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LIAPUNOV'S SECOND METHOD (*)

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1. Introduction.

In the preceding paper [1] an introduction to Liapunov's second or direct method was given based on ideas originally introduced by the author of this paper, and we wish to continue in that direction. We shall give first of all a statement of what can be called the fundamental theorem of stability (Theorem 1) which extends somewhat and includes the results of [1]. This fundamental theorem is based on a broader *definition of a Liapunov function* and makes use of the invariance property of limit sets of solutions of autonomous differential equations. It also has an important bearing on the extension of stability theory to more general dynamical systems and to applications of the theory.

By means of a simple example we will *illustrate* that this theorem takes us beyond the classical theory of Liapunov and shows how one may study the qualitative behavior of systems in the large. The techniques are not unknown but Theorem 1 now brings them within the domain of Liapunov's method while at the same time *unifying the whole theory*. The previous paper [1] demonstrated this unification for theorems on stability. We will indicate — and this is shown also by the example — how one obtains from this same fundamental theorem criteria for instability. Up to this point

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we shall have confined ourselves to autonomous systems and basic to Theorem 1 is the fact that the limit sets of solutions of autonomous systems are invariant sets. We certainly expect therefore that Theorem 1 will have to be modified for nonautonomous systems and following Yoshizawa in [2] we give in Theorem 2 the analogous theorem for nonautonomous systems. As to be expected the information given by Liapunov functions is now less precise but by means of an example it is shown that the conclusion of the theorem given here is, however, the « best possible ». There are types of nonautonomous systems where the limit sets of solutions have an invariance property that enables one to improve Theorem 2. This is discussed in Section 3.

More recently Hale in [3] has shown that properly interpreted the solutions of autonomous functional differential equations have limit sets which are invariant. With modifications this gives him a stability theory quite similar to that for autonomous differential equations. Functional differential equations which include delay differential equations are mathematical models for systems whose future behavior depends upon a portion or all of its past history. They can be expected to be of increasing importance in economics, biology, and control. Hale's work carries us so far beyond our geometric intuition that it is here that we can appreciate the necessity of a theory to guide us and his work suggests how the theory can be developed for more general dynamical systems. Since his paper [3] is complete, and is well illustrated by examples, his results are not summarized here.

2. Autonomous systems.

For the sake of simplicity we shall assume with some exceptions that all functions introduced are C^1 and as much as possible adopt the notations and definitions of [1]. With f an arbitrary C^1 function on R^n to R^n we consider first the ordinary differential equation $\left(\dot{x} = \frac{dx}{dt}\right)$

$$\dot{x} = f(x). \tag{1}$$

In order not to have to confine ourselves to bonded solutions we compactify R^n by adding the point at infinity where the distance $d(\infty, x)$ of x to infinity is $|x|^{-1}$. Thus, if Γ is a set in R^n and we define $\Gamma^* = \Gamma \cup \{\infty\}$, then a function $\varphi(t)$ is said to *approach* Γ^* if $d(\Gamma^*, \varphi(t)) \to 0$ as $t \to \infty$. This also gives a meaning to the statement that ∞ is a limit point of $\varphi(t)$, which is not necessarily the same as saying $\varphi(t) \to \infty$ as $t \to \infty$. When $\varphi(t)$ is a solution of (1) it may happen that its maximum positive interval of definition is $[o, \tau)$. This causes no difficulty. We need only replace ∞ by τ. Understanding this we will usually ignore this point and speak as though all solutions are defined on $(-\infty, \infty)$.

Let G be an arbitrary set in R^n and let V be a C^1 function on R^n to R. We shall say that V is a *Liapunov function on G* for the system (1) if $\dot{V} = (\text{grad } V) \cdot f$ does not change sign on G. We define ($\bar{G}$ is the closure of G)

$$E = \{x ;\ \dot{V}(x) = 0,\ x \in \bar{G}\};$$

M will denote the largest invariant set in E and $M^* = M \cup \{\infty\}$. It then follows easily from the invariance property of limit sets of solutions of (1) that

THEOREM 1. If V is a Liapunov function on G for the system (1) and if a solution $x(t)$ of (1) remains in G for all $t > 0$ ($t < 0$), then $x(t)$ approaches M^* as $t \to \infty$ ($t \to -\infty$). If M is bounded, then either $x(t) \to M$ or $x(t) \to \infty$ as $t \to \infty$ ($t \to -\infty$).

This theorem states that a Liapunov function V on G locates all possible positive and negative limit sets of solutions which remain in G for $t > 0$ or $t < 0$. The problem in applying the theorem is to find a « good » Liapunov function. A constant function V is always a Liapunov function for the whole state space R^n but, of course, gives no information. Here $E = M = R^n$. The theorem does, however, make it possible to obtain more information about the asymptotic behavior of systems with Liapunov functions not as severly restricted as those of the classical theory. It is also true that every C^1 function V is a Liapunov function on the region $\dot{V} \le 0$ (or $\dot{V} \ge 0$) but this may or may not be helpful. The following simple example illustrates some features of this result and how it may be applied and how one obtains additional information by using more than one Liapunov function. It is not always this easy, and this example was manufactured for this purpose. In actuality it is often easier using Liapunov functions to synthesize a system to have a particular behavior than it is to analyze a given system, and this is proving to be true in the design of control systems.

The second order system

$$\dot{x} = -2xy$$

$$(2) \qquad \dot{y} = -x + y + xy - y^3$$

has three equilibrium points: $P_1 = (0, 1)$, $P_2 = (0, -1)$ and $P_3 = (0, 0)$. The eigenvalues of the linear approximation about P_1 are -2, -2; about P_2 they are -2, 2 and about P_3 are $0, 1$. Thus P_1 is asymptotically stable. P_2 is a saddle point and is unstable, and P_3 is unstable. The linear approximation does not given any information about the region of attraction (of asymptotic stability) about P_1 or about the character of the equilibrium point P_3. (See Figure 1).

We have first of all for each of the four quadrants the obvious Liapunov function $V_1 = x$ since $\dot{V} = -2xy$. For each of these quadrants E_1 is the union of the x-and y-axes, and since $\dot{x} = 0$ when $x = 0$ and $\dot{y} = -x$ when $y = 0$, we see that M_1 is the y-axis. It is clear, for example, that no solution starting in the 4th quadrant can leave for $t > 0$ and cannot have a limit point on M_1. Hence all solutions starting in the 4th quadrant approach ∞ as $t \to \infty$.

Another Liapunov function is $V_2 = x - y^2$; $\dot{V}_2 = 2y^2(y^2 - x - 1) = -2y^2(V_2 + 1)$ and V_2 is a Liapunov function for the regions G_1: $V_2 < -1$ and G_2: $V_2 > -1$. Here E_2 is the x-axis and the parabola $V_2 = -1$, which is an integral, and M_2 is the curve $V_2 = -1$ and the origin P_3. The regions G_1 and G_2 are invariant sets. In G_1, $V_2 > 0$ and no solution can approach M_2 as $t \to -\infty$. Therefore every solution starting in G_1 approaches ∞ as $t \to -\infty$. Note next that each solution starting in $x < 0$ remains in this region and is bounded for $t > 0$. Therefore the only possible positive limit points are the intersection of M_1 and M_2 which consists of the three equilibrium points P_1, P_2 and P_3. To the left of P_2, $\dot{x} < 0$ and to the left of P_3, $\dot{V}_2 < 0$ so that every solution starting in the left-half plane $x < 0$ must approach P_1 as $t \to \infty$. Similarly, one can see that every solution starting in this half-plane inside G_2 approaches P_3 as $t \to -\infty$. Also it is easy to see that every solution in the *1st* quadrant above $V_2 = 0$ approaches P_1 as $t \to \infty$. Hence the *2nd* and *3rd* quadrants and this portion of the *1st* quadrant are in the region of attraction of P_1. Below $V_2 = 0$ in the *1st* quadrant there must be a solution which approaches P_3 as $t \to \infty$ and this solution is the boundary of the region of attraction of P_1. We know this must happen since the boundary of the region of attraction is an invariant set and the region of attraction does not include the *4th* quadrant.

The following corollary is a direct consequence of Theorem 1 and illustrates how instability results can be obtained:

COROLLARY 1. Assume inside a set G that $V\dot{V} > 0$ and on the boundary of G that $V = 0$. Then every solution of (1) starting in G approaches ∞ as $t \to \infty$ (or possibly in finite time).

PROOF: The assumptions imply that every solution starting in G remains inside G for $t > 0$ and in fact cannot even have a positive limit point on the boundary of G. Since $G \cap M$ is the empty set, it must be that every solution approaches ∞ as $t \to \infty$ (it could have finite escape time).

In a manner similar to the above proof one can obtain Četaev's instability theorem as a corollary of Theorem 1.

COROLLARY 2. Let G_0 be an open set, let p be an equilibrium point on the boundary of G_0, and let N be a neighborhood of p. If $V(x)\dot{V}(x) > 0$ for x in $G = G_0 \cap N$ and $V(x) = 0$ for x on the boundary of G_0 inside N, then p is unstable.

From the point of view of applications the following is one of the most useful results.

COROLLARY 3. Assume that a component G of the set defined by $V(x) < l$ is bounded, $\dot{V}(x) \leq 0$ for $x \in G$, and $M^0 \subset G$ where $M^0 = \overline{M \cap G}$. Then M^0 is an attractor as $t \to \infty$ and G is in the region of attraction to M^0. If V is constant on the boundary of M^0, then M^0 is a stable attractor (is asymptotically stable).

Thus in the above corollary when M^0 is a single point p, V is constant on M^0 and the point p will be asymptotically stable with G providing an estimate of its stability. This is without any assumption that V be positive definite. However, in applying this theorem where the Liapunov function is itself to provide a positively invariant set one will usually look for a Liapunov function that is positive definite relative to p. Unless the set E where $\dot{V}$ vanishes contains a positively invariant set other than p, the point p will be a minimum of V so for this purpose one might expect « good » Liapunov functions to be positive definite. On the other hand the simple example above demonstrated this may not always be the best procedure and one can often do better using more than one Liapunov function none of which need be positive definite.

3. Nonautonomous systems.

In this section we follow fairly closely the ideas of Yoshizawa in [2] although we will not present them with as great a generality as he acheived. We concern ourselves with the system

$$\dot{x} = f(t, x) \tag{3}$$

where f is continuous for (t, x) in $\mathcal{D} = [0, \infty) \times R^n$ and is C^1 on $\mathcal{D}$ with respect to x (or any other of the known conditions that imply existence and uniqueness of solutions). Here limit sets of solutions are still defined but they will not in general be invariant sets. Hence we cannot expect a result as strong as Theorem 1. Theorem 2 below is a modified version of Theorem 1 and is closely related to Yoshizawa's Theorem 6 in [2].

Let $V(t, x)$ be a C^1 function on $[0, \infty) \times R^n$ to R. We shall say that V is a *Liapunov function on* a set G of R^n if $V(t, x) \geqq 0$ and $\dot{V}(t, x) \leqq -W(x) \leqq 0$ for all $t > 0$ and all x in G where W is continuous on R^n to R. We define

$$E = \{x;\ W(x) = 0,\ x \in \bar{G}\}. \tag{4}$$

Here

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x).$$

We then have

THEOREM 2. If V is a Liapunov function on G for equation (3), then each solution $x(t)$ of (3) that remains in G for all $t > t_0$ approaches $E^* = E \cup \{\infty\}$ as $t \to \infty$, provided one of the following conditions is satisfied:

(*i*) For each $p \in \bar{G}$ there is a neighborhood N of p such that $|f(t, x)|$ is bounded for all $t > 0$ and all x in N

(*ii*) W is C^1 and $\dot{W}(t, x)$ is bounded from above or below along each solution which remains in G for all $t > 0$.

If E is bounded, then each solution of (3) remaining in G for $t > 0$ either approaches E or ∞ as $t \to \infty$.

Thus, this theorem is quite similar to Theorem 1 except that M is replaced by the set E. E is in general larger than M and the information given is not as precise. Condition (*i*) is essentially the same as that used by Yoshizawa. The following example illustrates a case where (*ii*) is satisfied and (*i*) is not and also shows that in general even for linear nonautonomous systems this is the best result one can hope to have.

Consider $\ddot{x} + p(t)\dot{x} + x = 0$ where $p(t) \geqq \delta > 0$. An equivalent system is

$$\dot{x} = y$$

$$\dot{y} = -x - p(t)\,y.$$

Since we do not assume that $p(t)$ is bounded from above, condition (*i*) is not satisfied. With $2V = x^2 + y^2$, $\dot{V} = -p(t)\,y^2 \leqq -\delta y^2$. Thus V is a Liapunov function on the entire state space R^2 and $W = \delta y^2$. It is then clear that each solution is bounded for $t > 0$. Now $\dot{W} = 2\delta\, y\dot{y} = -2\delta(xy + y^2 p(t)) < -2\delta xy$. Hence condition (*ii*) is satisfied. E corresponds to $y = 0$ and we can conclude that for each solution $y(t) = \dot{x}(t) \to 0$ as $t \to \infty$. Since the equation $\ddot{x} + (2 + e^t)\dot{x} + x = 0$ has a solution $x(t) = 1 + e^{-t}$, this we see is the best possible result without further restrictions on $p(t)$. It also shows that Theorem 1 is not true for nonautonomous systems. Here M is the origin and if Theorem 1 held for nonautonomous systems this would imply that the origin is asymptotically stable which in the example it certainly is not.

In using Theorem 2 it is necessary to be able to identify solutions which remain in G for all positive t. We now look at this problem. If the Liapunov function $V(x)$ does not depend on t, define

$$Q_l = \{x;\ V(x) \leqq l\}.$$

It is then clear that the following is true:

LEMMA 1. If $\dot{V}(x) \leqq 0$ for all $t \geqq 0$ and all x in G and Ω_l is a component of Q_l which is contained in G, then each solution $x(t)$ of (3) starting in Ω_l at some time $t_0 \geqq 0$ remains in Ω_l for all $t > t_0$.

If the Liapunov function $V(t, x)$ depends on t, define

$$Q_l = \{x;\ V(t, x) \leqq l \quad \text{for all } t \geqq 0\}$$

$$Q_l^0 = \{x;\ V(0, x) \leqq l\}$$

$$Q_l^+ = \{x;\ V(t, x) \leqq l \quad \text{for some } t \geqq 0\}.$$

It is clear that $Q_l \subset Q_l^0 \subset Q_l^+$. Let Ω_l denote a component of Q_l; then Ω_l^0 will be the component of Q_l^0 and Ω_l^+ will be the component of Q_l^+ which contain Ω_l. We then have

LEMMA 2. If $\dot V(t, x) \leqq 0$ for all $t \geqq 0$ and all $x \in G$ and Ω_l^+ is contained in G then

a. Each solution starting in Ω_l^0 at time $t = 0$ remains in Ω_l^+ for all $t \geqq 0$.

b. Each solution starting in Ω_l at any time $t_0 \geqq 0$ remains in Ω_l^+ for all $t > t_0$.

These two lemmas combined with Theorem 2 give methods for estimating the region of attraction of equilibrium points of non-autonomous systems and for studying their asymptotic behavior in general. One can also derive from these results sufficient conditions for instability but it still remains true that nonautonomous systems are more difficult to study and relatively few significant problems have been solved.

3. Special classes of nonautonomous systems.

Although we cannot in general expect to go beyond Theorem 2 for nonautonomous systems there are some types of such systems where the invariance properties of the limit sets of their solutions enable us to obtain precise information on their asymptotic behavior using Liapunov functions. The simpliest of these are periodic systems (see [4])

$$\dot x = f(t, x) \tag{5}$$

where $f(t + T, x) = f(t, x)$ for all t and x. Here the limit sets of solutions have an invariance property somewhat different from autonomous systems. Suppose that $\Gamma \subset R^n$ is a limit set of a solution $x(t)$ of (5). Then Γ is invariant in the following sense: if p is contained in Γ, then there is a solution of (5) which remains in Γ for all t in $(-\infty, \infty)$. This means that if one starts a solution at p at the proper time it will remain in Γ for all t. However, this is sufficient to obtain a theorem quite similar to Theorem 1.

If $V(t, x)$ is C^1 on $R \times R^n$ and is periodic of period T and G is an arbitrary set in R^n, we say that V is a *Liapunov function on* G for the periodic system (5) if $\dot V$ does not change sign for x in G and all t. Define $E = \{(t, x);\ \dot V(t, x) = 0,\ x \in \bar G\}$ and let M be the union of all solutions $x(t)$ of (5) with the property that $(t, x(t))$ is in E for all t. M is called the *largest invariant set relative to E*. One then obtains the following theorem for periodic systems:

THEOREM 3. If V is a Liapunov function on G for the periodic system (5), then each solution of (5) which remains in G for all $t > 0$ ($t < 0$) approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$ ($t \to -\infty$). If M is bounded, then either $x(t) \to M$ or $x(t) \to \infty$ as $t \to \infty$ ($t \to -\infty$).

Recently in [5] Miller has shown that the limit sets of almost periodic systems have an invariance property and one then obtains a similar theorem for almost periodic systems. These results provide improved methods for studying these classes of nonautonomous systems. This periodic version and Miller's almost periodic version of Theorem 1 are not as well known as they should be in spite of the fact it would seem that the difficulty in applying them is not much greater than for autonomous system.

A simple example is the following:

$$\dot x = y$$

$$\dot y = -(a + \cos t)x - by.$$

With

$$V = x^2 + (a + \cos t)^{-1} y^2,$$

$$\dot V = -\frac{1}{2}(a + \cos t)^{-1}\left(2b - \frac{\sin t}{a + \cos t}\right) y^2.$$

If $a > 1$ and $2b\sqrt{a^2 - 1} > 1$, then $\dot V \leqq 0$ and V is a Liapunov function on the plane R^2. The form of V implies that the origin is stable and that all solutions are bounded for $t > 0$. Here $E = \{(t, x, 0);\ -\infty < t < \infty,\ -\infty < x < \infty\}$ but M is simply the origin. Therefore for $a > 1$ and $2b\sqrt{a^2 - 1} > 1$ the origin is asymptotically stable in the large.

As has been shown by Opial in [6] and Markus in [7] the solution of what may be called « asymptotically autonomous » systems have limit sets with an invariance property which we will explain in a minute. In [2] Yoshizawa used this invariance property and obtained a result similar to Theorem 4 below.

A system of the form

$$\dot{x} = f(t, x) = F(x) + f_1(t, x) + f_2(t, x) \tag{6}$$

will be said to be ***asymptotically autonomous*** if (i) (Markus) $f_1(t, x)$ approaches zero as $t \to \infty$ uniformly for x in an arbitrary compact set of R^n, (ii) (Opial) $\int_0^\infty |f_2(t, \varphi(t))|\, dt < \infty$ for all $\varphi(t)$ continuous and bounded on $[0, \infty)$ to R^n. The combined results of Markus and Opial then state that the positive limit sets of solutions of (6) are invariant sets of $\dot{x} = F(x)$. This then leads immediately, as a consequence of Theorem 2 to the following:

THEOREM 4. If V is a Liapunov function on G for the asymptotically autonomous system (6), then each solution of (6) which remains in G for all $t > 0$ approaches $M^* = M \cup \{\infty\}$, where M is the largest invariant set of $\dot{x} = F(x)$ in E, provided f_2 satisfies condition (i) of Theorem 2 or W satisfies condition (ii) of Theorem 2.

It turns out to be useful in order to apply this result to nonautonomous systems (3) which are not asymptotically autonomous to give also the following version of this theorem.

THEOREM 4. If in addition to the conditions of Theorem 2 it is known that the positive limit set of $x(t)$ is an invariant set of $\dot{x} = G(x)$, then $x(t) \to M^* = M \cup \{\infty\}$ where M is the largest invariant set of $\dot{x} = G(x)$ in E.

The example

$$\dot{x} = y$$
$$\dot{y} = -x - p(t)\, y, \quad 0 < \delta \leqq p(t) \tag{7}$$

considered before can again be used to illustrate the above theorem and to show how it can be applied even when the original system is not asymptotically autonomous. Let $(\bar{x}(t), \bar{y}(t))$ be any solution of (7). As shown previously we know it is bounded for $t > 0$ and that $\bar{y}(t) \to 0$ as $t \to \infty$. Assume now in addition that $p(t)$ is bounded from above: $0 < \delta \leqq p(t) \leqq m$ for all $t > 0$. Then consider for this particular solution the system

$$\dot{x} = y$$
$$\dot{y} = -x - p(t)\, \bar{y}(t).$$

Certainly $\bar{x}(t), \bar{y}(t)$ is a solution, and this system is asymptotically autonomous to $(^*)$ $\dot{x} = y$, $\dot{y} = -x$. Therefore the positive limit set of $(\bar{x}(t), \bar{y}(t))$ is an invariant set of $(^*)$ and must also lie on the x-axis. Hence its positive limit set is the origin. This means that when $0 < \delta \leqq p(t) \leqq m$ for all $t > 0$ the system (7) is asymptotically stable in the large.

REFERENCES

[1] STRAUSS, ARON, « *A geometric introduction to Liapunov's second method* », this volume, (Proc. of NATO SUMMER Instit., Padua, Italy. 1965).

[2] YOSHIZAWA, T., « *Asymptotic behavior of solutions of a system of differential equations* », Contrib. to Diff. Eq., 1 (1963), 371-387.

[3] HALE, J., « *Sufficient conditions for stability and instability of autonomous functional differential equations* », J. of Differential Equations, 1 (1965), 452-482.

[4] LASALLE, J. P., « *Asymptotic stability criteria* », Proc. of Symposia in Applied Math., Vol. 13, Hydrodynamic Instability, Amer. Math. Soc., 1962, 299-307.

[5] MILLER, R., « *On almost periodic differential equations* », Bul. Amer. Math. Soc., 70 (1964), 792-794.

[6] OPIAL, Z., « *Sur la dépendence des solutions d'un système d'équations differentielles de leur second membres. Application aux systèmes presque autonomes* », Ann. Polonici Math., 8 (1960), 75-89.

[7] MARKUS, L., « *Asymptotically autonomous differential equations* », Contrib. to the Theory of Nonlinear Oscillations, Vol. III, Annals of Math. Studies No. 36, Princeton University, 1956, 75-89.

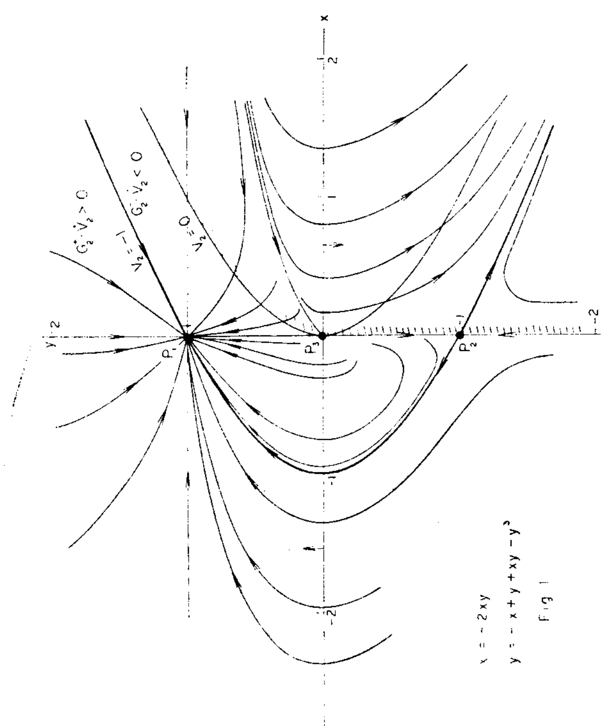

Fig 1