NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report No. 32-1018

Two-Dimensional Unsteady Flow in a Traveling Wave Plasma Accelerator

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ABSTRACT

Two-dimensional unsteady flow in a traveling wave plasma accelerator is studied by means of the small-perturbation theory under the conditions of small magnetic Reynolds number and weak electromagnetic interaction. The partial differential equation which determines the pressure is solved by using a Laplace transform. The other quantities are deduced from the pressure by integration of rather simple equations. Then, the analytical results are numerically computed in order to clarify the effect of different parameters.

I. INTRODUCTION

The acceleration of an ionized fluid is a problem of great interest. Its applications are numerous: from the plasma injector used in controlled thermonuclear fusion to electric propulsion systems for space missions.

The importance of these applications explains the many recent studies. Most of these are experimental or appeal to simplified theories. However, purely theoretical analysis of a plasma accelerator is useful and even necessary to predict and explain some phenomena appearing during the experiments.

The acceleration can be created by several different devices: pulsed or continuous; with electrodes or electrodeless; d.c. or a.c., etc. (Ref. 1). The plasma accelerator considered here is a traveling wave plasma accelerator. In such a system, which presents the advantage of being electrodeless, an unsteady traveling magnetic field is created and induces currents in the plasma. These currents interact with the magnetic field to create the acceleration of the ionized fluid, which remains nearly neutral.

Depending on the physical conditions, the analysis can be made from the microscopic point of view (plasma physics) or from the continuum point of view (magnetofluid-dynamics). It is the latter which is considered here.

So far, purely theoretical studies have been scarce. However some authors have studied traveling wave devices for different purposes. D. L. Turcotte and J. M. Lyons (Refs. 2, 3) studied the incompressible flow in an induction pump. The compressible one-dimensional case has been considered by E. E. Covert, L. R. Boedeker, and C. W. Haldeman (Ref. 4) and by the latter author for the axisymmetric case (Ref. 5). H. K. Messerle (Refs. 6, 7) was interested in these devices as an energy converter. J. L. Neuringer (Ref. 8) has calculated the traveling wave field for the study—with J. H. Turner (Ref. 9)—of an induction compressor. We must also mention the work of V. B. Baranov (Ref. 10), who considers the mean flow in an accelerator. J. Fabri and Th. Moulin (Refs. 11, 12), R. Peyret and Th. Moulin (Ref. 13), and R. Peyret (Refs. 14, 15) have considered some theoretical aspects of a plasma accelerator. Finally, in recent works, W. H. Braun (Ref. 16) has studied the problem of flow under certain physical conditions (slow variation of temperature, electrical conductivity proportional to density), and A. R. M. Rashad (Refs. 17, 18) has considered the case of an annular accelerator of infinite length.

The present study, which develops the results of (Ref. 15), is concerned with a system identical to that described in Refs. 11 and 12, and is intended to give an analysis of the flow which is established inside the accelerator.

The general equations, the fundamental hypotheses, and some previous results will be presented in Section II. Section III will be dedicated to the determination of the oscillatory flow, and the results will be discussed in Sec. IV, V, and VI.

II. GENERAL EQUATIONS

A. Fundamental Equations and Hypotheses

The accelerator considered here is essentially a semi-infinite plane channel with a constant cross section (Fig. 1); this plane duct is an idealization of an annular channel, the mean radius r_m of which would be large in comparison with the half-height h. The axis of the plane duct is selected as the $0\tilde{x}$ axis so that the walls (made with insulating material) are perpendicular to the $0\tilde{y}$ axis at $\tilde{y} = \pm h$ and infinite in the $0\tilde{z}$ direction; then, all the quantities are independent of $\tilde{z}$. The inlet of the channel is placed at $\tilde{x} = 0$.

Coils suitably placed and supplied by a polyphased current of frequency ω create, in the empty channel, a magnetic field which, in its turn, creates an electric field. When a conducting fluid is injected in the duct, the induced current $\tilde{\mathbf{J}}$ and the induction $\tilde{\mathbf{B}}$ define the electromagnetic force $\tilde{\mathbf{J}} \times \tilde{\mathbf{B}}$ which accelerates the fluid.

This being so, at $\tilde{x} = 0$, $(-h < y < +h)$, a perfectly compressible fluid of electrical conductivity σ is injected with uniform conditions. Let $\tilde{\mathbf{U}} = (\tilde{u}, \tilde{v}, 0)$ be the fluid velocity, and let $\tilde{p}$, $\tilde{\rho}$ and $\tilde{T}$ be respectively the pressure, density, and temperature of the fluid; the uniform conditions at the inlet of the accelerator will be $\tilde{\mathbf{U}}_0 = (\tilde{u}_0, 0, 0)$, $\tilde{p}_0$, $\tilde{\rho}_0$ and $\tilde{T}_0$. Then, from $\tilde{t} = 0$, this uniform flow is perturbed by the applied electromagnetic field.

The applied magnetic induction lies in the (x, y) plane (see Fig. 2) and its components along $0\tilde{x}$ and $0\tilde{y}$ are respectively

$$\tilde{B}_{\tilde{x}} = \frac{\tilde{B}_M}{\cosh h/\lambda} \sinh \frac{\tilde{y}}{\lambda} \sin \frac{1}{\lambda} (\tilde{x} - \tilde{V}\tilde{t}) \tag{1}$$

$$\tilde{B}_{\tilde{y}} = -\frac{\tilde{B}_M}{\cosh h/\lambda} \cosh \frac{\tilde{y}}{\lambda} \cos \frac{1}{\lambda} (\tilde{x} - \tilde{V}\tilde{t}) \tag{2}$$

where $\tilde{B}_M$ is a constant characterizing the magnitude of the magnetic induction. The constant $\tilde{V}$ is the velocity of the traveling field; λ is related to the wavelength Λ by $\lambda = \Lambda/2\pi$. Between these quantities and the frequency ω we have the following relation:

$$\tilde{V} = \lambda\omega = \frac{\Lambda\omega}{2\pi} \tag{3}$$

The electric field $\tilde{\mathbf{E}}$ has only one component: along the $0\tilde{z}$ axis perpendicular to the plane $(\tilde{x}, \tilde{y})$, namely

$$\tilde{E}_{\tilde{z}} = \frac{\tilde{B}_M \tilde{V}}{\cosh h/\lambda} \cosh \frac{\tilde{y}}{\lambda} \sin \frac{1}{\lambda} (\tilde{x} - \tilde{V}\tilde{t}) \tag{4}$$

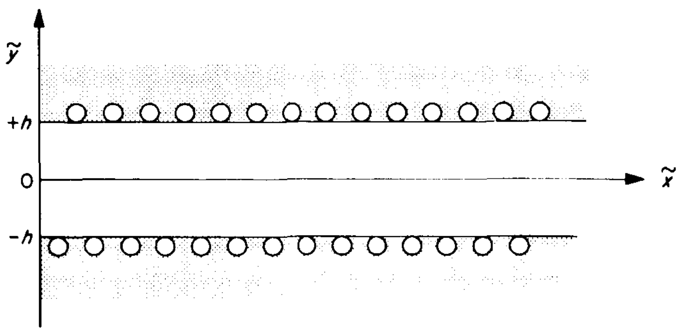

Fig. 1. Plane channel

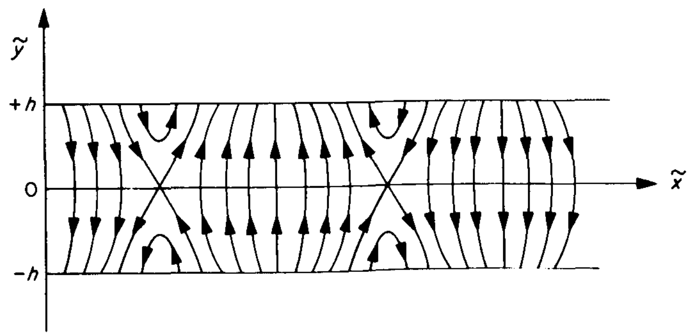

Fig. 2. Magnetic lines (at $t = 0$)

The only component $\tilde{J}_{\tilde{z}}$ of the current is obtained from Ohm's law:

$$\tilde{\mathbf{J}} = \sigma(\tilde{\mathbf{E}} + \tilde{\mathbf{U}} \times \tilde{\mathbf{B}}) \tag{5}$$

and hence

$$\tilde{J}_{\tilde{z}} = \frac{\sigma \tilde{B}_M}{\cosh h/\lambda}\left[(\tilde{v} - \tilde{u})\cosh\frac{\tilde{y}}{\lambda}\cos\frac{1}{\lambda}(\tilde{x} - \tilde{V}\tilde{t}) - \tilde{v}\sinh\frac{\tilde{y}}{\lambda}\sin\frac{1}{\lambda}(\tilde{x} - \tilde{V}\tilde{t})\right] \tag{6}$$

Now, we introduce the following dimensionless quantities

$$\left.\begin{array}{llll} x = \dfrac{\tilde{x}}{\lambda} & y = \dfrac{\tilde{y}}{h} & t = \dfrac{\tilde{C}_0 \tilde{t}}{\lambda} & T = \dfrac{C_p \tilde{T}}{\tilde{C}_0^2} \\ u = \dfrac{\tilde{u}}{\tilde{C}_0} & v = \dfrac{\lambda}{h}\dfrac{\tilde{v}}{\tilde{C}_0} & \rho = \dfrac{\tilde{\rho}}{\tilde{\rho}_0} & p = \dfrac{\tilde{p}}{\tilde{\rho}_0 \tilde{C}_0^2} \\ B_x = \dfrac{\tilde{B}_{\tilde{x}}}{\tilde{B}_M} & B_y = \dfrac{\tilde{B}_{\tilde{y}}}{\tilde{B}_M} & E_z = \dfrac{\tilde{E}_{\tilde{z}}}{\tilde{B}_M \tilde{C}_0} & J_z = \dfrac{\mu\lambda \tilde{J}_{\tilde{z}}}{\tilde{B}_M} \end{array}\right\} \tag{7}$$

and the dimensionless parameters

$$\left.\begin{array}{lll} \delta = \dfrac{h}{\lambda} & M_0 = \dfrac{\tilde{u}_0}{\tilde{C}_0} & \mathcal{M}_0 = \dfrac{\tilde{V}}{\tilde{C}_0} \\ R_m = \mu\sigma\tilde{C}_0\lambda & N = \dfrac{\sigma \tilde{B}_M^2 \lambda}{\tilde{\rho}_0 \tilde{C}_0} & \end{array}\right\} \tag{8}$$

$\tilde{C}_0$ is the isentropic sound velocity at the inlet of the duct. C_p is the specific heat at constant pressure and μ is the magnetic permeability of the fluid. The quantity δ is the ratio of duct half-height to the field reduced wavelength; M_0 is the Mach number at the inlet; $\mathcal{M}_0$ is one possible Mach number calculated from the local field velocity; R_m is the magnetic Reynolds number; and N is the interaction parameter which measures the relative importance of magnetic effects and dynamic effects.

Now, we assume that the magnetic Reynolds number is sufficiently small

$$R_m << 1 \tag{9}$$

so that the induced fields can be neglected.

Taking into account definitions (7) and (8) and hypothesis (9), as well as the usual assumptions of magnetofluid-dynamics, the equations governing the flow are

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{10}$$

$$\rho\frac{du}{dt} + \frac{\partial p}{\partial x} = N\left[(\mathcal{M}_0 - u)\cosh^2 \delta y \cos^2(x - \mathcal{M}_0 t) - \frac{\delta}{4} v \sinh 2\delta y \sin 2(x - \mathcal{M}_0 t)\right] \tag{11}$$

$$\delta^2\rho\frac{dv}{dt} + \frac{\partial p}{\partial y} = \delta N\left[\frac{1}{4}(\mathcal{M}_0 - u)\sinh 2\delta y \sin 2(x - \mathcal{M}_0 t) - \delta v \sinh^2 \delta y \sin^2(x - \mathcal{M}_0 t)\right] \tag{12}$$

$$\rho\frac{d}{dt}\left(T + \frac{u^2 + \delta^2 v^2}{2}\right) - \frac{\partial p}{\partial t} = N\mathcal{M}_0\left[(\mathcal{M}_0 - u)\cosh^2 \delta y \cos^2(x - \mathcal{M}_0 t) - \frac{\delta}{4} v \sinh 2\delta y \sin 2(x - \mathcal{M}_0 t)\right] \tag{13}$$

with

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{14}$$

Equation (10) is the equation of continuity; (11) and (12) are the momentum equations and (13) is the energy equation.

To these equations we must add the state law of the fluid which is assumed to be the law of perfect gases with constant specific heats:

$$p = \frac{\gamma - 1}{\gamma}\rho T \tag{15}$$

where γ is the adiabatic index.

Therefore, the five unknowns u, v, p, ρ, and T are a solution of Eqs. (10)–(15) with the following boundary conditions:

$$u = M_0 \quad v = 0 \quad p = \frac{1}{\gamma} \quad \rho = 1 \quad T = \frac{1}{\gamma - 1} \quad \text{at} \quad x = 0 \tag{16}$$

$$u = M_0 \quad v = 0 \quad p = \frac{1}{\gamma} \quad \rho = 1 \quad T = \frac{1}{\gamma - 1} \quad \text{at} \quad t = 0 \tag{17}$$

$$v = 0 \quad \text{at} \quad y = \pm 1 \tag{18}$$

B. Linearized Equations

This problem has already been solved in the two following particular cases:

(i) In Ref. 13, the ratio δ had been assumed sufficiently small to be taken equal to zero; hence the electromagnetic force was merely axial and we obtained a *quasi-one-dimensional solution* (v was identically zero). This solution was performed with the supplementary assumption

$$N << 1 \tag{19}$$

which allowed linearization of the flow about the uniform flow.

(ii) In Ref. 15 the assumption (19) was kept, but δ was arbitrary. Moreover R_m, although small, was not zero; hence, one determined the *two-dimensional perturbation flow in mean value with respect to the time.*

In the present analysis the assumptions on R_m ($R_m \approx 0$) and N ($N << 1$) are kept but any hypothesis may be made on the magnitude of δ; so, we calculate the *two-dimensional unsteady perturbation flow* due to the applied traveling field.

Now, we assume that for each unknown there exist asymptotic expansions in N of the following type:

$$u(x,y,t;N) = M_0 + Nu^{(1)}(x,y,t) + \cdots$$

$$v(x,y,t;N) = Nv^{(1)}(x,y,t) + \cdots$$

$$p(x,y,t;N) = \frac{1}{\gamma} + Np^{(1)}(x,y,t) + \cdots$$

$$\rho(x,y,t;N) = 1 + N\rho^{(1)}(x,y,t) + \cdots$$

$$T(x,y,t;N) = \frac{1}{\gamma - 1} + NT^{(1)}(x,y,t) + \cdots \tag{20}$$

Using these expansions in Eqs. (10)–(15), one obtains the linearized system:

$$\frac{\partial \rho^{(1)}}{\partial t} + M_0 \frac{\partial \rho}{\partial x} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0 \tag{21}$$

$$\frac{\partial u^{(1)}}{\partial t} + M_0 \frac{\partial u^{(1)}}{\partial x} + \frac{\partial p^{(1)}}{\partial x} = \frac{g_0}{4\cosh^2 \delta} (1 + \cosh 2\delta y)\,[1 + \cos 2\,(x - \mathcal{M}_0 t)] \tag{22}$$

$$\delta^2 \left(\frac{\partial v^{(1)}}{\partial t} + M_0 \frac{\partial v^{(1)}}{\partial x} \right) + \frac{\partial p^{(1)}}{\partial y} = \frac{\delta g_0}{4\cosh^2 \delta} \sinh 2\delta y \sin 2\,(x - \mathcal{M}_0 t) \tag{23}$$

$$\frac{\partial T^{(1)}}{\partial t} + M_0 \frac{\partial T^{(1)}}{\partial x} + M_0 \left(\frac{\partial u^{(1)}}{\partial x} + M_0 \frac{\partial u^{(1)}}{\partial x} \right) - \frac{\partial p^{(1)}}{\partial t} = \frac{\mathcal{M}_0 g_0}{4\cosh^2 \delta} (1 + \cosh 2\delta y)\,[1 + \cos 2\,(x - \mathcal{M}_0 t)] \tag{24}$$

$$\gamma p^{(1)} - \rho^{(1)} - (\gamma - 1)\, T^{(1)} = 0 \tag{25}$$

where g_0 is the slip coefficient defined by

$$g_0 = \mathcal{M}_0 - M_0 \tag{26}$$

It is possible to eliminate $\rho^{(1)}$ and to write

$$\frac{\partial u^{(1)}}{\partial t} + M_0 \frac{\partial u^{(1)}}{\partial x} + \frac{\partial p^{(1)}}{\partial x} = \frac{g_0}{4\cosh^2 \delta} (1 + \cosh 2\delta y)\,[1 + \cos 2\,(x - \mathcal{M}_0 t)] \tag{27}$$

$$\frac{\partial p^{(1)}}{\partial t} + M_0 \frac{\partial p^{(1)}}{\partial x} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = \frac{(\gamma - 1)\, g_0^2}{4\cosh^2 \delta} (1 + \cosh 2\,\delta y)\,[1 + \cos 2\,(x - \mathcal{M}_0 t)] \tag{28}$$

$$\delta^2 \left(\frac{\partial v^{(1)}}{\partial t} + M_0 \frac{\partial v^{(1)}}{\partial x} \right) + \frac{\partial p^{(1)}}{\partial y} = \frac{\delta g_0}{4\cosh^2 \delta} \sinh 2\delta y \sin 2\,(x - \mathcal{M}_0 t) \tag{29}$$

$$\frac{\partial T^{(1)}}{\partial t} + M_0 \frac{\partial T^{(1)}}{\partial x} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = \frac{\gamma g_0^2}{4\cosh^2 \delta} (1 + \cosh 2\delta y)\,[1 + \cos 2\,(x - \mathcal{M}_0 t)] \tag{30}$$

with the boundary conditions

$$u^{(1)} = v^{(1)} = p^{(1)} = T^{(1)} = 0 \quad \text{at} \quad x = 0 \tag{31}$$

$$u^{(1)} = v^{(1)} = p^{(1)} = T^{(1)} = 0 \quad \text{at} \quad t = 0 \tag{32}$$

$$v^{(1)} = 0 \quad \text{at} \quad y = \pm 1 \tag{33}$$

Equations (27)–(29) determine $u^{(1)}$, $v^{(1)}$, and $p^{(1)}$; then, it is possible to calculate $T^{(1)}$ and $\rho^{(1)}$.

Because of the linearity of system (27)–(28), each unknown can be decomposed in the following manner:

$$\left.\begin{aligned} u^{(1)}(x,y,t) &= u^{(1,0)}(x,y) + u^{(1,1)}(x,t) + u^{(1,2)}(x,y,t) \\ v^{(1)}(x,y,t) &= v^{(1,0)}(x,y) + v^{(1,2)}(x,y,t) \\ p^{(1)}(x,y,t) &= p^{(1,0)}(x,y) + p^{(1,1)}(x,t) + p^{(1,2)}(x,y,t) \end{aligned}\right\} \tag{34}$$

and analogous expressions for $T^{(1)}$ and $\rho^{(1)}$.

The quantities $u^{(1,0)}$, $v^{(1,0)}$, and $p^{(1,0)}$, which are characteristic of the steady, two-dimensional part of the solution, have been calculated in Ref. 14 (in Fig. 3 the variations of the axial velocity $u^{(1,0)}$ on the axis are represented for three

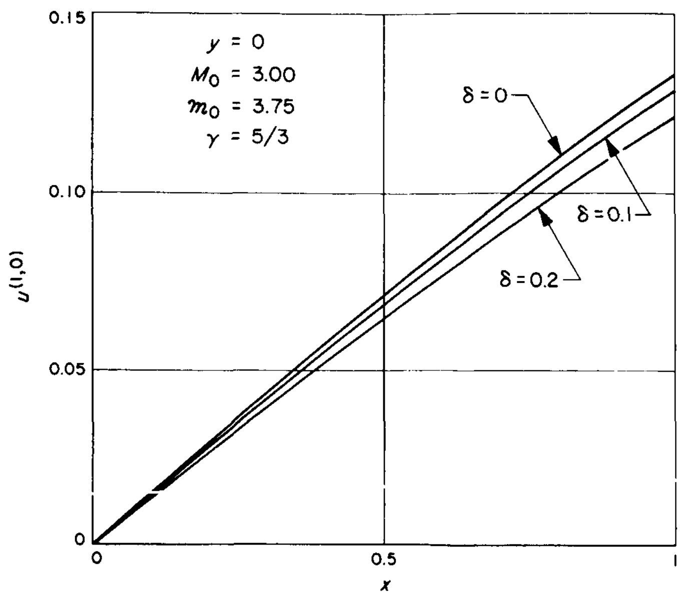

Fig. 3. Variation of axial velocity $u^{(1,0)}$ on the axis

values of δ). This solution shows, in particular, that the acceleration on the axis can be characterized by the coefficient (see also Ref. 8):

$$\frac{(\mathfrak{M}_0 - M_0)\,[\gamma M_0 - (\gamma - 1)\,\mathfrak{M}_0]}{(M_0^2 - 1)\cosh^2\delta} \tag{35}$$

which makes evident the effect of the duct height (by the ratio δ) and also the effect of Mach numbers M_0 and $\mathfrak{M}_0$. More precisely, in the supersonic case (the only case which is considered here),

$$M_0 > 1 \tag{36}$$

there will exist an acceleration if

$$\frac{\gamma - 1}{\gamma}\mathfrak{M}_0 < M_0 < \mathfrak{M}_0 \tag{37}$$

These conditions are assumed to hold in the continuation of the analysis.

Note that this steady solution is not valid for large values of x because it has been found that the perturbation quantities tend toward infinity with x.

Now, we are concerned only with the unsteady part of the solution. The quantities $u^{(1,1)}$ and $p^{(1,1)}$, the unsteady one-dimensional part, have been calculated in Refs. 11 and 13. In fact, the right-hand side of Eqs. (3)–(6) of Ref. 13 must be divided by $2\cosh^2\delta$ and the constant terms must be removed. The study of the system determining $u^{(1,1)}$ and $p^{(1,1)}$ leads to a division of the plane (x,t) in three regions delimited by the characteristic lines $x - (M_0 \pm 1)\,t = 0$. Within each of these regions, the solution—which is continuous everywhere—is represented by a different expression. Moreover, the calculation of $\rho^{(1,1)}$ and $T^{(1,1)}$ introduces a supplementary division of plane (x,t) by the streamline $x - M_0 t = 0$.

These regions are shown in Fig. 4; the region (IV) corresponds to the usual regime of operation of the accelerator. From the practical point of view it is the most interesting regime and, for that reason, it is the only one which is considered here.

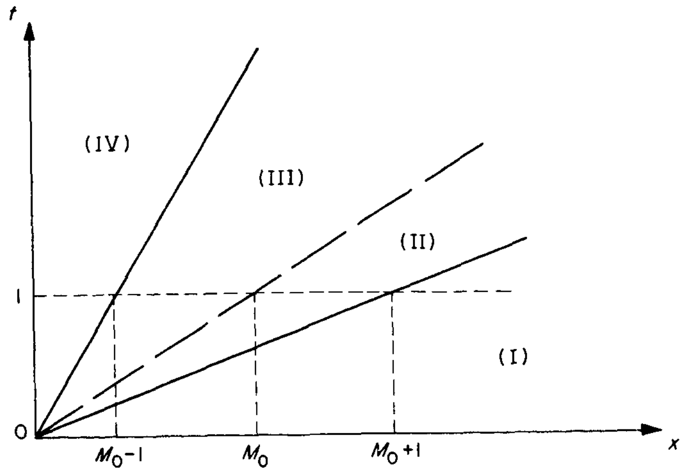

Fig. 4. Division of plane (x, t) into "regions of influence"

The solution in the region (IV) for $u^{(1,1)}$, $p^{(1,1)}$, and $\rho^{(1,1)}$ is

$$u^{(1,1)}(x,t) = X_1 \sin 2(x - \mathcal{M}_0 t) + X_2 \sin \frac{2\mathcal{M}_0}{M_0 - 1}[x - (M_0 - 1)t] + X_3 \sin \frac{2\mathcal{M}_0}{M_0 + 1}[x - (M_0 + 1)t] \tag{38}$$

$$p^{(1,1)}(x,t) = X_4 \sin 2(x - \mathcal{M}_0 t) - X_2 \sin \frac{2\mathcal{M}_0}{M_0 - 1}[x - (M_0 - 1)t] + X_3 \sin \frac{2\mathcal{M}_0}{M_0 + 1}[x - (M_0 + 1)t] \tag{39}$$

$$\rho^{(1,1)}(x,t) = \frac{X_1}{g_0} \sin 2(x - \mathcal{M}_0 t) - X_2 \sin \frac{2\mathcal{M}_0}{M_0 - 1}[x - (M_0 - 1)t] + X_3 \sin \frac{2\mathcal{M}_0}{M_0 + 1}[x - (M_0 + 1)t] + X_5 \sin \frac{2\mathcal{M}_0}{M_0}(x - M_0 t) \tag{40}$$

where

$$\left.\begin{aligned} X_1 &= -\frac{\gamma g_0^2}{8(g_0^2 - 1)\cosh^2\delta} & X_2 &= g_0 \frac{1 - (\gamma - 1) g_0}{16(g_0 + 1)\cosh^2\delta} \\ X_3 &= g_0 \frac{1 + (\gamma - 1) g_0}{16(g_0 - 1)\cosh^2\delta} & X_4 &= -g_0 \frac{1 + (\gamma - 1) g_0^2}{16(g_0^2 - 1)\cosh^2\delta} \\ X_5 &= -\frac{(\gamma - 1) g_0}{8\cosh^2\delta} \end{aligned}\right\} \tag{41}$$

In the case where $\mathcal{M}_0 - M_0 - 1 = 0$, a phenomenon of resonance appears: the amplitude of oscillations increases linearly with x.

Now, it remains to study the unsteady two-dimensional part of the solution, namely $u^{(1,2)}$, $v^{(1,2)}$, and $p^{(1,2)}$; this is the subject of the subsequent sections.

III. TWO-DIMENSIONAL UNSTEADY FLOW

This part of the solution must satisfy the equations

$$\frac{\partial u^{(1,2)}}{\partial t} + M_0 \frac{\partial u^{(1,2)}}{\partial x} + \frac{\partial p^{(1,2)}}{\partial x} = \frac{g_0}{4\cosh^2\delta}\cosh 2\delta y \cos 2(x - \mathcal{M}_0 t) \tag{42}$$

$$\frac{\partial p^{(1,2)}}{\partial t} + M_0 \frac{\partial p^{(1,2)}}{\partial x} + \frac{\partial u^{(1,2)}}{\partial x} + \frac{\partial v^{(1,2)}}{\partial y} = \frac{(\gamma - 1) g_0^2}{4\cosh^2\delta}\cosh 2\delta y \cos 2(x - \mathcal{M}_0 t) \tag{43}$$

$$\delta^2\left(\frac{\partial v^{(1,2)}}{\partial t} + M_0 \frac{\partial v^{(1,2)}}{\partial x}\right) + \frac{\partial p^{(1,2)}}{\partial y} = \frac{\delta g_0}{4\cosh^2\delta}\sinh 2\delta y \sin 2(x - \mathcal{M}_0 t) \tag{44}$$

with boundary conditions deduced from (31)–(33).

A. Determination of the Pressure

It is possible to eliminate $u^{(1,2)}$ and $v^{(1,2)}$ from Eqs. (42)–(44) in order to obtain an equation for $p^{(1,2)}$ only:

$$(M_0^2 - 1)\frac{\partial^2 p^{(1,2)}}{\partial x^2} + 2M_0 \frac{\partial^2 p^{(1,2)}}{\partial x\,\partial t} + \frac{\partial^2 p^{(1,2)}}{\partial t^2} - \frac{1}{\delta^2}\frac{\partial^2 p^{(1,2)}}{\partial y^2} = \frac{(\gamma - 1) g^3}{2\cosh^2\delta}\cosh 2\delta y \cos 2(x - \mathcal{M}_0 t) \tag{45}$$

The boundary conditions (31)–(33) are prescribed in the following planes of space (x, y, t):

(P_1)	$x = 0$	$-1 \leqq y \leqq 1$	$t \geqq 0$
(P_2)	$t = 0$	$-1 \leqq y \leqq 1$	$x \geqq 0$
(P_3)	$y = +1$	$x \geqq 0$	$t \geqq 0$
(P_4)	$y = -1$	$x \geqq 0$	$t \geqq 0$

Then it is necessary to consider the Mach cones (C_1) and (C_2) originating from points $x = 0$, $y = 1$, $t = 0$; and $x = 0$, $y = -1$, $t = 0$. The planes (P_1), (P_2), (P_3), (P_4); the Mach cones (C_1) and (C_2); and the cones reflected with respect to planes (P_3) and (P_4) determine regions in the space (x, y, t) so that, in each of these regions, the solution presents a different form (concept of "domain of dependence"). Therefore it is necessary to calculate the solution in each of these regions. Through the characteristic surfaces this solution must verify some "jump conditions," which are reduced here to conditions of continuity. However, if a method of solution by integral transform—say Laplace transform—is used, this division of space (x, y, t) automatically appears in the course of calculations of the inverse transform. In this Report we shall use such a Laplace transform.

This simplifies the problem but the calculations remain complicated because Eq. (45) is a partial differential equation with three independent variables and a forcing function which depends on these three variables. But, if we are concerned—as it has been already mentioned—with the established regime, the initial conditions at $t = 0$ need not be satisfied and the previous regions of space (x, y, z) simply become regions of plane (x, y) (see Fig. 5).

This being so, it is possible to look for a solution of Eq. (45) of the following form:

$$p^{(1,2)}(x, y, t) = \mathrm{Re}\left\{\Phi(x, y)\, e^{2i\mathcal{M}_0 t} + \Psi(x, y)\, e^{-2i\mathcal{M}_0 t}\right\} \tag{46}$$

where Re means "real part of." The complex functions $\Phi(x, y)$ and $\Psi(x, y)$ are conjugate; the expression (46) can be written as:

$$p^{(1,2)}(x, y, t) = 2\left[\Phi_r(x, y)\cos 2\mathcal{M}_0 t - \Phi_j(x, y)\sin 2\mathcal{M}_0 t\right] \tag{47}$$

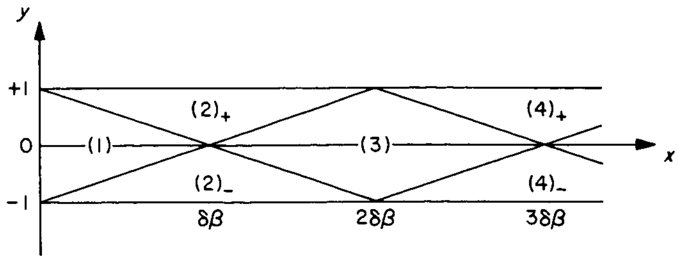

Fig. 5. Division of plane (x, y) into "regions of influence"

where $\Phi_r(x,y)$ and $\Phi_i(x,y)$ are respectively the real part and the imaginary part of the function $\Phi(x,y)$ which satisfies the equation

$$(M_0^2 - 1)\frac{\partial^2\Phi}{\partial x^2} - \frac{1}{\delta^2}\frac{\partial^2\Phi}{\partial y^2} + 4iM_0\mathcal{M}_0\frac{\partial\Phi}{\partial x} - 4\mathcal{M}_0^2\Phi = i\frac{(\gamma-1)\,g_0^3}{4\cosh^2\delta}e^{-2ix}\cosh 2\delta y \tag{48}$$

with the boundary conditions

$$\Phi(0,y) = 0 \tag{49}$$

$$\frac{\partial\Phi}{\partial x}(0,y) = -\frac{g_0}{8\beta^2\cosh^2\delta}\left[1 - (\gamma - 1)M_0 g_0\right]\cosh 2\delta y \tag{50}$$

$$\frac{\partial\Phi}{\partial y}(x,\pm 1) = \pm i\frac{\delta g_0 \tanh\delta}{4}e^{-2ix} \tag{51}$$

where

$$\beta^2 = M_0^2 - 1 \tag{52}$$

Now we introduce

$$\xi = \frac{x}{\beta} \qquad \eta = \delta y \tag{53}$$

and

$$\phi(\xi,\eta)\exp\left\{-2i\frac{M_0\mathcal{M}_0}{\beta}\xi\right\} = \Phi(x,y) \tag{54}$$

hence, Eq. (48) simply becomes

$$\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\eta^2} + \kappa^2\phi = i\frac{(\gamma-1)\,g_0^2}{4\cosh^2\delta}e^{ia_1\xi}\cosh 2\eta \tag{55}$$

where

$$\kappa = \frac{2\mathcal{M}_0}{\beta} \qquad a_1 = 2\frac{M_0 g_0 + 1}{\beta} \tag{56}$$

and the boundary conditions become

$$\phi(0,\eta) = 0 \tag{57}$$

$$\frac{\partial\phi}{\partial\xi}(0,\eta) = -\frac{g_0}{8\beta\cosh^2\delta}\left[1 - (\gamma-1)M_0 g_0\right]\cosh 2\eta \tag{58}$$

$$\frac{\partial \phi}{\partial \eta}(\xi_1, \pm \delta) = \pm i \frac{g_0 \tanh \delta}{4} e^{i a_1 \xi} \tag{59}$$

Equation (55) with the boundary conditions (57)–(59) is solved by means of a Laplace transform with respect to ξ. The function $\overline{\phi}(q, \eta)$, transformed from $\phi(\xi, \eta)$, is defined by

$$\overline{\phi}(q, \eta) = \int_0^\infty \phi(\xi, \eta) e^{-q\xi} d\xi \tag{60}$$

Assuming that there exists a real number α such that, for $\mathrm{Re}\{q\} > \alpha$, the expressions $\Phi e^{-q\xi}$ and $(\partial\phi/\partial\xi) e^{-q\xi}$ tend toward zero as ξ tends toward positive infinity, the equation transformed from Eq. (55) is

$$\frac{d^2\overline{\phi}}{d\eta^2} - r^2\overline{\phi} = \frac{g_0}{8\beta \cosh^2 \delta}\left[1 - (\gamma - 1) M_0 g_0 - 2i \frac{(\gamma - 1)\beta g_0^2}{q - i a_1}\right] \cosh 2\eta \tag{61}$$

where

$$r^2 = q^2 + \kappa^2 \tag{62}$$

In taking account of the transformed boundary conditions

$$\frac{d\overline{\phi}}{d\eta}(q, \pm\delta) = \pm i \frac{g_0 \tanh \delta}{4(q - i a_1)} \tag{63}$$

the solution of Eq. (61) is

$$\begin{aligned} \bar{\phi}(q, \eta) = {} & \frac{g_0 \tanh \delta}{4\beta}\left[\frac{i\gamma\beta}{q - i a_1} + \frac{2\gamma}{r^2 - 4} - i(\gamma - 1)\beta \frac{q}{r^2 - 4}\right] \frac{\cosh \eta r}{r \sinh \delta r} \\ & - \frac{g_0}{16\beta \cosh^2 \delta}\left[\frac{i(\gamma - 1)\beta}{q - i a_1} + \frac{2\gamma}{r^2 - 4} - i(\gamma - 1)\beta \frac{q}{r^2 - 4}\right] \cosh 2\eta \end{aligned} \tag{64}$$

The formal inversion of the second bracket of Eq. (64) is immediate. The first one is slightly more complicated because of the term $(\cosh \eta r)(r \sinh \delta r)^{-1}$. This term can be written as

$$\frac{\cosh \eta r}{r \sinh \delta r} = \frac{e^{(\eta - \delta) r} + e^{-(\eta + \delta) r}}{r(1 - e^{-2\delta r})} = \frac{1}{r} \sum_{n=0}^{\infty} (e^{-b_{n-} r} + e^{-b_{n+} r}) \tag{65}$$

where

$$b_{n\pm} = (2n + 1)\delta \pm \eta \geqq 0 \tag{66}$$

It remains to find out the inverse transforms of

$$\overline{f}^{(i)}(q) \frac{e^{-b_{n\pm} r}}{r} \tag{67}$$

where

$$\overline{f}^{(1)} = \frac{1}{q - i a_1} \qquad \overline{f}^{(2)} = \frac{1}{q^2 + (\kappa^2 - 4)} \qquad \overline{f}^{(3)} = \frac{q}{q^2 + (\kappa^2 - 4)} \tag{68}$$

The inverse transform of $e^{-b_{n\pm} r} r^{-1}$ is

$$\begin{cases} 0 & \text{if} \quad 0 < \xi < b_{n\pm} \\ J_0[\kappa(\xi^2 - b_{n\pm}^2)^{1/2}] & \text{if} \quad \xi > b_{n\pm} \end{cases} \tag{69}$$

where J_0 is the Bessel function of zero order. Using the theorem of convolution, one finds the inverse transform of (67) under the form

$$Y_{n\pm}\int_{b_{n\pm}}^{\xi} f^{(i)}(\xi - X)\, J_0\,[\kappa\,(X^2 - b_{n\pm}^2)^{1/2}]\, dX \tag{70}$$

where $f^{(i)}$ is the inverse transform of $f^{(i)}$ and $Y_{n\pm}$ the Heaviside function with argument $(\xi - b_{n\pm})$, that is

$$\left.\begin{aligned} Y_{n\pm} &= 0 \quad \text{if} \quad \xi - b_{n\pm} < 0 \\ Y_{n\pm} &= 1 \quad \text{if} \quad \xi - b_{n\pm} > 0 \end{aligned}\right\} \tag{71}$$

Then, the function $\phi(\xi, \eta)$, the solution of Eq. (55), is

$$\begin{aligned} \phi(\xi,\eta) = &-\frac{g_0}{16\beta\cosh^2\delta}\left[i(\gamma-1)\,\beta e^{ia_1\xi} + \frac{2\gamma}{a_2}\sin a_2\,\xi - i(\gamma-1)\,\beta\cos a_2\,\xi\right]\cosh 2\eta \\ &+ \frac{g_0\tanh\delta}{4\beta}\sum_{n=0}^{\infty}\Big\{-\gamma\beta\,[F^{s,a_1}_{n+} + F^{s,a_1}_{n-}] + i\gamma\beta\,[F^{c,a_1}_{n+} + F^{c,a_1}_{n-}] \\ &\qquad + 2\frac{\gamma}{a_2}[F^{s,a_2}_{n+} + F^{s,a_2}_{n-}] - i(\gamma-1)\,\beta\,[F^{c,a_2}_{n+} - F^{c,a_2}_{n-}]\Big\} \end{aligned} \tag{72}$$

where

$$a_2 = \frac{2}{\beta}(\mathcal{M}_0^2 - \beta^2)^{1/2} \tag{73}$$

and

$$F^{c,a_i}_{n\pm}{}_{s}\;(\xi,\eta) = Y_{n\pm}\int_{b_{n\pm}}^{\xi} J_0\,[\kappa\,(X^2 - b_{n\pm}^2)^{1/2}]\,{}^{\cos}_{\sin}\,\{a_i\,(\xi - X)\}\, dX \tag{74}$$

Finally, the solution $p^{(1,2)}(x,y,t)$ is

$$\begin{aligned} p^{(1,2)}(x,y,t) = &\frac{g_0}{16\cosh^2\delta}\,[-2\Gamma\sin 2\,(x - \mathcal{M}_0 t) + (\Gamma - S)\sin 2\,(A_+x - \mathcal{M}_0 t) \\ &+ (\Gamma + S)\sin 2\,(A_-x - \mathcal{M}_0 t)]\cosh 2\delta y \\ &+ \frac{g_0\tanh\delta}{4}\sum_{n=0}^{\infty}\{-2\gamma\,[(I^{c,a_1}_{n+} + I^{c,a_1}_{n-})\sin 2\,(x - \mathcal{M}_0 t) + (I^{s,a_1}_{n+} + I^{s,a_1}_{n-})\cos 2\,(x - \mathcal{M}_0 t)] \\ &- (\Gamma - S)\,[(I^{c,a_2}_{n+} + I^{c,a_2}_{n-})\sin 2\,(A_+x - \mathcal{M}_0 t) - (I^{s,a_2}_{n+} + I^{s,a_2}_{n-})\cos 2\,(A_+x - \mathcal{M}_0 t)] \\ &- (\Gamma + S)\,[(I^{c,a_2}_{n+} + I^{c,a_2}_{n-})\sin 2\,(A_-x - \mathcal{M}_0 t) + (I^{s,a_2}_{n+} + I^{s,a_2}_{n-})\cos 2\,(A_-x - \mathcal{M}_0 t)]\} \end{aligned} \tag{75}$$

with

$$A_\pm = \frac{1}{\beta^2}[M_0\mathcal{M}_0 \pm (\mathcal{M}_0^2 - \beta^2)^{1/2}] \tag{76}$$

$$\Gamma = \gamma - 1 \qquad S = \frac{\gamma}{(\mathcal{M}_0^2 - M_0^2)^{1/2}} \tag{77}$$

and the I_n's are functions of x and y defined by

$$I^{c,a_i}_{n\pm}{}_{s}\;(x,y) = Y_{n\pm}\int_{b_{n\pm}}^{x/\beta} J_0\,[\kappa\,(X^2 - b_{n\pm}^2)^{1/2}]\,{}^{\cos}_{\sin}\,\{a_iX\}\, dX \tag{78}$$

This solution will be discussed in Sections IV and V.

B. Determination of Flow Velocity

The pressure $p^{(1,2)}$ being known, the axial velocity $u^{(1,2)}$ is determined by Eq. (42) and the transverse velocity $v^{(1,2)}$ by Eq. (44). The calculations are tedious but without fundamental difficulty. We find for $u^{(1,2)}$:

$$u^{(1,2)}(x,y,t) = \frac{1}{16\cosh^2\delta}\left[-2\gamma\sin 2(x-\mathfrak{M}_0 t) + 2P\sin\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t) - \left(\frac{Q}{a_2}-R\right)\sin 2(A_+x-\mathfrak{M}_0 t) + \left(\frac{Q}{a_2}+R\right)\sin 2(A_-x-\mathfrak{M}_0 t)\right]\cosh 2\delta y$$
$$+\frac{\tanh\delta}{2}\sum_{n=0}^{\infty}\Bigg\{\gamma\left[(I^{c,a_1}_{n+}+I^{c,a_1}_{n-})\sin 2(x-\mathfrak{M}_0 t) + (I^{s,a_1}_{n+}+I^{s,a_1}_{n-})\cos 2(x-\mathfrak{M}_0 t)\right]$$
$$-\frac{\mathfrak{M}_0^2}{M_0^2}P\left[(I^{c,a_3}_{n+}+I^{c,a_3}_{n-})\sin\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t) + (I^{s,a_3}_{n+}+I^{s,a_3}_{n-})\cos\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t)\right]$$
$$+\frac{1}{2}\left(\frac{Q}{a_2}-R\right)\left[(I^{c,a_2}_{n+}+I^{c,a_2}_{n-})\sin 2(A_+x-\mathfrak{M}_0 t) - (I^{s,a_2}_{n+}+I^{s,a_2}_{n-})\cos 2(A_+x-\mathfrak{M}_0 t)\right]$$
$$-\frac{1}{2}\left(\frac{Q}{a_2}+R\right)\left[(I^{c,a_2}_{n+}+I^{c,a_2}_{n-})\sin 2(A_-x-\mathfrak{M}_0 t) + (I^{s,a_2}_{n+}+I^{s,a_2}_{n-})\cos 2(A_-x-\mathfrak{M}_0 t)\right]\Bigg\} \quad (79)$$

with

$$a_3 = 2\frac{\mathfrak{M}_0}{\beta M_0} \quad (80)$$

$$P = \frac{M_0}{\mathfrak{M}_0+M_0} \qquad Q = 2\frac{\gamma M_0+(\gamma-1)\mathfrak{M}_0(\mathfrak{M}_0^2-\beta^2)}{\beta(\mathfrak{M}_0+M_0)} \qquad R = \frac{\gamma\mathfrak{M}_0+(\gamma-1)M_0}{\mathfrak{M}_0+M_0} \quad (81)$$

and an analogous expression for $v^{(1,2)}$:

$$v^{(1,2)}(x,y,t) = \frac{1}{16\,\delta\cosh^2\delta}\left[2\gamma\cos 2(x-\mathfrak{M}_0 t) + 2P\cos\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t) - \left(\frac{Q'}{a_2}-R'\right)\cos 2(A_+x-\mathfrak{M}_0 t) + \left(\frac{Q'}{a_2}+R'\right)\cos 2(A_-x-\mathfrak{M}_0 t)\right]\sinh 2\delta y$$
$$+\frac{\tanh\delta}{4\delta^2}\sum_{n=0}^{\infty}\Bigg\{\gamma\left[\frac{\partial}{\partial y}(I^{c,a_1}_{n+}+I^{c,a_1}_{n-})\cos 2(x-\mathfrak{M}_0 t) - \frac{\partial}{\partial y}(I^{s,a_1}_{n+}+I^{s,a_1}_{n-})\sin 2(x-\mathfrak{M}_0 t)\right]$$
$$+P'\left[\frac{\partial}{\partial y}(I^{c,a_3}_{n+}+I^{c,a_3}_{n-})\cos\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t) - \frac{\partial}{\partial y}(I^{s,a_3}_{n+}+I^{s,a_3}_{n-})\sin\frac{2\mathfrak{M}_0}{M_0}(x-M_0 t)\right]$$
$$-\frac{1}{2}\left(\frac{Q'}{a_2}-R'\right)\left[\frac{\partial}{\partial y}(I^{c,a_2}_{n+}+I^{c,a_2}_{n-})\cos 2(A_+x-\mathfrak{M}_0 t) + \frac{\partial}{\partial y}(I^{s,a_2}_{n+}+I^{s,a_2}_{n-})\sin 2(A_+x-\mathfrak{M}_0 t)\right]$$
$$+\frac{1}{2}\left(\frac{Q'}{a_2}+R'\right)\left[\frac{\partial}{\partial y}(I^{c,a_2}_{n+}+I^{c,a_2}_{n-})\cos 2(A_-x-\mathfrak{M}_0 t) - \frac{\partial}{\partial y}(I^{s,a_2}_{n+}+I^{s,a_2}_{n-})\sin 2(A_-x-\mathfrak{M}_0 t)\right]\Bigg\}$$
$$(82)$$

with

$$P' = -\frac{\mathcal{M}_0}{\mathcal{M}_0 + M_0} \qquad Q' = -2\frac{\gamma\mathcal{M}_0 + (\gamma - 1)M_0(\mathcal{M}_0^2 - \beta^2)}{\beta(\mathcal{M}_0 + M_0)} \qquad R' = -\frac{\gamma M_0 + (\gamma - 1)\mathcal{M}_0}{\mathcal{M}_0 + M_0} \tag{83}$$

C. Determination of Density and Temperature

By subtraction of the equation for temperature deduced from (30) from the continuity equation corresponding to (21), the function $\phi_*(x,t)$ defined by

$$T^{(1,2)} - \rho^{(1,2)} = \frac{\gamma g_0^2}{4\cosh^2\delta}\phi_*(x,t)\cosh 2\delta y \tag{84}$$

is a solution of the equation

$$\frac{\partial\phi_*}{\partial t} + M_0\frac{\partial\phi_*}{\partial x} = \cos 2(x - \mathcal{M}_0 t) \tag{85}$$

with boundary condition, in the case of established regime:

$$\phi_*(0,t) = 0 \tag{86}$$

One finds:

$$\phi_*(x,t) = -\frac{1}{2g_0}\left[\sin 2(x - \mathcal{M}_0 t) - \sin\frac{2\mathcal{M}_0}{M_0}(x - M_0 t)\right] \tag{87}$$

Now, the relation (84) and the state law (25) give

$$\begin{aligned} \rho^{(1,2)} &= p^{(1,2)} - \frac{(\gamma - 1)g_0^2}{4\cosh^2\delta}\phi_*\cosh 2\delta y \\ T^{(1,2)} &= p^{(1,2)} + \frac{g_0^2}{4\cosh^2\delta}\phi_*\cosh 2\delta y \end{aligned} \tag{88}$$

where $p^{(1,2)}$ is given by Eq. (75), and ϕ_* by Eq. (87).

IV. DISCUSSION OF ANALYTICAL RESULTS

The relations (75), (79), and (82) show that the solution is made of two parts $\mathcal{Q}_1^{(1,2)}(x,t)$ and $\mathcal{Q}_2^{(1,2)}(x,y,t,\delta)$ so that

$$\mathcal{Q}^{(1,2)}(x,y,t;\delta) = (1-\tanh^2\delta)\,\mathcal{Q}_1^{(1,2)}(x,t)\cosh 2\delta y + \tanh\delta\,\mathcal{Q}_2^{(1,2)}(x,y,t;\delta) \qquad (89)$$

(for $v^{(1,2)}$ the term $\cosh 2\delta y$ is replaced by $\sinh 2\delta y$).

The function $\mathcal{Q}_1^{(1,2)}$ is periodic in x and t, and it keeps the same expression in the whole channel. This part makes evident several waves, say for the velocity $u^{(1,2)}$:

A wave $(x - \mathcal{M}_0 t)$ coming from the applied traveling field.

A merely dynamical wave: the entropy wave $(x - \mathcal{M}_0 t)$.

Two waves, the velocities of which depending on flow (by M_0) and field (by $\mathcal{M}_0$):

$$\left(x - \frac{\mathcal{M}_0}{A_\pm}t\right)$$

To these waves, we must add those coming from the one-dimensional part $\mathcal{Q}^{(1,1)}$; see Eqs. (38)–(40).

The part $\mathcal{Q}_2^{(1,2)}$ is a series of oscillatory functions, so that, in each region of plane (x,y) (Fig. 3), there is only a finite number of nonzero terms. From one region to another, the solution is continuous but not its derivatives. This part represents the same waves as the first one, but the amplitude of these waves is no longer constant; in fact, these amplitudes, which are functions of x and y, are essentially expressed in terms of integrals (78). These integrals are convergent for infinite x when $b_{n\pm}$ is kept fixed, except for the particular case $\mathcal{M}_0 - M_0 - 1 = 0$, which is just the case of resonance already encountered.

The complexity of this solution did not permit us to demonstrate the convergence (or nonconvergence) of the series when x becomes infinite. Moreover, we have pointed out that the steady part $\mathcal{Q}^{(1,0)}$ is a solution which is not valid for large values of x; but the fact that any real accelerator has a finite length assures us that the present solution gives a good representation of the flow in the accelerator. Note, however, that for a finite-length accelerator, "end effects" (Ref. 11) modify the flow in the neighborhood of the outlet of the accelerator, and have not been considered here.

V. DISCUSSION OF NUMERICAL RESULTS

The analytical solution has been numerically computed. Some parameters remain fixed during the calculations: $M_0 = 3$, $\gamma = 5/3$. The functions, which are periodic with respect to t (the period is equal to $\pi/\mathcal{M}_0$) are computed as functions of x, at fixed y, for two values of t: $t = 0$ and $t = \pi/4\mathcal{M}_0$.

The oscillating part only of the solution is considered and we introduce

$$\left.\begin{aligned} u_{II}(x,y,t;\delta) &= u^{(1,1)}(x,t;\delta) + u^{(1,2)}(x,y,t;\delta) \\ \rho_{II}(x,y,t;\delta) &= \rho^{(1,1)}(x,t;\delta) + \rho^{(1,2)}(x,y,t;\delta) \end{aligned}\right\} \qquad (90)$$

which will be compared with the quasi-one-dimensional solution calculated in Ref. 13. The latter solution can be defined from Eqs. (38)–(40) by

$$\left.\begin{aligned} u_I(x,t) &= 2u^{(1,1)}(x,t;0) \\ \rho_I(x,t) &= 2\rho^{(1,1)}(x,t;0) \end{aligned}\right\} \qquad (91)$$

(i) Figure 6 represents the variations of u_I and u_{II} on the axis $y = 0$ for $\mathcal{M}_0 = 3.75$ and several values of δ. In order to make clear the effect of δ, only the peaks are represented in Fig. 7. Figures 8 and 9 show the varia-

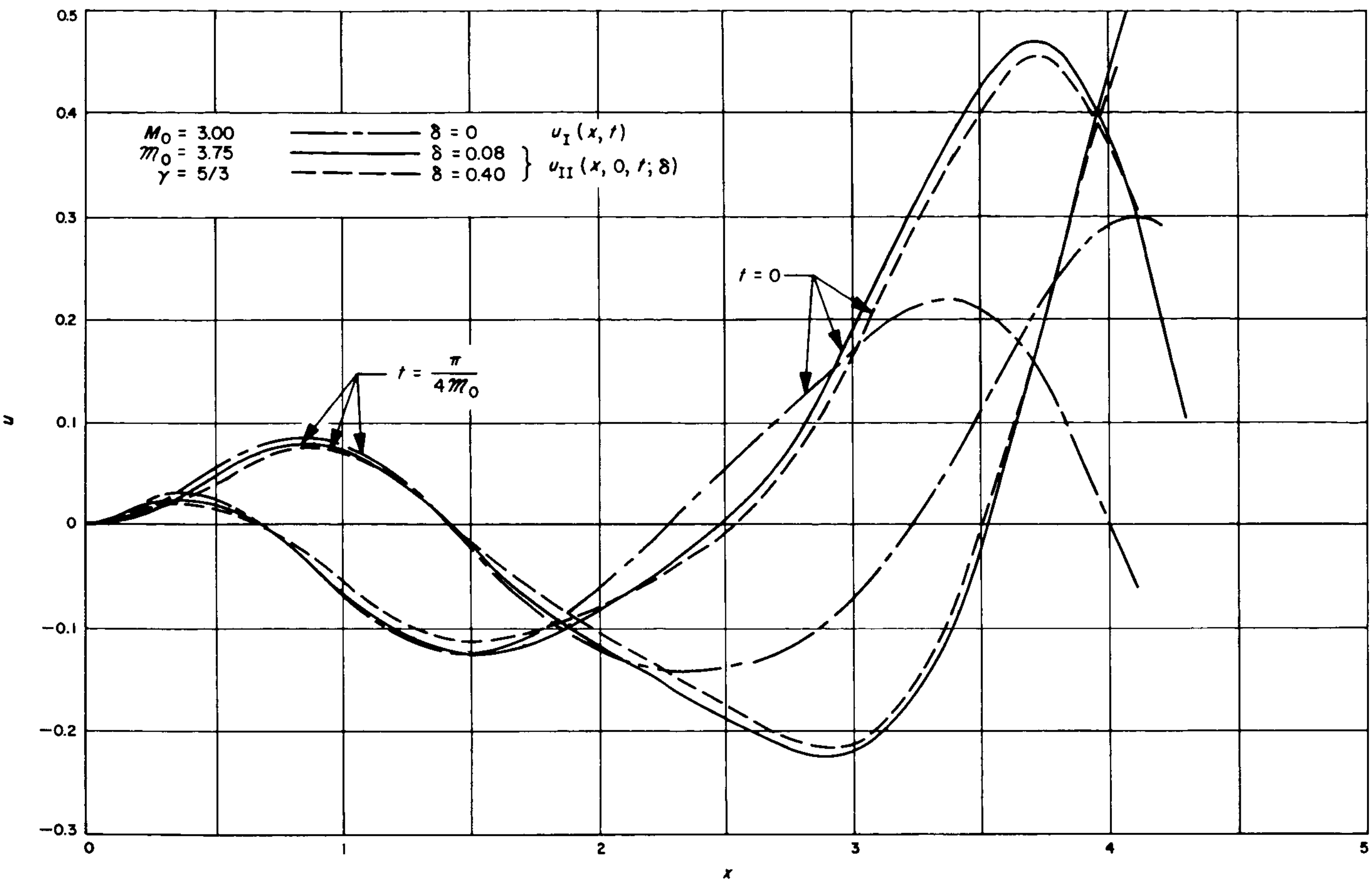

Fig. 6. Variations of velocity u_I and axial velocity u_{II} on the axis

tions of ρ_I and ρ_{II}. We can draw the following conclusions:

The quasi-one-dimensional solution (u_I, ρ_{II}) and the two-dimensional one (u_{II}, ρ_{II}) remain close for small δ, as long as x is smaller than some value which, in the present case, is of the order of 1.5. Beyond, this difference becomes important.

The amplitude of oscillations are decreasing functions of δ and we can remark that this phenomenon is stronger for ρ_{II} than for u_{II}.

At the wall (the curves are not drawn) the oscillations of u_{II} have an amplitude which is a decreasing function of δ, while those of ρ_{II} have an amplitude increasing with δ.

(ii) Figures 10 and 11 show respectively the variations of $u^{(1,2)}$ and $\rho^{(1,2)}$ on the axis compared to the analogous values at the wall. These curves, which are drawn for several values of δ, show again the two-dimensional character of the flow. Here too, the effect of δ is more important on density than on velocity.

(iii) The variations of mass flux $u^{(1,2)} + M_0\rho^{(1,2)}$ are represented in Fig. 12 for the same conditions. The oscillations at the axis are of greater amplitude than those at the wall.

(iv) To make evident the effect of the field of velocity $\tilde{V}$, we plotted on Figs. 13 and 14 the variations of the velocity u_{II} and density ρ_{II} for several values of $\mathcal{M}_0$. It must be recalled that $\delta = h/\lambda$ and $x = \tilde{x}/\lambda$ where λ is the reduced wavelength of the field which appears in $\mathcal{M}_0$ as $\mathcal{M}_0 = \tilde{V}/\tilde{c}_0 = \lambda\omega/\tilde{c}_0$; thus, in order to consider a channel of invariant geometry, to change $\mathcal{M}_0$ would mean to change the frequency ω in keeping the wavelength constant. The curves show how rapidly the oscillations grow with the field velocity.

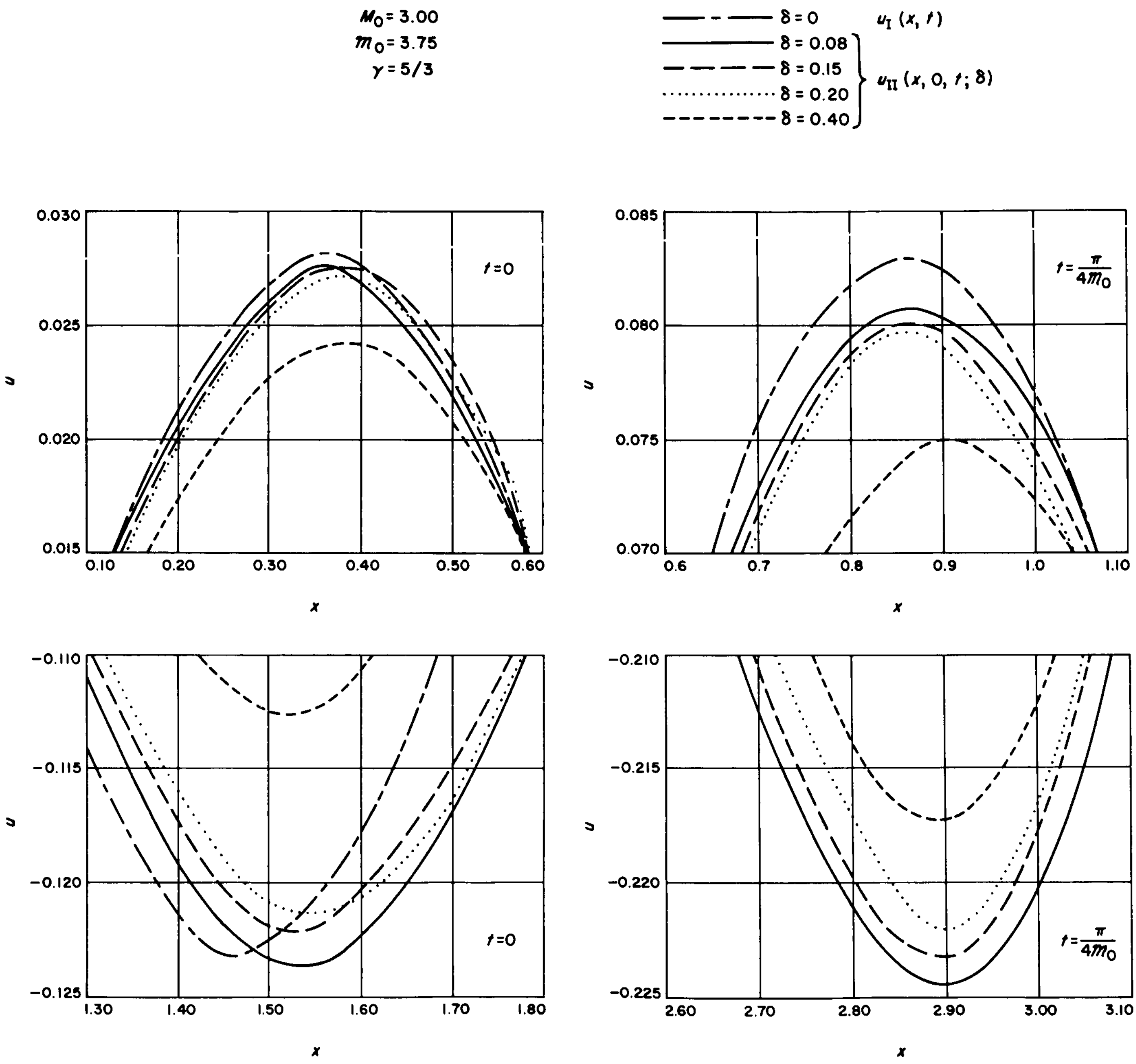

Fig. 7. Effect of δ on the amplitude of oscillations of the axial velocity on the axis (peaks of curves of Fig. 6)

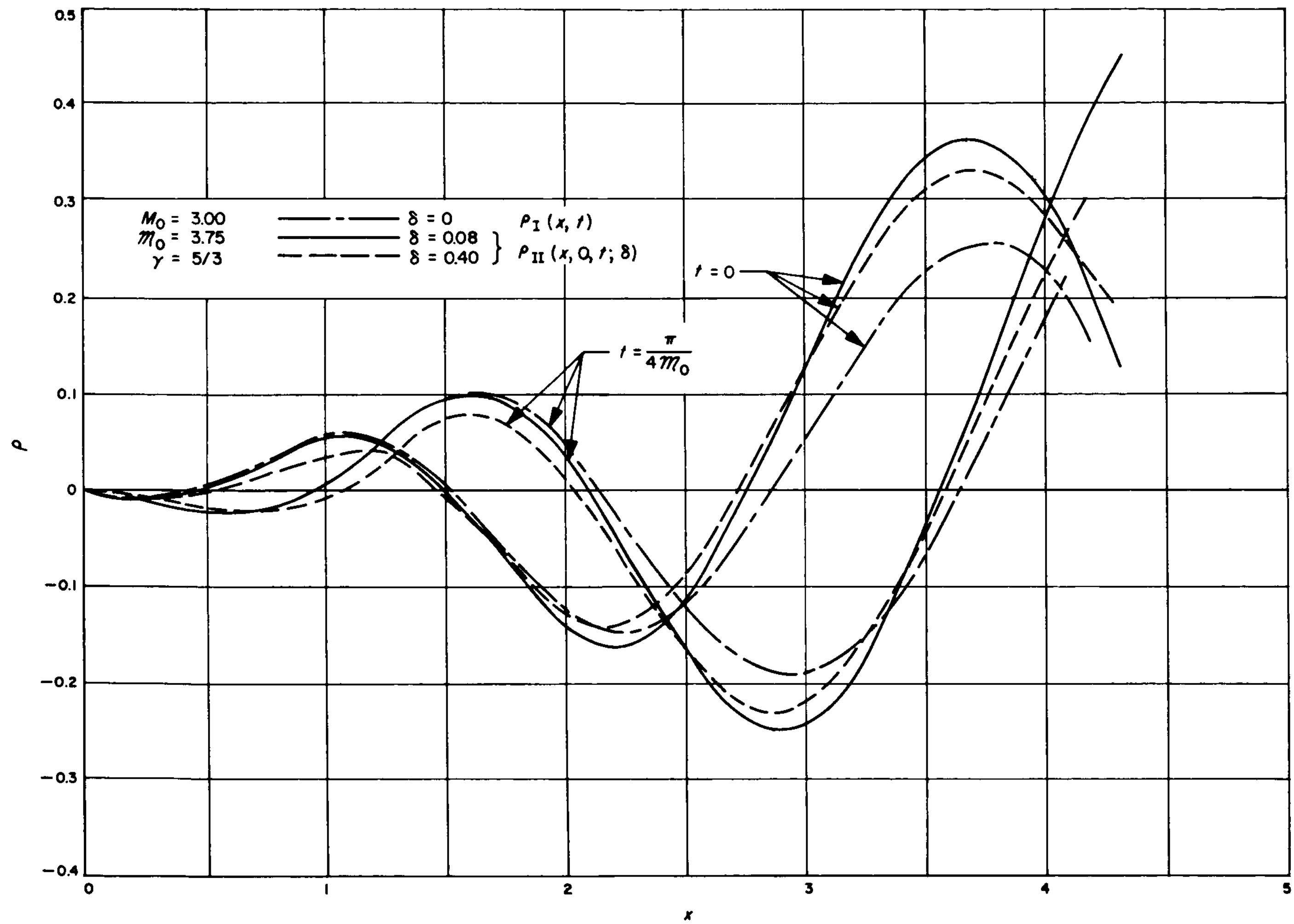

Fig. 8. Variations of density ρ_I and density ρ_{II} on the axis

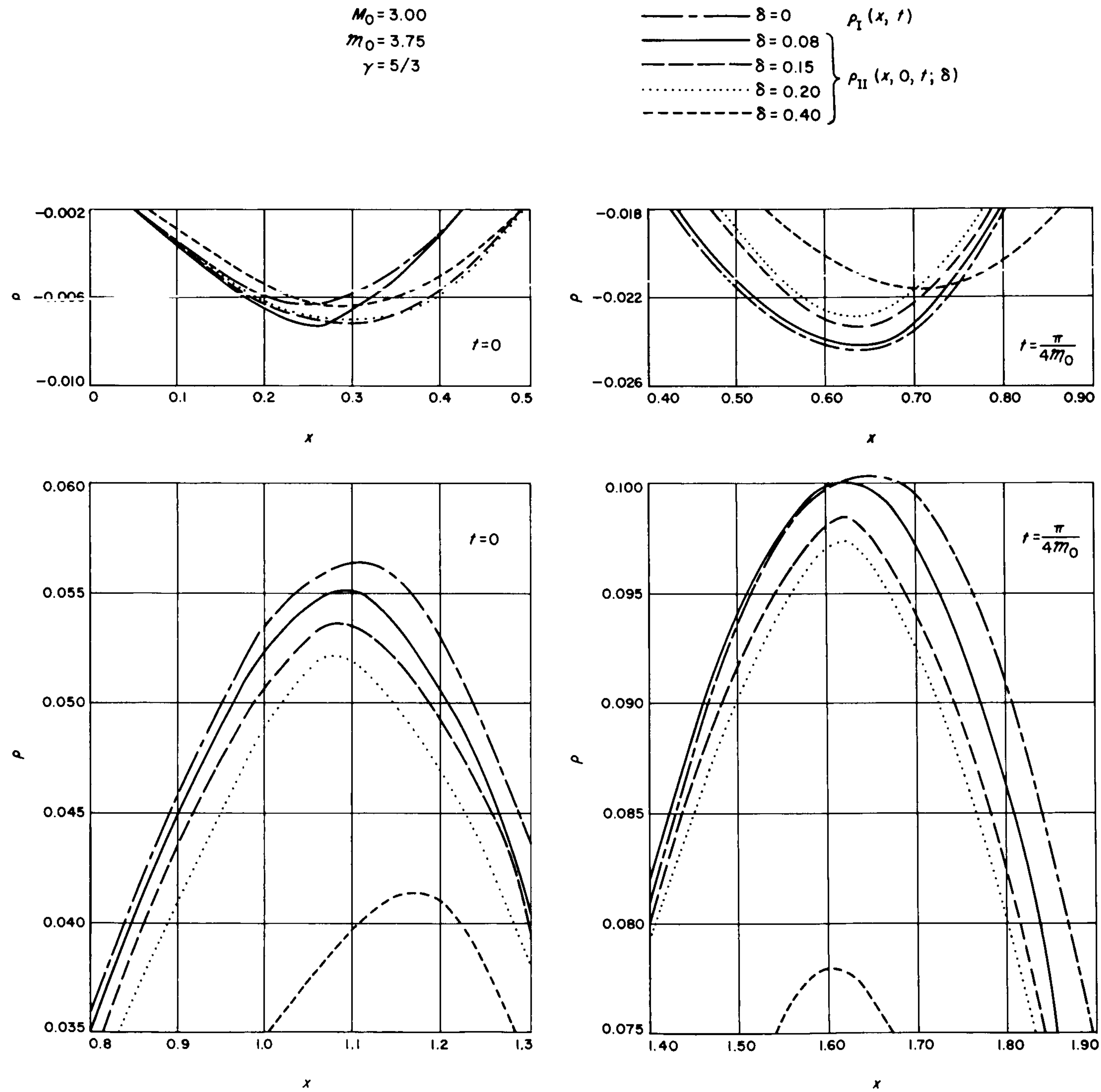

Fig. 9. Effect of δ on the amplitude of oscillations of the density on the axis (peaks of curves of Fig. 8)

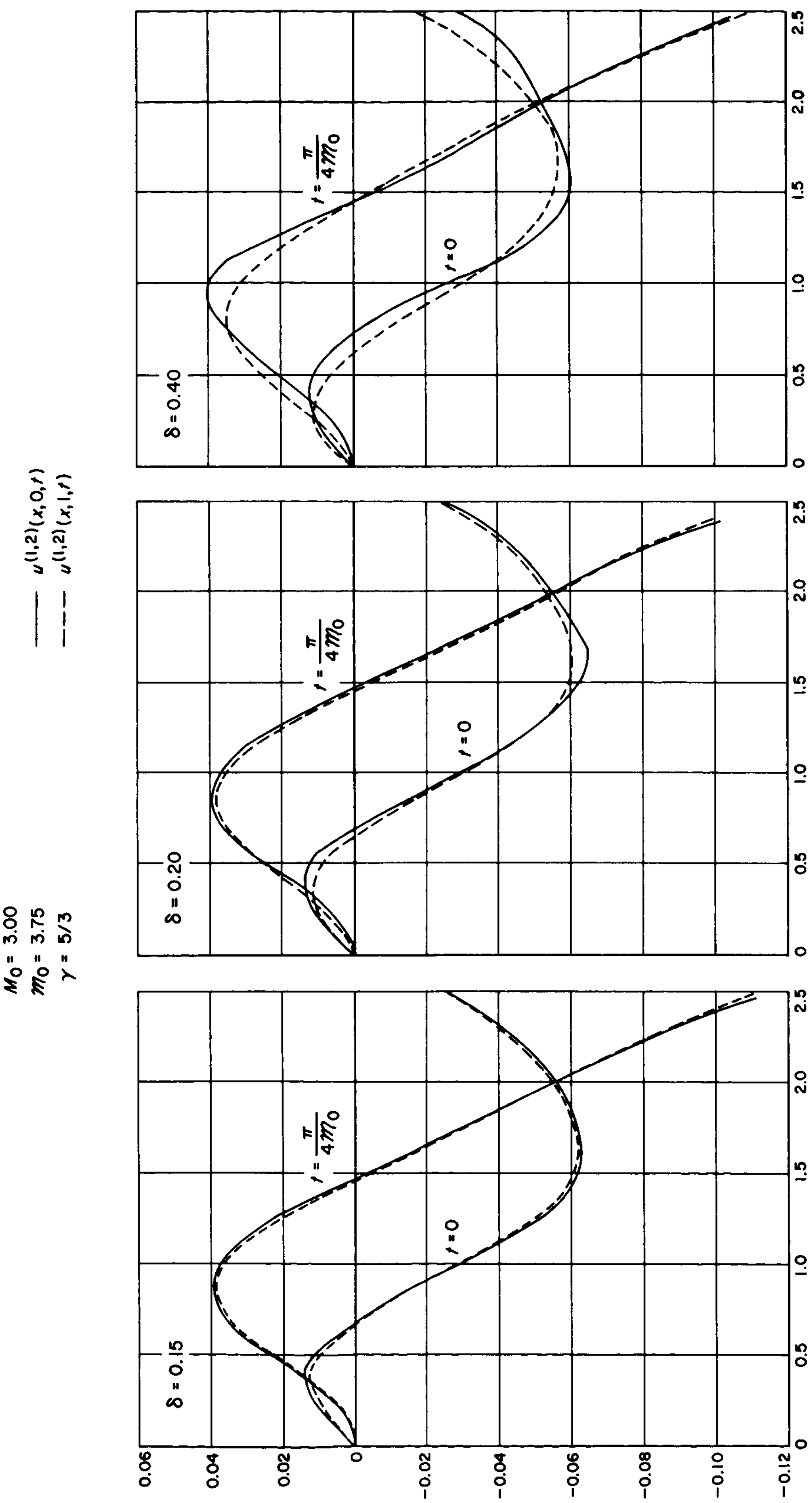

Fig. 10. Comparison between the values of axial velocity $u^{(1,2)}$ on the axis and its values at the wall

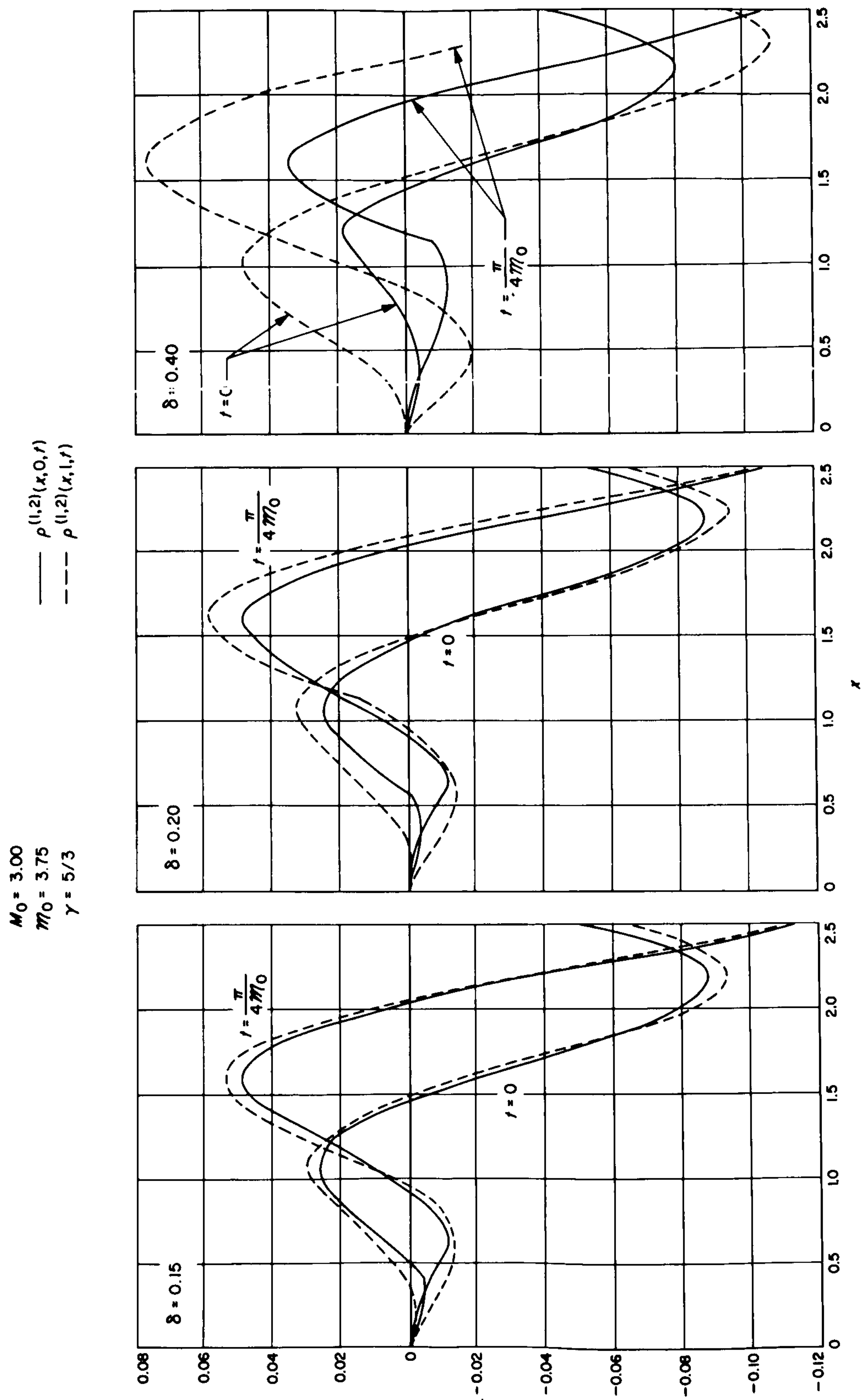

Fig. 11. Comparison between the values of density $\rho^{(1,2)}$ on the axis and its values at the wall

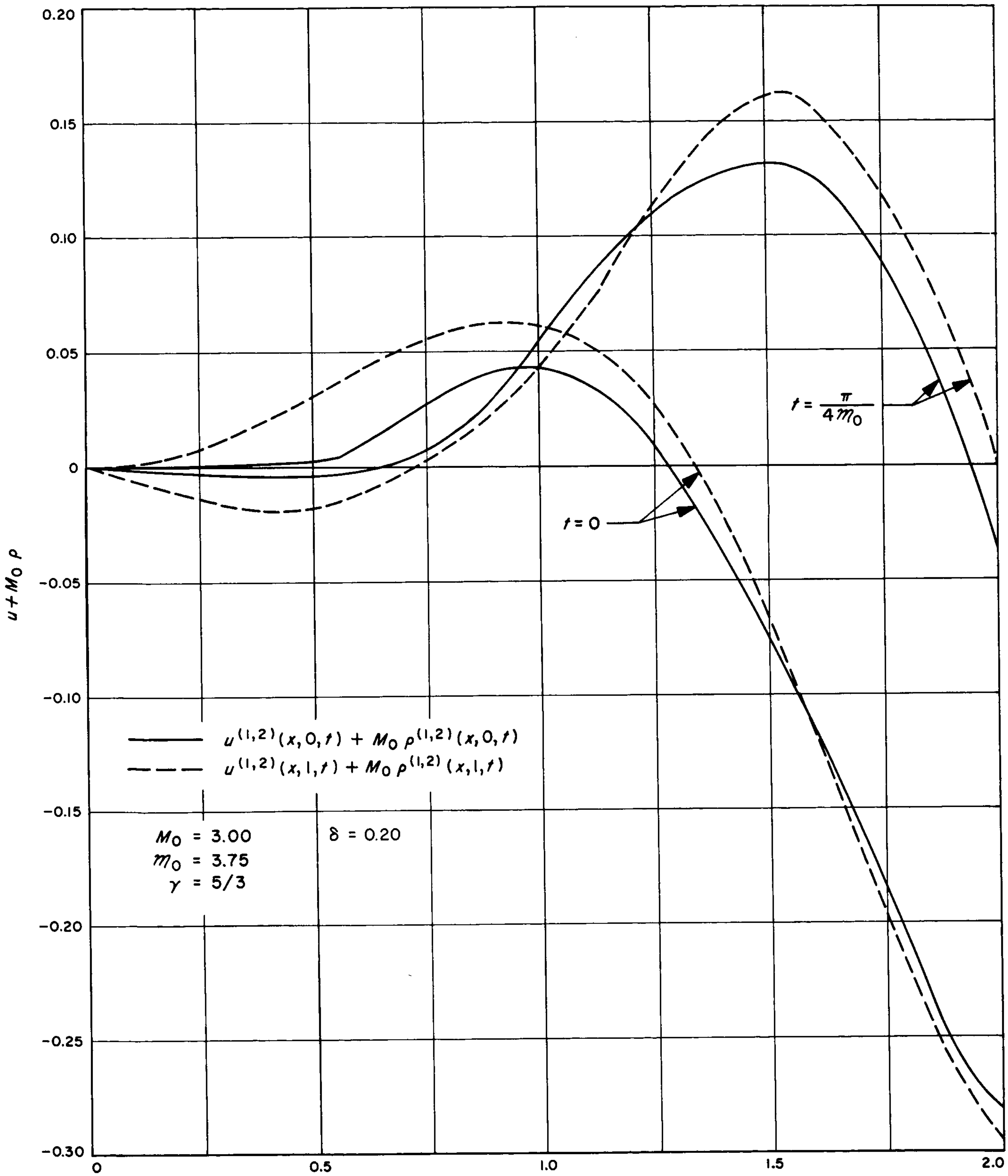

Fig. 12. Comparison of the values of mass flux $u^{(1,2)} + M_0\rho^{(1,2)}$ on the axis and its values at the wall

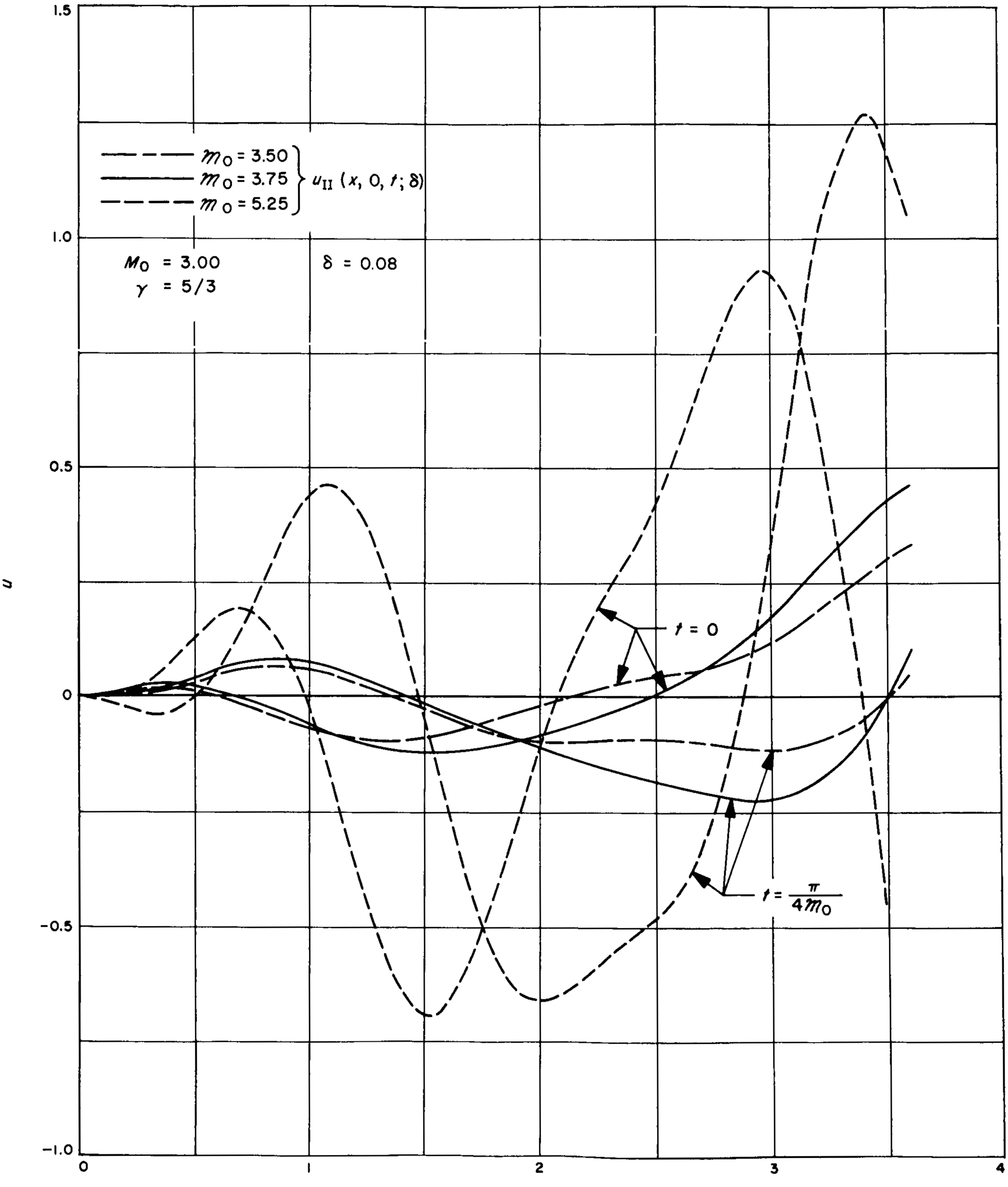

Fig. 13. Effect of field velocity (i.e., $\mathcal{M}_0$) on the axial velocity u_{II} on the axis

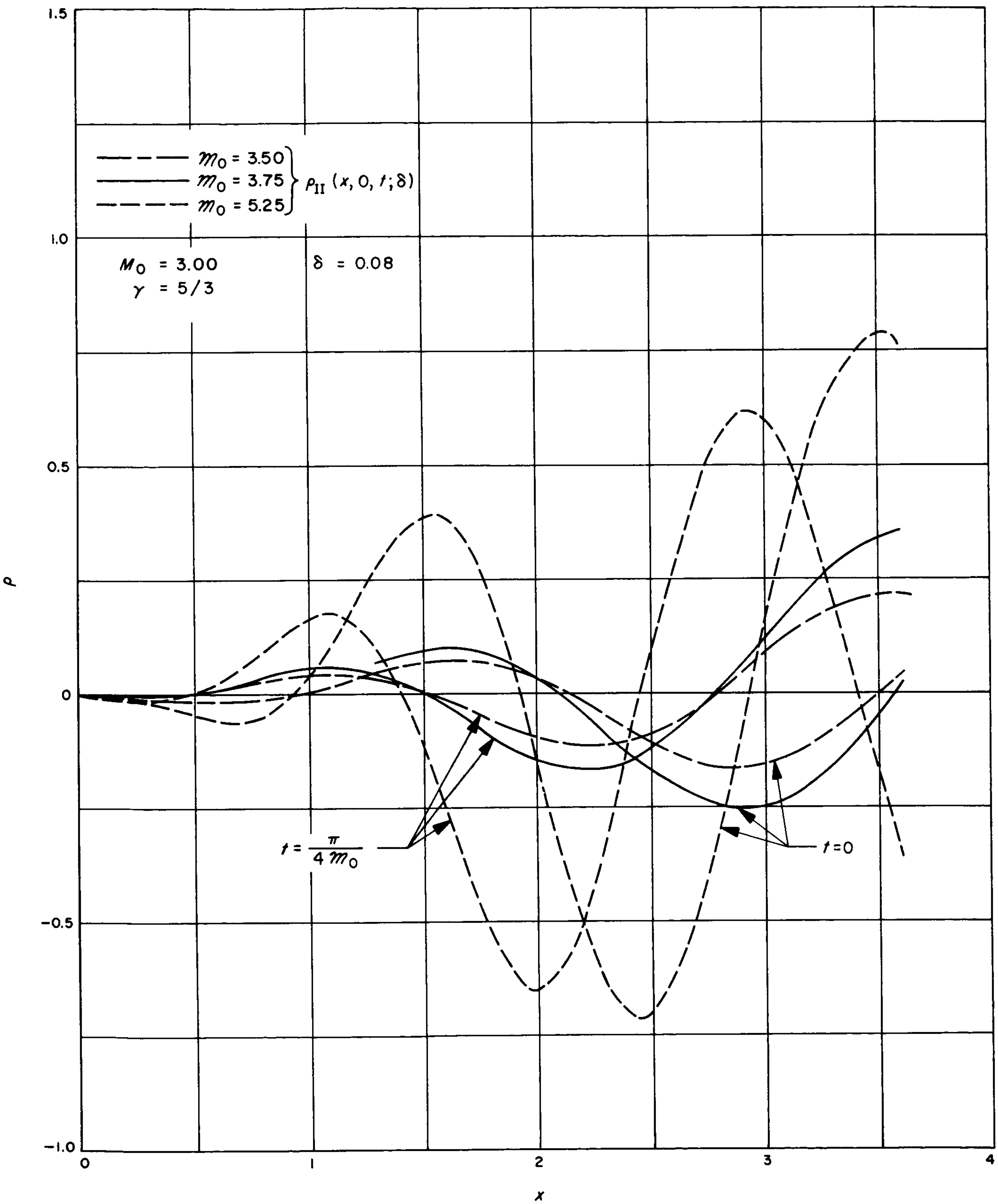

Fig. 14. Effect of field velocity (i.e., $\mathcal{M}_0$) on the density ρ_{II} on the axis

VI. COMPARISON WITH THE QUASI-ONE-DIMENSIONAL SOLUTION

Figures 6 and 8 show that, for nonsmall values of x, the two-dimensional solution calculated for $\delta = 0.08$–i.e., for small values of δ–and the quasi-one-dimensional solution of Ref. 13 are not close. More precisely, the limit of the two-dimensional solution for $\delta \to 0$ is not equal to the quasi-one-dimensional solution; that can be clearly seen in Fig. 15 where the variations of $u_{II}|_{\delta=0}$ are compared with those of $u_I(x,t)$. The reason for the nonidentity of these two solutions is the following: the limit $\delta \to 0$ of $u_{II}, v_{II}, p_{II}, \rho_{II}$ is a singular limit because the function $v_{II}|_{\delta=0} \equiv v^{(1,2)}|_{\delta=0}$ is nonzero and noncontinuous, while the functions $u_I, v_I = 0, p_I, \rho_I$ represent a continuous solution of the one-dimensional equations.

However, it is easy to see that the limit $\delta \to 0$ of the two-dimensional solution and the quasi-one-dimensional solution have the same asymptotic behavior when $\mathfrak{M}_0$ is infinitely large, so that it is possible, at the same time, to neglect M_0 with respect to $\mathfrak{M}_0$ (note that in this case the conditions of acceleration are violated).

Finally, it is possible to show that the limit $\delta \to 0$ of the average y of the steady part of the two-dimensional solution is exactly equal to the average t of the unsteady quasi-one-dimensional solution, and it would be possible to show the identity between the limit $\delta \to 0$ of the average y of the oscillatory part of the two-dimensional solution and the quasi-one-dimensional one, namely

$$\lim_{\delta \to 0} \frac{1}{2} \int_{-1}^{+1} \mathcal{Q}_{II}(x, y, t; \delta)\, dy = \mathcal{Q}_I(x, t) \qquad (92)$$

but the complexity of the expressions forbids such a verification.

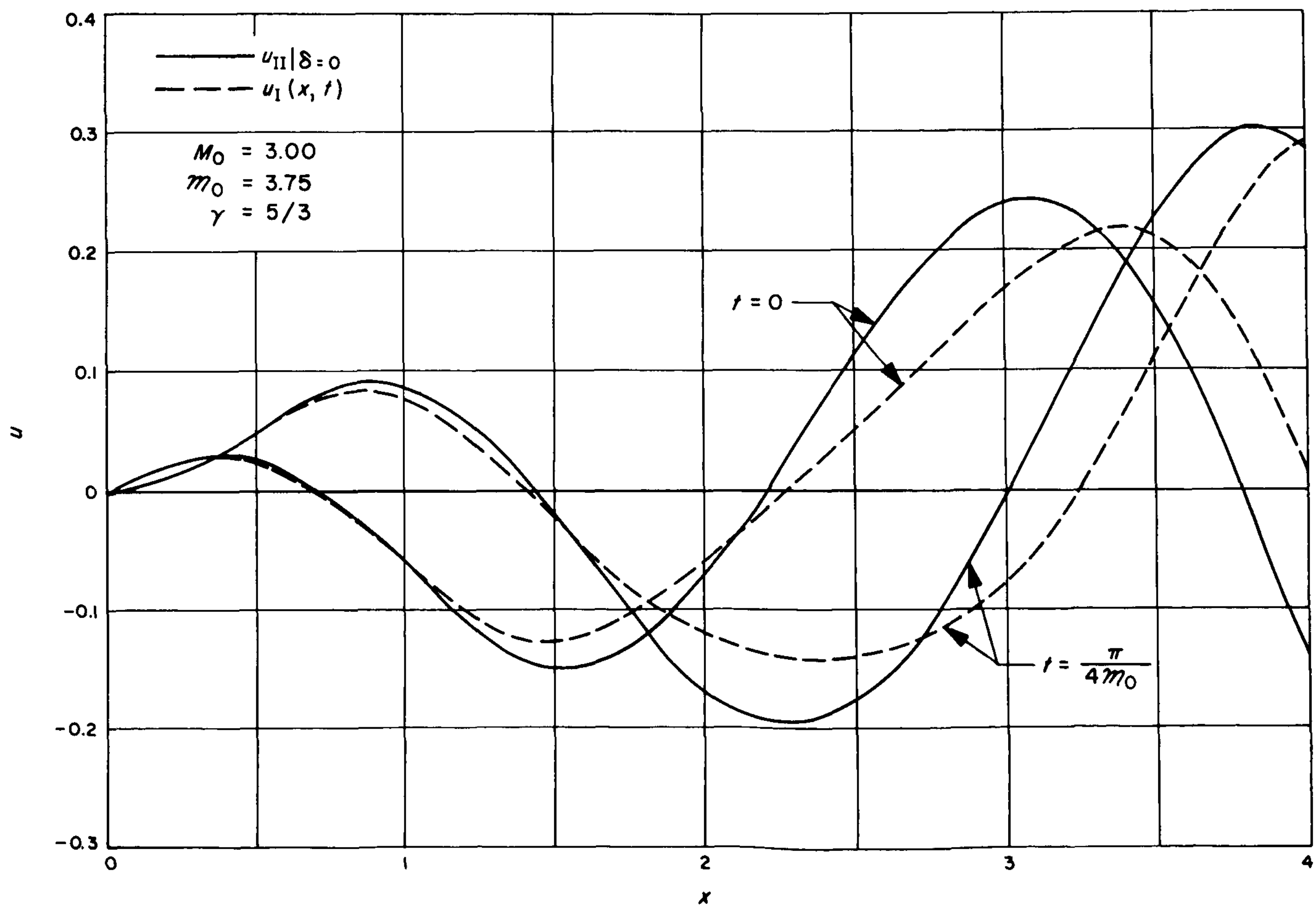

Fig. 15. Variations of $u_{II}|_{\delta=0}$ and u_I

NOMENCLATURE

a_1, a_2, a_3	coefficients defined by Eqs. (50), (73), and (80)
$A_\pm$	coefficients defined by Eq. (76)
$b_{n\pm}$	coefficients defined by Eq. (66)
$\mathbf{B}$	magnetic induction
B_M	characteristic value of magnetic induction
B_x, B_y	component of magnetic induction
c	sound velocity
C_p	specific heat at constant pressure
$\mathbf{E}$	electric field
E_z	component of electric field
$F_{n\pm}^{{c \atop s}, a_i}$	integrals defined by Eq. (74)
g_0	$= \mathcal{M}_0 - M_0$, slip coefficient
h	half-height of the duct
i	$= (-1)^{1/2}$
$I_{n\pm}^{{c \atop s}, a_i}$	integrals defined by Eq. (78)
J_0	Bessel function of zero order
$\mathbf{J}$	current density
J_z	component of the current density
M	Mach number
$\mathcal{M}$	Mach number of traveling field
N	$= \sigma \widetilde{B}_M^2 \lambda / \widetilde{\rho}_0 \widetilde{c}_0$, interaction parameter
p	pressure
q	variable of Laplace transform
P, Q, R	coefficients defined by Eq. (81)
P', Q', R'	coefficients defined by Eq. (83)
r	$= (q^2 + \kappa^2)^{1/2}$
r_m	mean radius of the annular channel
R_m	$= \mu \sigma \widetilde{c}_0 \lambda$, magnetic Reynolds number
Re	"real part of"
S	coefficient defined by Eq. (77)
t	time
T	temperature
u, v	components of the flow velocity
$\mathbf{U}$	flow velocity
V	velocity of the progressive field
x, y, z	spatial coordinates
X_1, X_2, X_3, X_4, X_5	coefficients defined by Eq. (41)
$Y_{n\pm}$	Heaviside function
β	$= (M_0^2 - 1)^{1/2}$
γ	adiabatic index
Γ	$= \gamma - 1$
δ	$= h/\lambda$, ratio of half-height of duct and reduced wavelength
η	$= \delta y$
κ	$= 2M_0/\beta$
λ	$= \Lambda/2\pi$, reduced wavelength
Λ	wavelength of traveling field
μ	magnetic permeability
ξ	$= x/\beta$
ρ	density
σ	electrical conductivity
ϕ	function deduced from Φ, Eq. (54)
Φ, Ψ	functions used to express pressure, Eq. (40)
ω	frequency of the progressive field

Symbols covered by a tilde ($\sim$) are generally dimensional quantities.

Superscripts

(1)	perturbation quantities
(1, 0)	steady, two-dimensional part of perturbation quantities
(1, 1)	unsteady, one-dimensional part of perturbation quantities
(1, 2)	unsteady, two-dimensional part of perturbation quantities

Subscripts

I	quasi-one-dimensional solution, Eq. (91)
II	sum of quantities (1, 1) and (1, 2)

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