Second Interim Technical Report

EFFECT OF THE TWO-PARTICLE SPIN-ORBIT NUCLEON-NUCLEON INTERACTION ON THE O^{16} and N^{16} NUCLEI

by

Byron L. Coulter

on research performed at

The University of Alabama

August, 1966

Supported as Project No. 4819-P by the U. S. Army Research Office-Durham, under Grant No. DA-ARO-D-31-124-G664.

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INTRODUCTION

The nonrelativistic description of a nucleus of A interacting particles is given by the solution of the Schrodinger equation

$$\left[-\frac{\hbar^2}{2M} \sum_{i=1}^{A} \nabla_i^2 + \sum_{i<j}^{A} v_{ij} \right] |\psi\rangle = E|\psi\rangle$$

where v_{ij} represents the interaction between the i-th and j-th nucleons and M is the nucleon mass. The problem of obtaining this description is seen to be twofold: (1) finding the interaction operator explicitly and (2) using the interaction operator to deduce the physical character of the various nuclear systems in order to compare them with experiment. Regrettably, the nucleon-nucleon interaction is sufficiently complicated that the explicit form of the interaction operator is as yet undetermined. Furthermore, even if v_{ij} were known, the Schrodinger equation for a system of A interacting particles could not be solved generally. Thus, to obtain an adequate explanation of the properties of various nuclei, theorists have been forced to introduce many simplifying assumptions, some of which can be justified only by their end result.

One approximation which has been remarkably successful in predicting many of the properties of nuclear ground states, but for which the theoretical justification was slow in being developed is the shell model of Mayer[1] and of Haxel, Jensen, and Suess.[2] The basic postulate of this theory is that the motion of an individual nucleon can be regarded as that of a particle in a spherically symmetric potential produced by the

rest of the nucleons. In its original form, this model is inadequate in dealing with excited states, and modification has been found necessary in two respects. First, many of the properties of light nuclei in particular have been found to arise from the reorientation of several particles relative to each other, leading to the individual-particle picture of nuclear structure in which a residual interaction between nucleons is assumed to induce a correlation in the motion of the particles in a central potential. Second, the central field has been shown in general to be neither static nor spherically symmetric, indicating a collective model of nuclear structure in which the motions of individual particles are regarded as being superposed upon the collective oscillations of the nucleus as a whole. This model, as first developed by Bohr and Mottelson,[3] has its greatest success in the region of heavy nuclei well away from closed shells. In light nuclei and in regions near to closed shells, the individual-particle aspects of nuclear structure are dominant, and for these nuclei the collective effects may be treated in terms of configuration mixing.

The first employment of the individual-particle model for a detailed analysis of the odd-parity states of O^{16} and N^{16} nuclei was by Elliott and Flowers.[4] They assumed that these states could be described primarily by a hole in a 1p state and a particle in a 1d or 2s state moving in a harmonic-oscillator potential. A particle-hole interaction consisting of a single-particle spin-orbit potential and a two-particle central interaction potential with a Rosenfeld exchange mixture was treated as a perturbation. Subsequent investigations by Duck[5] and by Gillet[6] were based on the same assumptions, and only the parameters in the central

interaction potential and the character of the exchange mixture were changed. This method produced a reasonable picture of the O^{16} and N^{16} nuclei, but some anomalous features did develop, primarily in the beta decay of the 2^- level of N^{16} to the 1^- level of O^{16} and in the O^{16} muon capture. Our study is directed toward eliminating these irregularities by employing a Serber exchange mixture for the central interaction potential and by replacing the single-particle spin-orbit potential in the residual interaction with a two-particle spin-orbit interaction potential. Thus, we are seeking not only information about these nuclei, but also knowledge of the general nucleon-nucleon interaction potential.

Occuring in our calculations are sixteen independent nuclear and nucleon parameters, the indiscriminate variation of which would lead to the fitting of almost any data desired. Since this would be of dubious significance, we assign to thirteen of the parameters values indicated by scattering and energy level data; the remaining three are varied about their experimental values to compensate partially for the inaccuracies resulting from theoretical approximations and experimental error. In every instance, however, we conform to experimentally-indicated results as closely as possible.

The main text of this paper is divided into three sections. In Chapter I the possible nucleon-nucleon interaction potentials are considered, and the form is selected which is thought necessary to obtain a reasonably accurate picture of the O^{16} and N^{16} nuclei without unnecessarily complicating the calculations. Then the assumptions made to approximate the solution of the Schrodinger equation are discussed, and the potential well introduced by these assumptions is chosen. A mathematical treatment of the methods used for constructing the energy matrices

is presented in Chapter II. The final chapter contains theoretical predictions of the energy-level splittings of O^{16} and N^{16} and of the beta-decay and muon-capture rates for the two nuclei. After a comparison of these results with experimental data, the nuclear state vectors giving the best experimental fit are chosen. The appendices contain three derivations too cumbersome to be included in the main text, but necessary for the completeness of the work, and a summary of formulas from Racah algebra which are used throughout our calculations.

CHAPTER I

THE MUTUAL INTERACTION BETWEEN NUCLEONS

Form of the Nucleon-Nucleon Interaction Potential

In describing a nucleus of interacting particles by the Schrodinger equation

$$\left[-\frac{\hbar^2}{2M} \sum_{i=1}^{A} \nabla_i^2 + \sum_{i<j}^{A} v_{ij} \right] |\psi\rangle = E|\psi\rangle \qquad \text{(I.1)}$$

we are making two assumptions about the mutual interaction between nucleons. First, and most basic, we are assuming that the interaction of nucleons through the meson field can be represented by an interaction potential between nucleons. Such an approximation is considered valid only if $(v/c)^2 \ll 1$ where v is the mean nucleon velocity. Taking the average kinetic energy of a nucleon in a nucleus to be 20 Mev,[7] we find $(v/c)^2 \approx 1/20$ so that the nonrelativistic approach is justified. The second assumption made in writing Eq. (I.1) is that the mutual interaction between nucleons is a two-body interaction. Although many-body forces may someday be required by meson theory, their necessity has not as yet been demonstrated.[8]

The interaction potential between nucleons has been found to satisfy many requirements which severely restrict its possible forms. In addition to being Hermitian, the interaction potential must be invariant under the following transformations:[9] (1) Galilean transformations, (2) rotations

in total angular momentum space, (3) spatial reflections, (4) time reversal, and (5) rotations in isotopic-spin space. Furthermore, Lorentz invariance arguments indicate that terms in the interaction involving powers of the momenta higher than the first must be very small.[7]

We also assume that the mutual interaction potential can be factored into separate orbital, intrinsic-spin, and isotopic-spin parts in the form

$$v_{ij} = [L^{\lambda}(r_i, r_j) \cdot S^{\lambda}(\sigma_i, \sigma_j)]\, D(\tau_i, \tau_j) \tag{I.2}$$

The isotopic-spin operator $D(\tau_i, \tau_j)$ must be scalar to satisfy the fifth invariance requirement and thus is restricted to being either 1 or $(\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j)$. The spatial part of the interaction potential is written as a scalar product of two tensors of degree λ which operate in different spaces. Since the spin operators are first-degree tensors, we can only form two-particle operators for which $\lambda \leq 2$. Thus the possible interaction potentials may be separated into three groups according to the degree of the spin operator tensor.

$\underline{\lambda = 0}$. The part of the mutual interaction which is scalar in both spin and orbital spaces is called the scalar interaction potential v^{o}_{ij}. From invariance requirements we find that $S^{o}(\sigma_i, \sigma_j)$ can take only the forms 1 or $(\underset{\sim}{\sigma}_i, \underset{\sim}{\sigma}_j)$ and $L^{o}(r_i, r_j)$ can only be a function of the distance between particles. The most general form of the scalar interaction potential is then

$$v_{ij} = [a_o + a_1 (\underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j) + a_2 (\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j) + a_3 (\underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j)(\underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j)] \times v^{o}(|\underset{\sim}{r}_i - \underset{\sim}{r}_j|) \tag{I.3}$$

The coefficients in this equation may be evaluated from scattering data more readily if the potential is rewritten in terms of permutation operators. From the relations[10]

$$P_\sigma(ij) = \frac{1}{2}\left[1 + \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j\right]$$

and

$$P_\tau(ij) = \frac{1}{2}\left[1 + \underset{\sim}{\tau}_i \cdot \underset{\sim}{\tau}_j\right]$$

where $P_\sigma(ij)$ and $P_\tau(ij)$ are the permutation, or exchange, operators in intrinsic-spin and isotopic-spin spaces respectively, and from the property of antisymmetric states

$$P_r(ij) = -P_\sigma(ij)\, P_\tau(ij)$$

where $P_r(ij)$ is the exchange operator in orbital space, it follows that the scalar interaction potential may be written

$$v^o_{ij} = v^o(|r_i - r_j|)\left[W + MP_r(ij) - HP_\tau(ij) + BP_\sigma(ij)\right] \qquad \text{(I.4)}$$

where

$$W = a_0 - a_1 - a_2 + a_3$$

$$M = -4a_3$$

$$H = -2a_2 + 2a_3$$

$$B = 2a_1 - 2a_3 \quad .$$

Low-energy neutron-proton scattering data[8] indicate that the Serber exchange mixture, defined by $W = M = 0.5$ and $H = B = 0$, or by $a_0 = 0.375$, $a_1 = a_2 = a_3 = -0.125$, closely approximates the actual case.

It has been found that the nuclear states are relatively insensitive to the radial dependence of the scalar interaction potential for reasonable well shapes.[9] Thus, letting $\underset{\sim}{r} = (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$, we assume for simplicity

that $v^o(r)$ has the form

$$v^o(r) = v^s(r)\ P^s + v^t(r)\ P^t$$

where $P^s = \frac{1}{4}(1 - \underset{\sim}{\sigma_1} \cdot \underset{\sim}{\sigma_2})$ and $P^t = \frac{1}{4}(3 + \underset{\sim}{\sigma_1} \cdot \underset{\sim}{\sigma_2})$ are the projection operators for the singlet and triplet states, respectively, and the $v^{s,t}(r)$ are taken to be Yukawa wells, i.e.

$$v^{s,t}(r) = V_{s,t} \frac{e^{-\frac{r}{a_{s,t}}}}{\frac{r}{a_{s,t}}} \quad .$$

The parameter values indicated by scattering experiments are:[11]

$$V_s = -46.9 \text{ MeV} \qquad a_s = 1.17 \text{ F.}$$

$$V_t = -52.1 \text{ MeV} \qquad a_t = 1.38 \text{ F.}$$

These values of the range parameters a_s and a_t will be accepted, but the interaction strengths V_s and V_t will be retained as free variables.

$\underline{\lambda = 1}$. The existence of an interaction potential with a vector spin component is necessary in first-order approximations to provide the doublet splitting found in nuclear states. As is well known, a single-particle potential which will give this splitting is

$$U_{so} = \sum_i \xi\ (r_i)\ \underset{\sim}{\ell_i} \cdot \underset{\sim}{S_i} \quad . \tag{I.5}$$

Many nuclear investigations, including all previous studies of the O^{16} and N^{16} nuclei, assume this single-particle potential to be the sole term in the mutual interaction potential with a vector spin dependence. There is evidence, however, that a potential of this type is not an actual nuclear potential, but rather that it is an effect of a more complicated potential.[12]

The only two-body potential which has a vector spin component and satisfies all the invariance requirements listed previously is

$$v'_{12} = [v'(r) + (\underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2)\, U'(r)](\underset{\sim}{s}_1 + \underset{\sim}{s}_2) \cdot \underset{\sim}{\ell}_{12} \qquad \text{(I.6)}$$

where

$$\underset{\sim}{\ell}_{12} = \frac{1}{2}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \times (\underset{\sim}{p}_1 - \underset{\sim}{p}_2) \quad .$$

Indications of the existence of this potential, which we will call the vector interaction potential, have been summarized by Signell and Marshak as follows:[12]

(1) Short-range spin-orbit interaction potentials originate in field theory if nucleon recoil is taken into account.

(2) For a nucleon outside a closed shell, the vector interaction potential produces on the nucleon an effective force of the type $\xi_1 \underset{\sim}{\ell} \cdot \underset{\sim}{s}$. This is, in fact, the only two-body interaction which in a first-order treatment will produce the correct spin-orbit splitting.

(3) An analysis of 310-MeV scattering data shows a large spin-orbit type scattering amplitude.

(4) The vector interaction potential is needed to explain a phase shift in 150-MeV scattering data.

(5) An extensive computer search for a good phenomenological interaction potential with no velocity dependence ended in failure. Previous calculations also show that the vector potential is attractive and that $U'(r)$ can be taken to be zero.[13]

Studies which employ only the single-particle spin-orbit potential are, in essence, ignoring the mutual interaction through the vector

potential of all particles outside closed shells. This has been found to be a good approximation for nuclei in which only the 1p shell is occupied.[14] However, the two-particle spin-orbit interaction has been found to play an important role in binding-energy calculations for nuclei with the particles in the 1d and 2s shells,[15] and consequently the vector interaction given by Eq. (I.6) with $U'(r) = 0$ is used in this study.

The nuclear states have been shown to be more sensitive to the radial dependence of the vector potential than to that of the scalar potential.[16] Therefore, we must choose the function $v'(r)$ more carefully than $v^o(r)$. The usual well shapes, such as the Yukawa well, the Gaussian well, the square well, etc., have been shown to give poor results in scattering experiments[17] and so are unacceptable. However, a reasonable approximation has been found to be a derivative of a Yukawa well with a straight cut-off, given by

$$v'(r) = V_{SO} \frac{1}{x_o} \frac{d}{dx} \left(\frac{e^{-x}}{x} \right) \Bigg|_{x = x_o} \qquad 0 \leq r \leq r_o \qquad \text{(I.7)}$$

$$v'(r) = V_{SO} \frac{1}{x} \frac{d}{dx} \left(\frac{e^{-x}}{x} \right) \qquad r_o \leq r \leq \infty$$

where $x = (r/a_{SO})$. The parameters in this form of the potential well have been found to be approximately given by[18] $V_{SO} = 30$ MeV, $a_{SO} = 1.07$ F., and $r_o = 0.21$ F. Once again, however, we will retain the interaction strength V_{SO} as a free parameter.

$\underline{\lambda = 2}$. The existence of an interaction potential (hereafter referred to as the tensor interaction potential) with the spin dependence given by a second-degree tensor is necessary to explain such properties as the quadrupole moment of the deuteron. However, previous calculations[9]

have shown its effects to be negligible in the first-order approximations used in this study, and it is omitted in our work.

Methods of Approximation

The basis of the nuclear shell model is the idea that the interaction of a nucleon with the rest of the nucleus can be replaced by the interaction of the nucleon with a central field. Obviously, however, a common single-particle central interaction cannot completely replace the actual mutual interactions in a system of many particles, and to be more practical we have to consider corrections to the central field. We are therefore interested in a system whose total Hamiltonian is given by $H = H_0 + H_1$, where H_0 is the Hamiltonian of the central field and H_1 is a residual interaction added to make the total Hamiltonian more realistic.

For a system having one particle outside a shell with one hole, the sum of the mutual interaction over all pairs of particles may be separated into three parts:

$$\sum_{i<j} v_{ij} = \sum_{m<n} v_{mn} + \sum_{m} (v_{mp} - v_{mh}) - v_{ph} \qquad (I.8)$$

where the sum over m and n is over all possible particle states in the closed shell, p denotes the particle outside the shell, and h represents the hole state. (Note that the sum over m and n does include h, but not p.) The first sum on the right-hand side of Eq. (I.8) merely gives the binding energy of the closed shells and we will arbitrarily take this to be zero. This assignment in no way affects our derivation of the nuclear states, but merely provides a reference point from which to measure energy levels. We now make the fundamental assumption of our model: the

interaction of nucleon with closed shells through the scalar interaction potential (see Eqs. (I.3) and (I.4)) may be replaced by the interaction of the nucleon with a central field. Thus the interaction of the particle and hole states with the closed shell through v^o is given by

$$\sum_m (v^o_{mp} - v^o_{mh}) = U(r_p) - U(r_h) \quad .$$

As noted in the previous section, the interaction of a nucleon with the closed shells through the vector interaction term (see Eq. (I.6)) can be replaced by the single-particle spin-orbit potential

$$U_{SO} = \sum \xi_i \underset{\sim}{\ell_i} \cdot \underset{\sim}{S_i}$$

where the sum is over the particle and the hole states. Thus the residual interaction consists only of U_{SO} and the interaction between the particle and hole states, i.e.,

$$H_1 = U_{SO} - v_{ph} \quad . \tag{I.9}$$

Although there is no rigorous theoretical justification for treating nuclei by first-order perturbation theory, such an approach has led to reasonable approximations in previous calculation,[4-6] and thus we use it here. We assume that the eigenstates $|\psi\rangle$ of the total Hamiltonian can be approximated by a linear combination of the degenerate eigenstates of the unperturbed Hamiltonian. Hence if the eigenstates $|E^o_i\alpha\rangle$ satisfy the relation

$$H_o|E^o_i\alpha\rangle = E^o_i|E^o_i\alpha\rangle \quad ,$$

then $|\psi\rangle$ is approximately equal to $|1\rangle$ where

$$|1\rangle = \sum_{\alpha} a_{\alpha} |E_a^o \alpha\rangle \quad . \tag{I.10}$$

Denoting the first-order energy correction by ε_1, then we find by use of degenerate perturbation theory that[19]

$$\begin{pmatrix} \mathcal{H}_{\alpha_1\alpha_2} & \mathcal{H}_{\alpha_1\alpha_1} & \mathcal{H}_{\alpha_1\alpha_3} & \cdots \\ \mathcal{H}_{\alpha_2\alpha_1} & \mathcal{H}_{\alpha_2\alpha_2} & \mathcal{H}_{\alpha_2\alpha_3} & \cdots \\ \mathcal{H}_{\alpha_3\alpha_1} & \mathcal{H}_{\alpha_3\alpha_2} & \mathcal{H}_{\alpha_3\alpha_3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} a_{\alpha_1} \\ a_{\alpha_2} \\ a_{\alpha_3} \\ \vdots \end{pmatrix} = \varepsilon_1 \begin{pmatrix} a_{\alpha_1} \\ a_{\alpha_2} \\ a_{\alpha_3} \\ \vdots \end{pmatrix} \tag{I.11}$$

where

$$\mathcal{H}_{\alpha\alpha'} = \langle E_a^o \alpha | H_1 | E_a^o \alpha' \rangle \quad .$$

Therefore by diagonalizing the matrix $(\mathcal{H}_{\alpha\alpha'})$, we may evaluate the first-order energy corrections and the corresponding constants a_{α}, which determine the eigenstate $|1\rangle$.

The Nuclear Potential Well

If the entire set of eigenstates $|E_i^o \alpha\rangle$ were used in the perturbation theory calculations, the source of the eigenstates, i.e., the potential well, would become irrelevant.[9] In practice, it is possible for one to deal only with the eigenstates corresponding to the lowest energy eigenvalue (or, in simple cases, eigenvalues). There is still no need to introduce the well explicitly, and only the radial shape of the well must be specified; no reference to its depth is necessary.

The nuclear potential well is known to be flat over the bulk of the nucleus and to climb steadily to zero at the edge of the nucleus, as one may represent quite well by the Saxon-Woods potential

$$V(r) = -V_o[1 + \exp\left(\frac{r - R}{d}\right)]^{-1} \quad .$$

Unfortunately, these features are not represented by any known well shape for which the eigenvalues and eigenstates are readily obtainable. Although numerical computations would give the desired information, such labor seems unnecessary because the actual well shape can be approximated by two idealized wells, the square well and the harmonic-oscillator well, for which the spectra are known. Strictly accurate results cannot be obtained by using these idealized wells. Nevertheless, qualitative features may be determined quite well; in fact, a numerical calculation using a Saxon-Woods potential of the wave functions and radial integrals for nuclei of mass 38 shows a quite negligible difference from those calculated in an oscillator field.[20] For computational ease, we assume that the nuclear potential can be approximated by a harmonic-oscillator potential

$$U(r) = \frac{1}{2} M\omega^2 r^2$$

where M is the nucleon mass and ω is the frequency of vibration of the classical oscillator.

The eigenstates of the harmonic-oscillator potential are

$$|\psi_{n\ell m}\rangle = \frac{R_{n\ell}(r)}{r} Y_\ell^m(\theta, \varphi) \qquad \text{(I.12)}$$

where $Y_\ell^m(\theta, \varphi)$ is a spherical harmonic and the radial function $R_{n\ell}(r)$ is given by

$$R_{n\ell}(r) = \frac{\left[2\left(\frac{1}{b}\right)^{2\ell+3} n!\right]^{\frac{1}{2}}}{\Gamma(n + \ell + \frac{3}{2})} r^{\ell+1} e^{-\frac{r^2}{2b^2}} L_n^{\ell+\frac{1}{2}}\left(\frac{r^2}{b^2}\right) \qquad \text{(I.13)}$$

where $L_n^{\ell+\frac{1}{2}}(x)$ is a Laguerre polynomial and $b = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}$ is the harmonic-oscillator length. (In this harmonic-oscillator notation, the minimum value of the principle quantum number n is 0 rather than 1 as in the usual nuclear notation.)

The energy eigenvalues for the harmonic-oscillator potential are

$$E_{n\ell} = (2n + \ell + \frac{3}{2})\,\omega\hbar \quad .$$

Thus for this potential the 1s and 0d states are degenerate. Since experimental data indicate that the 0d energy level is about 1.16 MeV lower than that of the 1s state, we add this energy to the $(0p)^{-1}$ (0d) configuration in computing the energy matrix of Eq. (I.11).

CHAPTER II

EVALUATION OF THE ENERGY MATRIX FOR HOLE-PARTICLE CONFIGURATIONS

Previous calculations[4] concerning O^{16} and N^{16} have shown that the nuclear states are better approximated by a pure j-j representation than by a pure L-S representation, and consequently j-j coupled wave functions are used in this paper. This choice is not critical, however, as we can transform the state vectors from one representation to the other by means of the A-coefficients discussed in Appendix A. If we denote by $|\overline{jm}\rangle$ a hole in a state with quantum numbers j and m (the quantum numbers n, ℓ, s, τ, m_τ being understood), then the energy matrix elements of Eq. (I.11) for hole-particle configurations are

$$H_{\alpha\alpha'} = \langle \overline{j_1} j_2 \; JM_J TM_T \; | U_{SO} - v \; | \overline{j_1'} j_2' \; JM_J TM_T \rangle$$

where U_{SO} is the single-particle spin-orbit interaction potential and v is the mutual-interaction potential between the hole and particle states. The matrix elements of the single-particle spin-orbit term are found to be[21]

$$\langle \overline{j_1} j_2 \; JM_J TM_T \; | U_{SO} | \overline{j_1'} j_2' \; JM_J TM_T \rangle \tag{II.1}$$

$$= \frac{1}{2} \delta(j_1, j_1') \, \delta(j_2, j_2') \, \delta(\ell_1, \ell_1') \, \delta(\ell_2, \ell_2') \, \delta(n_1, n_1') \, \delta(n_2, n_2')$$

$$\times \left\{ \xi_{n_2 \ell_2} \left[j_2(j_2+1) - \ell_2(\ell_2+1) - \tfrac{3}{4} \right] - \xi_{n_1 \ell_1} \left[j_1(j_1+1) - \ell_1(\ell_1+1) - \tfrac{3}{4} \right] \right\}$$

The parameters $\xi_{n_2 \ell_2}$ and $\xi_{n_1 \ell_1}$ can be determined experimentally from neighboring nuclei. The computation of the matrix elements for the

mutual-interaction potential is more more complex and requires detailed discussion.

Decomposition of Energy Matrix Elements into Orbital, Intrinsic Spin, and Isotopic Spin Components

As shown in Appendix B, the matrix elements of operators in the hole-particle configuration are reduced to matrix elements of operators in the particle-particle configuration with the result

$$\langle \overline{j_1} j_2 J M_J T M_T | \; v \; | \overline{j_1'} j_2' J M_J T M_T \rangle$$

$$= \sum_{J',T'} (2J'+1)(2T'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & T' \end{Bmatrix}$$

$$\times \left[\langle j_1' j_2 J' M_{J'}; \tfrac{1}{2}\tfrac{1}{2} T' M_{T'} | \; v \; | j_1 j_2' J' M_{J'}; \tfrac{1}{2}\tfrac{1}{2} T' M_{T'} \rangle \right.$$

$$\left. - (-1)^{j_1+j_2'+J'+T'+1} \langle j_1' j_2 J' M_{J'}; \tfrac{1}{2}\tfrac{1}{2} T' M_{T'} | \; v \; | j_2' j_1 J' M_{J'}; \tfrac{1}{2}\tfrac{1}{2} T' M_{T'} \rangle \right].$$

Writing the general term of the scalar interaction as

$$V^o = [U^K(\tau_1) \cdot U^K(\tau_2)] \; [W^q(\sigma_1) \cdot W^q(\sigma_2)] \; V(r)$$

and isolating the isotopic spin dependence of the matrix elements by use of Eqs. (A.14), (A.18D), (A.5), and (A.6), we find the above may be expressed as

$$\langle \overline{j_1 j_2} JM_J TM_T | \; V^o \; | \overline{j_1' j_2'} JM_J TM_T \rangle$$

$$= -\langle \tfrac{1}{2} \| \; U^K(\tau_1) \| \; \tfrac{1}{2} \rangle \; \langle \tfrac{1}{2} \| \; U^K(\tau_2) \| \; \tfrac{1}{2} \rangle \sum_{J'} \; (2J' + 1) \; \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} \tag{II.2}$$

$$\times \; [(-1)^{1-T-K} \; \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & K \end{Bmatrix} \quad \langle j_1' j_2 J' M_J' | \; [W^q(\sigma_1) W^q(\sigma_2)] v(r) | j_1 j_2' J' M_J' \rangle$$

$$- \; (-1)^{j_1 + j_2' + J'} \; \frac{\delta(T,K)}{2T+1} \; \langle j_1' j_2 J' M_J' | \; [W^q(\sigma_1) W^q(\sigma_2)] v(r) | j_2' j_1 J' M_J' \rangle].$$

The matrix elements in this equation could be evaluated by performing a tensor expansion of the function $V(|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|)$ and recoupling the angular momentum part of the potential operator into a product of a function of the coordinates of the first particle and a function of the coordinates of the second particle.[11] A somewhat easier approach, however, is to transform the state vectors from the j-j to the L-S coupled representation by means of Eq. (A.8). Accordingly, we find

$$\langle j_a j_b J' M_J' | [W^q(\sigma_1) \cdot W^q(\sigma_2)] V(r) | j_c j_d J' M_J' \rangle$$

$$= \sum_{\substack{LS \\ L'S'}} \; A \begin{pmatrix} \ell_a & \frac{1}{2} & j_a \\ \ell_b & \frac{1}{2} & j_b \\ L & S & J' \end{pmatrix} \quad A \begin{pmatrix} \ell_c & \frac{1}{2} & j_c \\ \ell_d & \frac{1}{2} & j_d \\ L' & S' & J' \end{pmatrix}$$

$$\times \; \langle (\ell_a \ell_b) L, (\tfrac{1}{2} \tfrac{1}{2}) S, J' M_J' | [W^q(\sigma_1) \cdot W^q(\sigma_2)] V(r) | (\ell_c \ell_d) L', (\tfrac{1}{2} \tfrac{1}{2}) S', J' M_J' \rangle .$$

The intrinsic-spin portion of the above matrix element may be evaluated from Eqs. (A.14), (A.18D), and (A.3) with the result

$$\langle j_a j_b J' M_J' | [W^q(\sigma_1)\cdot W^q(\sigma_2)] V(r) | j_c j_d J' M_J' \rangle$$

$$= \langle \tfrac{1}{2} \| W^q(\sigma_1) \| \tfrac{1}{2} \rangle \langle \tfrac{1}{2} \| W^q(\sigma_2) \| \tfrac{1}{2} \rangle \sum_{L,S} (-1)^{S+1} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & q \end{Bmatrix} \qquad \text{(II.3)}$$

$$\times A \begin{pmatrix} \ell_a & \frac{1}{2} & j_a \\ \ell_b & \frac{1}{2} & j_b \\ L & S & J' \end{pmatrix} \quad A \begin{pmatrix} \ell_c & \frac{1}{2} & j_c \\ \ell_d & \frac{1}{2} & j_d \\ \ell & S & J' \end{pmatrix} \langle \ell_a \ell_b L M_L | V(r) | \ell_c \ell_d L M_L \rangle .$$

From Eq. (A,15A) and (A.15B) the reduced matrix elements of the intrinsic spin and isotopic spin operators are evaluated to be

$$\left(\tfrac{1}{2} \| \tau \| \tfrac{1}{2} \right) = \left(\tfrac{1}{2} \| \sigma \| \tfrac{1}{2} \right) = \sqrt{6} , \qquad \text{(II.4A)}$$

$$\left(\tfrac{1}{2} \| 1 \| \tfrac{1}{2} \right) = \sqrt{2} . \qquad \text{(II.4B)}$$

Hence using Eqs. (II.2), (II.3), and (II.4), we find that the matrix elements of the general scalar interaction potential of Eq. (I.3) is given by

$$\langle \bar{j}_1 j_2 J M_J T M_T | v^o | \bar{j}_1' j_2' T M_T \rangle$$

$$= - \sum_{L,S,J'} (2J'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} A \begin{pmatrix} \ell_1' & \frac{1}{2} & j_1' \\ \ell_2 & \frac{1}{2} & j_2 \\ L & S & J \end{pmatrix} A \begin{pmatrix} \ell_1 & \frac{1}{2} & j_1 \\ \ell_2' & \frac{1}{2} & j_2' \\ L & S & J' \end{pmatrix}$$

$$\times \quad [a_o + (-1)^{S+1} 6a_1 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix}][\langle \ell_1' \ell_2 L M_L | V(r) | \ell_1 \ell_2' L M_L \rangle$$

$$+ (-1)^{\ell_1+\ell_2'+L+S} 2\delta(T,0) \langle \ell_1' \ell_2 L M_L | V(r) | \ell_2' \ell_1 L M_L \rangle] \qquad \text{(II.5)}$$

$$+ [a_2 + (-1)^{S+1} 6a_3 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix}] [(-1)^T 6 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix}$$

$$\times \langle \ell_1' \ell_2 L M_L | V(r) | \ell_1 \ell_2' L M_L \rangle$$

$$+ (-1)^{\ell_1+\ell_2'+L+S} 2\delta(T,1) \langle \ell_1' \ell_2 L M_L | V(r) | \ell_2' \ell_1 L M_L \rangle] \quad .$$

For the Serber exchange mixture, defined by $a_1 = a_2 = a_3 = -0.125$, $a_0 = 0.375$, an expression for the energy matrix elements which is not immediately obvious from Eq. (II.5) may be derived by use of the permutation operators. Letting

$$v_{SER} = \frac{1}{2} V(r)[1 + P_r(1,2)]$$

we find by the same methods used previously that

$$\langle \bar{j}_1 j_2 JM_J TM_T | v_{SER} | \bar{j}_1' j_2' JM_J TM_T \rangle$$

$$= \sum_{L,S,J'} (2J'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} A \begin{pmatrix} \ell_1' & \frac{1}{2} & j_1' \\ \ell_2 & \frac{1}{2} & j_2 \\ L & S & J' \end{pmatrix} A \begin{pmatrix} \ell_1 & \frac{1}{2} & j_1 \\ \ell_2' & \frac{1}{2} & j_2' \\ L & S & J' \end{pmatrix}$$

$$\times \left[\langle \ell_1' \ell_2 LM_L | v_{SER} | \ell_1 \ell_2' LM_L \rangle + (-1)^{\ell_1+\ell_2'+L+S} 2\delta(T,0) \langle \ell_1' \ell_2 LM_L | V_{SER} | \ell_2' \ell_1 LM_L \rangle \right].$$

Since

$$\langle \ell_1' \ell_2 LM_L | V(r) P_r(1,2) | \ell_1 \ell_2' LM_L \rangle = (-1)^{\ell_1+\ell_2'+L} \langle \ell_1' \ell_2 LM_L | v(r) | \ell_2' \ell_1 LM_L \rangle ,$$

it follows that the matrix elements for a scalar potential with a Serber exchange mixture is given by

$$\langle \bar{j}_1 j_2 JM_J TM_T | v_{SER} | \bar{j}_1' j_2' JM_J TM_T \rangle$$

$$= \frac{1}{2} \sum_{LSJ'} (2J'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} A \begin{pmatrix} \ell_1' & \frac{1}{2} & j_1' \\ \ell_2 & \frac{1}{2} & j_2 \\ L & S & J' \end{pmatrix} A \begin{pmatrix} \ell_1 & \frac{1}{2} & j_1 \\ \ell_2' & \frac{1}{2} & j_2' \\ L & S & J' \end{pmatrix}$$

$$\times \left[1 + (-1)^S 2\delta(T,0) \right] \left[\langle \ell_1' \ell_2 LM | v(r) | \ell_1 \ell_2' LM \rangle + (-1)^{\ell_1+\ell_2'+L} \right.$$

$$\left. \times \langle \ell_1' \ell_2 LM_L | v(r) | \ell_2' \ell_1 LM_L \rangle \right] .$$

The decomposition of the matrix elements of the vector interaction potential may be accomplished by the same procedures as those used for the scalar interaction. The major difference in the two cases occurs in

the evaluation of the reduced matrix element of the intrinsic-spin operator. For the spin component of the vector potential, we obtain from Eqs. (A.18B) and (A.18C)

$$\langle \tfrac{1}{2}\,\tfrac{1}{2}\,S\|\underset{\sim}{\sigma}_1+\underset{\sim}{\sigma}_2\|\tfrac{1}{2}\,\tfrac{1}{2}\,S'\rangle=[(-1)^S+(-1)^{S'}][(2S+1)(2S'+1)]^{\frac{1}{2}}\begin{Bmatrix}\frac{1}{2} & S & \frac{1}{2}\\ S' & \frac{1}{2} & 1\end{Bmatrix}\langle\tfrac{1}{2}\|\sigma\|\tfrac{1}{2}\rangle\ .$$

From the triangular conditions on the 6-j symbol and the $[(-1)^S+(-1)^{S'}]$ factor, it follows that the only nonzero value of this matrix element occurs for S = S' = 1. Hence we find

$$\langle \tfrac{1}{2}\,\tfrac{1}{2}\,S\|\underset{\sim}{\sigma}_1+\underset{\sim}{\sigma}_2\|\tfrac{1}{2}\,\tfrac{1}{2}\,S'\rangle = 2\sqrt{6}\ \delta\ (S,1)\delta(S',1).$$

Therefore, the matrix elements of the vector interaction potential are

$$\langle \bar{j}_1 j_2 JM_J TM_T|\,(\underset{\sim}{S}_1+\underset{\sim}{S}_2)\cdot\underset{\sim}{\ell}_{12}\ v'(r)\,|\bar{j}_1' j_2' JM_J TM_T\rangle$$

$$= \sum_{J'L'L}(-1)^{J'+L'+1}\,4\sqrt{6}\,(2J'+1)\begin{Bmatrix} j_1 & j_2 & J\\ j_1' & j_2' & J'\end{Bmatrix}\begin{Bmatrix} L & L' & 1\\ 1 & 1 & J'\end{Bmatrix} A\begin{pmatrix} \ell_1' & \frac{1}{2} & j_1'\\ \ell_2 & \frac{1}{2} & j_2\\ L & 1 & J'\end{pmatrix}$$

$$\times\ A\begin{pmatrix} \ell_1 & \frac{1}{2} & j_1\\ \ell_2' & \frac{1}{2} & j_2'\\ L' & 1 & J'\end{pmatrix}[\langle \ell_1'\ell_2 L|\,|\underset{\sim}{\ell}V(r)|\,|\ell_1\ell_2'L'\rangle$$

$$+\ (-1)^{\ell_1+\ell_2'-L'}2\delta(T,0)\langle \ell_1'\ell_2 L|\,|\underset{\sim}{\ell}V(r)|\,|\ell_2'\ell_1 L'\rangle].$$

The matrix elements for the hole-particle interaction have now been reduced to functions of matrix elements of the orbital portion of the potential operators. Thus far we have used no assumptions concerning the nature of the nuclear potential well. To evaluate the matrix elements of the orbital portion of the interaction potential, however, we will employ the harmonic-oscillator wave functions, and our calculations will be only approximately valid for the true nuclear potential.

Evaluation of Orbital Matrix Elements by Use of Transformation Brackets

The evaluation of the orbital matrix elements is complicated by the fact that the operators are functions of the relative coordinates, $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$ and $|\underset{\sim}{p}_1 - \underset{\sim}{p}_2|$, whereas the state vectors are essentially products of a function of $\underset{\sim}{r}_1$ and a function of $\underset{\sim}{r}_2$. One approach to the solution of this problem is to reduce the matrix elements to a sum of Slater integrals by using a tensor expansion of the interaction to factor the angular dependence of the potential operator into a function of (θ_1,φ_1) and a function of (θ_2,φ_2).[10] A simpler approach utilizes the transformation

$$|n_1\ell_1,n_2\ell_2,LM_L\rangle = \sum_{\substack{n\ell \\ \mathfrak{N}\mathfrak{L}}} |n\ell,\mathfrak{N}\mathfrak{L},LM\rangle\langle n\ell,\mathfrak{N}\mathfrak{L},L|n_1\ell_1,n_2\ell_2,L\rangle \tag{2.8}$$

where $|n\ell m_\ell\rangle$ and $|\mathfrak{N}\mathfrak{L}M_{\mathfrak{L}}\rangle$ are functions only of the relative and the center-of-mass coordinates respectively. This expansion can be made in closed form only for harmonic-oscillator wave functions.[22] The factor $\langle n\ell,\mathfrak{N}\mathfrak{L},L|n_1\ell_1,n_2\ell_2,L\rangle$ is called the Moshinsky transformation bracket and its evaluation is discussed in Appendix C.

Using Eq. (2.8), we find for a general potential, V(r,p),

$$\langle n_1\ell_1,n_2\ell_2,LM|V(r,p)|n_1'\ell_1',n_2'\ell_2',L'M'\rangle = \sum_{\substack{n\ell\,\mathfrak{N}\mathfrak{L} \\ n'\ell'\mathfrak{N}'\mathfrak{L}'}} \langle n\ell,\mathfrak{N}\mathfrak{L},L|n_1\ell_1,n_2\ell_2,L\rangle\langle n'\ell',\mathfrak{N}'\mathfrak{L}',L'|n_1'\ell_1',n_2'\ell_2'L'\rangle \times \langle n\ell,\mathfrak{N}\mathfrak{L},LM|V(r,p)|n'\ell',\mathfrak{N}'\mathfrak{L}',LM\rangle . \tag{2.9}$$

The matrix elements in the center-of-mass and relative coordinates configuration are easily evaluated by the techniques of Racah algebra. For

the orbital portion of the scalar interaction potential we find by using first Eqs. (A.14) and (A.18B) and then Eqs. (A.3) and (A.15A) that

$$\langle n\ell,\mathfrak{n}\mathfrak{l},LM|v^{o}(r)|n'\ell',\mathfrak{n}'\mathfrak{l}',LM\rangle$$
$$= (-1)^{\ell+\mathfrak{l}+L}(2L+1)^{\frac{1}{2}}\begin{Bmatrix}\ell & L & \mathfrak{l}\\ L & \ell' & 0\end{Bmatrix}\langle n\ell||v^{o}(r)||n'\ell'\rangle\delta(\mathfrak{l},\mathfrak{l}')\delta(\mathfrak{n},\mathfrak{n}')$$

$$= R^{o}(n\ell,n'\ell')\delta(\ell,\ell')\delta(\mathfrak{l},\mathfrak{l}')\delta(\mathfrak{n},\mathfrak{n}')$$

where

$$R^{K}(n\ell,n'\ell') = \int_{o}^{\infty} R_{n\ell}(r)v^{K}(r)R_{n'\ell'}(r)dr \qquad (2.10)$$

Therefore the matrix element of the orbital portion of the scalar interaction potential is

$$\langle n_1\ell_1,n_2\ell_2,LM|v^{o}(r)|n_1'\ell_1',n_2'\ell_2',LM\rangle \qquad (2.11)$$
$$= \sum_{\substack{n\ell\\ \mathfrak{n}\mathfrak{l}}} \langle n\ell,\mathfrak{n}\mathfrak{l},L|n_1\ell_1,n_2\ell_2,L\rangle\langle n'\ell,\mathfrak{n}\mathfrak{l},L|n_1'\ell_1',n_2'\ell_2',L\rangle$$

$$\times\ R^{o}(n\ell,n'\ell).$$

Applying conservation of energy to Eq. (2.8) we find that

$$2n_1+\ell_1+2n_2+\ell_2 = 2n+\ell+2\mathfrak{n}+\mathfrak{l},$$

and hence n and n' in Eq. (2.10) are related by

$$n'-n = n_1'-n_1+n_2'-n_2+\tfrac{1}{2}(\ell_1'+\ell_2'-\ell_1-\ell_2). \qquad (2.12)$$

For the vector interaction potential we find in a similar manner that

$$\langle n_1\ell_1,n_2\ell_2,L||\ \underset{\sim}{\ell}\ V(r)||n_1'\ell_1',n_2'\ell_2',L'\rangle \qquad (2.13)$$
$$= \sum_{\substack{n\ell\\ \mathfrak{n}\mathfrak{l}}} (-1)^{\ell+\mathfrak{l}+L'+1}\left[(2L+1)(2L'+1)\ell(\ell+1)(2\ell+1)\right]^{\frac{1}{2}}\begin{Bmatrix}\ell & L & \mathfrak{l}\\ L' & \ell & 1\end{Bmatrix}$$

$$\times\ \langle n\ell,\mathfrak{n}\mathfrak{l},L|n_1\ell_1,n_2\ell_2,L\rangle\langle n'\ell,\mathfrak{n}\mathfrak{l},L'|n_1'\ell_1',n_2'\ell_2',L'\rangle R^{1}(n\ell,n'\ell).$$

Once again n and n' are related by Eq. (2.12).

Evaluation of Radial Integrals

The components of the energy matrix have now been reduced to functions of the radial integrals $R^K(n\ell,n'\ell)$. These integrals can be evaluated in the form given by Eq. (2.10), but it is more convenient to reduce them to linear combinations of Talmi integrals, i.e., to expand the radial integrals by

$$R^K(n\ell,n'\ell) = \sum_p B(n\ell,n'\ell,p) I_p^K \tag{2.14}$$

where the Talmi integrals I_p^K are defined by

$$I_p^K = \int_o^\infty R_{op}^2 v^K(r) dr.$$

From Eq. (1.13) we find

$$I_p^K = \frac{2(\frac{1}{b})^{2p+3}}{\Gamma(p+\frac{3}{2})} \int_o^\infty r^{2p+2} e^{-\frac{r^2}{b'^2}} v^K(r) dr. \tag{2.15}$$

where $b' = \sqrt{2}\, b$.

The constants $B(n\ell,n'\ell,p)$ are found to be[23]

$$B(n\ell,n'\ell,p) = \frac{(-1)^{p-\ell}(2p+1)!}{2^{n+n'}(p!)} \left[\frac{n!n'!(2n+2\ell+1)!(2n'+2\ell+1)!}{(n+\ell)!(n'+\ell)!}\right]^{\frac{1}{2}}$$

$$\times \sum_{k=r}^{s} \frac{(\ell+k)!(p-k)!}{k!(2\ell+2k+1)!(n-k)!(2p-2k+1)!(n'-p+\ell+k)!(p-\ell-k)!}$$

where

$$r = \max(o, p-\ell-n)$$

$$s = \min(n, p-\ell)$$

To proceed further in the evaluation of the radial integrals we must introduce the explicit radial dependence of the mutual interaction potential. For the Yukawa well used in the scalar interaction potential, we find by substituting

$$v^{Y}(r) = V_{Y} \frac{e^{-\frac{r}{a}}}{\frac{r}{a}}$$

into Eq. (2.15) and letting

$$x = \frac{r}{b'} + \frac{b'}{2a} \quad ,$$

that the Talmi integral I_p^Y may be expressed as

$$I_p^Y = \frac{2^{p+1} V_Y e^{\mu^2}}{\sqrt{\pi}\mu (2p+1)!!} \int_{\mu}^{\infty} e^{-x^2} (x-\mu)^{2p+1} dx$$

where $\mu = b'/(2a)$ and

$$(2p+1)!! = 1 \times 3 \times 5 \cdots (2p+1).$$

For $p = 0$ and $p = 1$ we find

$$I_o^Y = \frac{V_Y}{2\mu} \left[\frac{2}{\sqrt{\pi}} - 2\mu e^{\mu^2} \operatorname{erfc}(\mu) \right] \tag{2.16A}$$

$$I_1^Y = \frac{V_Y}{3\mu} \left[\frac{2}{\sqrt{\pi}} (1+\mu^2) - 2\mu \left(\frac{3}{2}+\mu^2\right) e^{\mu^2} \operatorname{erfc}(\mu) \right] \tag{2.16B}$$

with

$$\operatorname{erfc}(\mu) = \frac{2}{\sqrt{\pi}} \int_{\mu}^{\infty} e^{-x^2} dx.$$

The remainder of the Talmi integrals for the Yukawa potential can be found from the recurrence relation[24]

$$I_{p+1}^Y(\mu) = \frac{2p}{2p+3} \left[\left(2 + \frac{2\mu^2+3}{2p}\right) Ip(\mu) - I_{p-1}(\mu) \right] \tag{2.16C}$$

The radial integrals for the vector interaction potential are more complex and satisfy no general recurrence relation. Fortunately, however, we are interested in only three of these integrals. Substituting Eq. (I.7) into Eq. (II.15), we find the Talmi integrals for the vector potential are

$$I_p^{SO} = \frac{2^{p+2}}{(2p+1)!!\sqrt{\pi}} V_{SO} \left\{ \frac{1}{\alpha^3} [1 + \alpha] e^{-\alpha} \int_o^{\frac{r_o}{b'}} e^{-x^2} x^{2p+2} dx \right.$$

$$\left. + \lambda^2 e^{\mu^2} \int_\gamma^\infty e^{-x^2} [x - \mu + \lambda] (x - \mu)^{2p-1} dx \right\}$$

where

$$\alpha = \frac{r_o}{a_{SO}} \quad , \lambda = \frac{a_{SO}}{b'} \quad , \mu = \frac{b'}{2a_{SO}} \quad , \gamma = \frac{r_o}{b'} + \mu \quad .$$

These integrals may be evaluated in terms of error functions.

Application to the O^{16} and N^{16} Nuclei

The formalism has now been developed for the computation of the energy matrix for nuclei that can be described by a particle-hole configuration. The parameters of the mutual interaction potential are the same for all such nuclei, so that only two types of constants must be determined for each individual nucleus: (1) the harmonic-oscillator length b which determines the nuclear radius, and (2) the spin-orbit splitting parameters ξ_p and ξ_h. (There may be more than two of the latter if the particle or hole states are in degenerate oscillator configurations.) For the O^{16} and N^{16} nuclei the harmonic-oscillator length has been found to be[8] b = 1.71 F. and the spin-orbit splitting parameters to be[5] ξ_{op} = -4.22 MeV and ξ_{od} = -2.03 MeV.

Within the framework of our assumptions, the only difference between the nuclei of O^{16} and N^{16} is that the isotopic spin projection quantum number M_T of O^{16} is 0 and of N^{16} is -1. Following from this, the lowest-lying excited states of O^{16} are found to have the total isotopic-spin

quantum number T = 0 and those of N^{16} to have T = 1. We also assume that both nuclei can be described completely in terms of the following basis states:

$$|\overline{0p}_{\frac{1}{2}}\ 1s_{\frac{1}{2}}\ JM_J\ TM_T\rangle \quad , \quad |\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{3}{2}}\ JM_J\ TM_T\rangle \quad ,$$

$$|\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{5}{2}}\ JM_J\ TM_T\rangle \quad , \quad |\overline{0p}_{\frac{3}{2}}\ 1s_{\frac{1}{2}}\ JM_J\ TM_T\rangle \quad ,$$

$$|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{3}{2}}\ JM_J\ TM_T\rangle \quad , \quad |\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{5}{2}}\ JM_J\ TM_T\rangle \quad .$$

One unusual feature entering our calculations concerns the correction which must be applied for the motion of the center of mass as a whole. It has been shown[25] that basis states of mixed configurations constructed according to the usual shell-model prescription may contain components describing different states of motion of the center-of-mass. Certain linear combinations of the basis states must be rejected because they refer to lower states of internal motion in overall translational motion. Hence, a new set of basis vectors which refer only to the lowest energy of the center of mass must be constructed.

It is shown in Appendix D that in the J = 1, T = 0 state of O^{16} an excited center-of-mass motion corresponds to the state vector

$$\begin{aligned} |\Psi_A\rangle = &- \frac{\sqrt{2}}{6} |\overline{0p}_{\frac{1}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle + \frac{\sqrt{10}}{6} |\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle \\ &+ \frac{1}{3} |\overline{0p}_{\frac{3}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle + \frac{\sqrt{2}}{6} |\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle \\ &+ \frac{\sqrt{2}}{2} |\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{5}{2}}\ 1M;00\rangle \quad . \end{aligned} \tag{II.18}$$

The acceptable basis vectors corresponding to lowest center-of-mass energy are

$$|\psi_B\rangle = \frac{\sqrt{10}}{6}\,|\overline{0p}_{\frac{1}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle + \frac{\sqrt{2}}{6}\,|\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle$$
$$- \frac{\sqrt{5}}{3}\,|\overline{0p}_{\frac{3}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle + \frac{\sqrt{10}}{30}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle \qquad \text{(II.19A)}$$
$$+ \frac{\sqrt{10}}{10}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{5}{2}}\ 1M;00\rangle\ ,$$

$$|\psi_C\rangle = \frac{\sqrt{6}}{3}\,|\overline{0p}_{\frac{1}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle + \frac{\sqrt{3}}{3}\,|\overline{0p}_{\frac{3}{2}}\ 1s_{\frac{1}{2}}\ 1M;00\rangle\ , \qquad \text{(II.19B)}$$

$$|\psi_D\rangle = \frac{\sqrt{6}}{6}\,|\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle + \frac{2\sqrt{30}}{15}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle$$
$$- \frac{\sqrt{30}}{10}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{5}{2}}\ 1M;00\rangle\ , \qquad \text{(II.19C)}$$

$$|\psi_E\rangle = \frac{\sqrt{2}}{2}\,|\overline{0p}_{\frac{1}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle - \frac{\sqrt{10}}{5}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{3}{2}}\ 1M;00\rangle$$
$$- \frac{\sqrt{10}}{10}\,|\overline{0p}_{\frac{3}{2}}\ 0d_{\frac{5}{2}}\ 1M;00\rangle\ . \qquad \text{(II.19D)}$$

One may correct for the center-of-mass motion either by calculating the energy matrix using the vectors given in Eq. (II.19) as basis vectors or by calculating the energy matrix in the original set of basis vectors and then transforming it so that the spurious state can be dropped.

With the exception of the spin-orbit splitting and the character of the exchange mixture associated with the scalar interaction potential, all parameters involved in calculating the energy matrices of O^{16} and N^{16} must be specified only during the evaluation of the radial integrals.

Therefore it is useful to express the matrix elements for the scalar interaction with a Serber exchange mixture and for the vector interaction as functions of the Talmi integrals. The coefficients N_i and C_i defined by

$$\langle \bar{j}_1 j_2 JM_J TM_T | v_{SER} | \overline{j_1}' j_2' JM_J TM_T \rangle \tag{II.20A}$$

$$= N_1 \{[C_0^s I_0^s + C_1^s I_1^s + C_2^s I_2^s][1 + 2\delta(T,0)]$$

$$+ [C_0^t I_0^t + C_1^t I_1^t + C_2^t I_2^t][1 - 2\delta(T,0)]\}$$

and by

$$\langle \bar{j}_1 j_2 JM_J TM_T | (\underset{\sim}{S}_1 + \underset{\sim}{S}_2) \cdot \underset{\sim}{\ell} V'(r) | \overline{j_1}' j_2' JM_J TM_T \rangle$$

$$= N_2 [C_1^{SO} I_1^{SO} + C_2^{SO} I_2^{SO} + C_3^{SO} I_3^{SO}) \tag{II.20B}$$

$$- 2\delta(T,0)(C_1^{SO} I_1^{SO} + C_4^{SO} I_2^{SO} + C_3^{SO} I_3^{SO})] .$$

may be evaluated by the methods outlined in this chapter. Tables for the necessary A-coefficients,[26] the 6-j symbols,[27] and the 3-j symbols,[28] have been published. The results of these calculations are listed in Table I.

The evaluation of the Talmi integrals in Eq. (II.20) for the range parameters listed in Chapter I may be performed by use of Eqs. (II.16) and (II.17). Retaining the interaction strengths as free variables, we find the integrals to be

TABLE I

COEFFICIENTS OF TAIMI INTEGRALS IN ENERGY MATRIX ELEMENTS

$n_1l_1j_1$	$n_2l_2j_2$	$n_1'l_1'j_1'$	$n_2'l_2'j_2'$	J	N_1	C_0^s	C_1^s	C_2^s	C_0^t	C_1^t	C_2^t	N_2	C_1^{SO}	C_2^{SO}	C_3^{SO}	C_4^{SO}
$0p_{1/2}$	$1s_{1/2}$	$0p_{1/2}$	$1s_{1/2}$	0	$\frac{1}{96}$	11	-2	15	11	-2	15	0	0	0	0	0
				1	$\frac{1}{288}$	11	-2	15	55	-10	75	$-\frac{1}{18}$	19	-30	35	-70
		$0p_{1/2}$	$0d_{3/2}$	1	$\frac{\sqrt{5}}{72}$	1	2	-3	-1	-2	3	$\frac{\sqrt{5}}{36}$	1	6	-7	14
		$0p_{3/2}$	$1s_{1/2}$	1	$\frac{\sqrt{2}}{144}$	11	-2	15	-11	2	-15	$-\frac{\sqrt{2}}{72}$	19	-30	35	-70
		$0p_{3/2}$	$0d_{3/2}$	0	$\frac{\sqrt{5}}{48}$	1	2	-3	1	2	-3	0	0	0	0	0
				1	$\frac{1}{144}$	5	10	-15	1	2	-3	$-\frac{1}{18}$	1	6	-7	14
		$0p_{3/2}$	$0d_{5/2}$	1	$-\frac{1}{12}$	0	0	0	1	2	-3	$\frac{1}{6}$	9	-16	7	-4
$0p_{1/2}$	$0d_{3/2}$	$0p_{1/2}$	$0d_{3/2}$	1	$-\frac{1}{144}$	7	-10	-9	-49	-2	-9	$-\frac{1}{18}$	1	-12	-49	8
				2	$\frac{1}{240}$	35	-50	51	35	-26	75	$\frac{1}{10}$	5	8	7	-12
		$0p_{1/2}$	$0d_{5/2}$	2	$\frac{\sqrt{6}}{30}$	0	0	1	0	-1	0	$-\frac{\sqrt{6}}{30}$	0	3	7	-7

TABLE I -- CONTINUED

$n_1 l_1 j_1$	$n_2 l_2 j_2$	$n_1' l_1' j_1'$	$n_2' l_2' j_2'$	J	N_1	C_0^s	C_1^s	C_2^s	C_0^t	C_1^t	C_2^t	N_2	C_1^{SO}	C_2^{SO}	C_3^{SO}	C_4^{SO}
$0p_{1/2}$	$0d_{3/2}$	$0p_{3/2}$	$1s_{1/2}$	1	$-\frac{\sqrt{10}}{288}$	1	2	-3	-7	-14	21	$\frac{\sqrt{10}}{72}$	31	66	35	-34
				2	$-\frac{\sqrt{10}}{480}$	1	2	-3	1	2	-3	$-\frac{\sqrt{10}}{40}$	3	-10	7	-10
		$0p_{3/2}$	$0d_{3/2}$	1	$\frac{\sqrt{5}}{360}$	7	-10	-9	35	-14	-9	$\frac{\sqrt{5}}{180}$	49	-84	119	32
				2	$-\frac{1}{120}$	7	-10	15	7	-22	3	$-\frac{1}{20}$	1	20	7	-8
		$0p_{3/2}$	$0d_{5/2}$	1	$-\frac{\sqrt{5}}{240}$	21	0	3	-35	8	3	$-\frac{\sqrt{5}}{30}$	3	-8	-7	4
				2	$-\frac{\sqrt{21}}{240}$	3	0	5	3	-8	-3	$\frac{\sqrt{21}}{30}$	3	0	1	-4
$0p_{1/2}$	$0d_{5/2}$	$0p_{1/2}$	$0d_{5/2}$	2	$\frac{1}{240}$	35	0	9	35	16	25	$-\frac{1}{15}$	5	2	-7	2
				3	$\frac{1}{48}$	1	0	3	13	-16	23	$-\frac{1}{3}$	1	-2	3	-2
		$0p_{3/2}$	$1s_{1/2}$	2	$\frac{\sqrt{15}}{120}$	1	2	-3	1	2	-3	$\frac{\sqrt{15}}{60}$	3	-10	7	-10
		$0p_{3/2}$	$0d_{3/2}$	2	$-\frac{\sqrt{6}}{240}$	7	-10	-5	7	-2	3	$\frac{\sqrt{6}}{60}$	9	-10	-7	-2

TABLE I -- CONTINUED

$n_1 l_1 j_1$	$n_2 l_2 j_2$	$n_1' l_1' j_1'$	$n_2' l_2' j_2'$	J	N_1	C_0^s	C_1^s	C_2^s	C_0^t	C_1^t	C_2^t	N_2	C_1^{SO}	C_2^{SO}	C_3^{SO}	C_4^{SO}
$0p_{\frac{1}{2}}$	$0d_{\frac{5}{2}}$	$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	3	$\frac{\sqrt{30}}{240}$	7	-10	11	-5	6	-9	$-\frac{\sqrt{30}}{60}$	3	2	3	-6
		$0p_{\frac{3}{2}}$	$0d_{\frac{5}{2}}$	2	$\frac{\sqrt{14}}{30}$	1	0	0	1	-1	-1	$\frac{\sqrt{14}}{30}$	2	5	-1	-1
				3	$\frac{\sqrt{5}}{60}$	1	0	3	-5	8	-7	$\frac{\sqrt{5}}{30}$	7	-2	7	-14
$0p_{\frac{3}{2}}$	$1s_{\frac{1}{2}}$	$0p_{\frac{3}{2}}$	$1s_{\frac{1}{2}}$	1	$-\frac{1}{288}$	11	-2	15	-77	14	-105	$\frac{1}{18}$	19	-30	35	-70
				2	$\frac{1}{96}$	11	-2	15	11	-2	15	0	0	0	0	0
		$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	1	$\frac{\sqrt{2}}{144}$	1	2	-3	5	10	-15	$\frac{\sqrt{2}}{72}$	37	-30	-7	50
				2	$-\frac{\sqrt{10}}{240}$	1	2	-3	1	2	-3	$-\frac{\sqrt{10}}{40}$	3	-10	7	-10
		$0p_{\frac{3}{2}}$	$0d_{\frac{5}{2}}$	1	$\frac{\sqrt{2}}{96}$	3	6	-9	-5	-10	15	$-\frac{\sqrt{2}}{24}$	3	-10	7	-10
				2	$\frac{\sqrt{210}}{480}$	1	2	-3	1	2	-3	$-\frac{\sqrt{210}}{120}$	3	-10	7	-10
$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	0	$\frac{1}{24}$	7	-1	0	7	-1	0	0	0	0	0	0

TABLE I -- CONTINUED

$n_1 l_1 j_1$	$n_2 l_2 j_2$	$n_1' l_1' j_1'$	$n_2' l_2' j_2'$	J	N_1	C_0^s	C_1^s	C_2^s	C_0^t	C_1^t	C_2^t	N_2	C_1^{SO}	C_2^{SO}	C_3^{SO}	C_4^{SO}
$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	$0p_{\frac{3}{2}}$	$0d_{\frac{3}{2}}$	1	$\frac{1}{360}$	49	11	18	77	7	18	$\frac{1}{45}$	25	-48	35	20
				2	$\frac{1}{120}$	7	-1	12	7	11	24	$-\frac{2}{5}$	1	-3	0	-1
				3	$\frac{1}{40}$	1	-1	2	13	-17	22	$\frac{1}{5}$	5	-2	5	-10
		$0p_{\frac{3}{2}}$	$0d_{\frac{5}{2}}$	1	$-\frac{1}{120}$	21	-6	-3	-7	-2	-3	$\frac{1}{60}$	45	-92	35	40
				2	$\frac{\sqrt{21}}{120}$	1	2	1	1	-2	-3	$-\frac{\sqrt{21}}{60}$	3	-4	5	-8
				3	$-\frac{\sqrt{6}}{30}$	1	-1	2	-2	3	-3	$-\frac{2\sqrt{6}}{15}$	0	1	0	0
$0p_{\frac{3}{2}}$	$0d_{\frac{5}{2}}$	$0p_{\frac{3}{2}}$	$0d_{\frac{5}{2}}$	1	$-\frac{1}{80}$	7	-2	-1	-49	6	-1	$-\frac{1}{10}$	5	-4	35	0
				2	$\frac{1}{240}$	31	-18	-9	31	38	47	$\frac{7}{30}$	1	-8	-5	4
				3	$-\frac{1}{240}$	11	-36	-3	-37	28	-83	$\frac{2}{15}$	5	2	5	-10

$I_0^s = 0.127\ V_s$ $\quad I_1^s = 0.036\ V_s$ $\quad I_2^s = 0.015\ V_s$

$I_0^t = 0.180\ V_t$ $\quad I_1^t = 0.057\ V_t$ $\quad I_2^t = 0.026\ V_t$

$I_1^{SO} = 0.029\ V_{SO}$ $\quad I_2^{SO} = 0.006\ V_{SO}$ $\quad I_3^{SO} = 0.002\ V_{SO}$

In accordance with their experimentally indicated values, V_s and V_t were varied from -30 MeV to -60 MeV and V_{SO} was varied from 0 MeV to 40 MeV. The eigenvalues and eigenstates of the resulting energy matrices were calculated on the Univac 1107 Computer at the University of Alabama Research Institute.

CHAPTER III

COMPARISON OF THEORETICAL PREDICTIONS WITH EXPERIMENTAL DATA

An analysis of the importance of the vector interaction potential to the O^{16} and N^{16} nuclei is complicated both by experimental ambiguities and by theoretical approximations. Nevertheless, qualitative conclusions may clearly be drawn from a comparison of theoretical predictions with experimental data. In this chapter the influence of the vector interaction potential on the energy levels and on the beta-decay and muon-capture transition rates is discussed. An investigation of the electromagnetic transitions was performed by Elliott and Flowers,[4] but corrections for the collective oscillations of the closed-shell core were found necessary. Hence, an accurate treatment of this effect is beyond the scope of this work.

Energy Levels

A study of the absolute energy of the nuclear states requires an accurate determination of the binding energies of the particle and hole states. Previous investigations of these nuclei have used binding energies found experimentally or have merely chosen the binding energy giving the best theoretical predictions. Obviously, however, these procedures do not add materially to the theory, and accordingly, we investigate only the splitting of the energy levels, since this may be

done entirely within the framework of our previous assumptions.

At the outset of the comparison of the energy levels with experiment, it must be noted that such neglected processes as the Coulomb interaction between protons have a greater influence on some of the energy levels than does the vector interaction potential. This is obvious from a comparison of the T = 1 states of O^{16} and N^{16}, which according to the assumptions of this work should be identical. However, there is a shift in the relative energies of the levels of as much as 0.3 MeV between the two nuclei. This is, in some cases, greater than the shift caused by the vector interaction potential.

The relative energy levels of the T = 0 states of O^{16} are presented in Table II. As can be seen, the theoretical prediction of the 0^- and 2^- energy levels using parameters indicated by scattering experiment are about 1 MeV too high. However, the energy levels of all states are predicted reasonably well for several sets of interaction strengths, two of which are

$$V_t = -40 \text{ MeV} \qquad V_s = -35 \text{ MeV} \qquad V_{SO} = 15 \text{ MeV}$$

and

$$V_t = -45 \text{ MeV} \qquad V_s = -40 \text{ MeV} \qquad V_{SO} = 20 \text{ MeV}.$$

The energy levels for $V_t = -45$ MeV and $V_s = -40$ MeV and extreme cases of the vector interaction strength are also listed in Table II. Although the influence of the vector interaction potential on the 0^- and 2^- states is too small to be of importance, its effect on the 1^- state is pronounced. This is significant because the largest error for $V_{SO} = 0$ occurs in the 1^- level, and a value of $V_{SO} = 15$-20 MeV provides a good approximation for the energy of the state.

TABLE II

ENERGY SPLITTINGS FOR T = 0

Interaction Strengths			Energy Relative to 3^- Level		
V_t	V_s (MeV)	V_{SO}	0^- (MeV)	1^- (MeV)	2^- (MeV)
-50	-45	30	5.76	0.97	3.77
-40	-35	15	4.49	1.00	2.82
-45	-40	20	5.08	0.82	3.11
-45	-40	0	5.33	1.87	3.10
-45	-40	30	4.95	0.17	3.12
Elliott-Flowers[a]			7.0	2.1	4.2
Duck[b]			. .	. .	5.8
Gillet[c]			3.8	-1.2	3.2
Experimental[d]			4.80	0.97	2.74

[a]See reference 4.

[b]See reference 5.

[c]See reference 6.

[d]See reference 30.

TABLE III

ENERGY SPLITTINGS FOR T = 1

Interaction Strengths			Energy Relative to 2^- level			Total Splitting
V_t	V_s (MeV)	V_{SO}	0^- (MeV)	1^- (MeV)	3^- (MeV)	(MeV)
-50	-45	30	0.50	0.38	0.57	0.57
-40	-35	15	0.21	0.21	0.19	0.21
-45	-40	20	0.30	0.05	0.33	0.33
-45	-40	0	-0.21	0.79	-0.02	1.00
-45	-40	30	0.72	-0.23	0.60	0.95
Elliott-Flowers[a]			-0.1	0.6	-0.2	0.8
Duck[b]			2.8	1.4	1.2	2.8
Gillet[c]			0.7	0.6	-0.2	0.9
Experimental[d]		O^{16}	-0.18	0.13	0.29	0.47
		N^{16}	0.12	0.39	0.29	0.39

[a]See reference 4.

[b]See reference 5.

[c]See reference 6.

[d]See reference 30.

The energy levels of the T = 1 states are too closely packed to make meaningful predictions about their order. However, the total energy splitting of the states is approximately the same for both O^{16} and N^{16}, and it is instructive to determine this splitting for various choices of the interaction strength parameters. As can be observed in Table III, the effect of increasing V_{SO} is first to decrease the total energy splitting and then to produce a spreading of the energy levels of the four states. For V_t = -45 MeV and V_s = -40 MeV, a value of V_{SO} of 10-25 MeV provides an approximately correct total energy splitting.

Primarily due to the form of the scalar interaction potential, the energy levels obtained by our model of the nucleon-nucleon interaction are a notable improvement over those of previous studies of O^{16} and N^{16}. The vector interaction potential has only a small influence on the energy levels, but it does enable us to decrease our largest error from about 1 MeV to about 0.4 MeV. The best theoretical predictions occur for values of the vector interaction strength of 15-20 MeV.

Beta Decay of 2^- Level of N^{16}

The 2^- level of N^{16} decays by a beta transition to the 0^+ (ground), 3^-, 2^-, and 1^- levels of O^{16} with branching ratios as shown in Fig. 1.

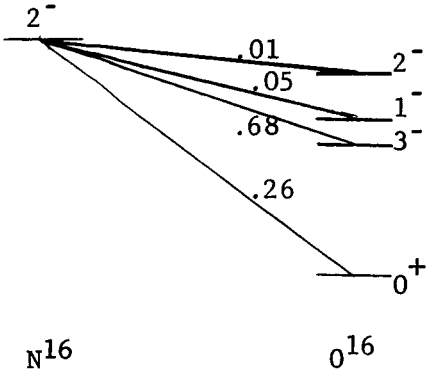

Fig. 1. Beta decay of N^{16}

The beta decays to the excited states of O^{16} are allowed transitions since $|J-J'| = 0,1$ and $\pi = \pi'$. However, the transition to the ground state of O^{16} is characterized by $|J-J'| = 2$ and a change in parity and hence is first-forbidden unique.

Employing a spherical-tensor notation for the beta-decay operators,[29] we find that the $f_o t$ value for an allowed transition from a state $|\bar{j}_1 j_2 'J'M_{J'};\ 1\text{-}1\rangle$ to a state $|\bar{j}_1 j_2 JM_J;\ 00\rangle$ is given by (in units determined by $\hbar = m_e = c = 1$)

$$f_o t = \frac{2\pi^2 \ln 2}{G^2} \{\delta(J,J')\delta(j_1,j_1')\delta(j_2,j_2')[(2j_2+1)^{-\frac{1}{2}} \langle j_2||Y_0||j_2\rangle - (2j_1+1)^{-\frac{1}{2}} \langle j_1||Y_0||j_1\rangle] \qquad (3.1)$$

$$- \frac{\lambda}{2J+1} [(-1)^{J+j_1+j_2'} \begin{Bmatrix} J & J' & 1 \\ j_2' & j_2 & j_1 \end{Bmatrix} \langle j_2||S_{1o}||j_2'\rangle + (-1)^{J'+j_1+j_2} \begin{Bmatrix} J & J' & 1 \\ j_1' & j_1 & j_2 \end{Bmatrix} \langle j_1'||S_{1o}||j_1\rangle]\}^{-2}$$

where G and λG are the vector and axial vector beta-decay coupling constants respectively, Y_ℓ^m is a spherical harmonic, and

$$S_{JL}^M = [Y_L^{M-m} x \sigma_1^m]_J^M .$$

In a like manner, the $f_1 t$ value for a first-forbidden beta transition from a state $|\bar{j}_1 'j_2 J'M_{J'},\ 1\text{-}1\rangle$ to a closed shell is found to be

$$f_1 t = \frac{\pi^2 \ln 2}{\lambda^2 G^2} (2J'+1) \langle j_1'||rS_{21}||j_2\rangle^{-2} . \qquad (3.2)$$

The values of the constants used in our calculation are $G = 3.09 \times 10^{-12}$ and $\lambda = 1.18$. The reduced matrix elements in Eqs. (3.1) and (3.2) are found to be

$$\langle j||r^n Y_J||j'\rangle = (-1)^{j+1}[\frac{(2j+1)\ (2j'+1)\ (2J+1)}{4\pi}] \begin{pmatrix} j & J & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \frac{[1+(-1)^{\ell+\ell'+J}]}{2} F_n$$

and

$$\langle j||r^n S_{JL}||j'\rangle = (-1)^{\ell} \left[\frac{(2\ell+1)\ (2\ell'+1)\ (2J+1)}{2\pi}\right]^{\frac{1}{2}} \begin{pmatrix} \ell & L & \ell' \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} A \begin{Bmatrix} \ell & \frac{1}{2} & j \\ \ell' & \frac{1}{2} & j' \\ L & 1 & J \end{Bmatrix} F_n$$

where

$$F_n = \int r^n RR'dr.$$

The log ft values for the beta decay from N^{16} to O^{16} are tabulated in Table IV for various sets of parameters. An increase in the vector interaction strength is seen to produce an improvement in the theoretical predictions of the $2^- \rightarrow 3^-$, the $2^- \rightarrow 2^-$, and the $2^- \rightarrow 0^+$ decays. However, the influence of the vector potential on these three decays is small compared to that of the scalar potential, and good approximations are obtained even for $V_{SO} = 0$. In contrast, the vector interaction has a decided effect on the $2^- \rightarrow 1^-$ beta decay, and herein is found perhaps the best argument for the inclusion of the two-particle spin-orbit interaction in the mutual interaction potential. Although, as seen in Fig. 2, the exact experimental results are not predicted for any reasonable value of the vector interaction strength, the theoretical ft value of the $2^- \rightarrow 1^-$ decay is reduced from about 800 times its experimental value to less than 4 times its experimental value as the vector interaction strength is increased from 0 to 20 MeV. Since this beta decay occurs in neither the j-j nor the L-S coupling extremes, it is very sensitive to the exact form of the nucleon-nucleon interaction, and the inclusion of such effects as the tensor interaction and the Coulomb interaction, in addition to a more accurate evaluation of all parameters, is obviously necessary to produce strictly correct results. Accordingly, it is difficult to determine V_{SO} exactly, but a value of 15-25 MeV is

TABLE IV

LOG ft VALUES FOR BETA DECAY OF N^{16}

Interaction Strengths (MeV)			Log f_0t (Sec)			Log f_1t (Sec)
V_t	V_s	V_{SO}	$2^- \rightarrow 1^-$	$2^- \rightarrow 2^-$	$2^- \rightarrow 3^-$	$2^- \rightarrow 0^+$
-50	-45	30	5.40	4.39	4.67	8.07
-40	-35	15	6.03	4.29	4.57	7.97
-45	-40	20	5.58	4.37	4.62	8.03
-45	-40	0	7.73	4.21	4.68	7.92
-45	-40	30	5.40	4.45	4.60	8.07
Elliott-Flowers[a]			7.9	4.2	4.65	7.88
Duck[b]			. .	4.81	4.15	8.38
Gillet[c]			7.13	4.32	4.74	7.87
Pure j-j			∞	3.95	4.43	7.68
Experimental[d]			5.1	4.4	4.6	8.0-8.2

[a]See reference 4.

[b]See reference 5.

[c]See reference 6.

[d]See reference 30.

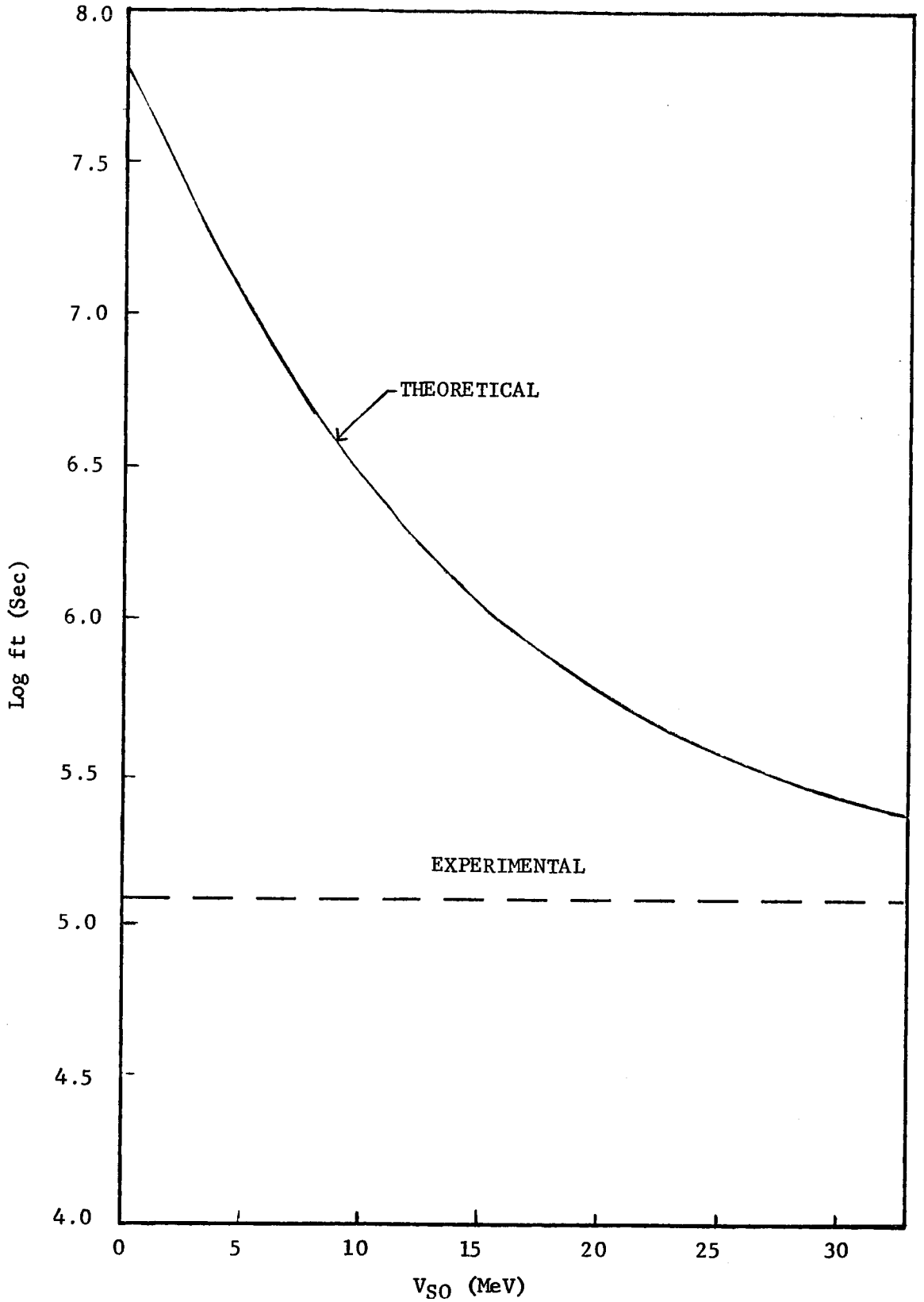

Fig. 2.-- Log ft for $2^- \rightarrow 1^-$ Beta Decay

needed to produce an overall good approximation to experimental data.

O^{16} Muon-Capture Rates

A single-level muon-capture calculation for O^{16} with corrections for effects of finite nuclear size and relativistic muon wave functions has been performed by Jacob.[31] Taking the induced tensor form factor g_T to be zero and the induced pseudoscalar form factor g_P to be 8λ, where $\lambda = 1.22$, we find the muon capture rates to the 0^-, 1^-, and 2^- levels of N^{16} to be, respectively,

$$\Lambda_0 = 1.807 \times 10^5 \, [0.1126 \, A_0 - 0.2441 \, E_0]^2 ,$$

$$\Lambda_1 = 1.797 \times 10^5 \, \{[0.0715 \, A_1 + 0.3544 \, B_1 + 0.2066 \, D_1 + 0.4085 \, E_1 - 0.0255 \, F_1]^2 + [0.1596 \, A_1 - 0.0520 \, B_1 - 0.0215 \, D_1 + 0.1435 \, E_1 - 0.4063 \, F_1]^2 \},$$

$$\Lambda_2 = 1.793 \times 10^5 \, \{[0.0998 \, B_2 + 0.3876 \, C_2 + 0.2117 \, D_2 - 0.1514 \, E_2 + 0.3600 \, F_2]^2 + [0.0776 \, B_2 + 0.1241 \, C_2 + 0.0218 \, D_2 - 0.0789 \, E_2 + 0.1129 \, F_2]^2 \}$$

where the coefficients are defined by

$$|\psi_{J,T=1}\rangle = A_J |\overline{0p_{\frac{1}{2}}} 1s_{\frac{1}{2}} JM\rangle + B_J |\overline{0p_{\frac{1}{2}}} 0d \; JM\rangle + C_J |\overline{0p_{\frac{1}{2}}} 0d \; JM\rangle + D_J |\overline{0p} \; 1s_{\frac{1}{2}} JM\rangle + E_J |\overline{0p} \; 0d \; JM\rangle + F_J |\overline{0p} \; 0d \; JM\rangle \; .$$

The resulting capture rates for various sets of interaction strengths are presented in Table V. Comparison of the theoretical rates with experiment is complicated by the fact that a determination of the capture rates by Astbury, et al.,[32] at the Lawrence Radiation Laboratory in Berkeley resulted in values different from those obtained by Cohen, Devons, and Kanaris[33] at Columbia University. This ambiguity occurs principally in the transition to the 0^- state, which is independent of the vector

TABLE V

O^{16} MUON CAPTURE RATES[a]

V_t	V_s	V_{SO}	Λ_o	Λ_1	Λ_2
-50	-45	30	1.71	1.44	13.3
-40	-35	15	1.78	1.87	16.3
-45	-40	20	1.73	1.50	14.4
-45	-40	0	1.73	3.12	17.7
-45	-40	30	1.73	1.08	13.5
Elliott-Flowers[b]			2.87	3.93	19.1
Duck[c]			1.26	1.87	5.78
Gillet[d]			2.27	2.12	19.5
Pure j-j			2.29	5.50	29.7
Experiment A[e]			1.6 ± 0.2	1.40 ± 0.20	. .
Experiment B[f]			1.1 ± 0.2	1.73 ± 0.10	6.3 ± 0.7

[a]All capture rates are in units of 10^3 sec^{-1}.

[b]See reference 4.

[c]See reference 5.

[d]See reference 6.

[e]See reference 32.

[f]See reference 33.

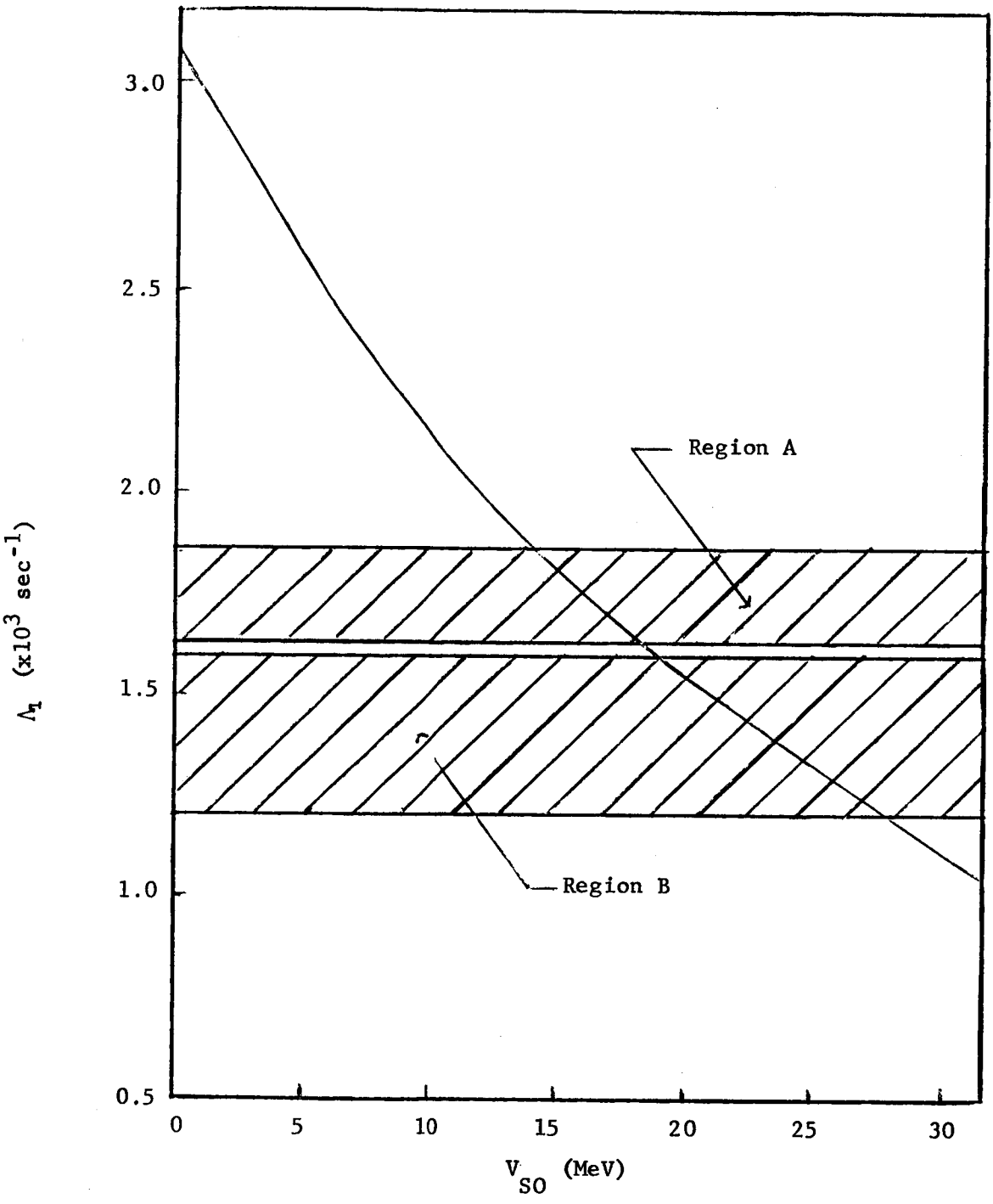

Fig. 3.-- Muon Capture Rate to 1^- Level of N^{16}

Region A corresponds to the experimental results of Cohen, Devons, and Kanaris and region B to those of Astbury et al.

interaction potential. For reasonable values of V_s and V_t the Λ_o predicted by our model agrees with that of Astbury, et al., but is about 1.5 times that of the Columbia group. A study of the transition to the 1^- state of N^{16} is more instructive, however, since the Λ_1's evaluated by the two experiments almost overlap. As seen in Fig. 3, this transition rate for V_t = -45 Mev, V_s = -40 MeV, and V_{SO} = 0 is about twice the experimental value. An increase in the vector interaction strength produces a decrease in the theoretical transition rate, and for values of V_{SO} between 15 and 25 MeV the 1^- capture rates of the two experiments are predicted. In contrast, the Λ_2 measured by the Columbia group is unattainable using our model of the nucleon-nucleon force. Although an increase in the vector interaction strength produces a definite improvement in the predicted Λ_2, the theoretical rate for all reasonable parameters remains about twice the experimental rate. The Berkeley group, whose Λ_o was predicted by our model, did not measure Λ_2.

The muon capture rates obtained in this investigation are in every case an improvement over those of Elliott and Flowers and of Gillet. However, the capture rates calculated by use of Duck's wave function provide a good approximation to the experimental results of Cohen, Devons, and Kanaris. The poor experimental agreement obtained by Duck in energy level and beta-decay computations indicates that his success in muon capture may be largely coincidental, but a more accurate experimental evaluation of the muon-capture rates is obviously necessary to eliminate completely the confusion.

Conclusions

The comparison of theoretical prediction with experiment shows

that the two-particle vector interaction exerts a significant influence on several properties of the O^{16} and N^{16} nuclei. The most pronounced of these is the reduction of the ft value of the $2^- \rightarrow 1^-$ beta decay from 800 times its experimental value for $V_{SO} = 0$ to less than 4 times its experimental value for $V_{SO} = 20$ MeV. Smaller, but still important, improvements in the theoretical results occur in the prediction of the 1^- energy level of O^{16}, the total energy splitting of N^{16}, and the rate of muon capture to the 1^- level of N^{16}. The effect of the vector interaction potential on all other properties investigated is small, but with the exception of the 2^- energy level of O^{16}, the best theoretical prediction of each property considered is obtained for V_{SO} greater than 15 MeV; in fact, with the possible exception of the muon capture to the 2^- level of N^{16}, a good approximation to all experimental data considered is obtained for the interaction strengths $V_t = -45$ MeV, $V_s = -40$ MeV, and $V_{SO} = 20$ MeV. These parameters compare favorably with the experimentally determined values listed in Chapter I. The corresponding state vectors for the low-lying odd parity states of O^{16} and N^{16} are tabulated in Table VI.

TABLE VI

COEFFICIENTS OF BASIS VECTORS FOR O^{16} AND N^{16}

T	J	$(0p_{\frac{1}{2}})^{-1}(1s_{\frac{1}{2}})$	$(0p_{\frac{1}{2}})^{-1}(0d_{\frac{3}{2}})$	$(0p_{\frac{1}{2}})^{-1}(0d_{\frac{5}{2}})$	$(0p_{\frac{3}{2}})^{-1}(1s_{\frac{1}{2}})$	$(0p_{\frac{3}{2}})^{-1}(0d_{\frac{3}{2}})$	$(0p_{\frac{3}{2}})^{-1}(0d_{\frac{5}{2}})$
0	0	0.999	. .	. .	. .	0.036	. .
0	1	0.870	0.124	. .	-0.322	-0.008	0.352
0	2	. .	-0.092	0.943	-0.003	0.009	-0.321
0	3	. .	. .	0.919	. .	-0.271	-0.288
1	0	0.998	. .	. .	. .	0.060	. .
1	1	0.980	-0.006	. .	-0.007	-0.019	0.197
1	2	. .	0.046	0.967	-0.067	0.079	-0.228
1	3	. .	. .	0.999	. .	0.042	0.032

APPENDICES

APPENDIX A

FORMULAS FROM RACAH ALGEBRA

Coupling Coefficients

Clebsch-Gordon coefficients and 3-j symbols. The Clebsch-Gordon coefficient $(j_1 m_1 j_2 m_2 | j_1 j_2 JM)$ is defined by

$$|j_1 j_2 JM\rangle = \sum_{m_1} (j_1 m_1 j_2 m_2 | j_1 j_2 JM) |j_1 m_1\rangle |j_2 m_2\rangle \ .$$

The more symmetrical 3-j symbol is related by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{(2j_3+1)^{\frac{1}{2}}} (j_1 m_1 j_2 m_2 | j_1 j_2 j_3 - m_3) . \qquad (A.1)$$

The components of the 3-j symbol must satisfy

$$m_1 + m_2 + m_3 = 0$$

and the triangular condition $\Delta (j_1 j_2 j_3)$, or

$$|j_1 - j_2| \le j_3 \le j_1 + j_2 \ .$$

The symmetry properties of this symbol are

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \begin{pmatrix} j_2 & j & j_1 \\ m_2 & m & m_1 \end{pmatrix} = (-1)^{j_1 + j_2 + j} \begin{pmatrix} j_1 & j & j_2 \\ m_1 & m & m_2 \end{pmatrix} ,$$

and

$$\begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix} = (-1)^{j_1 + j_2 + j} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \ .$$

The orthogonality relation of the 3-j symbol is

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta(j_3, j_3')\delta(m_3, m_3')}{2j_3+1} .$$

If $m_1 = m_2 = m_3 = 0$, then the value of the 3-j symbol satisfies

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^g \left[\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j_3)!}{(2g+1)!}\right]^{\frac{1}{2}} \frac{g!}{(g-j_1)!(g-j_2)!(g-j_3)!} \quad (A.2)$$

if $2g = j_1+j_2+j_3$ is even, and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

if $j_1+j_2+j_3$ is odd.

Racah coefficients and 6-j symbols. The Racah coefficient is defined by

$$|(j_1 j_3)J_{13}\ j_2\ JM\rangle = \sum_{J_{12}} [(2J_{13}+1)(2J_{12}+1)]^{\frac{1}{2}} W \begin{matrix} j_1 & j_2 & J & j_3 \\ J_{13} & & J_{12} & \end{matrix} |(j_1 j_2)J_{12}\ j_3\ JM\rangle .$$

The more symmetrical 6-j symbol satisfies the relation

$$\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} = (-1)^{j_1+j_2+j_1'+j_2'} W \begin{matrix} j_1 & j_2 & j_2' & j_1 \\ & JJ' & & \end{matrix} .$$

This must satisfy the triangular conditions

$$\Delta(j_1 j_2 J), \Delta(j_1' j_2' J), \Delta(j_1 j_2' J'), \Delta(j_1' j_2 J').$$

The symmetry properties are

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1' & j_2' & j_3' \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_2' & j_1' & j_3' \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ j_2' & j_3' & j_1' \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2' & j_3' \\ j_1' & j_2 & j_3 \end{Bmatrix} .$$

Two special values of the 6-j symbol are

$$\begin{Bmatrix} j_1 & j_1' & 0 \\ j_2 & j_2' & j \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j}}{\sqrt{(2j_1+1)(2j_2+1)}} \delta(j_1, j_1')\delta(j_2, j_2') . \quad (A.3)$$

and

$$\begin{Bmatrix} j_1 & j_2 & j_1+j_2 \\ j_1' & j_2' & J \end{Bmatrix} = (-1)^{j_1+j_2+j_1'+j_2'} \left[\frac{(2j_1)!(2j_2)!(j_1+j_2+j_1'+j_2'+1)!}{(2j_1+2j_2+1)!(j_1'+j_2'-j_1-j_2)!} \right]^{\frac{1}{2}} \tag{A.4}$$

$$\times \left[\frac{(j_1+j_2+j_2'-j_1')!(j_1+j_2+j_1'-j_2')!(j_2'+J-j_1)!(j_1+J-j_2)!}{(j_1+j_2'-J)!(j_1+J-j_2')!(j_1+j_2'+J+1)!(j_2+j_1'-J)(j_2+J-j_1')!(j_1'+j_2+J+1)!} \right]^{\frac{1}{2}}$$

The orthogonality of the 6-j symbol and other sum rules are given by

$$\sum_j (2j+1) \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & J \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & J' \end{Bmatrix} = \frac{\delta(J,J')}{2J+1} , \tag{A.5}$$

$$\sum_j (-1)^{j+J+J'} (2j+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_3 & j_4 & j \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & J' \\ j_2 & j_4 & j \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & J' \end{Bmatrix} , \tag{A.6}$$

$$\sum_j (2j+1) \begin{Bmatrix} j_1 & j_2 & j \\ j_1 & j_2 & j_1 \end{Bmatrix} = (-1)^{2(j_1+j_2)} \tag{A.7}$$

The 6-j symbol is related to the 3-j symbol by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix} = \sum_{m_1', m_2', m_3'} (-1)^{\ell_1+\ell_2+\ell_3+m_1'+m_2'+m_3'}$$

$$\begin{pmatrix} j_1 & \ell_2 & \ell_3 \\ m_1 & m_2' & -m_3' \end{pmatrix} \begin{pmatrix} \ell_1 & j_2 & \ell_3 \\ -m_1' & m_2 & m_3' \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & j_3 \\ m_1' & -m_2' & m_3 \end{pmatrix} .$$

A-coefficients and 9-j symbols. The A-coefficient is defined by

$$|(j_1 j_3) J_{13} (j_2 j_4) J_{24} JM\rangle = \sum_{J_{12} J_{34}} A \begin{pmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{pmatrix} \tag{A.8}$$

$$\times |(j_1 j_2) J_{12} (j_3 j_4) J_{34} JM\rangle .$$

The 9-j symbol is related by

$$\begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{Bmatrix} = [(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)]^{-\frac{1}{2}} A \begin{pmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{pmatrix} .$$

The symmetry properties of the 9-j symbol are shown by

$$\begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{Bmatrix} = \begin{Bmatrix} j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \\ j_1 & j_3 & J_{13} \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = (-1)^{\Sigma} \begin{Bmatrix} j_2 & j_4 & J_{24} \\ j_1 & j_3 & J_{13} \\ J_{12} & J_{34} & J \end{Bmatrix}$$

where

$$\Sigma = j_1 + j_2 + j_3 + j_4 + J_{12} + J_{34} + J_{13} + J_{24} + J .$$

Each row and column of the 9-j symbol must satisfy a triangular condition.

The 9-j symbol is related to the 3-j and the 6-j symbols by the equations

$$\begin{pmatrix} J_{12} & J_{34} & J \\ M_{12} & M_{34} & M \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_{m_1 m_2 m_3 m_4 M_{13} M_{24}} \begin{pmatrix} j_1 & j_2 & J_{12} \\ m_1 & m_2 & M_{12} \end{pmatrix} \times \begin{pmatrix} j_3 & j_4 & J_{34} \\ m_3 & m_4 & M_{34} \end{pmatrix} \begin{pmatrix} j_1 & j_3 & J_{13} \\ m_1 & m_3 & M_{13} \end{pmatrix} \begin{pmatrix} j_2 & j_4 & J_{24} \\ m_2 & m_4 & M_{24} \end{pmatrix} \begin{pmatrix} J_{13} & J_{24} & J \\ M_{13} & M_{24} & M \end{pmatrix} , \tag{A.9}$$

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_{J'} (-1)^{2J'-\Sigma} (2J'+1) \begin{Bmatrix} J_{12} & J_{34} & J \\ J_{24} & J_{13} & J' \end{Bmatrix} \begin{Bmatrix} j_2 & j_4 & J_{24} \\ J_{34} & J' & j_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ J' & J_{12} & j_2 \end{Bmatrix} , \tag{A.10}$$

$$\begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & K \end{Bmatrix} = (-1)^{j_2+J+j_1'+K} [(2J+1)(2K+1)]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J \\ K & K & 0 \end{Bmatrix} . \tag{A.11}$$

Reduced Matrix Elements

The Wigner-Eckart theorem states[34] that for irreducible tensors T_L^M the dependence of the matrix element $\langle jm|T_L^M|j'm'\rangle$ on the projection quantum numbers is entirely contained in the Clebsch-Gordon coefficient:

$$\begin{aligned} \langle jm|T_L^M|j'm'\rangle &= \frac{(j'm'LM|j'Ljm)}{2j'+1} \langle j||T_L||j'\rangle \\ &= (-1)^{j-m} \begin{pmatrix} j & L & j' \\ -m & M & m' \end{pmatrix} \langle j||T_L||j'\rangle . \end{aligned} \tag{A.12}$$

The quantity $\langle j||T||j'\rangle$ is called a reduced matrix element of the set of tensor operators T_L^M. If T_L^0 is a Hermitian operator, then

$$\langle J'||T_L||J\rangle = (-1)^{J-J'}\langle J||T_L||J'\rangle^{\dagger}. \tag{A.13}$$

If T^o_o is an irreducible tensor of degree zero, then

$$\langle JM|T^o_o|J'M'\rangle = \frac{1}{\sqrt{2J+1}}\ \langle J||T_o||J\rangle\delta(J,J')\delta(M,M'). \tag{A.14}$$

Reduced matrix elements of some operators in frequent use are

$$\langle J||1||J'\rangle = \sqrt{2J+1}\ \delta(J,J'), \tag{A.15A}$$

$$\langle J||\underset{\sim}{J}||J'\rangle = \sqrt{J(J+1)(2J+1)}, \tag{A.15B}$$

$$\langle \ell||Y_K||\ell'\rangle = (-1)^{\ell}\left[\frac{(2\ell+1)(2K+1)(2\ell'+1)}{4\pi}\right]^{\frac{1}{2}}\begin{pmatrix}\ell & K & \ell'\\ 0 & 0 & 0\end{pmatrix}, \tag{A.15C}$$

$$\langle \ell\tfrac{1}{2}j||Y_K||\ell'\tfrac{1}{2}j'\rangle = (-1)^{j+\frac{1}{2}}[(2j+1)(2j'+1)]^{\frac{1}{2}}\begin{pmatrix}j & K & j'\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\frac{1+(-1)^{\ell+\ell'+K}}{2}. \tag{A.15D}$$

For harmonic oscillator wave functions, we can evaluate the following reduced matrix elements

$$\begin{aligned}\langle n\ell||r^2||n'\ell'\rangle = {} & (2\ell+\tfrac{1}{2})^{\frac{1}{2}}\Big\{(2n+\ell+\tfrac{3}{2})\delta(n,n') \\ & - [n(n+\ell+\tfrac{1}{2})]^{\frac{1}{2}}\delta(n',n-1) - [(n+1)(n+\ell+\tfrac{3}{2})]^{\frac{1}{2}}\delta(n',n+1)\Big\}\ \delta(\ell,\ell'),\end{aligned} \tag{A.15E}$$

$$\begin{aligned}\langle n\ell||\underset{\sim}{r}||n'\ell'\rangle = {} & (2\ell'+\tfrac{1}{2})^{\frac{1}{2}}(\ell'010|\ell'1\ell0) \\ & \times\Big\{[(n+\ell+\tfrac{3}{2})\delta n,n') - \sqrt{n}\delta(n,n'+1)]\delta(\ell,\ell'-1) \\ & + [(n+\ell+\tfrac{1}{2})\delta(n,n') - \sqrt{n+1}\ \delta(n,n'-1)]\delta(\ell,\ell'+1)\Big\}\ .\end{aligned} \tag{A.15F}$$

Tensor Products and Their Reduced Matrix Elements

The tensor product of two tensors is defined by

$$[T^{k_1}_{K_1}\times T^{k_2}_{K_2}]^{k}_{K} = \sum_{k_1 k_2}(K_1 k_1 K_2 k_2|K_1 K_2 K\ k)T^{k_1}_{K_1}T^{k_2}_{K_2}\ . \tag{A.16}$$

The scalar product of two tensors is defined by

$$\begin{aligned}T_K\cdot U_K &= (-1)^K\sqrt{2K+1}[T_K\times U_K]^o_o \\ &= \sum_m (-1)^m T^m_K U^{-m}_K\ .\end{aligned} \tag{A.17}$$

If

$$T_K = [T_{K_1}(1) \times T_{K_2}(2)]_K ,$$

1 and 2 being different spaces, then the reduced matrix element of T_K can be decomposed as follows

$$\langle \alpha_1 j_1 \alpha_2 j_2 J || T_K || \alpha_1' j_1' \alpha_2' j_2' J' \rangle$$

$$= [(2J+1)(2K+1)(2J'+1)]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ K_1 & K_2 & K \end{Bmatrix} \langle \alpha_1 j_1 || T_{K_1} || \alpha_1' j_1' \rangle \langle \alpha_2 j_2 || T_{K_2} || \alpha_2' j_2' \rangle .$$

For cases in which $T_{K_2}(2) = 1$ or $T_{K_2}(1) = 1$. We have respectively

$$\langle \alpha_1 j_1 \alpha_2 j_2 J || T_{K_1}(1) || \alpha_1' j_1' \alpha_2' j_2' J' \rangle = (-1)^{j_1 + j_2 + J' + K_1} [(2J+1)(2J'+1)]^{\frac{1}{2}}$$

$$\times \begin{Bmatrix} j_1 & J & j_2 \\ J' & j_1' & K_1 \end{Bmatrix} \langle \alpha_1 j_1 || T_{K_1} || \alpha_1' j_1' \rangle \delta(\alpha_2, \alpha_2') \delta(j_2, j_2') , \tag{A.18B}$$

$$\langle \alpha_1 j_1 \alpha_2 j_2 J || T_{K_2}(2) || \alpha_1' j_1' \alpha_2' j_2' J' \rangle = (-1)^{j_1 + j_2' + J + K} [(2J+1)(2J'+1)]^{\frac{1}{2}}$$

$$\times \begin{Bmatrix} j_2 & J & j_1 \\ J' & j_2' & K_2 \end{Bmatrix} \langle \alpha_2 j_2 || T_{K_2} || \alpha_2' j_2' \rangle \delta(\alpha_1, \alpha_1') \delta(j_1, j_1') . \tag{A.18C}$$

The reduced matrix element of the scalar product of two tensors is

$$\langle \alpha_1 j_1 \alpha_2 j_2 J || T_K(1) \cdot T_K(2) || \alpha_1' j_1' \alpha_2' j_2' J' \rangle = (-1)^{j_2 + J + j_1'} \sqrt{2J+1}$$

$$\times \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & K \end{Bmatrix} \langle \alpha_1 j_1 || T_K(1) || \alpha_1' j_1' \rangle \langle \alpha_2 j_2 || T_K(2) || \alpha_2' j_2' \rangle \delta(J, J') . \tag{A.18D}$$

APPENDIX B

REDUCTION OF HOLE-PARTICLE MATRIX ELEMENTS TO PARTICLE-PARTICLE MATRIX ELEMENTS

The convenience of the occupation-number representation of Dirac[35] in evaluating matrix elements of sum operators between states differing from closed-shell configurations by a few particles was first noted by Brink and Satchler.[36] An outline of their theory is given here, and the results are used to reduce matrix elements of operators between hole-particle configurations to matrix elements in the particle-particle configurations.

Let $\eta_a^\dagger$ and η_a denote respectively the creation operator and the annihilation operator for the single particle state $|a\rangle$. For fermions these operators have the commutation relations

$$\eta_a \eta_b + \eta_b \eta_a = 0, \tag{B.1A}$$

$$\eta_a^\dagger \eta_b^\dagger + \eta_b^\dagger \eta_a^\dagger = 0, \tag{B.1B}$$

$$\eta_a \eta_b^\dagger + \eta_b^\dagger \eta_a = \delta_{ab}. \tag{B.1C}$$

The occupation number n_a of a state $|a\rangle$ is an eigenvalue of the operator $\eta_a^\dagger \eta_a$, i.e.,

$$\eta_a^\dagger \eta_a |a\rangle = n_a |a\rangle.$$

For fermions we have, of course,

$$n_a = 0,1.$$

A closed shell can be represented by a single vector $|n_{jm} = 1, \text{all} m\rangle$ which we will denote by $|C\rangle$; thus

$$\eta_a^\dagger \eta_a |C\rangle = |C\rangle \text{ for all } |a\rangle \epsilon |c\rangle. \tag{B.2}$$

The addition of one particle outside the closed shell, in the single

particle orbit $|j_b\ m_b\rangle$, gives the resultant state

$$|(C+1)\ j_b\ m_b\rangle = \eta^\dagger_{j_b\ m_b}|C\rangle .$$

On the other hand, a shell filled but for a particle in the state $|j_a m_a\rangle$ necessarily has a resultant angular momentum $(j_a,-m_a)$. This we can represent by the vector

$$|\ (C-1)j_a-m_a\rangle = (-1)^{j_a-m_a}\eta_{j_a m_a}|C\rangle .$$

The phase factor $(-1)^{j_a-m_a}$ is chosen to provide correct rotational properties.[37] For a particle in a state denoted by the number 2 outside a shell filled but for a hole in a state denoted by 1, we have

$$|\bar{j}_1 j_2 JM\rangle = \sum_{m_1} (-1)^{j_1-m_1}(j_1 -m_1\, j_2 m_2\,|\,j_1 j_2 JM)\eta_1\eta^\dagger_2|C\rangle .$$

If a closed shell contains a state $|a\rangle$, then, employing Eqs. (B.2) and (B.1B), we obtain

$$\eta^\dagger_a|C\rangle = \eta^\dagger_a\ \eta_a{}^\dagger\eta_a|C\rangle = -\eta^\dagger_a\ \eta^\dagger_a\ \eta_a|C\rangle = -\eta^\dagger_a|C\rangle .$$

Thus $\eta^\dagger_a|C\rangle = 0$, if $|a\rangle\epsilon|C\rangle$. (B.3A)

We similarly find, for $|a\rangle\epsilon|C\rangle$ and $|b\rangle\notin|C\rangle$ that

$$\eta_a\ \eta_a|C\rangle = 0, \tag{B.3B}$$

$$\eta^\dagger_b\ \eta^\dagger_b|C\rangle = 0, \tag{B.3C}$$

$$\eta_b\ |C\rangle = 0. \tag{B.3D}$$

The two-body operator $V = \frac{1}{2}\sum_{ij} v_{ij}$ is given in the occupation number representation by[35]

$$V = \tfrac{1}{2}\sum_{abcd}\eta^\dagger_a\ \eta^\dagger_b\ \eta_d\ \eta_c\ \langle ab|v_{ij}|cd\rangle$$

where, omitting isotopic spin,

$$\langle ab|v_{ij}|cd\rangle = \sum_{J,J'}{}'\ (j_a m_a j_b m_b|j_a j_b JM)$$
$$\times\ (j_c m_c j_d m_d|j_c j_d J'M')\langle j_a j_b JM|v_{ij}|j_c j_d J'M'\rangle .$$

Since the potentials we use are all scalar in J-space, we have J = J'. In this paper we are interested only in the interaction between the particle and hole states, so omitting isotopic spin considerations, we

have the matrix element of v_{12} is

$$\langle \bar{j}_1 j_2 JM | v_{12} | \overline{j_1'} j_2' JM \rangle = \sum_{m_1, m_1'} (-1)^{j_1 + j_1' - m_1 - m_1'} (j_1 - m_1, j_2 m_2 | j_1 j_2 JM)$$

$$(j_1' - m_1', j_2' m_2' | j_1' j_2' JM) \langle C | \eta_1^\dagger \eta_2 v_{12} \eta_2'^\dagger \eta_1'^\dagger | C \rangle \tag{B.4}$$

where

$$\langle C | \eta_1^\dagger \eta_2 v_{12} \eta_2'^\dagger \eta_1'^\dagger | C \rangle = \tfrac{1}{2} \sum_{abcd} \langle C | \eta_1^\dagger \eta_2 \eta_a^\dagger \eta_b^\dagger \eta_d \eta_c \eta_2'^\dagger \eta_1' | C \rangle$$

$$\times \langle ab | v_{12} | cd \rangle \tag{B.5}$$

with a,b,c,d being summed over 1, 1', 2, 2'. Using Eqs. (B.1) and (B.2) we find

$$\eta_d \eta_c \eta_2'^\dagger \eta_1' | C \rangle = \eta_d \eta_c \eta_2'^\dagger \eta_1' \eta_1^\dagger \eta_1 | C \rangle$$

$$= \delta(1,1') \eta_d \eta_c \eta_2' \eta_1 | C \rangle - \eta_d \eta_c \eta_2' \eta_1 \eta_1' \eta_1 | C \rangle . \tag{B.6}$$

Use of Eq. (B.3) leads to the result that

$$\delta(1,1') \eta_d \eta_c \eta_2'^\dagger \eta_1 | C \rangle = \delta(1,1') [\delta(2',c) \eta_d \eta_1 | C \rangle - \delta(2',d) \eta_c \eta_1 | C \rangle + \eta_2'^\dagger \eta_d \eta_c \eta_1 | C \rangle]$$

is zero for all values of c and d.

In a like manner we obtain

$$\eta_d \eta_c \eta_2'^\dagger \eta_1^\dagger \eta_1' \eta_1 | C \rangle = \delta(c,2') [\delta(d,1) - \eta_1^\dagger \eta_d] \eta_1' \eta_1 | C \rangle - [\delta(d,2') - \eta_2' \eta_d] [\delta(c,1) - \eta_1^\dagger \eta_c] \eta_1' \eta_1 | C \rangle ,$$

or

$$\eta_d \eta_c \eta_2'^\dagger \eta_1^\dagger \eta_1' \eta_1 | C \rangle = [\delta(c,2') \delta(d,1) - \delta(d,2') \delta(c,1)] \eta_1' \eta_1 | C \rangle .$$

Thus it follows from Eq. (B.6)

$$\eta_d \eta_c \eta_2'^\dagger \eta_1' | C \rangle = [\delta(c,1) \delta(d,2') - \delta(c,2') \delta(d,1)] \eta_1' \eta_1 | C \rangle .$$

Similarly, one finds

$$\langle C | \eta_1^\dagger \eta_2 \eta_a^\dagger \eta_b = -[\delta(a,1') \delta(b,2) - \delta(a,2) \delta(b,1')] \langle C | \eta_1^\dagger \eta_1'^\dagger ,$$

so that

$$\langle c|\eta_1^\dagger\ \eta_2\ \eta_a^\dagger\ \eta_b\ \eta_d\ \eta_c\ \eta_2^{\dagger\prime}\ \eta_1'\ |c\rangle = -[\delta(a,1')\delta(b,2)\delta(c,1)\delta(d,2')$$

$$+ \delta(a,2)\delta(b,1')\delta(c,2')\delta(d,1) - \delta(a,1')\delta(b,2)\delta(c,2')\delta(d,1)$$

$$- \delta(a,2)\delta(b,1')\delta(c,1)\delta(d,2')].$$

Inserting this expression into Eq. (B.5) and using $v_{21} = v_{12}$ we find

$$\langle c|\eta_1^\dagger\ \eta_2\ v_{12}\ \eta_2^{\dagger\prime}\ \eta_1'\ |c\rangle = -[\langle 1'2|v_{12}|12'\rangle - \langle 1'2|v_{12}|2'1\rangle]$$

$$= -\sum_{JJ'} (j_1'm_1'j_2m_2|j_1'j_2J'M')(j_1m_1j_2'm_2'|j_1j_2'J'M')$$

$$\times [\langle j_1'j_2J'M'|v_{12}|j_1j_2'J'M'\rangle - (-1)^{j_1+j_2'-J'}\langle j_1'j_2J'M'|v_{12}|j_2'j_1J'M'\rangle].$$

Substitution of this expression into Eq. (B.4) and manipulation of the coupling coefficients according to the formulas in Appendix A yields

$$\langle \bar{j}_1 j_2 JM|v_{12}|\overline{j_1'}j_2'JM\rangle = -\sum_{J'} (2J'+1)\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix}$$

$$\times [\langle j_1'j_2J'M'|v_{12}|j_1j_2'J'M'\rangle - (-1)^{j_1+j_2'+J'}\langle j_1'j_2J'M'|v_{12}|j_2'j_1J'M'\rangle].$$

Inclusion of the isotopic spin quantum number leads, by the same procedure, to the result that the hole-particle matrix element is

$$\langle \bar{j}_1 j_2 JM_JTM_T|v_{12}|\bar{j}_1'j_2'JM_JTM_T\rangle = -\sum_{J'T'} (2J'+1)(2T'+1) \tag{B.6}$$

$$\times\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J \end{Bmatrix}\begin{Bmatrix} \tfrac{1}{2} & \tfrac{1}{2} & T \\ \tfrac{1}{2} & \tfrac{1}{2} & T' \end{Bmatrix} \times [\langle j_1'j_2J'M_J';\tfrac{1}{2}\tfrac{1}{2}T'M_T'|v_{12}|j_1j_2'J'M_J';\tfrac{1}{2}\tfrac{1}{2}T'M_T'\rangle$$

$$- (-1)^{j_1+j_2'+J'+T'+1}\langle j_1'j_2J'M_J';\tfrac{1}{2}\tfrac{1}{2}T'M_T'|v_{12}|j_2'j_1J'M_J';\tfrac{1}{2}\tfrac{1}{2}T'M_T'\rangle].$$

APPENDIX C

MOSHINSKY TRANSFORMATION BRACKETS

The transformation from wave functions with coordinates $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$ to wave functions with center-of-mass coordinates $\underset{\sim}{R} = \frac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2)$ and relative coordinates $\underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2$ was first treated in detail by Moshinsky.[38] He defined the transformation bracket $\langle n\ell, \eta\lambda, L | n_1\ell_1, n_2\ell_2 L\rangle$ such that

$$|n_1\ell_1, n_2\ell_2, LM\rangle = \sum_{\substack{n\ell \\ \eta\lambda}} |n\ell, \eta\lambda, LM\rangle \langle n\ell, \eta\lambda, L | n_1\ell_1, n_2\ell_2, L\rangle$$

where $n\ell$ and $\eta\lambda$ are the quantum numbers associated with the relative and center-of-mass coordinates respectively. The wave function in relative coordinates contains a harmonic oscillator length $b' = \sqrt{2}\, b$, and associated with the center-of-mass coordinate is $b'' = \frac{b}{\sqrt{2}}$. This appendix is devoted to the two-step evaluation of these transformation brackets; first, we evaluate $\langle n\ell, \eta\lambda, L | 0\ell_1, 0\ell_2 L\rangle$, and then a recurrence relation for $n_1, n_2 > 0$ is derived.

Evaluation of $\langle n\ell, \eta\lambda, L | 0\ell_1, 0\ell_2, L\rangle$

From Eqs. (I.12) and (I.13) we find that the two particle harmonic oscillator wave function is

$$|0\ell_1, 0\ell_2, LM\rangle = \left[\frac{2\left(\frac{1}{b}\right)^{2\ell_1+3} 2\left(\frac{1}{b}\right)^{2\ell_2+3}}{\Gamma(\ell_1+\frac{3}{2})\Gamma(\ell_2+\frac{3}{2})}\right]^{\frac{1}{2}} e^{-\frac{1}{b^2}(r_1^2+r_2^2)} |\ell_1\ell_2 LM) \qquad (C.1)$$

where

$$|\ell_1\ell_2 LM) = \sum_{m_1 m_2} (\ell_1 m_1 \ell_2 m_2 | \ell_1\ell_2 LM) r_1^{\ell_1} Y_{\ell_1}^{m_1}(\theta_1, \varphi_1) r_2^{\ell_2} Y_{\ell_2}^{m_2}(\theta_2, \varphi_2).$$

Employing a formula for the translation of multiple fields which states[38] that if

$$\underset{\sim}{r} = \underset{\sim}{r}' - \underset{\sim}{r}''$$

then

$$r^{\ell} Y_{\ell}^{m}(\theta,\varphi) = \sum_{\ell_1 \ell''}^{\ell} \delta(\ell'+\ell'',\ell) G(\ell'\ell''\ell) \sum_{m'm''} (\ell' m' \ell'' m'' | \ell' \ell'' \ell m)$$

$$\times\, r'^{\ell'} Y_{\ell'}^{m'}(\theta',\varphi') r''^{\ell''} Y_{\ell''}^{m''}(\theta'',\varphi'')$$

where

$$G(\ell'\ell''\ell) = (-1)^{\ell''} \left[\frac{4\pi(2\ell+1)!}{(2\ell'+1)!\,(2\ell''+1)!} \right]^{\frac{1}{2}} \tag{C.2}$$

and a formula for the product of two spherical harmonics which states[34]

$$Y_{\ell'}^{n'}(\theta',\varphi') Y_{\ell''}^{n''}(\theta'',\varphi'') = \sum_{\ell m} H(\ell'\ell''\ell)(\ell' n' \ell'' n'' | \ell' \ell'' \ell m) Y_{\ell}^{m}(\theta,\varphi)$$

where

$$H(\ell'\ell''\ell) = \left[\frac{(2\ell'+1)(2\ell''+1)}{4\pi(2\ell+1)} \right]^{\frac{1}{2}} (\ell' 0 \ell'' 0 | \ell' \ell'' \ell 0), \tag{C.3}$$

we find

$$|\ell_1 \ell_2 LM) = \sum_{m_1 m_2} (\ell_1 m_1 \ell_2 m_2 | \ell_1 \ell_2 LM) \sum_{\substack{\ell' \lambda' \\ \ell'' \lambda''}} \sum_{\substack{m'M' \\ m''M''}} (-1)^{\ell} \delta(\ell'+\lambda', \ell_1)$$

$$\times\, \delta(\ell''+\lambda'', \ell_2) G(\ell'\lambda'\ell_1) G(\ell''\lambda''\ell_2) (\tfrac{1}{2}r)^{\ell'+\ell''} R^{\lambda'+\lambda''}$$

$$\times\, (\ell' m' \lambda' M' | \ell' \lambda' \ell_1 m_1)(\ell'' m'' \lambda'' m'' | \ell'' \lambda'' \ell_2 m_2)$$

$$\times \sum_{\ell_0 m_0} H(\ell'\ell''\ell_0)(\ell' m' \ell'' m'' | \ell' \ell'' \ell_0 m_0) Y_{\ell_0}^{m_0}(\theta,\varphi)$$

$$\times \sum_{\lambda_0 m_0} H(\lambda'\lambda''\lambda_0)(\lambda' m'' \lambda'' m'' | \lambda' \lambda'' \lambda_0 m_0) Y_{\lambda_0}^{m_0}(\Theta,\Phi).$$

By use of Eqs. (A.1) and (A.9) the vector coupling coefficients can be combined to yield

$$|\ell_1 \ell_2 L M) = \sum_{\ell_0 \lambda_0} \sum_{\substack{\ell' \lambda' \\ \ell'' \lambda''}} (-1)^{\ell'} \left(\tfrac{1}{2}\right)^{\lambda'+\lambda''} \delta(\ell'+\lambda', \ell_1)\, \delta(\ell''+\lambda'', \ell_2)\, G(\ell'\lambda'\ell_1)$$

$$\times G(\ell''\lambda''\ell_2) H(\ell'\ell''\ell_0)\, H(\lambda'\lambda''\lambda_0) \left[(2\ell_0+1)(2\lambda_0+1)(2\ell_1+1)(2\ell_2+1)\right]^{\frac{1}{2}}$$

$$\times\, r^{\ell'+\ell''-\ell_0} R^{\lambda'+\lambda''-\lambda_0} \begin{Bmatrix} \ell' & \ell'' & \ell_0 \\ \lambda' & \lambda'' & \lambda_0 \\ \ell_1 & \ell_2 & L \end{Bmatrix} |\ell_0 \lambda_0 L M).$$

If the H-function is to be nonzero, then $\ell' + \ell'' + \ell_0$ and $\lambda' + \lambda'' + \lambda_0$ must be positive even integers, thus, expanding $\left(\frac{r^2}{b'^2}\right)^{\frac{1}{2}(\ell'+\ell''-\ell_0)}$ and $\left(\frac{R^2}{b''^2}\right)^{\frac{1}{2}(\lambda'+\lambda''+\lambda_0)}$ into a series of Laguerre polynomials by the formula[39]

$$x^K = \sum_{m=0}^{K} (-1)^m \frac{K!}{(K-m)!} \frac{\Gamma(1+\alpha+K)}{\Gamma(1+\alpha+m)} L_m^{\alpha}(x),$$

we find that $|0\ell_1, 0\ell_2, LM\rangle$ is related to $|n_0\ell_0, n_0\lambda_0, LM\rangle$ by

$$|0\ell_1, 0\ell_2, LM\rangle = \sum_{\substack{n_0\ell_0 \\ n_0\lambda_0}} \sum_{\substack{\ell'\lambda' \\ \lambda''\ell''}} (-1)^{\ell'+n_0+n_0} \left(\tfrac{1}{2}\right)^{\frac{1}{2}(\ell_1+\ell_2)} \delta(\ell'+\lambda', \ell_1)\delta(\lambda'+\ell'', \ell_2)$$

$$\times G(\ell'\lambda''\ell_1)\, G(\lambda''\ell''\ell_2)\, H(\ell'\ell''\ell_0)\, H(\lambda'\lambda''\lambda_0)$$

$$\times \frac{\left[\frac{1}{2}(\ell'+\ell''-\ell_0)\right]!}{\left[\frac{1}{2}(\ell'+\ell''-\ell_0)-n_0\right]!} \frac{\left[\frac{1}{2}(\lambda'+\lambda''-\lambda_0)\right]!}{\left[\frac{1}{2}(\lambda'+\lambda''-\lambda_0)-n_0\right]!}$$

$$\times \left[\frac{(2\ell_0+1)(2\lambda_0+1)(2\ell_1+1)(2\ell_2+1)}{n_0!\, n_0!\, \Gamma(\ell_1+\tfrac{3}{2})\Gamma(\ell_2+\tfrac{3}{2})\Gamma(n_0+\ell_0+\tfrac{3}{2})\Gamma(n_0+\lambda_0+\tfrac{3}{2})}\right]^{\frac{1}{2}}$$

$$\times \Gamma\left[\tfrac{1}{2}(\ell'+\ell''+\ell_0+3)\right] \Gamma\left[\tfrac{1}{2}(\lambda'+\lambda''+\lambda_0+3)\right] \begin{Bmatrix} \ell' & \ell'' & \ell_0 \\ \lambda' & \lambda'' & \lambda_0 \\ \ell_1 & \ell_2 & L \end{Bmatrix} |n_0\ell_0, n_0\lambda_0, LM\rangle.$$

Therefore the transformation bracket satisfies the relation

$$\langle n\ell, n\lambda, L|0\ell_1, 0\ell_2, L\rangle = (2)^{-\frac{1}{2}(\ell_1+\ell_2)} \left[\frac{(2\ell+1)(2\lambda+1)(2\ell_1+1)(2\ell_2+1)}{n!\, n!\, \Gamma(\ell_1+\frac{3}{2})\Gamma(\ell_2+\frac{3}{2})\Gamma(n+\ell+\frac{3}{2})\Gamma(n+\lambda+\frac{3}{2})}\right]^{\frac{1}{2}}$$

$$\times \sum_{\substack{\ell'\ell'' \\ \lambda'\lambda''}} (-1)^{\ell'+n+n} \delta(\ell'+\lambda', \ell_1)\delta(\ell''+\lambda'', \ell_2)\, G(\ell'\lambda'\ell_1) G(\ell''\lambda''\ell_2) \qquad (C.4)$$

$$\times H(\lambda'\ell''\ell) H(\ell'\lambda''\lambda) \frac{\left[\frac{1}{2}(\ell'+\ell''-\ell)\right]!}{\left[\frac{1}{2}(\ell'+\ell''-\ell)-n\right]!} \frac{\left[\frac{1}{2}(\lambda'+\lambda''-\lambda)\right]!}{\left[\frac{1}{2}(\lambda'+\lambda''-\lambda)-n\right]!} \times$$

$$\times \Gamma\left[\tfrac{1}{2}(\ell'+\ell''+\ell+\tfrac{3}{2})\right]\Gamma\left[\tfrac{1}{2}(\lambda'+\lambda''+\lambda+3)\right] \begin{Bmatrix} \ell' & \ell'' & \ell \\ \lambda' & \lambda'' & \lambda \\ \ell_1 & \ell_2 & L \end{Bmatrix}.$$

This expression will be zero unless

$$\ell' + \ell'' - \ell - 2m \geq 0$$

and

$$\lambda' + \lambda'' - \lambda - 2n \geq 0 .$$

Adding these and using

$$\ell_1 = \ell' + \lambda' \quad , \quad \ell_2 = \ell'' + \lambda''$$

we find

$$\ell_1 + \ell_2 - \ell - 2m - \lambda - 2n \geq 0 .$$

But from the conservation of energy this must be exactly zero, and hence we must have

$$\ell' + \ell'' - \ell - 2m = 0$$
$$\lambda' + \lambda'' - \lambda - 2n = 0 .$$

This second expression may be derived from the first along with the δ-functions in Eq. (C.4). Thus, conservation of energy is ensured by the inclusion of $\delta(\ell'+\ell'', \ell+2m)$ in the summation in Eq. (C.4).

By using Eqs. (C.2), (C.3), (A.2), (A.10), and (A.4) we may reduce the expression for the transformation bracket to

$$\langle m\ell, n\lambda, L \mid 0\ell_1, 0\ell_2, L \rangle$$

$$= \left[\frac{(2\ell_1+1)(2\ell_2+1)(2\ell+1)(2\lambda+1)}{2^{\ell_1+\ell_2}\, m!\, n!\, \Gamma(\ell_1+\tfrac{3}{2})\,\Gamma(\ell_2+\tfrac{3}{2})\,\Gamma(m+\ell+\tfrac{3}{2})\,\Gamma(n+\lambda+\tfrac{3}{2})}\right]^{1/2}$$

$$\times \sum_{\substack{x, \lambda', \lambda' \\ \ell'', \lambda''}} (-1)^{m+n+\ell+\lambda-L+\ell''+\frac{1}{2}(\ell-\ell'-\ell'')+\frac{1}{2}(\lambda-\lambda'-\lambda'')}$$

$$\times \delta(\ell'+\lambda', \ell_1)\,\delta(\ell''+\lambda'', \ell_2)\,\delta(\ell'+\ell'', 2m+\ell)\,\Gamma\left[\tfrac{1}{2}(\ell'+\ell''+\ell+3)\right] \times$$

$$\times\ \Gamma\left[\tfrac{1}{2}(\lambda'+\lambda''+\lambda+3)\right]\left[\frac{(2\ell_1+1)!\,(2\ell_2+1)!\,(\ell''+\ell-\ell)!\,(\lambda'+\lambda-\lambda'')!}{(\lambda'+\ell''+\ell+1)!\,(\lambda'+\lambda''+\lambda+1)!}\right]$$

$$\times\ \frac{\left[\tfrac{1}{2}(\ell'+\ell''+\ell)\right]!\left[\tfrac{1}{2}(\lambda'+\lambda''+\lambda)\right]!}{\left[\tfrac{1}{2}(\ell''+\ell-\ell')\right]!\left[\tfrac{1}{2}(\ell'+\ell-\ell'')\right]!\left[\tfrac{1}{2}(\lambda''-\lambda-\lambda')\right]!\left[\tfrac{1}{2}(\lambda'+\lambda-\lambda'')\right]!}$$

$$\times\ \left[\frac{(\ell''+\lambda''+\chi+\ell+1)!\,(\ell''+\lambda''+\chi-\ell)!\,(\ell''+\lambda''+\lambda-\chi)!\,(\lambda'+\lambda-\lambda'')!}{(2\ell''+2\lambda''+1)!\,(\chi+\lambda-\ell''-\lambda'')!\,(\ell''+\ell'-\chi)!\,(\ell''+\chi+\ell'+1)!\,(\lambda''+\lambda+\lambda'+1)!}\right]^{1/2}$$

$$\times\ \left[\frac{(\ell'+\lambda'+\ell+\chi+1)!\,(\ell'+\lambda'+\ell-\chi)!\,(\ell'+\lambda'+\chi-\ell)!\,(\ell+\ell''-\ell')!}{(2\ell'+2\lambda'+1)!\,(\ell+\chi-\ell'-\lambda')!\,(\ell'+\ell+\ell''+1)!\,(\lambda'+\ell''-\chi)!\,(\lambda'+\ell''+\chi+1)!}\right]^{1/2}$$

$$\times\ W^{\ell\lambda\ell_1\ell_2}_{L\chi}\ .$$

Defining q by

$$q = \ell_1+\ell''-\ell'\ ,$$

and summing over ℓ'', λ', λ'', we obtain

$$\langle m\ell,\eta\ell,L|o\ell_1,o\ell_2,L\rangle = \sum_{\chi,q}(-1)^{m+\ell+\ell-L+\frac{1}{2}(\ell+q-\ell_1)}\frac{(2\chi+1)\,\Gamma(m+\ell+\tfrac{3}{2})\,\Gamma(\eta+\ell+\tfrac{3}{2})}{}$$

$$\times\left[\frac{(2\ell+1)(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{2^{(\ell_1+\ell_2)}\,m!\,\eta!\,\Gamma(\ell_1+\tfrac{3}{2})\,\Gamma(\ell_2+\tfrac{3}{2})\,\Gamma(m+\ell+\tfrac{3}{2})\,\Gamma(\eta+\ell+\tfrac{3}{2})}\right]^{1/2}$$

$$\times\left[\frac{(2\ell_1+1)!\,(2\ell_2+1)!\,(q+\ell-\ell_1)!\,(q+\ell-\ell_2)!\,(\ell_2+\chi+\ell+1)!\,(\ell_2+\chi-\ell)!}{(2m+2\ell+1)!\,(2\eta+2\ell+1)!\,(2\ell_2+1)!\,(\chi+\ell-\ell_2)!\,(q-\chi)!\,(q+\chi+1)!}\right]^{1/2}\times$$

$$\times \left[\frac{(\ell_1+\ell-\chi)!\,(\ell+q-\ell_2)!\,(\ell_1+\ell+\chi+1)!\,(\ell_1+\ell-\chi)!\,(\ell_1+\chi-\ell)!\,(\ell+q-\ell_1)!}{(2\eta+2\ell+1)!\,(2\ell_1+1)!\,(\ell-\ell_1+\chi)!\,(2\ell+2m+1)!\,(q-\chi)!\,(q+\chi+1)!}\right]^{1/2}$$

$$\times \frac{(m+\ell)!\,(\eta+\ell)!}{[\tfrac{1}{2}(\ell+q-\ell_1)]!\,[\tfrac{1}{2}(\ell+\ell_1-q)]!\,[\tfrac{1}{2}(\ell+q-\ell_2)]!\,[\tfrac{1}{2}(\ell+\ell_2-q)]!}\, W_{L\chi}^{\ell\ell\ell_1\ell_2} .$$

Finally, using

$$\Gamma\left[(m+1)+\tfrac{1}{2}\right] = \frac{(2m+1)!}{2^{2m+1}(m!)}\sqrt{\pi}$$

we find the transformation bracket is given by

$$\langle m\ell, \eta\ell, L \,|\, o\ell_1, o\ell_2, L\rangle \qquad (C.5)$$

$$= (-1)^{m+\ell_1+\ell_2-L}\left[\frac{\ell_1!\,\ell_2!}{(2\ell_1)!\,(2\ell_2)!}\,\frac{(2\ell+1)(2\ell+1)}{2^{\ell+\ell}}\,\frac{(m+\ell)!\,(\eta+\ell)!}{m!\,\eta!\,(2m+2\ell+1)!\,(2\eta+2\ell+1)!}\right]^{1/2}$$

$$\times \sum_{\chi} (2\chi+1)\, A(\ell_1\,\ell;\ \ell_2\,\ell;\ \chi) \begin{Bmatrix} \ell & \ell & L \\ \ell_2 & \ell_1 & \chi \end{Bmatrix}$$

where

$$A(\ell_1\,\ell;\ \ell_2\,\ell;\chi) = \left[\frac{(\ell_1+\ell+\chi+1)!\,(\ell_1+\ell-\chi)!\,(\ell_1+\chi-\ell)!}{(\ell+\chi-\ell_1)!}\right]^{1/2} \qquad (C.6)$$

$$\times \left[\frac{(\ell_2+\ell+\chi+1)!\,(\ell_2+\ell-\chi)!\,(\ell_2+\chi-\ell)!}{(\ell+\chi-\ell_2)!}\right]^{1/2} \sum_{q} (-1)^{\frac{1}{2}(\ell+q-\ell_1)}\,\frac{1}{(q-\chi)!\,(q+\chi+1)!}$$

$$\times \frac{(\ell+q-\ell_1)!}{[\tfrac{1}{2}(\ell+q-\ell_1)]!\,[\tfrac{1}{2}(\ell+\ell_1-q)]!} \cdot \frac{(\ell+q-\ell_2)!}{[\tfrac{1}{2}(\ell+q-\ell_2)]!\,[\tfrac{1}{2}(\ell+\ell_2-q)]!} .$$

Since

$$\ell + q_f - \ell_1 = \ell'' - \ell' + \ell$$

and

$$\mathcal{L} + q_f - \ell_2 = \mathcal{L}'' - \mathcal{L}' + \mathcal{L}$$

must be even integers, q is restricted to values for which the arguments of the factorials in Eq. (C.6) are integral. In order to have nonzero values of the 6-j symbols, the summation over χ is limited to values such that

$$|\ell - \ell_1| \leq \chi \leq \ell + \ell_1$$

and

$$|\mathcal{L} - \ell_2| \leq \chi \leq \mathcal{L} + \ell_2 .$$

Recurrence Relation for Transformation Brackets

Consider the wave function

$$|(m_1+1)\ell_1, m_2\ell_2, LM\rangle = \sum_{m_1 m_2} (\ell_1 m_1, \ell_2 m_2 | \ell_1 \ell_2 L M) \frac{1}{r_1 r_2}$$

$$\times R_{(m_1+1)\ell_1}(r_1)\ Y_{\ell_1}^{m_1}(\theta_1, \varphi_1)\ R_{m_2\ell_2}(r_2)\ Y_{\ell_2}^{m_2}(\theta_2, \varphi_2)$$

where

$$R_{m\ell}(r) = \left[\frac{2\left(\frac{1}{b}\right)^{2\ell+3} m!}{\Gamma(m+\ell+3/2)}\right]^{1/2} r^{\ell+1} e^{-\frac{r^2}{2b^2}} L_m^{\ell+\frac{1}{2}}\left(\frac{r^2}{b^2}\right).$$

The Laguerre polynomials satisfy the relation[39]

$$L_{m+1}^{\ell+\frac{1}{2}}(x^2) = \frac{(2m+\ell+3/2-x^2)}{m+1} L_m^{\ell+\frac{1}{2}}(x^2) - \frac{(m+\ell+1/2)}{m+1} L_{m-1}^{\ell+\frac{1}{2}}(x^2).$$

Hence we find

$$R_{(m+1)\ell}\left(\frac{r^2}{b^2}\right) = \left[\frac{2\left(\frac{1}{b}\right)^{2\ell+3}(m+1)!}{\Gamma(m+\ell+5/2)}\right]^{1/2} r^{\ell+1}\ e^{-\frac{r^2}{2b^2}}\ \times$$

$$\times \left[\frac{(2n+\ell+3/2-\frac{r^2}{b^2})}{n+1} L_n^{\ell+\frac{1}{2}}\left(\frac{r^2}{b^2}\right) - \frac{(n+\ell+\frac{1}{2})}{n+1} L_{n-1}^{\ell+\frac{1}{2}}\left(\frac{r^2}{b^2}\right)\right]$$

and it follows that

$$|(n_1+1)\ell_1, n_2\ell_2, LM\rangle = \frac{(2n_1+\ell_1+\frac{3}{2}-\frac{r_1^2}{b^2})}{[(n_1+1)(n_1+\ell_1+3/2)]^{1/2}} |n_1\ell_1, n_2\ell_2, LM\rangle$$

$$- \left[\frac{n_1(n_1+\ell_1+\frac{1}{2})}{(n_1+1)(n_1+\ell_1+3/2)}\right]^{1/2} |(n_1-1)\ell_1, n_2\ell_2, LM\rangle.$$

Thus we obtain the relation

$$\langle n\ell, N\mathcal{L}, LM|(n_1+1)\ell_1, n_2\ell_2, LM\rangle \tag{C.7}$$

$$= \frac{2n_1+\ell_1+\frac{3}{2}}{[(n_1+1)(n_1+\ell_1+3/2)]} \langle n\ell, N\mathcal{L}, LM|n_1\ell_1, n_2\ell_2, LM\rangle$$

$$- \left[\frac{n_1(n_1+\ell_1+\frac{1}{2})}{(n_1+1)(n_1+\ell_1+\frac{3}{2})}\right]^{1/2} \langle n\ell, N\mathcal{L}, LM|(n_1-1)\ell_1, n_2\ell_2, LM\rangle$$

$$- \frac{1}{[(n_1+1)(n_1+\ell_1+3/2)]^{1/2}} \langle n\ell, N\mathcal{L}, LM|\frac{r_1^2}{b^2}|n_1\ell_1, n_2\ell_2, LM\rangle.$$

From conservation of energy considerations we note that if the left-hand side of Eq. (C.7) is to be nonzero, the first two terms on the right-hand side must be zero. Accordingly, we obtain the recurrence relation

$$\langle n\ell, N\mathcal{L}, L|(n_1+1)\ell_1, n_2\ell_2, L\rangle = [(n_1+1)(n_1+\ell_1+3/2)]^{-1/2}$$

$$\times \sum_{\substack{n'\ell' \\ N'\mathcal{L}'}} \langle n\ell, N\mathcal{L}, LM|-\frac{r_1^2}{b^2}|n'\ell', N'\mathcal{L}', LM\rangle \tag{C.8}$$

$$\times \langle n'\ell', N'\mathcal{L}', L|n_1\ell_1, n_2\ell_2, L\rangle$$

The nonzero values of the matrix element in this equation are given in Table VII.

If we wish to evaluate $\langle n\ell, N\mathcal{L}, L|n_1\ell_1, (n_2+1)\ell_2, L\rangle$, we need only change the $[(n_1+1)(n_1+\ell_1+3/2)]^{-1/2}$ in Eq. (C.8) to $[(n_2+1)(n_2+\ell_2+3/2)]^{-1/2}$ and change the sign of the last four lines in Table VIII, since the sign of the $\underset{\sim}{r}\cdot\underset{\sim}{R}$ in r_2^2 is opposite that in r_1^2.

TABLE VII

Nonzero components of $\langle m\ell, n\lambda, LM|-\frac{r_1^2}{b^2}|m'\ell', n'\lambda', LM\rangle$[a]

m'	ℓ'	n'	λ'	$\langle m\ell, n\lambda, LM\|-\frac{r_1^2}{b^2}\|m'\ell', n'\lambda', LM\rangle$
$m-1$	ℓ	n	λ	$\frac{1}{2}[m(m+\ell+\frac{1}{2})]^{1/2}$
m	ℓ	$n-1$	λ	$\frac{1}{2}[n(n+\lambda+\frac{1}{2})]^{1/2}$
$m-1$	$\ell+1$	$n-1$	$\lambda+1$	$[mn(\ell+1)(\lambda+1)]^{1/2} W_{\ell\,\lambda+1}^{\lambda L\,\ell+1}$
$m-1$	$\ell+1$	n	$\lambda-1$	$[m\lambda(n+\lambda+\frac{1}{2})(\ell+1)]^{1/2} W_{\ell\,\lambda-1}^{\lambda L\,\ell+1}$
m	$\ell-1$	$n-1$	$\lambda+1$	$[m\ell(m+\ell+\frac{1}{2})(\lambda+1)]^{1/2} W_{\ell\,\lambda+1}^{\lambda L\,\ell-1}$
m	$\ell-1$	n	$\lambda-1$	$[\ell\lambda(m+\ell+\frac{1}{2})(n+\lambda+\frac{1}{2})]^{1/2} W_{\ell\,\lambda-1}^{\lambda L\,\ell-1}$

[a]The quantum numbers are subject to the condition

$$2m+\ell+2n+\lambda = 2m'+\ell'+2n'+\lambda'+2.$$

APPENDIX D

CENTER-OF-MASS MOTION OF NUCLEONS

When working with nucleons in a harmonic-oscillator-potential well, one must remember that the center of mass is also in motion governed by the harmonic-oscillator potential. If all nucleons occupy the lowest possible oscillator levels according to the Pauli principle, the center-of-mass motion can be ignored; only if some nucleons are excited to higher oscillator levels must the center of mass motion be carefully taken into account. In particular, certain excited states, called spurious states, have the same intrinsic wave functions as the ground state, but the center of mass is not in the lowest oscillator level. As shown by Elliott and Skyrme,[16] the spurious states can be eliminated by diagonalizing the matrix of the operator

$$R^2 = \frac{1}{A^2}\left[\sum_i r_i^2 + 2\sum_{i<j} \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j\right]$$

and retaining only those combinations of states with the minimum eigenvalue. This is equivalent to evaluating the energy of the mass center and rejecting those states which do not have energy $\frac{3}{2}\hbar\omega$, corresponding to the 0s state.

For the hole-particle configuration we find from Eq. (2.2) that

$$\langle \bar{j}_1 j_2 J M_J T M_T \mid \hat{R}^2 \mid \bar{j}_1' j_2' J M_J T M_T \rangle$$

$$= -\sum_{J'} (2J'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} \left[\langle j_1' j_2 J' M' \mid R^2 \mid j_1 j_2' J' M' \rangle -\right.$$

$$- (-1)^{j_1 + j_2' + J'} 2\delta(T,0) \langle j_1' j_2 J' M' | R^2 | j_2' j_1 J' M' \rangle]$$

where

$$R^2 = \frac{1}{4}\left(r_1^2 + r_2^2 + 2\,\underset{\sim}{r_1} \cdot \underset{\sim}{r_2} \right)$$

Using Eq. (A.18) we may expand the above expression to the form

$$\langle J_1\, j_2\, J\, M_J\, T M_T | R^2 | J_1'\, j_2'\, J M_J\, T M_T \rangle$$

$$= -\frac{1}{4} \sum_{J'} (2J'+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} \left\{ \left[(-1)^{j_1' + j_2 + J'} \sqrt{2J'+1} \begin{Bmatrix} j_1' & J' & j_2 \\ J' & j_1 & 0 \end{Bmatrix} \right.\right.$$

$$\times \langle j_1' \| r_1^2 \| j_1 \rangle \delta(2,2') + (-1)^{j_1' + j_2' + J'} \sqrt{2J'+1} \begin{Bmatrix} j_2 & J' & j_1 \\ J' & j_2' & 0 \end{Bmatrix} \langle j_2 \| r_2^2 \| j_2' \rangle \delta(1,1')$$

$$\left. + (-1)^{j_1 + j_2 + J'} 2 \begin{Bmatrix} j_1' & j_2 & J' \\ j_2' & j_1 & 1 \end{Bmatrix} \langle j_1' \| \underset{\sim}{r_1} \| \underset{\sim}{j_1} \rangle \langle j_2 \| \underset{\sim}{r_2} \| j_2' \rangle \right]$$

$$+ (-1)^{j_1 + j_2' + J'} 2\delta(T,0) \left[(-1)^{j_1' + j_2 + J'} \sqrt{2J'+1} \begin{Bmatrix} j_1' & J' & j_2 \\ J' & j_2' & 0 \end{Bmatrix} \right.$$

$$\times \langle j_1' \| r_1^2 \| j_2' \rangle \delta(1,2) + (-1)^{j_1' + j_1 + J'} \sqrt{2J'+1} \begin{Bmatrix} j_2 & J' & j_1' \\ J' & j_1 & 0 \end{Bmatrix} \langle j_2 \| r_2^2 \| j_1 \rangle \delta(1',2')$$

$$\left. \left. + (-1)^{j_2 + j_2' + J'} 2 \begin{Bmatrix} j_1' & j_2 & J' \\ j_1 & j_2' & 1 \end{Bmatrix} \langle j_1' \| \underset{\sim}{r_1} \| j_2' \rangle \langle j_2 \| \underset{\sim}{r_2} \| j_1 \rangle \right] \right\}$$

The first two terms in the exchange part of this formula will always be zero since we will never have the particle and hole in the same state.

Employing Eqs. (A.3), (A.5), (A.6), (A.7), and (A.13), we can reduce the previous expression to

$$\langle \bar{J}_1\ j_2\ J M_J T M_T \,|\, R^2 \,|\, \bar{J}_1'\ j_2'\ J M_J T M_T \rangle$$

$$= -\tfrac{1}{4}\Big[(2j_1+1)^{-\frac{1}{2}} \langle j_1' \| r_1^2 \| j_1 \rangle \delta(j_1, j_1') \delta(2,2') + (2j_2+1)^{-\frac{1}{2}} \langle j_2 \| r_2^2 \| j_2' \rangle$$

$$\times \delta(j_2, j_2') \delta(1,1') + (-1)^{j_1+j_2+J+1} 2 \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & 1 \end{Bmatrix} \langle j_1' \| \underset{\sim}{r}_1 \| j_1 \rangle \langle j_2 \| \underset{\sim}{r}_2 \| j_2' \rangle$$

$$- \tfrac{4}{3} \delta(T,0) \delta(J,1) \langle j_1' \| \underset{\sim}{r}_1 \| j_2' \rangle \langle j_1 \| \underset{\sim}{r}_2 \| j_2 \rangle \Big].$$

Utilization of Eqs. (A.18A), (A.11), and (A.3) leads to the relations

$$\langle \ell\ \tfrac{1}{2}\ j \| r^2 \| \ell'\ \tfrac{1}{2}\ j' \rangle = \left[\frac{2j+1}{2\ell+1}\right]^{\frac{1}{2}} \langle n\ell \| r^2 \| n'\ell' \rangle \delta(j,j') \delta(\ell,\ell')$$

and

$$\langle \ell\ \tfrac{1}{2}\ j \| \underset{\sim}{r} \| \ell'\ \tfrac{1}{2}\ j' \rangle$$

$$= (-1)^{\ell+j'-\frac{1}{2}} \left[(2j+1)(2j'+1)\right]^{\frac{1}{2}} \begin{Bmatrix} \ell & j & \frac{1}{2} \\ j' & \ell' & 1 \end{Bmatrix} \langle n\ell \| \underset{\sim}{r} \| n'\ell' \rangle,$$

and thus the evaluation of the matrix of R^2 may be completed by use of Eqs. (A.15E) and (A.15F).

In the case of O^{16} and N^{16} we find in every case except the J = 1, T = 0 state that for all linear combinations of the simple harmonic-oscillator wave functions, the center-of-mass energy is a minimum and thus need not be considered in choosing our basis vectors. In the J = 1, T = 0 state, however, (R^2) is not diagonal and we find that a certain linear combination of the harmonic-oscillator wave functions refer to a state in which the center-of-mass motion is in an excited state.

If we let

$$|\psi_1\rangle = |\overline{0p}_{1/2}\ 1s_{1/2}\ 1M; 00\rangle, \quad |\psi_2\rangle = |\overline{0p}_{1/2}\ 0d_{3/2}\ 1M; 00\rangle,$$

$$|\psi_3\rangle = |\overline{0p}_{3/2}\ 1s_{1/2}\ 1M; 00\rangle, \quad |\psi_4\rangle = |\overline{0p}_{3/2}\ 0d_{3/2}\ 1M; 00\rangle,$$

$$|\psi_5\rangle = |\overline{0p}_{3/2}\ 0d_{5/2}\ 1M; 00\rangle,$$

then the R^2 matrix element in this representation is

$$(\langle \psi_i | R^2 | \psi_j \rangle) =$$

$$\begin{pmatrix}
-\frac{3}{2}+\frac{2}{9} & -\frac{2\sqrt{5}}{9} & -\frac{2\sqrt{2}}{9} & \frac{2}{9} & -\frac{2}{3} \\
-\frac{2\sqrt{5}}{9} & -\frac{3}{2}+\frac{10}{9} & \frac{2\sqrt{10}}{9} & \frac{2\sqrt{5}}{9} & \frac{2\sqrt{5}}{3} \\
-\frac{2\sqrt{2}}{9} & \frac{2\sqrt{10}}{9} & -\frac{3}{2}+\frac{4}{9} & \frac{2\sqrt{2}}{9} & \frac{2\sqrt{2}}{3} \\
-\frac{2}{9} & \frac{2\sqrt{5}}{9} & \frac{2\sqrt{2}}{9} & -\frac{3}{2}+\frac{2}{9} & \frac{2}{3} \\
-\frac{2}{3} & \frac{2\sqrt{5}}{3} & \frac{2\sqrt{2}}{3} & \frac{2}{3} & -\frac{3}{2}+2
\end{pmatrix}$$

This matrix has eigenvalues $-\frac{3}{2}$, $-\frac{3}{2}$, $-\frac{3}{2}$, $-\frac{3}{2}$, and $\frac{5}{2}$.

The eigenstate corresponding to this last eigenvalue,

$$|\psi_A\rangle = -\frac{\sqrt{2}}{6}|\psi_1\rangle + \frac{\sqrt{10}}{6}|\psi_2\rangle + \frac{1}{3}|\psi_3\rangle + \frac{\sqrt{2}}{6}|\psi_4\rangle + \frac{\sqrt{2}}{2}|\psi_5\rangle ,$$

must be removed from our shell model calculations.

The admissible state vectors corresponding to the $-\frac{3}{2}$ eigenvalues are

$$|\psi_B\rangle = \frac{\sqrt{10}}{6}|\psi_1\rangle + \frac{\sqrt{2}}{6}|\psi_2\rangle - \frac{\sqrt{5}}{3}|\psi_3\rangle + \frac{\sqrt{10}}{30}|\psi_4\rangle + \frac{\sqrt{10}}{10}|\psi_5\rangle ,$$

$$|\psi_C\rangle = \frac{\sqrt{6}}{3}|\psi_1\rangle + \frac{\sqrt{3}}{3}|\psi_3\rangle ,$$

$$|\psi_D\rangle = \frac{\sqrt{6}}{6}|\psi_2\rangle + \frac{2\sqrt{30}}{15}|\psi_4\rangle - \frac{\sqrt{30}}{10}|\psi_5\rangle ,$$

$$|\psi_E\rangle = \frac{\sqrt{2}}{2}|\psi_2\rangle - \frac{\sqrt{10}}{5}|\psi_4\rangle - \frac{\sqrt{10}}{10}|\psi_5\rangle .$$

LIST OF REFERENCES

REFERENCES

1. M. G. Mayer, Phys. Rev. 78, 16 (1950).

2. O. Haxel, J. H. D. Jensen, and H. E. Suess, Phys. Rev. 75, 1766 (1949).

3. A. Bohr and B. R. Mottelson, Math-fys. Medd. 27, no. 16 (1953).

4. J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) 242, 57 (1957).

5. I. Duck, Nuclear Phys. 35, 27 (1962).

6. V. Gillet and N. Vinh Mau, Nuclear Phys. 54, 321 (1964).

7. L. Hulten and M. Sugawara, "The Two-Nucleon Problem," Handbuck der Physik, ed. S. Flugge, XXXIX, 1 (1957).

8. P. Goldhammer, Revs. Modern Phys. 35, 40 (1963).

9. J. P. Elliott and A. M. Lane, "The Nuclear Shell-Model," Handbuck der Physik, ed. S. Flugge, XXXIX, 241 (1957).

10. A. de Shalit and I. Talmi, Nuclear Shell Theory (Academic Press, New York, 1963).

11. F. H. Lewis, Jr., and J. D. Walecka, Phys. Rev. 133, B 849 (1964).

12. P. S. Signell and R. E. Marchak, Phys. Rev. 109, 1220 (1958).

13. J. P. Elliott and A. M. Lane, Phys. Rev. 96, 1160 (1956).

14. R. Muthukrishnan, Bull. Am. Phys. Soc. 11, 305 (1966).

15. S. Sack, L. C. Biedenharn, and G. Breit, Phys. Rev. 93, 321 (1954).

16. V. V. Balashov, J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 1123 (1959).

17. K. M. Case and A. Pais, Phys. Rev. 80, 203 (1950).

18. B. P. Niger, Revs. Modern Phys. 35, 117 (1963).

19. A. Messiah, Quantum Mechanics (North-Holland Publishing Co., Amsterdam, 1962), Vol. II, Chap. XVI.

20. J. P. Elliott, Selected Topics in Nuclear Theory (International Atomic Energy Agency, Vienna, 1963).

21. E. U. Condon and G. H. Shortly, The Theory of Atomic Spectra (University Press, Cambridge, 1953), p. 299.

22. I. Talmi, Helv. Phys. Acta, 25, 533 (1956/57).

23. T. A. Brody, G. Jacob, and M. Moshinsky, Nuclear Phys. 17, 16 (1960).

24. R. Thieberger, Nuclear Phys. 2, 533 (1956).

25. J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London), A 232, 561 (1955).

26. G. E. Lee-Whiting, Can. J. Phys. 36, 1199 (1958).

27. L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Revs. Modern Phys. 24, 249 (1952).

28. M. Abramowitz and I. A. Stegun (eds.), Handbook of Mathematical Functions, National Bureau of Standards, Applied Math Series 55, (1964), p. 1006.

29. J. N. Huffaker, Ph.D. thesis (unpub.), Duke University, (1962).

30. F. Ajzenberg-Selove and T. Lauritsen, Nuclear Phys. 11, 1 (1959).

31. R. F. Jacob, Ph.D. thesis (unpub.), University of Alabama, (1966); and private communication.

32. A. Astbury, et al., Nuovo Cimento 33, 1020 (1964).

33. R. Cohen, S. Devons, and A. Kanaris, Phys. Rev. Letters 11, 134 (1963); correction in Nuclear Phys. 57, 255 (1964).

34. M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, New York, 1957).

35. P. A. M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, 1930), Chap. X.

36. D. M. Brink and G. R. Satchlet, Nuovo Cimento 6, 549 (1956).

37. M. E. Rose, Multipole Fields (John Wiley and Sons, New York, 1955)

38. M. Moshinsky, Nuclear Phys. 13, 104 (1959).

39. E. D. Rainville, Special Functions (The Macmillan Company, New York, 1960), Chap. XII.

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13. ABSTRACT

The main text of this paper is divided into three sections. In Chapter I the possible nucleon-nucleon interaction potentials are considered, and the form is selected which is thought necessary to obtain a reasonably accurate picture of the O^{16} and N^{16} nuclei without necessarily complicating the calculations. Then the assumptions made to approximate the solution of the Schrodinger equation are discussed, and the potential well introduced by these assumptions is chosen. A mathematical treatment of the methods used for constructing the energy matrices is presented in Chapter II. The final chapter contains theoretical predictions of the energy-level splittings of O^{16} and N^{16} and of the beta-decay and muon-capture rates for the two nuclei. After a comparison of these results with experimental data, the nuclear state vectors giving the best experimental fit are chosen. The appendices contain three derivations too cumbersome to be included in the main text, but necessary for the completeness of the work, and a summary of formulas from Racah algebra which are used throughout our calculations.

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