in the form

$$f_o = n_o \left(\frac{\alpha_e}{\pi}\right)^{3/2} \exp(-\alpha_e|\vec{v}-\vec{u}|^2) \quad , \tag{49}$$

where the drift velocity $\vec{u}$ is given by

$$\vec{u} = \frac{(\vec{E}_a \times \vec{B}_o)}{|\vec{B}_o|^2} \quad . \tag{50}$$

Since the drift velocity depends neither on the ratio e/m nor on the initial velocities, it is the same for ions and electrons regardless of their energy. Crossed magnetic and electric fields produce a collective displacement of all of the electrical charges in the direction of $\vec{E}_a \times \vec{B}_o$. Thus F_o can be also given by

$$F_o = N_o \left(\frac{\alpha_i}{\pi}\right)^{3/2} \exp\left(-\alpha_i |\vec{v}-\vec{u}|^2\right) \quad . \tag{51}$$

It should also be noted that the distribution functions f_o and F_o given by Eqs. 44 and 45, respectively, are adequate for the study of the case where there is no externally applied electrostatic field; since $\nabla_v f_o = -2\alpha_e f_o \vec{v}$ and $(\partial\Phi/\partial z) = -E_s$, it can easily be shown that f_o, given by Eq. 44, indeed satisfies Eq. 6a. However, for the case where the externally applied electrostatic field is present, f_o, given by Eq. 44, is not adequate since it does not satisfy Eq. 6a and must be modified. Suppose that $\vec{E}_o = \vec{i}E_{ax} + \vec{j}E_{ay} + \vec{k}E_s$ and $\vec{B}_o = \vec{k}B_o$ are considered, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along the coordinate axes respectively, E_{az} and E_{ay} are the components of $\vec{E}_a$, and E_s is the space-charge field. Then it is not difficult to show that the following form of f_o satisfies Eq. 6a:

$$f_o = n_o \left(\frac{\alpha_e}{\pi}\right)^{3/2} \exp\left\{-\alpha_e \left[(v_x-u_x)^2 + (v_y-u_y)^2 + v_z^2\right] + 2\alpha \frac{e}{m} \Phi(z)\right\} \quad , \tag{52}$$

where $\vec{u} = (\vec{i}u_x + \vec{j}u_y)$ is the drift velocity as defined by Eq. 50.

For the consideration of the case where interpenetrating plasmas such as electrons drifting through ions to form the configuration of a plasma carrying a current along the lines of force, the drift velocity along the direction of static magnetic field must be taken into account. If this drift velocity u_o is much greater than the transverse drift velocity due to the transverse electrostatic field, which is the case for weak static fields, then w_e, associated with f_o of Eq. 44, can be expressed as

$$w_e = [v_\perp^2 + (v_z - u_o)^2] - \frac{2e}{m}\Phi(z) \quad . \tag{53}$$

Thus the time-independent distribution functions f_o and F_o must be properly chosen according to the type of problem under consideration.

V. MAXWELLIAN PLASMA

The two cases defined in Section III are examined for a homogeneous plasma. As an illustration of the method of analysis a homogeneous Maxwellian plasma is considered in this section. f_o and F_o can be written as

$$f_o = n_o \left(\frac{\alpha_e}{\pi}\right)^{3/2} \exp\,(-\alpha_e w_e) \tag{54}$$

and

$$F_o = N_o \left(\frac{\alpha_i}{\pi}\right)^{3/2} \exp\,(-\alpha_i w_i) \quad . \tag{55}$$

For Case 1:

$$w_e = [v_\perp^2 + (v_z - u_{oe})^2] \quad ; \quad w_i = [v_\perp^2 + (v_z - u_{oi})^2] \quad . \tag{56}$$

For Case 2:

$$w_e = (v_\perp^2 + v_z^2) = w_i \quad . \tag{57}$$

Having specified the form of f_o and F_o, the coefficients $K_{p,q}$ and $k_{p,q}$ in Eqs. 24, 25, 29 and 30 now can be determined. For the forms of f_o and F_o given by Eqs. 54 and 55, these coefficients are independent of φ, and the evaluation of the $R_{p,q}$ integrals can be carried out. Thus dispersion relationship (36) gives, for Case 1 and Case 2 respectively, (see Appendix A for details):

For Case 1:

$$D_+D_- \left\{D_z + 4\mu^2 \left[\left(\gamma_+ - \frac{\alpha_+\beta_+}{D_+}\right) + \left(\gamma_- - \frac{\alpha_-\beta_-}{D_-}\right) \right]\right\} = 0 \ , \qquad (58)$$

where $D_\pm \triangleq [R_1 Z_o G_o(U_\pm) + r_1 z_o g_o(u_\pm) - 1]$,

$$D_z \triangleq (d_z - 1) \ ,$$

$$d_z \triangleq R_2 \left[U_o(U_o - u_{oi})G_o(U_o) + j\frac{U_o}{\sqrt{\alpha_i}}\right] + r_2 \left[U_o(U_o - u_{oe})g_o(U_o) + j\frac{U_o}{\sqrt{\alpha_e}}\right] ,$$

$$\alpha_\pm \triangleq \mp R_2 Z_o [U_\pm G_o(U_\pm) - U_o G_o(U_o)] \mp r_2 z_o [u_\pm g_o(u_\pm) - U_o g_o(U_o)] \ ,$$

$$\beta_\pm \triangleq \mp R_1 \alpha_i [U_\pm - u_{oi})G_o(U_\pm) - (U_o - u_{oi})G_o(U_o)] \mp r_1 \alpha_e [(u_\pm - u_{oe}) \cdot g_o(u_\pm) - (U_o - u_{oe})g_o(U_o)] \ ,$$

$$\gamma_\pm \triangleq R_2 \alpha_i \left[U_\pm(U_\pm - u_{oi})G_o(U_\pm) - U_o(U_o - u_{oi})G_o(U_o) \pm \frac{j\Omega_z}{\sqrt{\alpha_i}\,k}\right] + r_2 \alpha_e \left[u_\pm(u_\pm - u_{oe})g_o(u_\pm) - U_o(U_o - u_{oe})g_o(U_o) \pm \frac{j\omega_z}{\sqrt{\alpha_e}\,k}\right] ,$$

$$\mu^2 \triangleq \frac{1}{4}\left(\frac{E_{ox}^2 + E_{oy}^2}{B_{oz}^2}\right) ,$$

$$G_o(Y) \triangleq \frac{j}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\alpha_i(v_z - u_{oi})^2}}{(v_z - Y)}\, dv_z \ ,$$

$$g_o(Y) \triangleq \frac{j}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\alpha_e(v_z - u_{oe})^2}}{(v_z - Y)}\, dv_z \ . \qquad (59)$$

For Case 2: $4\nu^2 \triangleq [(B_{ox}^2+B_{oy}^2)/B_{oz}^2] \ll 1$:

$$D_+D_-D_z = \nu^2\omega \{D_z(D_-+D_+)(M_--M_+) + 2N_+ [D_-M_++\nu^2\omega(M_+^2-M_-^2)] - 2N_-[D_+M_- + \nu^2\omega(M_-^2-M_+^2)]\} , \quad (60)$$

where $D_\pm \triangleq [R_1G_o(U_\pm) - r_1g_o(u_\pm) - 1]$,

$$D_z \triangleq \left[R_2\left(U_o^2G_o(U_o) + j\frac{U_o}{\sqrt{\alpha_i}}\right) + r_2\left(U_o^2g_o(U_o) + j\frac{U_o}{\sqrt{\alpha_e}}\right) - 1\right] ,$$

$$N_\pm \triangleq \mp R_2\left[U_\pm^2G_o(U_\pm) - U_o^2G_o(U_o) \pm j\frac{1}{\sqrt{\alpha_i}}\frac{\Omega_z}{k}\right] \mp r_2\left[u_\pm^2g_o(u_\pm) - U_o^2g_o(U_o) \pm j\frac{1}{\sqrt{\alpha_e}}\frac{\omega_z}{k}\right] ,$$

$$M_\pm \triangleq \frac{R_1}{\Omega_z}[G_o(U_\pm) - G_o(U_o) \pm \Lambda] + \frac{r_1}{\omega_z}[g_o(u_\pm) - g_o(U_o) \pm \lambda] ,$$

$$\Lambda \triangleq \frac{\Omega_z}{j\sqrt{\pi}\,k}\int_{-\infty}^{\infty}\frac{e^{-\alpha_i v_z^2}}{(v_z-U_o)^2}\,dv_z ,$$

$$\lambda \triangleq \frac{\omega_z}{j\sqrt{\pi}\,k}\int_{-\infty}^{\infty}\frac{e^{-\alpha_e v_z^2}}{(v_z-U_o)^2}\,dv_z ,$$

$$G_o(Y) \triangleq \frac{j}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\alpha_i v_z^2}}{(v_z-Y)}\,dv_z ,$$

$$g_o(Y) \triangleq \frac{j}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\alpha_e v_z^2}}{(v_z-Y)}\,dv_z .$$

The R's and U's in Eqs. 58 and 59 are defined as

$$R_1 \triangleq j\sqrt{\alpha_i}\,\frac{\Omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{\omega}{k}\right) \quad , \quad R_2 \triangleq j2\sqrt{\alpha_i}\,\alpha_i\left(\frac{\Omega_p^2}{\omega^2}\right)\left(\frac{\omega}{k}\right) \quad ,$$

$$r_1 \triangleq j\sqrt{\alpha_e}\,\frac{\omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{\omega}{k}\right) \quad , \quad r_2 \triangleq j2\sqrt{\alpha_e}\,\alpha_e\left(\frac{\omega_p^2}{\omega^2}\right)\left(\frac{\omega}{k}\right) \quad ,$$

$$U_\pm = \left(\frac{\omega\pm\Omega_z}{k}\right) \quad , \quad U_o = \left(\frac{\omega}{k}\right) \quad , \quad u_\pm = \left(\frac{\omega\pm\omega_z}{k}\right) \quad ,$$

$$Z_o = \left(1 - \frac{u_{oi}}{U_o}\right) \quad , \quad z_o = \left(1 - \frac{u_{oe}}{U_o}\right) \quad .$$

It should be noted that integrals $G_o(Y)$ and $g_o(Y)$ defined in Eqs. 59 have been discussed in detail by Stix[3] and his results can be applied in the present investigation. Y, appearing in Eqs. 59, may be complex in general and takes the values $(\omega\pm\Omega_z/k)$, $(\omega\pm\omega_z/k)$ and (ω/k). Let

$$\zeta^2 = \alpha_i(v_z-u_{oi})^2 \quad . \tag{61}$$

G_o may be written, for $Im(\omega) < 0$, as follows:

$$G_o = \frac{j}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\zeta^2}}{\zeta-\alpha_n}\,d\zeta \quad , \tag{62}$$

where

$$\alpha_n = \sqrt{\alpha_i}\,(Y-u_{oi}) \quad .$$

The contour of integration may be deformed and analytic continuation used to evaluate this integral in such a way that it is valid over the entire ω-plane:

$$G_o = \frac{j}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{e^{-\zeta^2}}{\zeta - \alpha_n} d\zeta + \frac{\sqrt{\pi}\, k}{|k|} \exp(-\alpha_n^2) \quad , \tag{63}$$

where the principal value integration is to be carried through the pole at α_n.

On the other hand, when $S(\xi)$ is written as

$$S(\xi) = \frac{1}{2\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{e^{-\tau^2}}{\tau + \xi} d\tau \; , \tag{64}$$

the asymptotic expansion of $S(\xi)$ exhibits a Stokes phenomenon; that is, different asymptotic expansions are required for validity in different portions of the ξ-plane. The Stokes phenomenon is a characteristic of the asymptotic expansion of analytic functions. For the expansion of $S(\xi)$,

$$S(\xi) = T(\xi) + U(\xi) \quad , \tag{65}$$

where

$$T(\xi) = \frac{1}{2\xi} + \frac{1}{2^2\xi^3} + \frac{1\cdot 3}{2^3\xi^5} + \frac{1\cdot 3\cdot 5}{2^4\xi^7} + \ldots$$

$$U(\xi) = 0 \quad , \quad \text{for} \quad |\mathrm{Re}\ \xi| > |\mathrm{Im}\ \xi| \quad ,$$

$$= \frac{-j\sqrt{\pi}}{k} e^{-\xi^2} \operatorname{sgn}(\mathrm{Im}\ \xi) \quad , \quad \text{for} \quad |\mathrm{Re}\ \xi| < |\mathrm{Im}\ \xi| \quad . \tag{66}$$

It should be noted that in Eq. 63, the Gaussian term in G_o diverges whenever $|\mathrm{Re}\ \alpha_n| < |\mathrm{Im}\ \alpha_n|$. However, relation (62) for G_o shows that G_o, in fact, converges to zero as $|\mathrm{Im}\ \alpha_n|^{-1}$ in the unstable half-plane ($\mathrm{Im}\ \omega < 0$). It is the $U(\xi)$ term in $S(\xi)$ which reconciles this apparent difference. The entire result is best summarized with the aid of a quadrant diagram for the α_n (sgn k) plane (see Fig. 1).

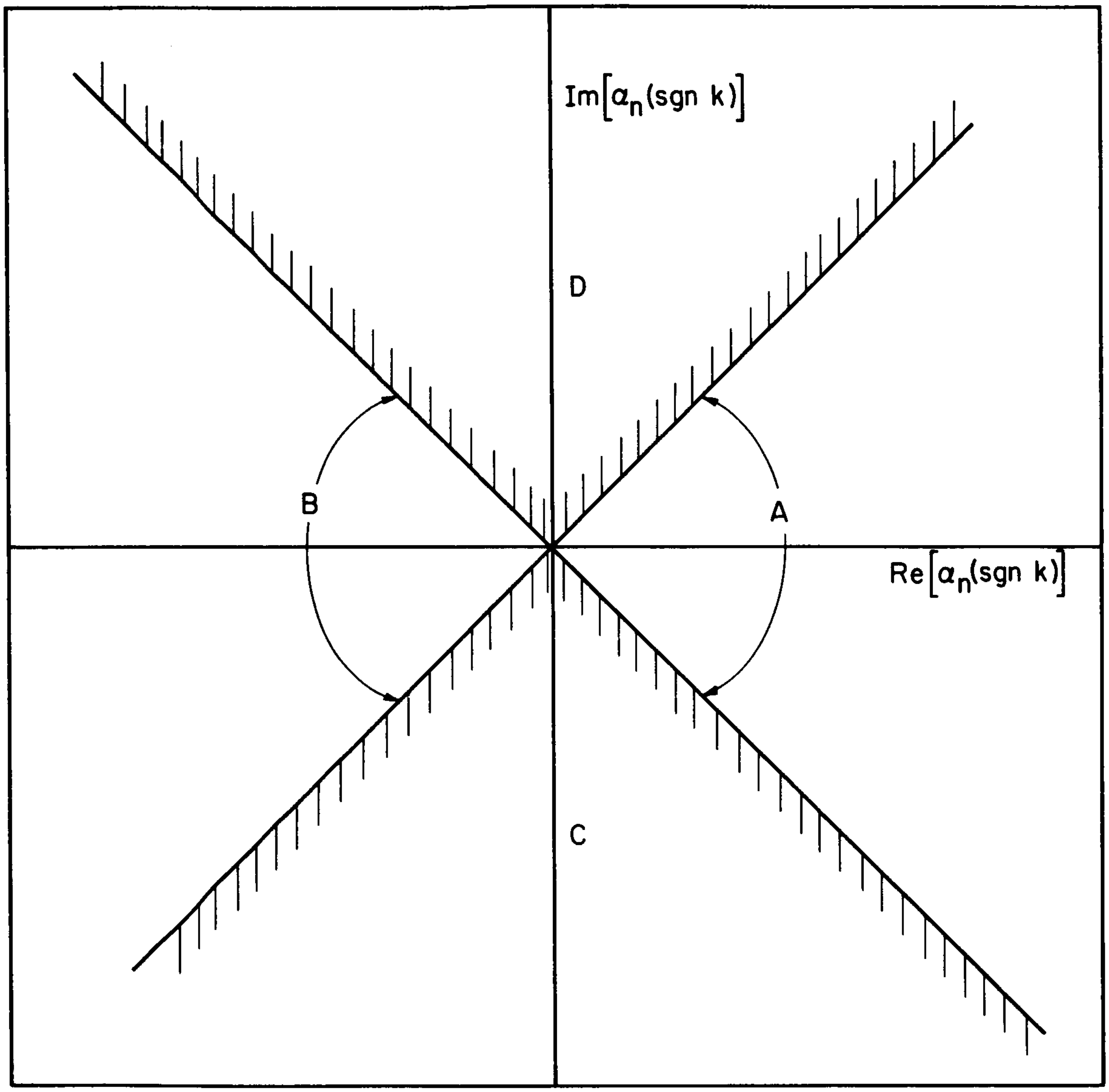

FIG. 1 QUADRANT DIAGRAM FOR THE α_n (sgn k) PLANE.

The following asymptotic expansions for G_o are appropriate in the quadrants indicated:

Quadrants A and B:

$$G_o = \frac{\sqrt{\pi}\, k}{|k|} \exp(-\alpha_n^2) - j2T(\alpha_n) \quad ,$$

Quadrant D:

$$G_o = \frac{2\sqrt{\pi}\, k}{|k|} \exp(-\alpha_n^2) - j2T(\alpha_n) \quad ,$$

Quadrant C:

$$G_o = -j2T(\alpha_n) \quad . \tag{67}$$

In Quadrant D, G_o diverges as $\alpha_n \rightarrow \infty$. Equation 63 shows that the Gaussian term may be neglected in Quadrants A and B because it is small for large values of α_n, and it is noted that this term is rigorously absent in Quadrant C.

If G_o is expanded in the first few terms of its asymptotic expansion in Quadrants A, B and C, i.e.,

$$G_o(\alpha_n) \simeq \left[\frac{\sqrt{\pi}\, k}{|k|} \exp(-\alpha_n^2) - \frac{j}{\alpha_n} - \frac{j}{2\alpha_n^3} \right] \quad , \tag{68}$$

then various factors appearing in dispersion equations (58) and (60) can be determined and, in principle, a detailed study of the propagation characteristics of various waves can be made. It should be noted that in Eq. 68 the three terms on the right-hand side represent, respectively, the Landau or cyclotron damping term, the cold plasma, and that due to a finite thermal spread.

VI. SPECIAL CASES

Suppose that the difference between the phase velocities of various modes in the system and the drift velocity are large in comparison to the electron thermal velocity ($1/\sqrt{\alpha_e}$), which is also greater than the ion thermal velocity ($1/\sqrt{\alpha_i}$), so that $G_o(Y)$ and $g_o(Y)$ may be approximately written as

$$G_o(Y) \simeq -j \frac{1}{\sqrt{\alpha_i}\,(Y-u_{oi})}\left[1+\frac{1}{2\alpha_i(Y-u_{oi})^2}\right] ,$$

$$g_o(Y) \simeq -j \frac{1}{\sqrt{\alpha_e}\,(Y-u_{oe})}\left[1+\frac{1}{2\alpha_e(Y-u_{oe})^2}\right] . \qquad (69)$$

Then the various factors appearing in Eq. 58 can be written as follows:

$$D_\pm = \left[\frac{\Omega_p^2(\omega-ku_{oi})}{(\omega^2-c^2k^2)(\omega\pm\Omega_z-ku_{oi})}\left(1+\frac{1}{2\alpha_i(U_\pm-u_{oi})^2}\right)\right.$$

$$\left.+\frac{\omega_p^2(\omega-ku_{oe})}{(\omega^2-c^2k^2)(\omega\pm\omega_z-ku_{oe})}\left(1+\frac{1}{2\alpha_e(u_\pm-u_{oe})^2}\right)-1\right] , \qquad (70a)$$

$$D_z = \left[\frac{\Omega_p^2}{(\omega-ku_{oi})^2}+\frac{\omega_p^2}{(\omega-ku_{oe})^2}-1\right] , \qquad (70b)$$

$$\alpha_{\pm} = \mp 2\alpha_i \left(\frac{\Omega_p^2}{\omega^2}\right)\left[\frac{(U_o-u_{oi})}{(U_{\pm}-u_{oi})} U_{\pm}\left(1+\frac{1}{2\alpha_i(U_{\pm}-u_{oi})^2}\right)\right.$$

$$\left. - U_o\left(1+\frac{1}{2\alpha_i(U_o-u_{oi})^2}\right)\right] \mp 2\alpha_e\left(\frac{\omega_p^2}{\omega^2}\right)$$

$$\cdot\left[\frac{(U_o-u_{oe})}{(u_{\pm}-u_{oe})} u_{\pm}\left(1+\frac{1}{2\alpha_e(u_{\pm}-u_{oe})^2}\right) - U_o\left(1+\frac{1}{2\alpha_e(U_o-u_{oe})^2}\right)\right] ,$$

(70c)

$$\beta_{\pm} = \mp \frac{\Omega_p^2 U_o}{2(\omega^2-c^2k^2)}\left(\frac{1}{(U_{\pm}-u_{oi})^2} - \frac{1}{(U_o-u_{oi})^2}\right)$$

$$\mp \frac{\omega_p^2 U_o}{2(\omega^2-c^2k^2)}\left(\frac{1}{(u_{\pm}-u_{oe})^2} - \frac{1}{(U_o-u_{oe})^2}\right) ,$$

(70d)

$$\gamma_{\pm} = \alpha_i\left(\frac{\Omega_p}{\omega}\right)^2 U_o\left[\frac{U_{\pm}}{(U_{\pm}-u_{oi})^2} - \frac{U_o}{(U_o-u_{oi})^2}\right]$$

$$+ \alpha_e\left(\frac{\omega_p}{\omega}\right)^2 U_o\left[\frac{u_{\pm}}{(u_{\pm}-u_{oe})^2} - \frac{U_o}{(U_o-u_{oe})^2}\right] .$$

(70e)

It should be noted that when the static transverse electric field is absent, $\mu = 0$ and Eq. 58 becomes $(D_-D_+D_z) = 0$. Then the dispersion equation for the uncoupled longitudinal mode is given by $D_z = 0$, i.e.,

$$\frac{\Omega_p^2}{(\omega-ku_{oi})^2} + \frac{\omega_p^2}{(\omega-ku_{oe})^2} = 1 , \qquad (71)$$

which is the familiar expression for a two-stream system.

Similarly the various factors appearing in Eq. 60 can be written as

$$D_\pm = \left[\frac{\Omega_p^2 \omega}{(\omega^2 - c^2k^2)(\omega \pm \Omega_z)} \left(1 + \frac{1}{2\alpha_i U_\pm^2}\right) + \frac{\omega_p^2 \omega \left(1 + \frac{1}{2\alpha_e U_\pm^2}\right)}{(\omega^2 - c^2k^2)(\omega \pm \omega_z)} - 1 \right] ,$$

$$D_z = \left(\frac{\Omega_p^2}{\omega^2} + \frac{\omega_p^2}{\omega^2} - 1 \right) ,$$

$$N_\pm = \pm \left(\frac{\Omega_p}{\omega}\right)^2 \left(1 - \frac{U_o}{U_\pm}\right) \pm \left(\frac{\omega_p}{\omega}\right)^2 \left(1 - \frac{U_o}{u_\pm}\right) ,$$

$$M_\pm = \frac{\Omega_o^2 U_o}{\Omega_z(\omega^2 - c^2k^2)} \left[\frac{1}{U_\pm}\left(1 + \frac{1}{2\alpha_i U_\pm^2}\right) - \frac{1}{U_o}\left(1 + \frac{1}{2\alpha_i U_o^2}\right) \pm \left(\frac{\Omega_z}{k}\right)\frac{1}{U_o^2} \right]$$

$$+ \frac{\omega_p^2 U_o}{\omega_z(\omega^2 - c^2k^2)} \left[\frac{1}{u_\pm}\left(1 + \frac{1}{2\alpha_e u_\pm^2}\right) - \frac{1}{U_o}\left(1 + \frac{1}{2\alpha_e U_o^2}\right) \pm \left(\frac{\omega_z}{k}\right)\frac{1}{U_o^2} \right] . \tag{72}$$

It should be observed that when a transverse static magnetic field is absent, $\nu = 0$ and Eq. 60 becomes $(D_- D_+ D_z) = 0$. Then the dispersion equations for the uncoupled transverse modes are given by $D_\pm = 0$, i.e.,

$$1 = \frac{\Omega_p^2 \omega}{(\omega^2 - c^2k^2)(\omega \pm \Omega_z)} \left(1 + \frac{1}{2\alpha_i U_\pm^2}\right) + \frac{\omega_p^2 \omega}{(\omega^2 - c^2k^2)(\omega \pm \omega_z)} \left(1 + \frac{1}{2\alpha_e u_\pm^2}\right) . \tag{73}$$

Furthermore if $|\alpha_i v_\pm^2| \gg 1$, and $|\alpha_e u_\pm^2| \gg 1$, then Eq. 73 becomes

$$\frac{\Omega_p^2 \omega}{(\omega^2 - c^2k^2)(\omega \pm \Omega_z)} + \frac{\omega_p^2 \omega}{(\omega^2 - c^2k^2)(\omega \pm \omega_z)} = 1 , \tag{74}$$

which is a familiar expression in the cold-plasma theory. Equation 74 is that given by Denisse and Delcroix[2] and is simply the Appleton-Hartree equation of the magnetoionic theory. It should be noted from Eq. 17 and Eq. 27 that Ω_z is opposite in sign to ω_z, i.e., $\omega_z = [(e/m) B_{oz}]$ and $\Omega_z = [-(e/M) B_{oz}]$.

VII. CONCLUDING REMARKS

With the aid of the coupled Boltzmann-Vlasov-Maxwell equations, under a small-signal, one-dimensional analysis, the dispersion relation for a finite temperature, homogeneous, magnetoactive plasma has been derived. Equation 36 is applicable to the case of longitudinal propagation in the presence of a transverse static applied electric field, as well as to the case of oblique propagation in the absence of a static electric field. Once the time-independent parts of the distribution functions of constituent plasma particles and applied static electric and magnetic fields are known, the $R_{p,q}$ elements of the determinant in Eq. 36 are specified and the dispersion equation can be solved for the propagation constants.

Although various forms of the time-independent distribution functions may be considered and used in the evaluation of the elements of the determinants in Eq. 36, the present report considers a Maxwellian distribution function. For a homogeneous Maxwellian plasma the dispersion equations for Cases 1 and 2 are given by Eqs. 58 and 60, respectively.

It should be pointed out that the formulation of the dispersion relations in the form given by Eq. 36 has certain advantages since the various characteristic modes (i.e., the right-hand and left-hand circularly polarized transverse modes and the longitudinal modes) can easily be identified and their possible mutual coupling caused by the presence of static transverse electric and magnetic fields is clearly indicated. Furthermore, a detailed study of these derived dispersion relations should provide useful information with regard to: (a) the effect of transverse static electric or magnetic fields on the propagation characteristics of electromagnetic waves in a magnetoactive plasma, as well as on the polarization of the wave, and (b) the question of energy conversion between the modes (with the aid of Eqs. 32).

The dispersion relation given in Eq. 36 is particularly suitable for the study of the coupling of the longitudinal mode to the transverse modes due to the transverse static electric or magnetic field present in the system. A natural important question then arises as to how effective is this type of coupling. This question is being investigated presently and will be discussed in a future report. For example, by this type of coupling mechanism, the energy carried by a longitudinal plasma oscillation may be converted into the transverse electromagnetic wave energy in the solar corona, thus leading to the escape of solar radio noise from the solar corona.

There are also phenomena found in the earth's ionosphere, e.g., the cutoff, amplification, and Landau damping of a whistler propagation in the ionospheric plasma, which may be explained by this type of coupling mechanism. In addition, the triggering of VLF emissions by a whistler in the ionospheric plasma, recently observed by Helliwell[10], might also be explained. Finally this type of coupling mechanism may be at work in some laboratory devices involving the interaction of the transverse cyclotron wave and longitudinal space-charge waves.

APPENDIX A. DERIVATION OF VARIOUS EQUATIONS

A.1 Derivation of Eqs. 27 and 29

From Eq. 25, for Case 1 ($\omega_+ = \omega_- = 0$ and $a_z = 0$):

$$\frac{\partial f_-}{\partial v_\perp} = \frac{1}{v_\perp} f_- \quad \text{and} \quad \frac{\partial f_+}{\partial v_\perp} = \frac{1}{v_\perp} f_+ \ , \tag{A.1}$$

$$j(b+\omega_z)f_- - a_- \frac{\partial g}{\partial v_\perp} = \eta M_{o_-}(f_o)E_- \ ,$$

$$j(b-\omega_z)f_+ - a_+ \frac{\partial g}{\partial v_\perp} = \eta M_{o_+}(f_o)E_+ \ ,$$

$$jbg - \frac{2a_-}{v_\perp} f_+ - \frac{2a_+}{v_\perp} f_- = \eta \frac{\partial f_o}{\partial v_z} E_{1z} \ , \tag{A.2}$$

where $b \equiv (\omega - kv_z)$ and $\eta \equiv (e/m)$.

When the fact that $\partial/\partial v_\perp(f_+/v_\perp) = 0$ and $\partial/\partial v_\perp(f_-/v_\perp) = 0$ is used, with the aid of Eq. A.1, differentiation of Eq. A.2 with respect to $v_\perp$ gives $(\partial g/\partial v_\perp) = (\eta/jb)(\partial/\partial v_\perp)(\partial f_o/\partial v_z)E_{1z}$ so that Eq. A.2 can be written as

$$f_- = \frac{\eta M_{o_-}(f_o)}{j(b+\omega_z)} E_- - \frac{\eta a_- \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b+\omega_z)} E_{1z} ,$$

$$f_+ = \frac{\eta M_{o_+}(f_o)}{j(b-\omega_z)} E_+ - \frac{\eta a_+ \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b-\omega_z)} E_{1z} ,$$

$$g = \frac{-2\left(\frac{a_+}{v_\perp}\right)\eta M_{o_-}(f_o)}{b(b+\omega_z)} E_- - \frac{2\left(\frac{a_-}{v_\perp}\right)\eta M_{o_+}(f_o)}{b(b-\omega_z)} E_+$$

$$+ \left[j \frac{4a_+a_-}{v_\perp} \frac{\eta \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b^2-\omega_z^2)} + \frac{\eta \frac{\partial f_o}{\partial v_z}}{jb} \right] E_{1z} . \qquad (A.3)$$

For Case 2 ($a_+ = a_- = a_z = 0$), from Eqs. 25,

$$D(f_\pm) = -\frac{v_z}{v_\perp} f_\pm ,$$

$$j(b+\omega_z)f_- - j\omega_- D(g) = \eta M_o(f_o)E_- , \qquad (A.4)$$

$$j(b-\omega_z)f_+ + j\omega_+ D(g) = \eta M_o(f_o)E_+ \qquad (A.5)$$

and

$$jbg + j\omega_- \, 2\left(\frac{v_z}{v_\perp}\right) f_+ - j\omega_+ \, 2\left(\frac{v_z}{v_\perp}\right) f_- = \eta \frac{\partial f_o}{\partial v_z} E_{1z} . \qquad (A.6)$$

Using the fact that

$$D\left(\frac{v_z}{v_\perp} f_\pm\right) = f_\pm \quad \text{and} \quad D(bg) = -(kv_\perp)g + bD(g) , \qquad (A.7)$$

and then operating D on Eq. A.6 yields

$$D(g) = \left(\frac{kv_\perp}{b}\right) g + 2\left(\frac{\omega_+}{b}\right) f_- - 2\left(\frac{\omega_-}{b}\right) f_+ - j\,\frac{\eta}{b}\, D\left(\frac{\partial f_o}{\partial v_z}\right) E_{1z} \quad . \tag{A.8}$$

Substituting Eq. A.8 into Eqs. A.4 and A.5, and solving algebraically for f_-, f_+ and g in terms of E_-, E_+ and E_{1z} gives

$$f_- = k_{11}E_- + k_{12}E_+ + k_{13}E_{1z}\,,$$

$$f_+ = k_{21}E_- + k_{22}E_+ + k_{23}E_{1z}\,,$$

$$g = k_{31}E_- + k_{32}E_+ + k_{33}E_{1z}\,, \tag{A.9}$$

where $k_{11} = (b_{11}/\Delta_o)[(\omega_z-b)b^2 + 2\omega_+\omega_-\omega]$,

$$k_{12} = (b_{22}/\Delta_o)(2\omega_-^2\omega),$$

$$k_{13} = (b_{13}/\Delta_o)[(\omega_z-b)b^2] + (b_{33}/\Delta_o)[\omega_-(kv_z)b(b-\omega_z)],$$

$$k_{21} = (b_{11}/\Delta_o)(2\omega_+^2\omega),$$

$$k_{22} = (b_{22}/\Delta_o)[-b^2(b+\omega_z) + 2\omega_+\omega_-\omega],$$

$$k_{23} = (-b_{23}/\Delta_o)[b^2(b+\omega_z) - 2\omega_+\omega_-\omega] - (b_{33}/\Delta_o)[\omega_+(kv_z)b(b+\omega_z)],$$

$$k_{31} = (-b_{11}/\Delta_o)[2\omega_+b(b-\omega_z)],$$

$$k_{32} = (b_{22}/\Delta_o)[2\omega_-b(b+\omega_z)],$$

$$k_{33} = (1/\Delta_o)[-2b_{13}\omega_+b^2 + 2b_{23}\omega_-b^2] + (b_{33}/\Delta_o)[b^2(b^2-\omega_z^2-4\omega_+\omega_-)],$$

where

$$\Delta_o \equiv b^2[4\omega_+\omega_-\omega + (\omega_z^2-b^2)b] \equiv b^2\delta_o \quad ,$$

$$b_{11} = -\, jb\eta M_o(f_o) \quad , \quad b_{12} = 0 \quad , \quad b_{13} = -\, j\omega_-\eta D\left(\frac{\partial f_o}{\partial v_z}\right) \quad ,$$

$$b_{21} = 0 \quad , \quad b_{22} = -\, jb\eta M_o(f_o) \quad , \quad b_{23} = j\omega_+\eta D\left(\frac{\partial f_o}{\partial v_z}\right) \quad ,$$

$$b_{31} = 0 \quad , \quad b_{32} = 0 \quad , \quad b_{33} = j\eta\left(\frac{v_\perp}{v_z}\right)\frac{\partial f_o}{\partial v_z} \quad .$$

A.2 Determination of $R_{p,q}$ (for Case 1 with Weak Transverse Static Electric Field)

Assuming that

$$f_o = ne^{-\alpha_e[(v_z-u_{oe})^2+v_\perp^2]} \quad ,$$

$$F_o = Ne^{-\alpha_i[(v_z-u_{oi})^2+v_\perp^2]} \quad , \qquad \text{(A.10)}$$

then

$$\frac{\partial f_o}{\partial v_z} = -\, 2\alpha_e(v_z-u_{oe})f_o \quad , \quad \frac{\partial f_o}{\partial v_\perp} = -\, 2\alpha_e v_\perp f_o \quad ,$$

$$\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right) = (2\alpha_e)^2\,(v_z-u_{oe})v_\perp f_o \quad ,$$

$$D(f_o) = 2\alpha_e u_{oe} v_\perp f_o \quad ,$$

$$M_o(f_o) = \frac{\partial f_o}{\partial v_\perp} + \frac{k}{\omega}D(f_o) = -\, 2\alpha_e v_\perp f_o\left(1 - \frac{ku_{oe}}{\omega}\right) \quad .$$

Furthermore, from Eq. 28,

$$C_{11} = \frac{-2\alpha_e\eta_e\left(1-\frac{ku_{oe}}{\omega}\right)(v_\perp f_o)}{j(b+\omega_z)} ,$$

$$C_{12} = 0 ,$$

$$C_{13} = \frac{-\eta_e(2\alpha_e)^2\, a_-(v_z-u_{oe})(v_\perp f_o)}{b(b+\omega_z)} ,$$

$$C_{21} = 0 ,$$

$$C_{22} = \frac{-2\alpha_e\eta_e\left(1-\frac{ku_{oe}}{\omega}\right)(v_\perp f_o)}{j(b-\omega_z)} ,$$

$$C_{23} = \frac{-\eta_e(2\alpha_e)^2\, a_+\,(v_z-u_o)(v_\perp f_o)}{b(b-\omega_z)} ,$$

$$C_{31} = \frac{4\alpha_e\eta_e a_+\left(1-\frac{ku_{oe}}{\omega}\right)f_o}{b(b+\omega_z)} ,$$

$$C_{32} = \frac{4\alpha_e\eta_e a_-\left(1-\frac{ku_{oe}}{\omega}\right)f_o}{b(b+\omega_z)} ,$$

$$C_{33} = \frac{-2\alpha_e\eta_e(v_z-u_{oe})f_o}{jb} + j\,\frac{4a_+a_-\eta_e(2\alpha_e)^2(v_z-u_{oe})f_o}{b(b^2-\omega_z^2)} . \qquad (A.11)$$

$C_{p,q}$ can be obtained by replacing f_o, α_e, η_e, u_{oe}, $a_\pm$ and ω_z by F_o, α_i, $-\eta_i$, u_{oi}, $A_\pm$ and Ω_z respectively in Eq. A.11. Let

$$R_{p,q} \equiv \frac{j\pi\left(\frac{\omega e}{\epsilon_o}\right)}{(\omega^2-c^2k^2)} r_{p,q} \quad ; \quad p = 1, 2 \quad ; \quad q = 1, 2, 3$$

$$\equiv \frac{j2\pi e}{\omega\epsilon_o} r_{p,q} \quad ; \quad p = 3 \quad ; \quad q = 1, 2, 3 \quad , \qquad \text{(A.12)}$$

with

$$r_{p,q} \equiv \int_{-\infty}^{\infty} \xi_{p,q}(v_z)dv_z \quad ; \quad p = 1, 2 \quad ; \quad q = 1, 2, 3$$

$$\equiv \int_{-\infty}^{\infty} v_z \, \eta_{p,q}(v_z)dv_z \quad ; \quad p = 3 \quad ; \quad q = 1, 2, 3 \, , \qquad \text{(A.13)}$$

where

$$\xi_{p,q}(v_z) \equiv \int_0^{\infty} v_\perp^2(C_{p,q}-C_{p,q})dv_\perp \quad ,$$

$$\eta_{p,q}(v_z) \equiv \int_0^{\infty} v_\perp(C_{p,q}-C_{p,q})dv_\perp \quad , \qquad \text{(A.14a)}$$

and let

$$f_o = w_o n_o \ell(v_z)e^{-\alpha_e v_\perp^2} \quad ,$$

$$F_o = W_o N_o L(v_z)e^{-\alpha_i v_\perp^2} \quad , \qquad \text{(A.14b)}$$

in which

$$w_o \equiv \left(\frac{\alpha_e}{\pi}\right)^{3/2} e^{e\Phi/KT_e} \quad , \quad W_o \equiv \left(\frac{\alpha_i}{\pi}\right)^{3/2} e^{-(e\Phi/KT_i)} \quad ,$$

$$\ell(v_z) \equiv e^{-\alpha_e(v_z-u_{oe})^2} \quad , \quad L(v_z) \equiv e^{-\alpha_i(v_z-u_{oi})^2} \quad ,$$

$$\alpha_e \equiv \frac{m}{2KT_e} , \quad \alpha_i \equiv \frac{M}{2KT_i} ,$$

$$I_{3e} \equiv \int_0^\infty v_\perp^3 e^{-\alpha_e v_\perp^2} dv_\perp = \frac{1}{2\alpha_e^2} , \quad I_{3i} \equiv \int_0^\infty v_\perp^2 e^{-\alpha_i v_\perp^2} dv_\perp = \frac{1}{2\alpha_i^2} ,$$

$$I_{1e} \equiv \int_0^\infty v_\perp e^{-\alpha_e v_\perp^2} dv_\perp = \frac{1}{2\alpha_e} , \quad I_{1i} \equiv \int_0^\infty v_\perp e^{-\alpha_i v_\perp^2} dv_\perp = \frac{1}{2\alpha_i} .$$

Define the integral

$$Q_o[Y] \equiv \int_{-\infty}^{\infty} Y(v_z) dv_z .$$

Then substituting Eqs. A.11 into Eqs. A.14, with the aid of Eqs. A.12 and A.13, yields

$$R_{11} = \left(1 - \frac{ku_{oi}}{\omega}\right) D_1 Q_o \left[\frac{L(v_z)}{b+\Omega_z}\right] + \left(1 - \frac{ku_{oe}}{\omega}\right) d_1 Q_o \left[\frac{\ell(v_z)}{b+\omega_z}\right] , \qquad (A.15a)$$

$$R_{12} = 0 , \qquad (A.15b)$$

$$R_{13} = j2\alpha_i A_- D_1 Q_o \left[\frac{(v_z-u_{oi})L(v_z)}{b(b+\Omega_z)}\right] + j2\alpha_e a_- d_1 Q_o \left[\frac{(v_z-u_{oe})\ell(v_z)}{b(b+\omega_z)}\right] , \qquad (A.15c)$$

$$R_{21} = 0 , \qquad (A.15d)$$

$$R_{22} = \left(1 - \frac{ku_{oi}}{\omega}\right) D_1 Q_o \left[\frac{L(v_z)}{b-\Omega_z}\right] + \left(1 - \frac{ku_{oe}}{\omega}\right) d_1 Q_o \left[\frac{\ell(v_z)}{b-\omega_z}\right] , \qquad (A.15e)$$

$$R_{23} = j2\alpha_i A_+ D_1 Q_o \left[\frac{(v_z - u_{oi}) L(v_z)}{b(b-\Omega_z)} \right] + j2\alpha_e a_+ d_1 Q_o \left[\frac{(v_z - u_{oe}) \ell(v_z)}{b(b-\omega_z)} \right] , \tag{A.15f}$$

$$R_{31} = -j\left(1 - \frac{ku_{oi}}{\omega}\right) 2A_+ D_2 Q_o \left[\frac{v_z L(v_z)}{b(b+\Omega_z)} \right] - j\left(1 - \frac{ku_{oe}}{\omega}\right) 2a_+ d_2 Q_o \cdot \left[\frac{v_z \ell(v_z)}{b(b+\omega_z)} \right] , \tag{A.15g}$$

$$R_{32} = -j\left(1 - \frac{ku_{oi}}{\omega}\right) 2A_- D_2 Q_o \left[\frac{v_z L(v_z)}{b(b-\Omega_z)} \right] - j\left(1 - \frac{ku_{oe}}{\omega}\right) 2a_- d_2 Q_o \cdot \left[\frac{v_z \ell(v_z)}{b(b-\omega_z)} \right] , \tag{A.15h}$$

$$R_{33} = D_2 Q_o \left[\frac{v_z (v_z - u_{oi}) L(v_z)}{b} \right] + d_2 Q_o \left[\frac{v_z (v_z - u_{oe}) \ell(v_z)}{b} \right] + 8\alpha_i A_+ A_- D_2 Q_o \cdot \left[\frac{v_z (v_z - u_{oi}) L(v_z)}{b(b^2 - \Omega_z^2)} \right] + 8\alpha_e a_+ a_- d_2 Q_o \left[\frac{v_z (v_z - u_{oe}) \ell(v_z)}{b(b^2 - \omega_z^2)} \right] , \tag{A.15i}$$

where

$$D_1 \equiv \frac{(2\pi\alpha_i W_o I_{3i})\omega\Omega_p^2}{(\omega^2 - c^2 k^2)} , \quad D_2 \equiv \frac{(4\pi\alpha_i W_o I_{1i})\Omega_p^2}{\omega} ,$$

$$d_1 \equiv \frac{(2\pi\alpha_e w_o I_{3e})\omega\omega_p^2}{(\omega^2 - c^2 k^2)} , \quad d_2 \equiv \frac{(4\pi\alpha_e w_o I_{1e})\omega_p^2}{\omega} .$$

Let

$$K_\pm \equiv Q_o \left(\frac{L}{b \pm \Omega_z} \right) , \qquad k_\pm \equiv Q_o \left(\frac{\ell}{b \pm \omega_z} \right) ,$$

$$S_\pm \equiv Q_o \left[\frac{v_z L \Omega_z}{b(b \pm \Omega_z)} \right] , \qquad s_\pm \equiv Q_o \left[\frac{v_z \ell \omega_z}{b(b \pm \omega_z)} \right] ,$$

$$P_\pm \equiv Q_o \left[\frac{(v_z - u_{oi}) L \Omega_z}{b(b \pm \Omega_z)} \right] , \qquad p_\pm \equiv Q_o \left[\frac{(v_z - u_{oe}) \ell \omega_z}{b(b \pm \omega_z)} \right] ,$$

$$T \equiv Q_o \left[\frac{v_z (v_z - u_{oi}) L}{b} \right] , \qquad \tau \equiv Q_o \left[\frac{v_z (v_z - u_{oe}) L}{b} \right] ,$$

$$V \equiv Q_o \left[\frac{v_z (v_z - u_{oi}) L \Omega_z^2}{b(b^2 - \Omega_z^2)} \right] , \qquad v \equiv Q_o \left[\frac{v_z (v_z - u_{oe}) \ell \omega_z^2}{b(b^2 - \omega_z^2)} \right] ,$$

$$Z_o \equiv \left(1 - \frac{u_{oi}}{U_o} \right) , \quad z_o \equiv \left(1 - \frac{u_{oe}}{U_o} \right) , \quad U_o \equiv \left(\frac{\omega}{k} \right) ,$$

$$\mu_1 \equiv \left(\frac{A_+}{\Omega_z} \right) = \left(\frac{a_+}{\omega_z} \right) , \quad \mu_2 \equiv \left(\frac{A_-}{\Omega_z} \right) = \left(\frac{a_-}{\omega_z} \right) ,$$

$$D_\pm \equiv [(Z_o D_1 K_\pm + z_o d_1 k_\pm) - 1] ,$$

$$D_z \equiv [(D_2 T + d_2 \tau) - 1] .$$

Since V and v can be written as

$$V = V_+ + V_- \quad \text{and} \quad v = v_+ + v_- ,$$

where

$$V_\pm \equiv \frac{1}{2} Q_o \left[v_z^2 L \left(\frac{1}{b \pm \Omega_z} - \frac{1}{b} \right) \right] - \frac{1}{2} u_{oi} Q_o \left[v_z L \left(\frac{1}{b \pm \Omega_z} - \frac{1}{b} \right) \right]$$

and

$$v_\pm \equiv \frac{1}{2} Q_o \left[v_z^2 \ell \left(\frac{1}{b \pm \omega_z} - \frac{1}{b} \right) \right] - \frac{1}{2} u_{oe} Q_o \left[v_z \ell \left(\frac{1}{b \pm \omega_z} - \frac{1}{b} \right) \right] ,$$

Eqs. A.15 can be written as

$$R_{11} = (D_+ + 1) \quad ,$$

$$R_{12} = 0 \quad ,$$

$$R_{13} = j2\mu_2\beta_+ \quad ,$$

$$R_{21} = 0 \quad ,$$

$$R_{22} = (D_- + 1) \quad ,$$

$$R_{23} = j2\mu_1\beta_- \quad .$$

$$R_{31} = -j2\mu_1\alpha_+ \quad ,$$

$$R_{32} = -j2\mu_2\alpha_- \quad ,$$

$$R_{33} = (D_z + 1) + 4\mu_1\mu_2(\gamma_+ + \gamma_-) \quad , \tag{A.16}$$

where

$$\alpha_\pm \equiv (Z_o D_2 S_\pm + z_o d_2 s_\pm) \quad ,$$

$$\beta_\pm \equiv (\alpha_i D_1 P_\pm + \alpha_e d_1 p_\pm) \quad ,$$

$$\gamma_\pm \equiv 2(\alpha_i D_2 V_\pm + \alpha_e d_2 v_\pm) \quad . \tag{A.17}$$

Substitution of Eqs. A.16 into Eq. 36, with the aid of Eqs. A.17, yields

$$D_+ D_- D_z = -4\mu_1\mu_2[D_+(D_-\gamma_- - \alpha_-\beta_-) + D_-(D_+\gamma_+ - \alpha_+\beta_+)] \quad , \tag{A.18}$$

which is Eq. 58.

Define the integrals $G_p(Y)$ and $g_p(Y)$ as

$$G_p(Y) \equiv \frac{j}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{v_z^p L(v_z)}{(v_z - Y)} dv_z \quad ,$$

$$g_p(Y) \equiv \frac{j}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{v_z^p \ell(v_z)}{(v_z - Y)} dv_z \quad , \tag{A.19}$$

which has been discussed by Stix[3]. By simple manipulation of the integrand in Eqs. A.19 it can be shown that

$$G_1(Y) = j \frac{1}{\sqrt{\alpha_i}} + YG_o(Y) \quad ,$$

$$G_2(Y) = j(Y + u_{oi}) \frac{1}{\sqrt{\alpha_i}} + Y^2 G_o(Y) \quad ,$$

$$g_1(Y) = j \frac{1}{\sqrt{\alpha_e}} + Yg_o(Y) \quad ,$$

$$g_2(Y) = j(Y + u_{oe}) \frac{1}{\sqrt{\alpha_e}} + Y^2 g_o(Y) \quad , \tag{A.20}$$

so that

$$K_{\pm} = j \frac{\sqrt{\pi}}{k} G_o(U_{\pm}) \quad ,$$

$$S_{\pm} = \mp j \frac{\sqrt{\pi}}{k} [U_{\pm}G_o(U_{\pm}) - U_oG_o(U_o)] \quad ,$$

$$P_{\pm} = \mp j \frac{\sqrt{\pi}}{k} [U_{\pm}-u_{oi})G_o(U_{\pm}) - (U_o-u_{oi})G_o(U_o)] \quad ,$$

$$2V_{\pm} = j \frac{\sqrt{\pi}}{k} \left[U_{\pm}(U_{\pm}-u_{oi})G_o(U_{\pm}) - U_o(U_o-u_{oi})G_o(U_o) + j\,(U_{\pm}-U_o)\frac{1}{\sqrt{\alpha_i}}\right] \quad ,$$

$$T = j \frac{\sqrt{\pi}}{k} \left[U_o(U_o-u_{oi})G_o(U_o) + j\frac{U_o}{\sqrt{\alpha_i}}\right] \quad ,$$

$$k_{\pm} = j \frac{\sqrt{\pi}}{k} g_o(u_{\pm}) \quad ,$$

$$s_{\pm} = \mp j \frac{\sqrt{\pi}}{k} [u_{\pm}g_o(u_{\pm}) - U_og_o(U_o)] \quad ,$$

$$p_{\pm} = \mp j \frac{\sqrt{\pi}}{k} [(u_{\pm}-u_{oe})g_o(u_{\pm}) - (U_o-u_{oe})g_o(U_o)] \quad ,$$

$$2v_{\pm} = j \frac{\sqrt{\pi}}{k} \left[u_{\pm}(u_{\pm}-u_{oe})g_o(u_{\pm}) - U_o(U_o-u_{oe})g_o(U_o) + j(u_{\pm}-U_o)\frac{1}{\sqrt{\alpha_e}}\right] \quad ,$$

$$\tau = j \frac{\sqrt{\pi}}{k} \left[U_o(U_o-u_{oe})g_o(U_o) + j\frac{U_o}{\sqrt{\alpha_e}}\right] \quad . \qquad (A.21)$$

Upon substituting Eqs. A.21 into Eqs. A.16 and A.17,

$$D_z = \left\{R_2\left[U_o(U_o-u_{oi})G_o(U_o) + j\,\frac{U_o}{\sqrt{\alpha_i}}\right] + r_2\left[U_o(U_o-u_{oe})g_o(U_o) + j\,\frac{U_o}{\sqrt{\alpha_e}}\right] - 1\right\} ,$$

$$D_{\pm} = \left\{R_1\,[Z_oG_o(U_{\pm})] + r_1[z_og_o(u_{\pm})] - 1\right\} ,$$

$$\alpha_{\pm} = \mp R_2Z_o[U_{\pm}G_o(U_{\pm}) - U_oG_o(U_o)] \mp r_2z_o[u_{\pm}g_o(u_{\pm}) - U_og_o(U_o)] ,$$

$$\beta_{\pm} = \mp R_1\alpha_i[(U_{\pm}u_{oi})G_o(U_{\pm}) - (U_o-u_{oi})G_o(U_o)] \mp r_1\alpha_e[(u_{\pm}-u_{oe})g_o(u_{\pm}) - (U_o-u_{oe})g_o(U_o)] ,$$

$$\gamma_{\pm} = R_2\alpha_i\left[U_{\pm}(U_{\pm}-u_{oi})G_o(U_{\pm}) - U_o(U_o-u_{oi})G_o(U_o) \pm j\,\frac{\Omega_z}{\sqrt{\alpha_i}\,k}\right]$$

$$+ r_2\alpha_e\left[u_{\pm}(u_{\pm}-u_{oe})g_o(u_{\pm}) - U_o(U_o-u_{oe})g_o(U_o) \pm j\,\frac{\omega_z}{\sqrt{\alpha_e}\,k}\right] , \quad (A.22)$$

where

$$R_1 \equiv \left(j\,\frac{\sqrt{\pi}}{k}\,D_1\right) = j\sqrt{\alpha_i}\,\frac{W_1\,\Omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{\omega}{k}\right) ,$$

$$R_2 \equiv \left(j\,\frac{\sqrt{\pi}}{k}\,D_2\right) = j2\alpha_i\sqrt{\alpha_i}\,W_1\left(\frac{\Omega_p^2}{\omega^2}\right)\left(\frac{\omega}{k}\right) ,$$

$$r_1 \equiv \left(j\,\frac{\sqrt{\pi}}{k}\,d_1\right) = j\sqrt{\alpha_e}\,w_1\,\frac{\omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{\omega}{k}\right) ,$$

$$r_2 \equiv \left(j\,\frac{\sqrt{\pi}}{k}\,d_2\right) = j2\alpha_i\sqrt{\alpha_i}\,w_1\left(\frac{\omega_p^2}{\omega^2}\right)\left(\frac{\omega}{k}\right) ,$$

with

$$W_1 \equiv e^{-(e\Phi/KT_i)} \quad \text{and} \quad w_1 \equiv e^{(e\Phi/KT_e)} \quad ,$$

and for a homogeneous plasma Φ is independent of z.

A.3 Determination of $R_{p,q}$

For Case 2,

$$f_o = n_o w_1 \, e^{-\alpha_e(v_z^2 + v_\perp^2)} \quad ,$$

$$F_o = N_o W_1 \, e^{-\alpha_i(v_z^2 + v_\perp^2)} \quad , \qquad \text{(A.23)}$$

$$\frac{\partial f_o}{\partial v_z} = -2\alpha_e v_z f_o \quad , \quad \frac{\partial f_o}{\partial v_\perp} = -2\alpha_e v_\perp f_o \quad ,$$

$$\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right) = (2\alpha_e)^2 \, v_z v_\perp f_o \quad , \quad D(f_o) = 0 \quad , \quad M_o(f_o) = -2\alpha_e v_\perp f_o \quad .$$

Then from Eqs. 25,

$$k_{11} = \frac{j2\alpha_e\eta_e(v_\perp f_o)}{b\delta_o}\left[(\omega_z - b)b^2 + 2\omega_+\omega_-\omega\right] \quad , \qquad \text{(A.24a)}$$

$$k_{12} = \frac{j2\alpha_e\eta_e(v_\perp f_o)}{b\delta_o}(2\omega_-^2\omega) \quad , \qquad \text{(A.24b)}$$

$$k_{13} = \frac{j2\alpha_e\eta_e(v_\perp f_o)}{b\delta_o}\left[\omega_-\omega(\omega_z - b)\right] \quad , \qquad \text{(A.24c)}$$

$$k_{21} = \frac{j2\alpha_e\eta_e(v_\perp f_o)}{b\delta_o}(2\omega_+^2\omega) \quad , \qquad \text{(A.24d)}$$

$$k_{22} = \frac{-j2\alpha_e\eta_e(v_\perp f_o)}{b\delta_o}\left[b^2(b+\omega_z) - 2\omega_+\omega_-\omega\right] \quad , \qquad \text{(A.24e)}$$

$$k_{23} = \frac{j2\alpha_e\eta_e(v_\perp f_o)\omega_+}{b\,\delta_o}[\omega(b+\omega_z)] \quad , \tag{A.24f}$$

$$k_{31} = \frac{-j2\alpha_e\eta_e(v_z f_o)}{\delta_o}[2\omega_+(b-\omega_z)] \quad , \tag{A.24g}$$

$$k_{32} = \frac{j2\alpha_e\eta_e(v_z f_o)}{\delta_o}[2\omega_-(b+\omega_z)] \quad , \tag{A.24h}$$

$$k_{33} = \frac{-j2\alpha_e\eta_e(v_z f_o)}{\delta_o}(b^2-\omega_z^2) \quad , \tag{A.24i}$$

where

$$\delta_o \equiv [(\omega_z^2-b^2)b + 4\omega_+\omega_-\omega] \quad .$$

$K_{p,q}$ can be obtained by replacing f_o, α_e, η_e, $\omega_\pm$, ω_z and δ_o by F_o, α_i, $-\eta_i$, $\Omega_\pm$, Ω_z and Δ_o in Eqs. A.24. Letting

$$\ell(v_z) = e^{-\alpha_e v_z^2} \quad \text{and} \quad L(v_z) = e^{-\alpha_i v_z^2}$$

in Eq. A.14b, and defining

$$\xi_{p,q} \equiv \int_0^\infty v_\perp^2(K_{p,q}-k_{p,q})dv_\perp \quad ; \quad p = 1, 2 \quad ; \quad q = 1, 2, 3 \quad ,$$

$$\eta_{p,q} \equiv \int_0^\infty v_\perp(K_{p,q}-k_{p,q})dv_\perp \quad ; \quad p = 3 \quad ; \quad q = 1, 2, 3 \quad ,$$

with the aid of Eqs. A.12 and A.13, gives

$$R_{11} = D_1Q_o\left\{\frac{L(v_z)}{\Delta b}\left[(\Omega_z-b)b^2 + 2\Omega_+\Omega_-\omega\right]\right\} + d_1Q_o\left\{\frac{\ell(v_z)}{\delta_o b}\left[(\omega_z-b)b^2 + 2\omega_+\omega_-\omega\right]\right\},$$

$$R_{12} = D_1Q_o\left[\frac{L(v_z)2\Omega_-^2\omega}{\Delta b}\right] + d_1Q_o\left[\frac{\ell(v_z)2\omega_-^2\omega}{\delta_o b}\right],$$

$$R_{13} = D_1Q_o\left[\frac{L(v_z)}{\Delta b}\,\Omega_-\omega(\Omega_z-b)\right] + d_1Q_o\left[\frac{\ell(v_z)}{\delta_o b}\,\omega_-\omega(\omega_z-b)\right],$$

$$R_{21} = D_1Q_o\left[\frac{L(v_z)2\Omega_+^2\omega}{\Delta b}\right] + d_1Q_o\left[\frac{\ell(v_z)2\omega_+^2\omega}{\delta_o b}\right],$$

$$R_{22} = -D_1Q_o\left\{\frac{L(v_z)}{\Delta b}\left[b^2(b+\Omega_z) - 2\Omega_+\Omega_-\omega\right]\right\} - d_1Q_o\left\{\frac{\ell(v_z)}{\delta_o b}\left[b^2(b+\omega_z) - 2\omega_+\omega_-\omega\right]\right\},$$

$$R_{23} = D_1Q_o\left\{\frac{L(v_z)\Omega_+}{\Delta b^2}\left[\omega b(b+\Omega_z)\right]\right\} + d_1Q_o\left\{\frac{\ell(v_z)\omega_+}{\delta_o b^2}\left[\omega b(b+\omega_z)\right]\right\},$$

$$R_{31} = -D_2Q_o\left[\frac{v_z^2L(v_z)}{\Delta}\,2\Omega_+(b-\Omega_z)\right] - d_2Q_o\left[\frac{v_z^2\ell(v_z)}{\delta_o}\,2\omega_+(b-\omega_z)\right],$$

$$R_{32} = D_2Q_o\left[\frac{v_z^2L(v_z)}{\Delta}\,2\Omega_-(b+\Omega_z)\right] + d_2Q_o\left[\frac{v_z^2\ell(v_z)}{\delta_o}\,2\omega_-(b+\omega_z)\right],$$

$$R_{33} = -D_2Q_o\left[\frac{v_z^2L(v_z)}{\Delta}\,(b^2-\Omega_z^2)\right] - d_2Q_o\left[\frac{v_z^2\ell(v_z)}{\delta_o}\,(b^2-\omega_z^2)\right], \qquad \text{(A.25)}$$

where Q_o, D_1, D_2, d_1 and d_2 are as defined previously in Eqs. A.15. It should be observed that the integrand of $R_{p,q}$, p,q = 1, 2, 3 in Eqs. A.25, has singularities at v_z for which $b = 0$, $\Delta = 0$ or $\delta_o = 0$, i.e.,

$$b^3 - \Omega_z^2 b - 4\Omega_+\Omega_-\omega = 0 \tag{A.26a}$$

and

$$b^3 - \omega_z^2 b - 4\omega_+\omega_-\omega = 0 \tag{A.26b}$$

which are the cubic equations in b, whose discriminants are given, respectively, by

$$\Delta_2 \equiv 27(4\Omega_+\Omega_-\omega)^2 - 4(\Omega_z^2)^3 \tag{A.27a}$$

and

$$\delta_2 \equiv 27(4\omega_+\omega_-\omega)^2 - 4(\omega_z^2)^3 \quad . \tag{A.27b}$$

It is not difficult to see that Eqs. A.26a and A.26b have either only one real root and two complex conjugate pair roots or three real roots according to whether $\Delta_2 > 0$ or $\Delta_2 < 0$, and $\delta_2 > 0$ or $\delta_2 < 0$, respectively; in other words, according to whether

$$108\left(\frac{\Omega_+\Omega_-}{\Omega_z^2}\right)^2\left(\frac{\omega}{\Omega_z}\right)^2 > 1 \text{ or } < 1 \tag{A.28a}$$

and

$$108\left(\frac{\omega_+\omega_-}{\omega_z^2}\right)^2\left(\frac{\omega}{\omega_z}\right)^2 > 1 \text{ or } < 1 \quad . \tag{A.28b}$$

Suppose that the following conditions are satisfied:

$$4\left(\frac{\Omega_+\Omega_-}{\Omega_z^2}\right) = 4\left(\frac{\omega_+\omega_-}{\omega_z^2}\right) = \left(\frac{B_{ox}^2+B_{oy}^2}{B_{oz}^2}\right) \ll 1 \quad . \tag{A.29}$$

Then

$$\Delta = b(\Omega_z^2-b^2) \quad \text{and} \quad \delta_o = b(\omega_z^2-b^2) \quad . \tag{A.30}$$

If

$$B_\pm \equiv Q_o \left[\frac{L(v_z)}{b \pm \Omega_z} \right] , \qquad b_\pm \equiv Q_o \left[\frac{\ell(v_z)}{b \pm \omega_z} \right] ,$$

$$C \equiv Q_o \left[\frac{\Omega_z^2 L(v_z)}{b^2(b^2 - \Omega_z^2)} \right] , \qquad c \equiv Q_o \left[\frac{\omega_z^2 \; \ell(v_z)}{b^2(b^2 - \omega_z^2)} \right] ,$$

$$X_\pm \equiv Q_o \left[\frac{\Omega_z \; L(v_z)}{b^2(b \pm \Omega_z)} \right] , \qquad x_\pm \equiv Q_o \left[\frac{\omega_z \; \ell(v_z)}{b^2(b \pm \omega_z)} \right] ,$$

$$F \equiv Q_o \left[\frac{\Omega_z^3 \; L(v_z)}{b^3(b^2 - \Omega_z^2)} \right] , \qquad f \equiv Q_o \left[\frac{\omega_z^3 \; \ell(v_z)}{b^3(b^2 - \omega_z^2)} \right] ,$$

$$G_\pm \equiv Q_o \left[\frac{\Omega_z v_z^2 L(v_z)}{b(b \pm \Omega_z)} \right] , \qquad g_\pm \equiv Q_o \left[\frac{\omega_z v_z^2 \ell(v_z)}{b(b \pm \omega_z)} \right] ,$$

$$H \equiv Q_o \left[\frac{v_z^2 \; L(v_z)}{b} \right] , \qquad h \equiv Q_o \left[\frac{v_z^2 \; \ell(v_z)}{b} \right] \qquad \text{(A.31)}$$

and

$$\nu_1 \equiv (\Omega_+/\Omega_z) = (\omega_+/\omega_z) ,$$

$$\nu_2 \equiv (\Omega_-/\Omega_z) = (\omega_-/\omega_z) ,$$

$$D_+ \equiv [(D_1 B_+ + d_1 b_+) - 1] ,$$

$$D_z \equiv [(D_2 H + d_2 h) - 1] ,$$

$$D_- \equiv [(D_1 B_- + d_1 b_-) - 1] ,$$

then Eqs. A.25 can be written as

$$R_{11} = (D_+ + 1) - 2\nu_1\nu_2\,\omega I \quad ,$$

$$R_{12} = -2\nu_2^2\,\omega I \quad ,$$

$$R_{13} = \nu_2\,\omega\,M_+ \quad ,$$

$$R_{21} = -2\nu_1^2\,\omega I \quad ,$$

$$R_{22} = (D_- + 1) - 2\nu_1\nu_2\,\omega I \quad ,$$

$$R_{23} = -\nu_1\,\omega\,M_- + 2\nu_1^2\nu_2\,\omega\,Q \quad ,$$

$$R_{31} = 2\nu_1\,N_+ \quad ,$$

$$R_{32} = -2\nu_2\,N_- \quad ,$$

$$R_{33} = (D_z + 1) \quad , \qquad \text{(A.32)}$$

where

$$I \equiv (D_1C + d_1c) \quad ,$$

$$M_\pm \equiv (D_1X_\pm + d_1x_\pm) \quad ,$$

$$N_\pm \equiv (D_2G_\pm + d_2g_\pm) \quad ,$$

$$Q \equiv (D_1F + d_1f) \quad . \qquad \text{(A.33)}$$

Substitution of Eqs. A.32 into Eq. 43 yields

$$(D_+ - 2\nu_1\nu_2\omega I)[D_z(D_- - 2\nu_1\nu_2\omega I) - 2\nu_1\nu_2\omega M_-N_- + 4\nu_1^2\nu_2^2\omega QN_-]$$

$$+ 2\nu_2^2\omega I(-2\nu_1^2\omega I D_z + 2\nu_1^2\omega M_-N_+ - 4\nu_1^3\nu_2^2\omega QN_+)$$

$$+ \nu_2\omega M_+ (4\nu_1^2\nu_2\omega IN_- - 2\nu_1 D_-N_+ + 4\nu_1^2\nu_2\omega I N_+) = 0 ,$$

$$D_+D_-D_z = 2\nu_1\nu_2\omega [D_-(ID_z + M_+N_+) + D_+(ID_z + M_-N_-) - 2\nu_1\nu_2 QD_+N_-]$$

$$- 4\nu_1^2\nu_2^2\omega^2 \{(N_+ + N_-) [I(M_+ + M_-) + 2\nu_1\nu_2 IQ]\} . \quad (A.34)$$

It should be noted that since

$$C = Q_o \left[\frac{\Omega_z L}{2b^2} \left(\frac{1}{b-\Omega_z} - \frac{1}{b+\Omega_z} \right) \right] = \frac{1}{2}(X_- - X_+)$$

and

$$F = Q_o \left\{ \frac{\Omega_z L}{b^2} \left[\frac{1}{2} \left(\frac{1}{b-\Omega_z} + \frac{1}{b+\Omega_z} \right) - \frac{1}{b} \right] \right\} = \frac{1}{2}(X_- + X_+) - H_3 ,$$

where

$$H_3 \equiv Q_o \left(\frac{\Omega_z L}{b^3} \right) ,$$

I and Q can be written as

$$I = \frac{1}{2}(M_- - M_+)$$

and

$$Q = \frac{1}{2}(M_- + M_+) - H_3' , \quad (A.35)$$

where

$$H_3' \equiv (D_1 H_3 + d_1 h_3) .$$

Since $4\nu_1\nu_2 \ll 1$ is assumed, and M_- and Q are of the same order of magnitude, Eqs. A.34 can be reduced with the aid of Eqs. A.35 to

$$D_+D_-D_z = \nu_1\nu_2\omega\,[D_-M_+(2N_+ - D_z) - D_+M_-(2N_- - D_z) + D_z(D_-M_- - D_+M_+)]$$

$$+ 2\nu_1^2\nu_2^2\omega^2\,[(M_+^2 - M_-^2)\,(N_+ + N_-)] \quad , \qquad (A.36)$$

which is Eq. 60 for $\nu^2 \equiv (\nu_1\nu_2)$.

On the other hand, with the aid of Eqs. A.19, $X_\pm$ can be written as

$$X_+ = Q_o\left[\frac{L}{b}\left(\frac{1}{b} - \frac{1}{b+\Omega_z}\right)\right] = Q_o\left(\frac{L}{b^2}\right) - \frac{Q_o}{\Omega_z}\left[L\left(\frac{1}{b} - \frac{1}{b+\Omega_z}\right)\right] \quad ,$$

so that

$$X_+ = \frac{j\sqrt{\pi}}{k\Omega_z}\,[G_o(U_+) - G_o(U_o)] + H_2 \quad ,$$

where

$$H_2 \equiv Q_o\left(\frac{L}{b^2}\right) \quad .$$

Similarly

$$X_- = \frac{j\sqrt{\pi}}{k\Omega_z}\,[G_o(U_-) - G_o(U_o)] - H_2 \quad ,$$

where the G_o's are as defined in Eqs. A.19.

However, since

$$G_+ = Q_o\left[v_z^2 L\left(\frac{1}{b} - \frac{1}{b+\Omega_z}\right)\right] = \frac{j\sqrt{\pi}}{k}\,[G_2(U_o) - G_2(U_+)] \quad ,$$

then from Eqs. A.20,

$$G_+ = \frac{j\sqrt{\pi}}{k}\left[U_o^2G_o(U_o) - U_+^2G_o(U_+) + j\,\frac{1}{\sqrt{\alpha_i}}\,(U_o - U_+)\right] \quad .$$

Similarly,

$$G_- = \frac{j\sqrt{\pi}}{k}\left[U_-^2G_o(U_-) - U_o^2G_o(U_o) + j\,\frac{1}{\sqrt{\alpha_i}}\,(U_- - U_o)\right] \quad .$$

Furthermore,

$$B_{\pm} = j \frac{\sqrt{\pi}}{k} G_o(U_{\pm})$$

and

$$H = j \frac{\sqrt{\pi}}{k} \left[U_o^2 G_o(U_o) + j \frac{1}{\sqrt{\alpha_i}} U_o \right] .$$

Substituting these expressions of $X_{\pm}$, $G_{\pm}$, etc., into Eqs. A.32 and A.33 gives

$$D_z = \left\{ R_2 \left[U_o^2 G_o(U_o) + j \frac{U_o}{\sqrt{\alpha_i}} \right] + r_2 \left[U_o^2 g_o(U_o) + j \frac{U_o}{\sqrt{\alpha_e}} \right] - 1 \right\} ,$$

$$D_{\pm} = [R_1 G_o(U_{\pm}) + r_1 g_o(u_{\pm}) - 1] ,$$

$$N_{\pm} = \mp R_2 \left[U_{\pm}^2 G_o(U_{\pm}) - U_o^2 G_o(U_o) \pm j \frac{1}{\sqrt{\alpha_i}} \frac{\Omega_z}{k} \right] \mp r_2 \left[u_{\pm}^2 g_o(u_{\pm}) - U_o^2 g_o(U_o) \pm j \frac{1}{\sqrt{\alpha_e}} \frac{\omega_z}{k} \right] ,$$

$$M_{\pm} = \frac{R_1}{\Omega_z} [G_o(U_+) - G_o(U_o)] + \frac{r_1}{\omega_z} [g_o(u_+) - g_o(U_o)] \pm (D_1 H_2 + d_1 h_2) .$$

Since $R_1 = (j\sqrt{\pi}/k)D_1$ and $r_1 = (j\sqrt{\pi}/k)d_1$, and by defining

$$\Lambda = \frac{\Omega_z}{jk\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{L(v_z)}{(v_z - U_o)^2} dv_z ,$$

$$\lambda = \frac{\omega_z}{jk\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\ell(v_z)}{(v_z - U_o)^2} dv_z , \qquad (A.37)$$

$M_{\pm}$ can be expressed as

$$M_\pm = \frac{R_1}{\Omega_z}\left[G_o(U_\pm) - G_o(U_o) \pm \Lambda\right] + \frac{r_1}{\omega_z}\left[g_o(u_\pm) - g_o(U_o) \pm \lambda\right] .$$

A.4 Derivation of Eqs. 70

From Eqs. 69

$$G_o(U_\pm) \simeq -j\,\frac{1}{\sqrt{\alpha_i}\,(U_\pm - u_{oi})}\left[1 + \frac{1}{2\alpha_i(U_\pm - u_{oi})^2}\right] ,$$

$$g_o(u_\pm) \simeq -j\,\frac{1}{\sqrt{\alpha_e}\,(u_\pm - u_{oe})}\left[1 + \frac{1}{2\alpha_e(u_\pm - u_{oe})^2}\right] \qquad (A.38)$$

and

$$R_1 Z_o G_o(U_\pm) \simeq \frac{\Omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{U_o - u_{oi}}{U_\pm - u_{oi}}\right)\left[1 + \frac{1}{2\alpha_i(U_\pm - u_{oi})^2}\right] ,$$

$$r_1 z_o g_o(u_\pm) \simeq \frac{\omega_p^2}{(\omega^2-c^2k^2)}\left(\frac{U_o - u_{oe}}{u_\pm - u_{oe}}\right)\left[1 + \frac{1}{2\alpha_e(u_\pm - u_{oe})^2}\right] .$$

Therefore

$$D_\pm = \left\{\frac{\Omega_p^2(\omega-ku_{oi})}{(\omega^2-c^2k^2)(\omega\pm\Omega_z-ku_{oi})}\left[1 + \frac{1}{2\alpha_i(U_\pm - u_{oi})^2}\right]\right.$$

$$\left. + \frac{\omega_p^2(\omega-ku_{oe})}{(\omega^2-c^2k^2)(\omega\pm\omega_z-ku_{oe})}\left[1 + \frac{1}{2\alpha_e(u_\pm - u_{oe})^2}\right] - 1\right\} ,$$

$$R_2\left[U_o(U_o-u_{oi})G_o(U_o) + j\,\frac{U_o}{\sqrt{\alpha_i}}\right] = \frac{\Omega_p^2}{k^2}\,\frac{1}{(U_o-u_{oi})^2} = \frac{\Omega_p^2}{(\omega-ku_{oi})^2} ,$$

$$r_1\left[U_o(U_o-u_{oe})g_o(U_o) + j\,\frac{U_o}{\sqrt{\alpha_e}}\right] = \frac{\omega_p^2}{k^2}\,\frac{1}{(U_o-u_{oe})^2} = \frac{\omega_p^2}{(\omega-ku_{oe})^2} ,$$

and consequently

$$D_z = \left[\frac{\Omega_p^2}{(\omega - ku_{oi})^2} + \frac{\omega_p^2}{(\omega - ku_{oe})^2} - 1 \right] ,$$

$$R_2 Z_o [U_\pm G_o(U_\pm) - U_o G_o(U_o)] = 2\alpha_i \left(\frac{\Omega_p^2}{\omega^2} \right) \left\{ \frac{(U_o - u_{oi})U_\pm}{(U_\pm - u_{oi})} \left[1 + \frac{1}{2\alpha_i (U_\pm - u_{oi})^2} \right] \right.$$

$$\left. - U_o \left[1 + \frac{1}{2\alpha_i (U_o - u_{oi})^2} \right] \right\} ,$$

$$r_2 z_o [u_\pm g_o(u_\pm) - U_o g_o(U_o)] = 2\alpha_e \left(\frac{\omega_p^2}{\omega^2} \right) \left\{ \frac{(U_o - u_{oe})u_\pm}{(u_\pm - u_{oe})} \left[1 + \frac{1}{2\alpha_e (u_\pm - u_{oe})^2} \right] \right.$$

$$\left. - U_o \left[1 + \frac{1}{2\alpha_e (U_o - u_{oe})^2} \right] \right\} ,$$

$$R_1 \alpha_i [(U_\pm - u_{oi}) G_o(U_\pm) - (U_o - u_{oi}) G_o(U_o)] = \frac{\Omega_p^2 U_o}{2(\omega^2 - c^2 k^2)} \left[\frac{1}{(U_\pm - u_{oi})^2} \right.$$

$$\left. - \frac{1}{(U_o - u_{oi})^2} \right] ,$$

$$r_1 \alpha_e [(u_\pm - u_{oe}) g_o(u_\pm) - (U_o - u_{oe}) g_o(U_o)] = \frac{\omega_p^2 U_o}{2(\omega^2 - c^2 k^2)} \left[\frac{1}{(u_\pm - u_{oe})^2} \right.$$

$$\left. - \frac{1}{(U_o - u_{oe})^2} \right] ,$$

$$R_2\alpha_i\left[U_\pm(U_\pm-u_{oi})G_o(U_\pm) - U_o(U_o-u_{oi})G_o(U_o) \pm j\,\frac{\Omega_z}{\sqrt{\alpha_i}\;k}\right]$$

$$= \alpha_i U_o\left(\frac{\Omega_p^2}{\omega^2}\right)\left[\frac{U_\pm}{(U_\pm-u_{oi})^2} - \frac{U_o}{(U_o-u_{oi})^2}\right] ,$$

$$r_2\alpha_e\left[u_\pm(u_\pm-u_{oe})g_o(u_\pm) - U_o(U_o-u_{oe})g_o(U_o) \pm j\,\frac{\omega_z}{\sqrt{\alpha_e}\;k}\right]$$

$$= \alpha_e U_o\left(\frac{\omega_p^2}{\omega^2}\right)\left[\frac{u_\pm}{(u_\pm-u_{oe})^2} - \frac{U_o}{(U_o-u_{oe})^2}\right] . \qquad (A.39)$$

Upon substituting the above expressions into Eq. 58, Eqs. 70 are obtained:

$$R_2\left[U_\pm^2G_o(U_\pm) - U_o^2G_o(U_o) \pm j\,\frac{1}{\sqrt{\alpha_i}}\;\frac{\Omega_z}{k}\right]$$

$$= 2\alpha_i\left(\frac{\Omega_p^2}{\omega^2}\right)U_o\left[U_\pm\left(1+\frac{1}{2\alpha_iU_\pm^2}\right) - U_o\left(1+\frac{1}{2\alpha_iU_o^2}\right) \mp \frac{\Omega_z}{k}\right]$$

$$= \left(\frac{\Omega_p^2}{\omega^2}\right)\left(\frac{U_o}{U_\pm} - 1\right) , \qquad (A.40a)$$

$$r_2\left[u_\pm^2g_o(u_\pm) - U_o^2g_o(U_o) \pm j\,\frac{1}{\sqrt{\alpha_e}}\;\frac{\omega_z}{k}\right] = \left(\frac{\omega_p^2}{\omega^2}\right)\left(\frac{U_o}{u_\pm} - 1\right) , \qquad (A.40b)$$

$$\Lambda = \frac{\Omega_z}{j\sqrt{\pi}\;k}\int_{-\infty}^{\infty}\frac{e^{\alpha_i v_z^2}}{(v_z-U_o)^2}\,dv_z = \frac{j\Omega_z 2\alpha_i}{\sqrt{\pi}\;k}\int_{-\infty}^{\infty}\frac{v_z e^{-\alpha_i v_z^2}}{(v_z-U_o)}\,dv_z ,$$

$$= 2\alpha_i\left(\frac{\Omega_z}{k}\right)G_1(U_o) ,$$

$$= 2\alpha_i\left(\frac{\Omega_z}{k}\right)\left[\frac{j}{\sqrt{\alpha_i}} + U_oG_o(U_o)\right] , \qquad (A.40c)$$

$$\frac{R_1}{\Omega_z}\,[G_o(U_\pm) - G_o(U_o) \pm \Lambda]$$

$$= \frac{1}{\Omega_z}\,\frac{\Omega_p^2}{(\omega^2-c^2k^2)}\,U_o\left[\frac{1}{U_\pm}\left(1+\frac{1}{2\alpha_i U_\pm^2}\right) - \frac{1}{U_o}\left(1+\frac{1}{2\alpha_i U_o^2}\right) \pm \left(\frac{\Omega_z}{k}\right)\frac{1}{U_o^2}\right]\ ,$$

(A.40d)

$$\frac{r_1}{\omega_z}\,[g_o(u_\pm) - g_o(U_o) \pm \lambda]$$

$$= \frac{1}{\omega_z}\,\frac{\omega_p^2}{(\omega^2-c^2k^2)}\,U_o\left[\frac{1}{u_\pm}\left(1+\frac{1}{2\alpha_e u_\pm^2}\right) - \frac{1}{U_o}\left(1+\frac{1}{2\alpha_e U_o^2}\right) \pm \left(\frac{\omega_z}{k}\right)\frac{1}{U_o^2}\right]\ .$$

(A.40e)

Substituting Eqs. A.40 into Eq. 60 yields Eqs. 72. $D_\pm$ and D_z appearing in Eqs. 72 are obtained by setting $u_{oi} = u_{oe} = 0$ in $D_\pm$ and D_z, given by Eqs. 70.

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