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LYAPUNOV FUNCTIONS FOR A CLASS OF nth ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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CONTENTS

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Lyapunov Functions for a Class of n^{th} Order Nonlinear Differential Equations

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Abstract

A method is described for sequentially developing a quadratic polynomial into a possible Lyapunov function for a class of nonlinear differential equations. The equations are n^{th} order in one dependent variable, with nonlinearities appearing as polynomial functions of the variable and its n-1 derivatives. The procedure utilizes higher order homogeneous polynomials in the dependent variable and its derivatives to generate sign definite functions.

LYAPUNOV FUNCTIONS FOR A CLASS OF n^{th} ORDER NONLINEAR DIFFERENTIAL EQUATIONS

I. INTRODUCTION

The problem of determining the stability of solutions to nonlinear differential equations by the direct method of Lyapunov is often an exercise in guessing suitable Lyapunov functions. The sufficiency condition that is characteristic of this method, however, may leave the analyst without a conclusive statement of either stability or instability. Techniques have not been widely developed to provide a sequence of steps that can be followed to correct or modify an initial choice of a possible Lyapunov function to gradually approach a satisfactory one. This report introduces one procedure for doing this under stated conditions. Specifically a method is given for sequentially developing a quadratic polynomial into a possible Lyapunov function for a class of nonlinear differential equations. The inherent limitations of the Lyapunov method (the sufficiency condition and the poorly defined region of stability) are not circumvented.

The construction of sign definite polynomials in the proposed procedure is not limited to quadratic forms. As a result modifications to an initial trial quadratic polynomial can progressively compensate for nonlinear terms in the equation. As one considers increasingly more complex nonlinearities, the philosophy of this approach is intuitively satisfying.

The class of nonlinear differential equations considered in this report is the following:

$$x^{(n)} + F(x,\dot{x},\ldots,x^{(n-1)}) = 0$$

where $$F = \sum_{i=1}^{\ell} f_i(x,\dot{x},\ldots,x^{(n-1)})$$

with f_i homogeneous polynomials of order i and real coefficients.

Equivalently

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\cdots\cdots\cdots \qquad (1)$$

$$\cdots\cdots\cdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1}x_2 \cdots\cdots\cdots -a_1 x_n - f_2(x_1,x_2,\cdots,x_n)$$

$$- f_3(x_1,x_2,\cdots,x_n) \cdots\cdots -f_\ell(x_1,x_2,\cdots,x_n) ,$$

or $\underline{\dot{X}} = \underline{\underline{A}}\underline{X} + \underline{F}$,

$$\text{where } \underline{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_2 & -a_1 \end{pmatrix} ,$$

$$\underline{F} = \begin{bmatrix} & & \underline{0} & \\ -f_2 & -f_3 & \cdots\cdots\cdots\cdots & -f_\ell \end{bmatrix} ,$$

and $$\underline{X} = [x_1 \; x_2 \; x_3 \; \ldots \; x_n]^t .$$

Without loss of generality, assume an equilibrium point at the origin of the state space, i.e., $f_i(0,0,\ldots,0) = 0$ [1]*.

The objective of the direct method of Lyapunov is to determine the stability characteristics of the equilibrium point utilizing the given form of the differential equations but without explicit knowledge of the solutions [1]. According to this method, if it is possible to generate a scalar function $V(\underline{X})$ within a region Ω around the equilibrium point such that:

(1) $V(\underline{X})$ is continuous together with its first partial derivatives in Ω,

(2) $V(\underline{0}) = 0$,

(3) $V(\underline{X}) > 0$, for $\underline{X} \neq 0$, and

(4) $\frac{dV(\underline{X})}{dt} = \dot{V}(\underline{X}) \leq 0$ in Ω,

then the equilibrium point is stable in the sense originally defined by Lyapunov [1]. The generation of this scalar function, V, called a Lyapunov function, for most nonlinear system equations, including those of the form of equation (1), has been found to depend critically on the ingenuity of the analyst [3, 4, 5]. For the class of equations (1), a Lyapunov function can always be found for an ε region around the equilibrium point, as long as the eigenvalues of $\underline{A}$ have negative real parts [1]. The purpose here is to find a finite region around the equilibrium point, larger than an ε region, within which the polynomial is positive definite and the time derivative is either negative definite or negative semidefinite.

To summarize the procedure one initially assumes a quadratic form in $\underline{X}$, $V_0(\underline{X},\underline{X})$, as the elementary polynomial of a possible Lyapunov function for the given differential equation. If $V_0(\underline{X},\underline{X})$ is not a Lyapunov function, then a sequence of correction polymonials is added to V_0 until the time derivative of the modified function is made either negative definite or negative semidefinite in a finite region around the equilibrium point. The method is described in greater detail in Section III and by illustration in Section IV.

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*Definitions of the terminology used in this report are referenced for the unfamiliar reader.

II. Sufficient Conditions for the Sign Definiteness of an Even Order Homogeneous Polynomial in n Variables

A major difficulty in generating Lyapunov functions has been the lack of a general method for proving or disproving the sign definiteness of scalar functions containing terms of higher than second order. This difficulty can be partially resolved by using the procedure illustrated in Section IV.

Suppose $V_{2m}(\underline{X})$ is a $(2m)^{th}$ order homogeneous polynomial in n variables, $x_1, x_2, \ldots, x_n$ (m a real integer > 0). It can be shown that $V_{2m}(\underline{X})$ can be reduced to a quadratic form $V(\underline{Y},\underline{Y}) = \underline{Y}^t \underline{K} \underline{Y}$ with at most $\left(\frac{n(n+1)\ldots(n+m-1)}{m!}\right)$ variables, y_i, where $\underline{K}$ is a constant symmetric matrix. It can also be shown that at most $\left(\frac{n(n+1)\ldots\ldots(n+m-1)}{m!}\right)$ conditions are needed on the principal minors of $\underline{K}$ to prove the sign definiteness of $V(\underline{Y},\underline{Y})$ and hence of $V_{2m}(\underline{X})$.

III. Description of the Procedure

1. Assume a complete quadratic form $V_0(\underline{X},\underline{X}) = \underline{X}^t \underline{P} \underline{X}$, where $\underline{P}$ is an n x n constant symmetric matrix containing $\frac{n(n+1)}{2}$ unknown variables p_i. Then

$$\dot{V}_o(\underline{X},\underline{X}) \triangleq \frac{dV_o}{dt} = \frac{\partial V_o}{\partial x_1}\dot{x}_1 + \frac{\partial V_o}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial V_o}{\partial x_n}\dot{x}_n .$$

Subject to the differential equation constraint

$$\dot{V}_0 = -\underline{X}^t \underline{Q} \underline{X} - F_{03}(\underline{X}) - F_{04}(\underline{X}) \ldots - F_{0(2m+1)}(\underline{X}),$$

where $\underline{Q}$ is an n x n constant symmetric matrix and $F_{0i}(\underline{X})$,

$3 \leq i \leq 2m + 1$, contains only i^{th} order terms in $\underline{X}$. An attempt is made to determine the $\frac{n(n+1)}{2}$ unknown constants, p_i, from the following conditions:

(a) $\underline{X}^t \underline{P} \underline{X}$ positive definite,

(b) $\underline{X}^t \underline{Q} \underline{X}$ at least positive semidefinite,

(c) the odd order terms in $\dot{V}_0$, $F_{03} = F_{05} = \dots F_{0(2m+1)}$, identically zero,

(d) the even order terms in $\dot{V}_0$, F_{04}, F_{06}, ..., $F_{(2m)}$, at least positive semidefinite.

Condition (a) requires n necessary and sufficient conditions such that all n principal minors of $\underline{P}$ are positive. Condition (b) also requires n necessary and sufficient conditions such that all n principal minors of $\underline{Q}$ are non-negative. Condition (c) is satisfied if each coefficient of F_{03}, F_{05}, ..., $F_{0(2m+1)}$ is set to zero.

For condition (d), as stated in Section II, at most $\frac{n(n+1) \dots\dots (n+q-1)}{q!}$ sufficient conditions are needed to show positive semidefiniteness for $F_{0(q)}$ (q: even integer $2 \leq q \leq m$). Since these four conditions indicate a number of equations greater than the number of unknowns, a consistent set of solutions for all p_i may or may not be found. If a set of values for p_i is found, the V_0 is positive definite everywhere, and $\dot{V}_0$ is either negative definite or negative semidefinite in a finite region, the region of stability [1]. If a consistent solution set for p_i doesn't exist, then correction terms are added until a consistent set of solutions is obtained. For example:

2. Add a complete third order homogeneous polynomial $G_{13}(\underline{X})$ to V_0, and define

$$V_1(\underline{X}) \triangleq V_0(\underline{X},\underline{X}) + G_{13}(\underline{X}) = \underline{X}^t \, \underline{P} \, \underline{X} + G_{13}(\underline{X})$$

Then

$$\dot{V}_1(\underline{X}) = -\underline{X}^t\underline{\underline{Q}}\underline{X} - F_{13}(\underline{X}) - F_{14}(\underline{X}) - \dots - F_{1(2m+2)}(\underline{X})$$

where $F_{1i}(\underline{X})$, $3 \leq i \leq 2m + 1$, contains only i^{th} order terms in $\underline{X}$.

Next one attempts to determine the $\frac{n(n+1)}{2}$ unknown constants p_i and the constants introduced in $G_{13}(\underline{X})$ from the following conditions:

(a) $\underline{X}^t\underline{\underline{P}}\underline{X}$ positive definite,

(b) $\underline{X}^t\underline{\underline{Q}}\underline{X}$ at least positive semidefinite,

(c) the odd order terms in $\dot{V}_1$ identically zero,

(d) the even order terms in $\dot{V}_1$ at least negative semidefinite.

3. This procedure can be continued, adding higher order terms to V_i at each step, if a consistent solution set for the unknown constants of V_i is not found.

IV. EXAMPLES

Example 1. Consider the nonlinear differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2-x_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix}$$

with an equilibrium point at (0,0).

1. Assume a quadratic form V_0 as the fundamental term of a possible Lyapunov function,

$$V_0(\underline{X},\underline{X}) = K_{01}x_1^2 + K_{02}x_1x_2 + K_{03}x_2^2 = \underline{X}^t\underline{\underline{P}}\underline{X},$$

where $\underline{P} = \begin{bmatrix} K_{01} & \frac{K_{02}}{2} \\ \frac{K_{02}}{2} & K_{03} \end{bmatrix}$, a symmetric matrix.

Then $\dot{V}_0(\underline{X},\underline{X}) = \frac{\partial V_0}{\partial x_1}\dot{x}_1 + \frac{\partial V_0}{\partial x_2}\dot{x}_2$

$$= (2K_{01}x_1+K_{02}x_2)x_2 + (K_{02}x_1+2K_{03}x_2)(-x_2-x_1^3)$$

$$= -\{(K_{02}-2K_{01})x_1x_2 + (2K_{03}-K_{02})x_2^2\} - \{K_{02}x_1^4 +$$

$$2k_{03}x_1^3x_2\} = -\underline{X}^t\underline{\underline{Q}}\underline{X} - F_{04}(\underline{X})$$

$$\text{where } \underset{\sim}{Q} = \begin{bmatrix} 0 & \dfrac{K_{02}-2K_{01}}{2} \\ \dfrac{K_{02}-K_{01}}{2} & 2K_{03} - K_{02} \end{bmatrix}, \text{ a symmetric matrix.}$$

According to Section II, reduce

$F_{04}(\underline{X}) = K_{02}x_1^4 + 2K_{03}x_1^3x_2$ to a quadratic form in $\underline{Y}$.

Define $y_1 = x_1^2$, $y_2 = x_1x_2$,

$$F_{04} = [y_1, y_2] \begin{bmatrix} K_{02} & K_{03} \\ K_{03} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and determine the unknown constants K_{01}, K_{02}, and K_{03} from the following conditions:

(a) $V_0(\underline{X},\underline{X})$ be positive definite,

(b) $\underline{X}^t\underset{\sim}{Q}\underline{X}$ be at least positive semidefinite,

(c) F_{04} be at least positive semidefinite.

Condition (a) requires:

$$K_{01} > 0, \tag{I-1}$$

$$K_{01}K_{03} - \left(\frac{K_{02}}{2}\right)^2 > 0, \tag{I-2}$$

Condition (b) requires:

$$-\left(\frac{K_{02}-2K_{01}}{2}\right)^2 \geq 0 \quad . \tag{I-3}$$

Condition (c) requires:

$$K_{02} \geq 0, \tag{I-4}$$

$$K_{03} = 0 \tag{I-5}$$

Condition (I-5) contradicts (I-2), so correction terms added to V_0 are required.

A complete fourth order homogeneous polynomial with unknown coefficients can be added to V_0, but inspection of F_{04} reveals that the coefficient K_{03} is associated with $y_1 y_2$, or $x_1^3 x_2$. Consequently we may only need to add a term $G_{14}(\underline{X})$ to V_0 such that its time derivative will contain $x_1^3 x_2$. As such a term is x_1^4, set $G_{14} = K_{11}x_1^4$.

2. Define

$$V_1 \triangleq V_0 + G_{14} = \underline{X}^t \underline{\underline{P}}\underline{X} + K_{11}x_1^4$$

Then

$$\dot{V}_1 \triangleq \dot{V}_0 + \dot{G}_{14} = - \underline{X}_t \underline{\underline{Q}}\underline{X} - \{ K_{02}x_1^4 + (2K_{03}-4K_{11})x_1^3 x_2 \}$$

$$= - \underline{X}^t \underline{\underline{Q}}\underline{X} - F_{14}(\underline{X})$$

where Q is unchanged. Reduce the bracketed term to a quadratic form of $\underline{Y}$ by defining $y_1 = x_1^2, y_2 = x_1 x_2$, i.e.,

$$F_{14} = [y_1 y_2] \begin{bmatrix} K_{02} & K_{03}-2K_{11} \\ K_{03}-2K_{11} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then attempt to determine the unknown constants K_{01}, K_{02}, K_{03} and K_{11} from conditions (a) ∿ (c) as given in step 1. Again

(a) $K_{01} > 0,$ (I-1)

$$K_{01}K_{03} - \left(\frac{K_{02}}{2}\right)^2 > 0, \qquad \text{(I-2)}$$

(b) $$-\left(\frac{K_{02}-2K_{01}}{2}\right)^2 \geq 0, \qquad \text{(I-3)}$$

(c) for F_{14} to be at least positive semidefinite,

$K_{02} \geq 0,$ (I-4)

$$-(K_{03} - 2K_{11})^2 \geq 0.$$

From (I-1), arbitrarily choose $K_{01} = 1$ and substitute (I-3) and $K_{01} = 1$ into (I-2),

$$K_{01}K_{03} - 4(K_{01})^2 = K_{03} - 4 \geq 0.$$

Let $K_{03} = 5$. From (I-3) and $K_{01} = 1$, $K_{11} = \frac{5}{2}$, and $K_{02} = 2$. Therefore

$$V_1 = \underline{X}^t \underline{P} \underline{X} + K_{11}X_1^4 = x_1^2 + 2x_1x_2 + \frac{5}{2}x_1^4 + 5x_2^2,$$

and $\dot{V}_1 = -\underline{X}^t \underline{Q} \underline{X} - F_{14}(\underline{X}) = -8x_2^2 - 2x_1^4$.
Thus V_1 is positive definite and $\dot{V}_1$ is negative definite everywhere in the x_1 ∿ x_2 plane and the equilibrium point is global asymptotically stable.

The stability of this differential equation has been examined in the literature by using the variable gradient method [2] to get the Lyapunov function

$$V = \frac{x_1^2}{2} + x_1 x_2 + x_2^2 + \frac{x_1^4}{2}$$

with $\dot{V} = -x_2^2 - x_1^4$

Example 2. Consider the differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -2x_2 - x_1^3 - 6x_1 x_2^2 - 3x_1^2 x_3 \end{bmatrix}$$

with an equilibrium point at the origin.

1. Assume a fundamental term of a possible Lyapunov function

$$V_0(\underline{X},\underline{X}) = \underline{X}^t \underline{P}\underline{X}$$

where $\underline{P} = \begin{bmatrix} K_{01} & K_{02} & K_{03} \\ K_{02} & K_{04} & K_{05} \\ K_{03} & K_{05} & K_{06} \end{bmatrix}$, a symmetric matrix.

Then $\dot{V}_0(\underline{X},\underline{X}) = -\underline{X}^t \underline{Q}\underline{X} - F_{04}(\underline{X})$

where $\underline{Q} = \begin{bmatrix} 0 & 2K_{03}-K_{01} & -K_{02} \\ 2K_{03}-K_{01} & 2(2K_{05}-K_{02}) & 2K_{06}-K_{03}-K_{04} \\ -K_{02} & 2K_{06}-K_{03}-K_{04} & -2K_{05} \end{bmatrix}$, and

$$F_{04}(\underline{X}) = 2K_{03}x_1^4 + 2K_{05}x_1^3x_2 + (6K_{03}+2K_{06})\ x_1^3x_3$$

$$+ 6K_{05}x_1^2x_2x_3 + 6K_{06}x_1^2x_3^2 + 12K_{03}x_1^2x_2^2$$

$$+ 12K_{05}x_1x_2^3 + 12K_{06}x_1x_2^2x_3.$$

Reduce $F_{04}(\underline{X})$ to a quadratic form in $\underline{Y}$ by defining

$$y_1 = x_1^2,\ y_2 = x_2^2,\ y_3 = x_1x_2,\ y_4 = x_1x_3,\ y_5 = x_2x_3.$$

Then $$F_{04}(\underline{X}) = F_{04}(\underline{Y},\underline{Y}) = \underline{Y}^t\underline{K}_0\underline{Y}$$

with $$\underline{K}_0 = \begin{bmatrix} 2K_{03} & 0 & K_{05} & 3K_{05}+K_{06} & 0 \\ 0 & 0 & 6K_{05} & 0 & 0 \\ K_{05} & 6K_{05} & 12K_{03} & 3K_{05} & 6K_{06} \\ 3K_{03}+K_{06} & 0 & 3K_{05} & 6K_{06} & 0 \\ 0 & 0 & 6K_{06} & 0 & 0 \end{bmatrix}$$

To determine the unknown constants K_{0i}, the following conditions are to be satisfied.

(a) $\underline{P}$ a positive definite matrix,
(b) the principal minors of $\underline{Q}$ be non-negative,
(c) the principal minors of $\overline{\underline{K}}_0$ be non-negative.

Condition (a) requires:

$$K_{01} > 0, \qquad \text{(II-1)}$$

$$K_{01}K_{04} - K_{02}^2 > 0, \qquad \text{(II-2)}$$

$$\begin{vmatrix} K_{01} & K_{02} & K_{03} \\ K_{02} & K_{04} & K_{05} \\ K_{03} & K_{05} & K_{06} \end{vmatrix} > 0. \qquad \text{(II-3)}$$

Condition (b) requires:

$$\begin{vmatrix} 0 & 2K_{03} - K_{01} \\ 2K_{03} - K_{01} & 2(2K_{05}-K_{02}) \end{vmatrix} = - (2K_{03}-K_{01})^2 \geq 0$$

which implies $K_{03} = \frac{K_{01}}{2}$, (II-4)

$$\begin{vmatrix} 0 & 2K_{03}-K_{01} & -K_{02} \\ 2K_{03} - K_{01} & 2(2K_{05}-K_{02}) & 2K_{06} - K_{03} - K_{04} \\ -K_{02} & 2K_{06} - K_{03} - K_{04} & -2K_{05} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -K_{02} \\ 0 & 2(2K_{05}-K_{02}) & 2K_{06}-K_{03}-K_{04} \\ -K_{02} & 2K_{06}-K_{03}-K_{04} & 2K_{05} \end{vmatrix}$$

$$= - K_{02}^2 \cdot 2 \cdot (2K_{05}-K_{02}) \geq 0,$$

which implies $K_{02} = 0$ (II-5)

Condition (c) requires:

$$2K_{03} \geq 0, \qquad \text{(II-6)}$$

$$\begin{vmatrix} 2K_{03} & 0 & K_{05} \\ 0 & 0 & 6K_{05} \\ K_{05} & 6K_{05} & 12K_{03} \end{vmatrix} = -\ 72K_{03}K_{05}^{2} \geq 0.$$

Since K_{03} must be greater than zero (II-1 and 4)

$$K_{05} = 0 \qquad \text{(II-7)}$$

$$F_{04} = \underline{Y}^{t} \begin{bmatrix} 2K_{03} & 0 & 0 & 3K_{03}+K_{06} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12K_{03} & 0 & 6K_{06} \\ 3K_{03}+K_{06} & 0 & 0 & 6K_{06} & 0 \\ 0 & 0 & 6K_{06} & 0 & 0 \end{bmatrix} \underline{Y}.$$

Since the second row and second column of the matrix associated with F_{04} are zero, the second element of $\underline{Y}$ can be dropped, and F_{04} becomes

$$F_{04} = \begin{bmatrix} y_1 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}^{t} \begin{bmatrix} 2K_{03} & 0 & 3K_{03}+K_{06} & 0 \\ 0 & 12K_{03} & 0 & 6K_{06} \\ 3K_{03}+K_{06} & 0 & 6K_{06} & 0 \\ 0 & 6K_{06} & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

The requirements for F_{04} to be positive semidefinite are now

$$2K_{03} \geq 0 \qquad \text{(II-6)}$$

$$\begin{vmatrix} 2K_{03} & 0 \\ 0 & 12K_{03} \end{vmatrix} = 24\ K_{03}^{\ 2} \geq 0,$$

$$\begin{vmatrix} 2K_{03} & 0 & 3K_{03}+K_{06} \\ 0 & 12K_{03} & 0 \\ 3K_{03}+K_{06} & 0 & 6K_{06} \end{vmatrix}$$

$$= 2K_{03}\ \{\ 72K_{03}K_{06} - 6(3K_{03}+K_{06})^2\} \geq 0, \qquad \text{(II-8)}$$

and

$$\begin{vmatrix} 2K_{03} & 0 & 3K_{03}+K_{06} & 0 \\ 0 & 12K_{03} & 0 & 6K_{06} \\ 3K_{03}+K_{06} & 0 & 6K_{06} & 0 \\ 0 & 6K_{06} & 0 & 0 \end{vmatrix}$$

$$= -\ 36K_{06}^{\ 2}\ \{\ 12K_{03}K_{06} - (3K_{03}+K_{06})^2\ \} \geq 0. \qquad \text{(II-9)}$$

Substituting (II-7) into (II-9)

$$12K_{03}K_{06} - (3K_{03}+K_{06}^{\ 2}) \geq 0, \qquad \text{(II-10)}$$

and (II-10) into (I-9),

$$-36K_{06}^{\ 2} \geq 0.$$

This is possible only if $K_{06} = 0$. (II-11)

But if $K_{06} = 0$, V_0 becomes

$$\underline{X}^t \begin{bmatrix} K_{01} & K_{02} & K_{03} \\ K_{02} & K_{04} & K_{05} \\ K_{03} & K_{05} & 0 \end{bmatrix} \underline{X},$$

which can never be positive definite. Therefore K_{06} can't be set to zero, and we need higher order correction terms added to V_0.

2. A complete fourth order homogeneous polynomial with unknown coefficients can be added to V_0, but an inspection of F_{04} reveals that the coefficient K_{06} is associated with $x_1^3x_3$, $x_1x_2^2x_3$, $x_1^2x_3^2$. Therefore it is sufficient to add only those terms of $G_{14}(\underline{X})$ to V_0 such that $\frac{dG_{14}}{dt}$ contains the terms $x_1^3x_3$, $x_1x_2^2x_3$, $x_1^2x_3^2$. Such terms are $x_1^3x_2$ and $x_1^2x_2x_3$. Therefore define

$$G_{14}(\underline{X}) \triangleq K_{11}x_1^3x_2 + K_{12}x_1^2x_2x_3$$

and $$V_1 \triangleq V_0 + G_{14}(\underline{X}) = \underline{X}^t\underline{P}\underline{X} + K_{11}x_1^3 x_2 + K_{12}x_1^2x_2x_3$$

$$= \underline{X}^t \begin{bmatrix} K_{01} & K_{02} & K_{03} \\ K_{02} & K_{04} & K_{05} \\ K_{03} & K_{05} & K_{06} \end{bmatrix} \underline{X}$$

$$+ K_{11}x_1^3x_2 + K_{12}x_1^2x_2x_3.$$

From (II-4,5,7)

$$V_1 = \underline{X}^t \begin{bmatrix} K_{01} & 0 & \frac{K_{01}}{2} \\ 0 & K_{04} & 0 \\ \frac{K_{01}}{2} & 0 & K_{06} \end{bmatrix} \underline{X}$$

$$+ K_{11}x_1^{\,3}x_2 + K_{12}x_1^{\,2}x_2x_3$$

$$\dot{V}_1 = - \underline{X}^t\underline{Q}\underline{X} - F_{04}(\underline{X}) + 3K_{11}x_1^{\,2}x_2^{\,2} + K_{11}x_1^{\,3}x_3 + 2K_{12}x_1x_2^{\,2}x_3$$

$$+ K_{12}x_1^{\,2}x_3^{\,2} - 2K_{12}x_1^{\,2}x_2^{\,2} - K_{12}x_1^{\,2}x_2(x_1^{\,3}+6x_1x_2^{\,2}+3x_1^{\,2}x_3)$$

$$= - \underline{X}_t\underline{Q}\underline{X} - \{ K_{01}x_1^{\,4} + (3K_{01}+2K_{06}-K_{11})x_1^{\,3}x_3$$

$$+ (6K_{06}-K_{12})\ x_1^{\,2}x_3^{\,2} + (6K_{01}-3K_{11}+2K_{12})x_1^{\,2}x_2^{\,2}$$

$$+ (12K_{06}-2K_{12})x_1x_2^{\,2}x_3 \} - \{ K_{12}x_1^{\,5}x_2 + 6K_{12}x_1^{\,4}x_2x_3 + 3K_{12}x_1^{\,4}x_2x_3\}$$

$$= - \underline{X}^t\underline{Q}\underline{X} - F_{14}(\underline{Y},\underline{Y}) - F_{16}(\underline{X})$$

$$\text{where } \underline{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2K_{06}-\frac{K_{01}}{2} -K_{04} \\ 0 & 2K_{06}-\frac{K_{01}}{2} -K_{04} & 0 \end{bmatrix}.$$

Since the first row and first column of $\underset{\approx}{Q}$ are zero, the first component of $\underline{X}$ is dropped, and $\underline{X}^t \underset{\approx}{Q} \underline{X}$ becomes

$$[x_2, x_3] \begin{bmatrix} 0 & 2K_{06} - \frac{K_{01}}{2} - K_{04} \\ 2K_{06} - \frac{K_{01}}{2} - K_{04} & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} .$$

Also

$$F_{14}(\underline{X}) = K_{01}x_1^{\,4} + (3K_{01}+2K_{06}-K_{11})x_1^{\,3}x_3 + (6K_{06}-K_{12})x_1^{\,2}x_3^{\,2}$$

$$+ (6K_{01}-3K_{11}+2K_{12})x_1^{\,2}x_2^{\,2} + (12K_{06}-2K_{12})x_1x_2^{\,2}x_3 ,$$

$$F_{16}(\underline{X}) = K_{12}x_1^{\,5}x_2 + 6K_{12}x_1^{\,3}x_2^{\,3} + 3K_{12}x_1^{\,4}x_2x_3 \quad .$$

Following Section II, $F_{14}(\underline{X})$ and $F_{16}(\underline{X})$ are reduced to quadratic forms as

$$F_{14} = \begin{Bmatrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{Bmatrix}^t \begin{bmatrix} K_{01} & 0 & \frac{3}{2}K_{01}+K_{06} - \frac{K_{01}}{2} & 0 \\ 0 & 6K_{01}-3K_{11} + 2K_{12} & 0 & 6K_{06}-K_{12} \\ \frac{3}{2}K_{01}+K_{06} - \frac{K_{11}}{2} & 0 & 6K_{06}-K_{12} & 0 \\ 0 & 6K_{06}-K_{12} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{Bmatrix}$$

and

$$F_{16} = \begin{Bmatrix} x_1^3 \\ x_1x_2^2 \\ x_1x_2x_3 \\ x_1^3 \end{Bmatrix}^t \begin{bmatrix} 0 & \frac{K_{12}}{2} & \frac{3}{2}K_{12} & 3K_{12} \\ \frac{K_{12}}{2} & 0 & 0 & 0 \\ \frac{3}{2}K_{12} & 0 & 0 & 0 \\ 3K_{12} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2x_3 \\ x_2^3 \end{Bmatrix}$$

To determine the unknown constants, condition (a) in Section III requires:

$$K_{01} > 0, \qquad \text{(II-1)}$$

$$K_{01}K_{04} > 0, \qquad \text{(II-12)}$$

$$\begin{vmatrix} K_{01} & 0 & \frac{K_{01}}{2} \\ 0 & K_{04} & 0 \\ \frac{K_{01}}{2} & 0 & K_{06} \end{vmatrix} = K_{01}K_{04}\left(K_{06} - \frac{K_{01}^2}{4}\right) > 0. \qquad \text{(II-13)}$$

Condition (b) requires:

$$\left(2K_{06} - \frac{K_{01}}{2} - K_{04}\right)^2 \geq 0,$$

which implies

$$2K_{06} - \frac{K_{01}}{2} - K_{04} = 0. \qquad \text{(II-14)}$$

Condition (c) requires:

$$K_{01} \geq 0, \qquad \text{(II-1)}$$

$$K_{01}(6K_{01} - 3K_{11} + 2K_{12}) \geq 0, \qquad \text{(II-15)}$$

$$\begin{vmatrix} K_{01} & 0 & \frac{3}{2}K_{01} + K_{06} - \frac{K_{11}}{2} \\ 0 & 6K_{01} - 3K_{11} + 2K_{12} & 0 \\ \frac{3}{2}K_{01} + K_{06} - \frac{K_{11}}{2} & 0 & 6K_{06} - K_{12} \end{vmatrix} \geq 0$$

or

$$(6K_{01}-3K_{11}+2K_{12})\ \{K_{01}(6K_{06}-K_{12}) - (\tfrac{3}{2}K_{01}+K_{06}-\frac{K_{11}}{2})^2\} \geq 0, \quad \text{(II-16)}$$

and finally the deteminant of the matrix of $F_{14} \geq 0$, which gives

$$-(6K_{06}-K_{12})^2\ \{K_{01}(6K_{06}-K_{12}) - (\tfrac{3}{2}K_{01}+K_{06}-\frac{K_{11}}{2})^2\} \geq 0. \quad \text{(II-17)}$$

From (II-1) and (II-15), $6K_{01} - 3K_{11} + 2K_{12} \geq 0.$ (II-18)

From (II-18) and (II-16), $K_{01}(6K_{06}-K_{12}) - (\tfrac{3}{2}K_{01}+K_{06}-\frac{K_{11}}{2})^2 \geq 0.$ (II-19)

From (II-10) and (II-17), $-(6K_{06}-K_{12})^2 \geq 0.$

This last expression can be satisfied only if

$$K_{12} = 6K_{06}.$$

Substitute this into (II-16)

$$-(6K_{01}-3K_{11}+2K_{12})\ (\tfrac{3}{2}K_{01}+K_{06}-\frac{K_{11}}{2})^2 \geq 0. \quad \text{(II-20)}$$

Since $6K_{01} - 3K_{11} + 2K_{12} \geq 0$ from (II-18), (II-20) is possible only if

$$\tfrac{3}{2}K_{01} + K_{06} - \frac{K_{11}}{2} = 0.$$

Therefore $K_{11}=3K_{01} + 2K_{06}.$ (II-21)

Condition (d) in Section II requires only that

$$\begin{vmatrix} 0 & \frac{K_{12}}{2} \\ \frac{K_{12}}{2} & 0 \end{vmatrix} = -\frac{K_{12}^{2}}{2} \geq 0. \qquad \text{(II-22)}$$

Equation (II-14) is possible if $K_{12} = 0$, but then the purpose of adding G_{14} to V_0 is negated. Therefore further correction terms are added to V_1.

We can add a complete 6^{th} order homogeneous polynomial with unknown coefficient to V_1, but again an inspection of F_{16} suggests that the coefficient K_{12} is associated with terms like $x_1^5x_2$, $x_1^3x_2^3$ and $x_1^4x_2x_3$. Therefore it will be sufficient to add only those terms of $G_{26}(\underline{X})$ to V_1 such that $\frac{dG_{26}}{dt}$ contains terms $x_1^5x_2$, $x_1^3x_2^3$ and $x_1^4x_2x_3$. Such terms obviously are x_1^6, $x_1^4x_2^2$.

3. Add $G_{26}(\underline{X}) = K_{21}x_1^6 + K_{22}x_1^4x_2^2$ to V_1 and define

$$V_2 \triangleq V_1 + G_{26} = \underline{X}^t\underline{\underline{P}}\underline{X} + G_{14}(\underline{X}) + G_{26}(\underline{X})$$

$$= \underline{X}^t \begin{bmatrix} K_{01} & 0 & \frac{K_{01}}{2} \\ 0 & 2K_{06}-\frac{K_{01}}{2} & 0 \\ \frac{K_{01}}{2} & 0 & K_{01} \end{bmatrix} \underline{X}$$

$$+ (3K_{01}+2K_{06})x_1^3x_2 + K_{12}x_1^2x_2x_3 + K_{21}x_1^6 + K_{22}x_1^4x_2^2.$$

Then $\dot{V} \triangleq F_{14} + F_{16} + \frac{dG_{26}}{dt}$

$$= -[x_1^2, x_1x_2] \begin{bmatrix} K_{01} & 0 \\ 0 & 6K_{01}-3K_{11}+2K_{12} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \end{bmatrix} -$$

$$\begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2x_3 \\ x_2^3 \end{bmatrix}^t \begin{bmatrix} 0 & \frac{K_{12}}{2}-3K_{21} & \frac{3}{2}K_{12}-K_{22} & 3K_{12}-2K_{22} \\ \frac{K_{12}}{2}-3K_{21} & 0 & 0 & 0 \\ \frac{3}{2}K_{12}-K_{22} & 0 & 0 & 0 \\ 3K_{12}-2K_{22} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1x_2x_3 \\ x_2^3 \end{bmatrix}$$

$$= - F_{24}(\underline{X}) - F_{26}(\underline{X}).$$

To determine the unknown constants, the following conditions are to be satisfied.

(a) $\underline{P}$ at least positive semidefinite

(b) F_{24} at least positive semidefinite

(c) F_{26} at least positive semidefinite.

Condition (a) requires:

$$K_{01} \geq 0, \qquad \text{(II-1)}$$

$$K_{01}(2K_{06}-\frac{K_{01}}{2}) \geq 0, \tag{II-23}$$

$$\begin{vmatrix} K_{01} & 0 & \frac{K_{01}}{2} \\ 0 & 2K_{06} - \frac{K_{01}}{2} & 0 \\ \frac{K_{01}}{2} & 0 & K_{06} \end{vmatrix} \geq 0. \tag{II-24}$$

Condition (b) requires:

$$K_{01} \geq 0, \tag{II-25}$$

$$K_{01}(6K_{01}-3K_{11}+2K_{12}) \geq 0. \tag{II-26}$$

Condition (c) requires:

$$\frac{K_{12}}{2} - 3K_{21} = 0, \tag{II-27}$$

$$\frac{3}{2}K_{12} - K_{22} = 0. \tag{II-28}$$

A set of solutions satisfying all of the above conditions is

$$K_{01} = 0,\ K_{06} = 1,\ K_{11} = 2,\ K_{12} = 6,\ K_{21} = 1,\ K_{22} = 9.$$

Then

$$V_2 = 2x_2^{\,2} + x_3^{\,2} + 2x_1^{\,3}x_2 + 6x_1^{\,2}x_2x_3 + x_1^{\,6} + 9x_1^{\,4}x_2^{\,2}$$

$$= (x_3+3x_1^{\,2}x_2)^2 + (x_2+x_1^{\,3})^2 + x_2^{\,2},$$

positive definite, and

$$\dot{V}_2 = -\,6x_1^{\,2}x_2^{\,2},$$

negative semidefinite.

Hence, Lyapunov's direct method indicates that the system is stable but not asymptotically stable. For this particular equation, however, no solution other than the origin exists such that V is positive difinite and $\dot{V}$ is negative semidefinite, indicating that the equilibrium point is asymptotically stable.

This example has also been considered in the literature by the variable gradient method [2]. The Lyapunov function was given as

$$V = (x_3+3x_1^{\,2}x_2)^2 + (x_2+x_1^{\,3})^2 + x_2^{\,2}$$

$$\dot{V} = -\,6x_1^{\,2}x_2^{\,2}.$$

Using the variable gradient method, one starts by assuming a particular form of $\underline{\nabla V}$, calculates $\dot{V} = (\underline{\nabla V})^t \cdot \dot{\underline{X}}$, and then constrains $\dot{V}$ to be at least semidefinite. Finally V is found by integration. The V so obtained is not guaranteed to be definite. Thus, as with most of the other known methods, a difficult step exists in the procedure that involves nonanalytical work. Alternately the procedure described here (a) provides a direction for developing a sequence of correction polynomials starting with an elementary quadratic form until a Lyapunov function is obtained, (b) presents the problem of determining the sign definiteness of a polynomial as one of solving a set of algebraic equations and inequalities, and (c) appears to offer a greater degree of flexibility for choosing the unknown constants in the V function.

The two examples where chosen to illustrate the procedure without encumbering the reader with algebraic details. More complex examples have been considered. These clearly indicate that while the algebra increases the sequence of steps is straight forward and does not require greater insight or familiarity with the solutions.

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