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THE SHORT-PERIOD DRAG PERTURBATIONS OF THE ORBITS OF ARTIFICIAL SATELLITES

L. Sehnal and Sara B. Mills

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ABSTRACT

The aim of the paper is the computation of short-period perturbations of the elements of the orbits of artificial satellites, caused by atmospheric drag. The density function of the earth's atmosphere includes the effect of the atmospheric bulge, described by a formula given by Jacchia. The method of the solution is in essence the method of construction of the disturbing function on an electronic computer; the formulas needed for such computations and some numerical results are given.

THE SHORT-PERIOD DRAG PERTURBATIONS OF THE ORBITS OF ARTIFICIAL SATELLITES[1]

L. Sehnal[2] and Sara B. Mills[3]

1. THE DENSITY FUNCTION

The variation of the density of the earth's atmosphere can be described by a formula given by Jacchia (1960):

$$\rho(z) = \rho_0 \left(1 + \alpha \cos^{n'} \frac{\psi'}{2}\right) \quad , \tag{1}$$

where

$\rho(z)$ denotes the density in the height z (km) above the earth's surface,

ψ' is the angle made by the radius vector of the satellite and the axis of the atmospheric bulge,

α and n' are arbitrary numbers, and

ρ_0 is given by the relation

[1] This work was supported in part by Grant No. NsG 87-60 from the National Aeronautics and Space Administration.

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$$\log \rho_0 = \bar{a} + bz + c \exp(-0.01\ z) \quad , \tag{2}$$

where $\bar{a}$, b, and c are empirical coefficients.

The coefficients α, $\bar{a}$, b, and c and the exponent n' are found for a given date and given satellite as follows: The nighttime exospheric temperature T_0 is found from the measured solar flux at 10.7 cm. Then, for the height of the perigee of the satellite's orbit, we find the temperatures $T_{1/2}$ and T_m (daytime maximum) as

$$T_{1/2} = 1.14\ T_0 \quad ; \quad T_M = 1.28\ T_0 \quad .$$

For the temperatures T_0, $T_{1/2}$, and T_M we find from the tables of Jacchia (1964) the densities ρ_0, $\rho_{1/2}$, and ρ_M. The coefficient α and the exponent n' are then found from the formula

$$\rho = \rho_0 \left(1 + \frac{\rho_M - \rho_0}{\rho_0} \cos^{n'} \frac{\psi'}{2}\right) \quad . \tag{3}$$

We have then the equations

$$\alpha = \frac{\rho_M - \rho_0}{\rho_0}$$

and

$$n' = 6.644 \left[\log(\rho_M - \rho_0) - \log(\rho_{1/2} - \rho_0)\right] \quad .$$

We then take from the same tables the values of the densities $\rho_0(z_i)$ for the temperature T_0 at three different points z_i above the perigee height, so that we have three equations

$$\log \rho_0(z_i) = \bar{a} + bz_i + c \exp(-0.01\ z_i) \qquad (i = 1, 2, 3) \quad , \quad (4)$$

from which we obtain the coefficients $\bar{a}$, b, and c.

Thus the density function is computed separately for each date and orbit.

2. THE EQUATIONS FOR THE VARIATIONS OF THE ELEMENTS

The change of an arbitrary orbital element σ, caused by the atmospheric drag, is given by the equation

$$\frac{d\sigma}{dt} = K_\sigma F_\sigma(E) \rho \frac{dE}{dt} \quad , \tag{5}$$

where K_σ is a constant different for each element and dependent mainly on the physical characteristics of the satellite. The function $F_\sigma(E)$ is a function of the eccentric anomaly E and can be derived from the equations given by Sterne (1960) or in a transcription by Izsak (1960). This function contains in itself the effect of the rotation of the atmosphere. It can be developed in the Fourier series; it will be better for our purposes to transform it into a series with powers of $\cos^t E$ instead of the series with multiples of argument of trigonometric functions.

The coefficients of the terms of the series can be again expressed as the series of the powers of the eccentricity e. We have then generally two kinds of the function $F_\sigma(E)$:

$$F_\sigma(E) = \sum_{t=0}^{T} \cos^t E \sum_{u=0}^{L} {}^\sigma g_u^{(t)} e^u \quad , \tag{6a}$$

and

$$F_\sigma(E) = \sum_{t=0}^{T} \cos^t E \sin E \sum_{u=0}^{L} {}^\sigma g_u^{(t)} e^u \quad . \tag{6b}$$

The coefficients ${}^{\sigma}g_u^{(t)}$ depend on the small parameter d introduced by Sterne (1960),

$$d = \frac{\omega_E}{n} \sqrt{1 - e^2} \cos i \quad , \tag{7}$$

where ω_E is the angular velocity of the earth, n is the mean motion of the satellite, and i is the orbital inclination. The summation limit L depends then on the precision in powers of eccentricity wanted in the computation.

We have now to express the density function ρ as the function of the eccentric anomaly and multiply it with one of the series (6a) or (6b), to be able to integrate equation (5). We shall transform the function ρ as follows:

$$\rho = \rho_0 + \rho_0 \, a \cos^{n'} \frac{\psi'}{2} = \rho_0 \, a_1 \cos^{n_1'} \frac{\psi'}{2} + \rho_0 a_2 \cos^{n_2'} \frac{\psi'}{2}$$
$$= \sum_{i=1}^{2} \rho_0 \, a_i \cos^{n_i'} \frac{\psi'}{2} \quad , \tag{8}$$

where

$$a_1 = 1 \qquad n_1' = 0$$

$$a_2 = a \qquad n_2' = n' \quad .$$

The result will then consist of two parts, the second one corresponding to the effect of the atmospheric bulge. For the analytical treatment of the problem, we can take only one of the two terms in equation (8), thus dropping the index i. We shall introduce into this term the substitution

$$z - z_0 = ae - ae\cos E \quad ,$$

z_0 being the height of the perigee and a the semimajor axis. Introducing new constants k_j $(j = 0, 1, 2, 3)$ by equations

$$k_{i0} = \alpha_i \, 10^{\overline{a}+b(z_0 + ae)} \qquad (i = 1, 2) \quad ,$$

$$k_1 = - \ln 10 \times bae \quad ,$$

$$k_2 = \ln 10 \; c \exp\left[-0.01\,(z_0 + ae)\right] \quad ,$$

and

$$k_3 = +0.01\,ae \quad , \tag{9}$$

we have for the general term of the density function the expression

$$\rho \sim k_0 \exp(k_1 \cos E) \exp\left[k_2 \exp(k_3 \cos E)\right] \cos^{n'} \frac{\psi'}{2} \; . \tag{10}$$

We shall now introduce some development of the functions contained in the density function (10):

$$\exp(k_1 \cos E) = \sum_{i=0}^{\infty} \frac{k_1^i}{i!} \cos^i E \quad , \tag{11}$$

$$\exp\left[k_2 \exp(k_3 \cos E)\right] = \sum_{\ell=0}^{\infty} \cos^{\ell} E \frac{k_3^{\ell}}{\ell!} \sum_{j=0}^{\infty} \frac{k_2^j}{j!} j^{\ell} \quad . \tag{12}$$

Multiplying equations (11) and (12), we obtain the expression for the exponential part of the density function in the form

$$\exp(k_1 \cos E) \exp\left[k_2 \exp(k_3 \cos E)\right] = \sum_{p=0}^{\infty} \cos^p E \cdot b_p \quad , \tag{13}$$

where

$$b_p = \sum_{i=0}^{p} \frac{k_1^i}{i!} \frac{k_3^{p-i}}{(p-i)!} \sum_{j=0}^{\infty} \frac{k_2^j}{j!} j^{p-i} \quad . \tag{14}$$

The actual limits that must be used in the computation of the coefficient (14) depend on the numerical values of the constants k_i and are best determined numerically. The coefficients b_p make a convergent series, growing from the beginning to a certain limit, then decreasing monotonically after this limit or a second maximum, depending on the values of k_1, k_2, and k_3.

The trigonometric term in equation (10) is expressed using some simple relations:

$$\cos^{n'} \frac{\psi'}{2} = \left(\frac{1}{2}\right)^{\frac{n'}{2}} (1 + \cos \psi')^{\frac{n'}{2}} = \left(\frac{1}{2}\right)^{\frac{n'}{2}} \sum_{\ell=0}^{J} n_\ell \cos^\ell \psi \quad , \qquad (15)$$

where

$$n_\ell = \frac{n'}{2}\left(\frac{n'}{2} - 1\right)\left(\frac{n'}{2} - 2\right) \ldots \left(\frac{n'}{2} - \ell + 1\right) \quad . \qquad (16)$$

We shall now suppose that the position of the orbital plane and the axis of the atmospheric bulge remain constant; this is nearly true for one revolution of the satellite, so that the short-period perturbations can be determined, assuming this hypothesis, with sufficient accuracy. The whole theory could be, of course, developed considering the real changes of the elements of the satellite orbit and the motion of the sun. In this case, we should obtain a good description of the behavior of the orbit during longer time intervals, including the long-period perturbations.

The angle ψ' can be found from the relations of spherical trigonometry, which were given by Cook and King-Hele (1965). We shall use a similar notation, writing

$$\cos \psi' = A \cos v + B \sin v \quad ,$$

v being the true anomaly. The coefficients A and B will then be constants, given by the equations

$$A = \frac{1}{2} \sin \epsilon \sin i [\cos(\omega - L) - \cos(\omega + L)]$$

$$+ \sin^2 \frac{\epsilon}{2} [\cos^2 \frac{i}{2} \cos(\omega + \Omega - L - \lambda) + \sin^2 \frac{i}{2} \cos(\omega - \Omega - L + \lambda)]$$

$$+ \cos^2 \frac{\epsilon}{2} [\cos^2 \frac{i}{2} \cos(\omega + \Omega - L - \lambda) + \sin^2 \frac{i}{2} \cos(\omega + \Omega + L - \lambda)] \quad ,$$

$$B = \frac{1}{2} \sin \epsilon \sin i [\sin(\omega - L) + \sin(\omega + L)]$$

$$+ \sin^2 \frac{\epsilon}{2} [- \sin^2 \frac{i}{2} \sin(\omega - \Omega - L + \lambda) - \cos^2 \frac{i}{2} \sin(\omega + \Omega + L - \lambda)]$$

$$+ \cos^2 \frac{\epsilon}{2} [- \sin^2 \frac{i}{2} \sin(\omega - \Omega + L + \lambda) - \cos^2 \frac{i}{2} \sin(\omega + \Omega - L - \lambda)] \quad , \tag{17}$$

where ϵ is the obliquity of the ecliptic, ω is the argument of perigee, Ω is the longitude of the ascending node, L is the longitude of the sun, and λ is the angle of which the axis of the bulge lags behind the direction to the sun in right ascension.

Introducing the eccentric instead of the true anomaly, we have

$$\cos \psi' = \frac{a}{r} \left(A \cos E - Ae + B \sqrt{1 - e^2} \sin E \right) = \frac{a}{r} C \quad , \tag{18}$$

r being the radius vector of the satellite.

Then we can write

$$\cos^{n'} \frac{\psi'}{2} = \left(\frac{1}{2}\right)^{n'} \sum_{\ell=0}^{J} \left(\frac{a}{r}\right)^{\ell} P_{\ell} \tag{19}$$

and

$$P_{\ell} = n_{\ell} C^{\ell} \quad .$$

The coefficients P_{ℓ} can be then expressed from equation (18) as the series in the powers of the eccentricity:

$$P_{\ell} = \sum_{v=0}^{L} X_{v}^{(\ell)} e^{v} \quad , \tag{20}$$

where the coefficients $X_{v}^{(\ell)}$ are given as series in powers of the trigonometric functions of the eccentric anomaly E:

$$X_{v}^{(\ell)} = \sum_{k=0}^{\ell} x_{k}^{\ell,v} \cos^{k} E + \sum_{k=0}^{\ell-1} y_{k}^{\ell,v} \cos^{k} E \sin E \quad . \tag{21}$$

The indices k will never be greater than ℓ, so that the limit of summation can be written as J . The ratio $(a/r)^{\ell}$ is easily expressed as

$$\left(\frac{a}{r}\right)^{\ell} = (1 - e \cos E)^{-\ell} = \sum_{j=0}^{L} k_{j}^{(\ell)} e^{j} \cos^{j} E \quad . \tag{22}$$

Multiplying equations (21) and (22) and introducing the result into equation (19), we have finally

$$\cos^{n'}\frac{\psi'}{2} = \left(\frac{1}{2}\right)^{\frac{n'}{2}} \sum_{\ell=0}^{J}\sum_{j=0}^{L} k_j^{(\ell)} e^j \cos^j E \sum_{v=0}^{L} X_v^{(\ell)} e^v$$

$$= \left(\frac{1}{2}\right)^{\frac{n'}{2}} \sum_{t=0}^{T} \cos^t E \sum_{p=0}^{L} e^p \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} x_{t-j}^{\ell, p-j}$$

$$+ \left(\frac{1}{2}\right)^{\frac{n'}{2}} \sum_{t=0}^{T-1} \cos^t \sin E \sum_{p=0}^{L} e^p \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} y_{t-j}^{\ell, p-j} \quad . \tag{23}$$

This expression must be now multiplied by expression (13), the exponential part of the density function. Introducing the substitution

$$k_5^{(p)} = \left(\frac{1}{2}\right)^{\frac{n'}{2}} b_p \quad , \tag{24}$$

and establishing the fixed summation limit in equation (13) as P, we shall have

$$\exp(k_1 \cos E) \exp[k_2 \exp(k_3 \cos E)] \cos^{n'}\frac{\psi'}{2}$$

$$= \sum_{s=0}^{S} \cos^s E \sum_{p=0}^{L} e^p C_p^{(s)} + \sum_{s=0}^{S-1} \cos^s E \sin E \sum_{p=0}^{L} e^p S_p^{(s)} \quad , \tag{25}$$

where we have introduced the factors $C_p^{(s)}$ and $S_p^{(s)}$ by

$$C_p^{(s)} = \sum_{t=0}^{T} k_5^{(s-t)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \, x_{t-j}^{\ell, p-j}$$

and

$$S_p^{(s)} = \sum_{t=0}^{T-1} k_5^{(s-t)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \, y_{t-j}^{\ell, p-j} \quad . \tag{26}$$

The expression (25) will now be multiplied by the function $F_\sigma(E)$, given by (6a) or (6b). We shall take the formula (6a) for the detailed explanation. After the multiplication, we obtain the expression

$$F_\sigma(E) \cdot \rho = k_0 \sum_{r=0}^{R} \cos^r E \sum_{q=0}^{L} e^q \sum_{s=0}^{M} \sum_{p=0}^{L} {}^{\sigma}g_{q-p}^{(r-s)} \, C_p^{(s)}$$

$$+ k_0 \sum_{r=0}^{R-1} \cos^r E \sin E \sum_{q=0}^{L} e^q \sum_{s=0}^{M} \sum_{p=0}^{L} {}^{\sigma}g_{q-p}^{(r-s)} \, S_p^{(s)} \quad . \tag{27}$$

The following formulas are known:

$$\cos^r E = \sum_{i=0}^{r} d_i^{(r)} \cos iE \tag{28a}$$

and

$$\cos^r E \sin E = \sum_{i=1}^{r+1} c_i^{(r)} \sin iE \quad , \tag{28b}$$

where the coefficients $d_i^{(r)}$ and $c_i^{(r)}$ are constant numbers. Using formula (28a), we have finally

$$F_\sigma(E) \cdot \rho = k_0 \sum_{i=0}^{R} \cos iE\, Z_i + k_0 \sum_{i=1}^{R} \sin iE\, W_i \quad .$$

To obtain the changes of elements, we shall find it sufficient to integrate this expression and multiply it by the constant coefficient K_σ. We shall make use of the relation $E = nt + e \cdot \sin E$ to introduce time into the secular term and shall imply the condition that the periodic changes of elements will vanish in perigee. We have then the change of an arbitrary element σ given as

$$\Delta\sigma = K_\sigma k_0 Z_0 nt + K_\sigma k_0 \sum_{i=1}^{R} \frac{Z_i'}{i} \sin iE$$

$$+ K_\sigma k_0 \sum_{i=1}^{R} \frac{W_i}{i} (1 - \cos iE) \quad , \tag{29}$$

where we denoted

$$Z_1' = e Z_0 + Z_1$$

and

$$Z_i' = Z_i \quad , \qquad \text{if } i > 1 \quad .$$

The coefficients Z_i and W_i of the different terms can be written as

$$Z_i = \sum_{q=0}^{L} e^q \sum_{r=0}^{R} d_i^{(r)} \sum_{s=0}^{M} \sum_{p=0}^{L} \sigma_{g_{q-p}}^{(r-s)} \sum_{t=0}^{T} k_5^{(s-t)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \, x_{t-j}^{\ell, p-j}$$

and

$$W_i = \sum_{q=0}^{L} e^q \sum_{r=0}^{R-1} c_i^{(r)} \sum_{s=0}^{M-1} \sum_{p=0}^{L} \sigma_{g_{q-p}}^{(r-s)} \sum_{t=0}^{T-1} k_5^{(s-t)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \, y_{t-j}^{\ell, p-j} \quad . \tag{30}$$

We used formula (6a) for our computation. This is sufficient in the case of the semimajor axis and the eccentricity. In the case of the argument of perigee, we have to use development (6b). The result will then be as follows:

$$\Delta\omega = K_\omega k_0 Z_0 nt + K_\omega k_0 \sum_{i=1}^{R+2} \frac{Z_i'}{i} \sin iE + K_\omega k_0 \sum_{i=1}^{R+1} \frac{W_i}{i} (1 - \cos iE) \quad , \tag{31}$$

where again

$$Z_1' = Z_1 + e Z_0$$

and

$$Z_i' = Z_i \quad , \qquad \text{if } i > 1$$

and the coefficients are given as

$$W_i = \sum_{q=0}^{L} e^q \sum_{r=0}^{R} c_i^{(r)} \sum_{s=0}^{M} \sum_{p=0}^{L} {}^{\sigma}g_{q-p}^{(r-s)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \sum_{t=0}^{T} k_5^{(s-t)} x_{t-j}^{\ell,p-j} \quad ,$$

$$Z_i = \sum_{r=0}^{R+2} d_i^{(r)} \left(H_r - H_{r-2}\right) \quad , \text{ and}$$

$$H_r = \sum_{q=0}^{L} e^q \sum_{s=0}^{M} \sum_{p=0}^{L} {}^{\sigma}g_{q-p}^{(r-s)} \sum_{\ell=0}^{J} \sum_{j=0}^{L} k_j^{(\ell)} \sum_{t=0}^{T} k_5^{(s-t)} y_{t-j}^{\ell,p-j} \quad . \qquad (32)$$

3. THE COMPUTATION OF THE INDIVIDUAL COEFFICIENT OF THE SERIES

The problem was thus solved in Section 2. To be able to use the expression practically, we would have to compute the individual terms. It is obvious that it is almost impossible to obtain by analytical hand computation even the expressions for the secular terms. We shall therefore try to find some recurrent formulas for the coefficients that appear in expressions (30) or (32). It would be best if we could find the values of the coefficients as functions of the indices. This is not always possible, so we have to use some recurrent formulas. We obtain for the coefficients these expressions:

A. The coefficients $d_i^{(r)}$:

a. if $i = 0$, then $$d_0^{(r)} = \frac{1}{2^r}\binom{r}{\frac{r}{2}} ,$$

b. if $i \neq 0$, then $$d_i^{(r)} = \frac{1}{2^{r-1}}\binom{r}{\frac{r-1}{2}} ,$$

when $r/2$ or $(r-1)/2$ are integers; otherwise, $d_i^{(r)} \equiv 0$. Moreover, the indices must satisfy the conditions $i, r \geq 0$, $r \geq i$; otherwise, again $d_i^{(r)} \equiv 0$.

B. The coefficients $c_i^{(r)}$:

a. if $i = 0$, then $$c_0^{(r)} \equiv 0 ,$$

b. if $i \neq 0$, then $$c_i^{(j+1)} = \frac{1}{2}\left(c_{i+1}^{(j)} + c_{i-1}^{(j)}\right) .$$

The indices must satisfy the conditions $i, j \geq 0$, $j+1 \geq i$; otherwise, $c_i^{(j)} \equiv 0$.

C. The coefficients $k_j^{(\ell)}$ are given as

$$k_j^{(\ell)} = \frac{\ell(\ell+1)(\ell+2)\ \ldots\ (\ell+j-1)}{j!} \quad ,$$

and the condition holds: $j, \ell \geq 0$, $k_0^{(\ell)} = 1$.

D. The coefficients $k_5^{(s-t)}$ are given according to fomulas (24) and (14) as

$$k_5^{(s-t)} = \left(\frac{1}{2}\right)^{\frac{n'}{2}} \sum_{i=0}^{s-t} \frac{k_1^i}{i!} \frac{k_3^{s-t-i}}{(s-t-i)!} \sum_{j=0}^{\infty} \frac{k_2^j}{j!} \, j^{s-t-i} \quad . \tag{33}$$

The computation of these coefficients is not too complicated, the actual limits of the infinite series depending on the precision wanted. It can be shown that the coefficients $k_5^{(s-t)}$ make a convergent series, several first terms being of increasing value. The rapidity of convergence depends on the values of the coefficients k_1, k_2, k_3, which are again given as a combination of the initial orbital elements and the coefficients in the density function, according to the formulas (9). The best way to obtain a sufficient number of coefficients is to determine this limit numerically, since the analytical conditions, imposed on the number of terms in the series (33), could lead to an immense number of terms.

E. The coefficients $x_{t-j}^{\ell, p-j}$ and $y_{t-j}^{\ell, p-j}$ are given by some recurrent formulas, different for individual values of p-j, which is in essence the power of the orbital eccentricity. We have to keep in mind that according to formulas (19) and (20), the coefficient n_ℓ from expressions (15) and (16) is included in the terms $X_v^{(\ell)}$ (21), and so in the expressions for x and y, too. The upper index p-j must be ≥ 0, and if we want to be precise to the second power of eccentricity, we have the following recurrent formulas, where we shall write, for the sake of brevity, a subscript i instead of t-j:

$$x_i^{\ell+1,0} = \frac{n_{\ell+1}}{n_\ell}\left(A\, x_{i-1}^{\ell,0} + B\, y_i^{\ell,0} - B\, y_{i-2}^{\ell,0}\right)$$

$$x_i^{\ell+1,1} = \frac{n_{\ell+1}}{n_\ell}\left(A\, x_{i-1}^{\ell,1} + B\, y_i^{\ell,1} - B\, y_{i-2}^{\ell,2} - A\, x_i^{\ell,0}\right)$$

$$x_i^{\ell+1,2} = \frac{n_{\ell+1}}{n_\ell}\left(A\, x_{i-1}^{\ell,2} + B\, y_i^{\ell,2} - B\, y_{i-2}^{\ell,2} - A\, x_i^{\ell,1} + \frac{1}{2} B\, y_{i-2}^{\ell,0} - \frac{1}{2} B\, y_i^{\ell,0}\right)$$

$$y_i^{\ell+1,0} = \frac{n_{\ell+1}}{n_\ell}\left(A\, y_{i-1}^{\ell,0} + B\, x_i^{\ell,0}\right)$$

$$y_i^{\ell+1,1} = \frac{n_{\ell+1}}{n_\ell}\left(A\, y_{i-1}^{\ell,1} + B\, x_i^{\ell,1} - A\, y_i^{\ell,0}\right)$$

$$y_i^{\ell+1,2} = \frac{n_{\ell+1}}{n_\ell}\left(A\, y_{i-1}^{\ell,2} + B\, x_i^{\ell,2} - A\, y_i^{\ell,1} - \frac{1}{2} B\, x_i^{\ell,0}\right) \quad . \quad (34)$$

We have the initial values

$$x_0^{0,0} = 1 \qquad y_0^{0,0} = 0$$

$$x_0^{0,1} = 0 \qquad y_0^{0,1} = 0$$

$$x_0^{0,2} = 0 \qquad y_0^{0,2} = 0$$

and the conditions for the values of indices

$$p-j, \quad t-j, \quad \ell \geq 0 \quad .$$

The coefficients n_ℓ are given by (16) and the values of A and B are given by (17). Thus we can obtain, from formulas (34), the values of all coefficients x and y needed.

F. The coefficients ${}^\sigma g_{q-p}^{(r-s)}$ are different for different elements σ and must be computed from the series (6a) or (6b). Fortunately, the number of these coefficients is not too high, and if we want precision to the second power of eccentricity, we have the expressions:

a. the semimajor axis a:

$$\begin{array}{lll} {}^a g_0^{(0)} = 1 - 2d & g_0^{(1)} = 0 & g_0^{(2)} = 0 \\ g_1^{(0)} = 0 & g_1^{(1)} = 2 & g_1^{(2)} = 0 \\ g_2^{(0)} = 0 & g_2^{(1)} = 0 & g_2^{(2)} = \frac{3}{2} \end{array} \qquad (35a)$$

b. the eccentricity e:

$$
\begin{array}{llll}
{}^{e}g_0^{(0)} = 0 & g_0^{(1)} = 0 & g_0^{(2)} = 0 & g_0^{(3)} = 0 \\
g_1^{(0)} = \frac{1}{2}\,d & g_1^{(1)} = 1 + \frac{5}{2}\,d & g_1^{(2)} = 1 + \frac{3}{2}\,d & g_1^{(3)} = 0 \\
g_2^{(0)} = 0 & g_2^{(1)} = -\frac{3}{8} & g_2^{(2)} = 0 & g_2^{(3)} = \frac{1}{2}
\end{array}
\qquad (35b)
$$

c. the argument of perigee ω:

$$
\begin{array}{lll}
{}^{\omega}g_0^{(0)} = 1 - 2d & g_0^{(1)} = 0 & g_0^{(2)} = 0 \\
g_1^{(0)} = 0 & g_1^{(1)} = 1 + \frac{3}{2}\,d & g_1^{(2)} = 0 \\
g_2^{(0)} = 0 & g_2^{(1)} = 0 & g_2^{(2)} = \frac{1}{2}
\end{array}
\quad . \qquad (35c)
$$

The small parameter d was introduced by Sterne (1960) and contains the ratio of the velocity of the revolution of the satellite to the rotation of the earth. It is given by (7). We have omitted the terms $0(d^2)$ and $0(e^3)$ in expressions (35).

G. The coefficients k_0 are given by (9) and we have for different elements σ different values of coefficient K_σ, according to

$$K_a = -C_d \frac{A'}{m} a^2$$

$$K_e = K_a \cdot \frac{1 - e^2}{a}$$

$$K_\omega = K_a \cdot \frac{(1 - e^2)^{1/2}}{a \cdot e} \quad ,$$

where A'/m is the cross-section area-to-mass ratio of the satellite and C_d is the drag coefficient.

4. THE CHANGES OF THE INDIVIDUAL ELEMENTS

According to (8), the whole computation of the perturbations of any element splits into two parts, corresponding to the two parts of the density function, the second of which contains the description of the atmospheric bulge. It is obvious, with respect to (8), that the first part of the density function (i. e., if i = 1), does not give rise to the coefficients W_i in the case of the semimajor axis a and the eccentricity e, nor to the coefficients Z_i in the case of the argument of perigee. The secular terms Z_0 arise in the case of a and e from both the first and the second parts of the density function; in the case of ω we obtain the secular change only from the second part of the disturbing function. Thus, the pattern of coefficients in (29) will be in the case of a and ω as follows:

$$
\begin{array}{ccccccc}
 & \Delta a & & & \Delta\omega & & \\
{}^{a}Z_0^{(1)} & {}^{a}Z_0^{(2)} & & {}^{\omega}Z_0^{(2)} & & & \\
{}^{a}Z_1^{(1)} & {}^{a}Z_1^{(2)} & {}^{a}W_1^{(2)} & {}^{\omega}Z_1^{(2)} & {}^{\omega}W_1^{(1)} & {}^{\omega}W_1^{(2)} & \\
{}^{a}Z_2^{(1)} & {}^{a}Z_2^{(2)} & {}^{a}W_2^{(2)} & {}^{\omega}Z_2^{(2)} & {}^{\omega}W_2^{(1)} & {}^{\omega}W_2^{(2)} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
{}^{a}Z_i^{(1)} & {}^{a}Z_i^{(2)} & {}^{a}W_i^{(2)} & {}^{\omega}Z_i^{(2)} & {}^{\omega}W_i^{(1)} & {}^{\omega}W_i^{(2)} & .\ (36)
\end{array}
$$

The left upper indices denote the element to which change the coefficients belong.

The time derivative of the mean anomaly is

$$\frac{dM}{dt} = n - \sqrt{1 - e^2}\,\frac{d\omega}{dt} - 2\,\frac{Sr}{na^2} \quad ,$$

where S is the radial component of the disturbing function. It was shown by Izsak (1960) that the last term can be neglected, since it is e^2 times smaller than the second term. The change of the mean anomaly is then given as a combination of changes of the semimajor axis and the argument of perigee. Using the pattern of results given in equation (36), we can write for the change of the mean anomaly the expression

$$\Delta M = \left(1 + \frac{3}{8}\,\frac{e}{a}\,{}^{a}W_1^{(2)} - \sqrt{1 - e^2}\,{}^{\omega}Z_0^{(2)}\right) nt$$

$$- \frac{3}{4}\,\frac{1}{a}\left({}^{a}Z_0^{(1)} + {}^{a}Z_0^{(2)}\right) n^2 t$$

$$- \sum_{n=1}^{2}\left[\frac{3}{2}\,\frac{1}{a}\sum_{i=1}^{R}\frac{1}{i}\,P_i^{(n)} + \sqrt{1 - e^2}\sum_{i=1}^{R+1}\frac{1}{i}\,{}^{\omega}W_i^{(n)}\right](1 - \cos iE)$$

$$- \sum_{n=1}^{2}\left[\frac{3}{2}\,\frac{1}{a}\sum_{i=1}^{R}\frac{1}{i}\,Q_i^{(n)} + \sqrt{1 - e^2}\sum_{i=1}^{R+2}\frac{1}{i}\,{}^{\omega}Z_i'^{(2)}\right]\sin iE \quad ,$$

where

$$P_1^{(n)} = {}^{a}Z_1'^{(n)} - \frac{e}{4}\,{}^{a}Z_2'^{(n)}$$

$$P_i^{(n)} = {}^aZ_i^{(n)} - \frac{e}{2}\left({}^aZ_{i-1}'^{(n)} + {}^aZ_{i+1}'^{(n)}\right) \quad , \qquad \text{if } i > 1$$

$$Q_1 = \left(1 - \frac{e^2}{2}\right) {}^aW_1^{(2)} - \frac{e}{4} \; {}^aW_2^{(2)}$$

$$Q_i = {}^aW_1^{(2)} - \frac{e}{2}\left({}^aW_{i-1}^{(2)} + {}^aW_{i+1}^{(2)}\right) \quad , \qquad \text{if } i > 1 \quad .$$

The changes of the longitude of the ascending node Ω and of the inclination i are much smaller than the changes of a, e, ω, and M, since the equations for their changes contain the quantity d as a factor, which then appears in the coefficients ${}^\sigma g_j^i$, given in (35). The short-period perturbations of those elements are thus negligible in our analysis.

5. NUMERICAL RESULTS

We chose the orbit of the satellite 1958 gamma (Explorer 3), since the short-period perturbations of the orbit of this satellite were computed by Izsak (1960) and we could compare both results. However, the density model of the atmosphere used by Izsak was a very simple one, but the satellite had a low perigee and small eccentricity, so that it moved in a region where the differences in the course of the atmospheric density determined by Izsak's or Jacchia's density function were not too great. We shall see that our results, obtained from the first part of the density function (8), are in very good agreement with those obtained by Izsak. Of course, the perturbations corresponding to the second part of the density function are of different shape and value.

The changes of the semimajor axis during one revolution are plotted in Figure 1. The short-period perturbations are superimposed on the secular change. The dotted curves correspond to the changes computed from the first part of the density function (8), the full line being the whole change, both parts of the density function included. We see that the results do not differ substantially from those of Izsak, either in the shape or in the numerical values. The secular change of the semimajor axis is -1.58×10^5 cm. The part of the density function corresponding to the effect of the bulge contributes to this change with a value -4.89×10^3 cm. The short-period perturbations of the semimajor axis, corresponding to the effect of the bulge, are plotted separately in Figure 2. The sums of cos (W_i) and sin (Z_i) are shown, and the scale is enlarged.

The contribution of the second (bulge) part of the density function to the short-period perturbations of the argument of perigee is of the

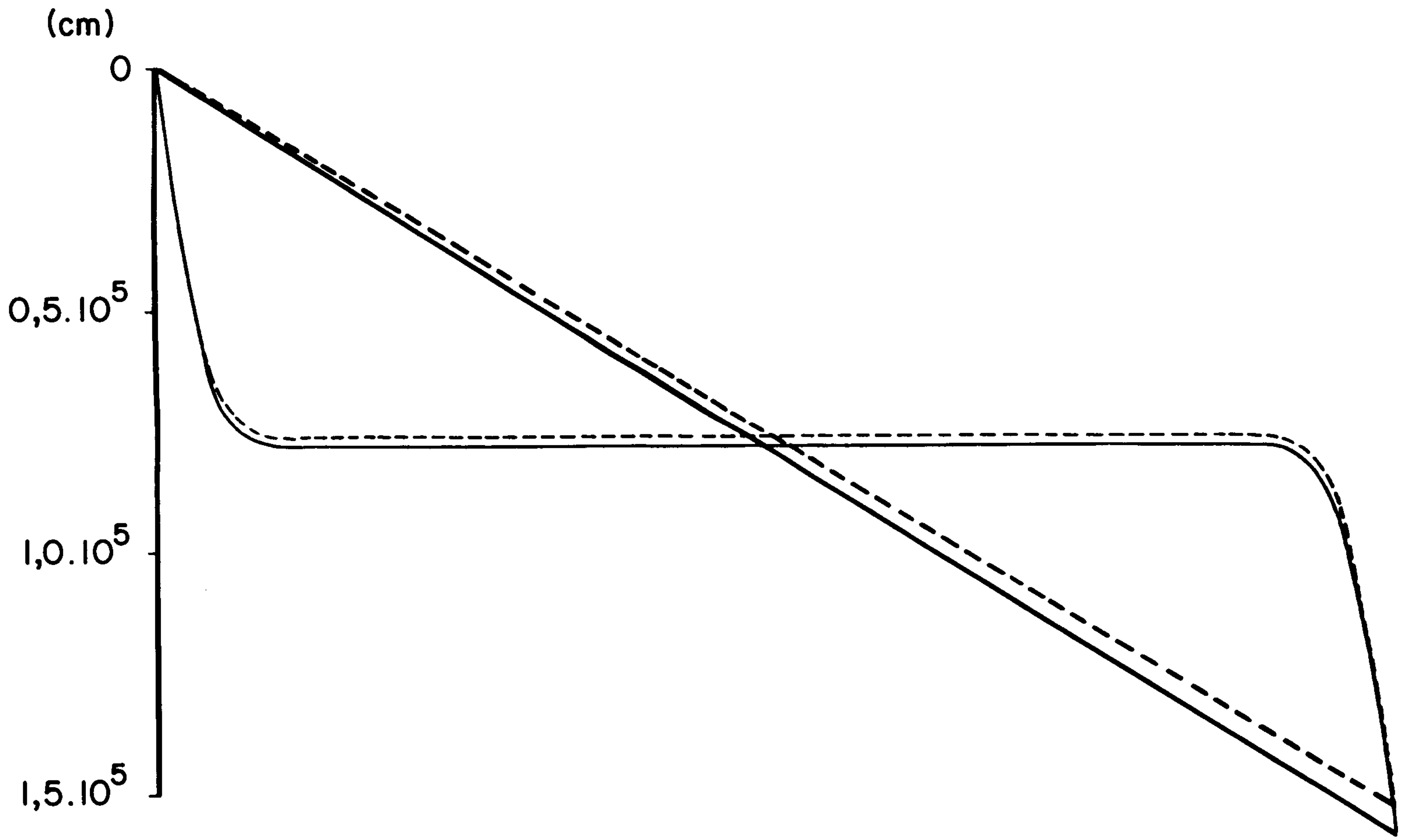

Figure 1. The short-period and secular perturbations of the semimajor axis during one revolution. The dashed line denotes the perturbations without considering the effect of the atmospheric bulge (the first term from equation (8) only).

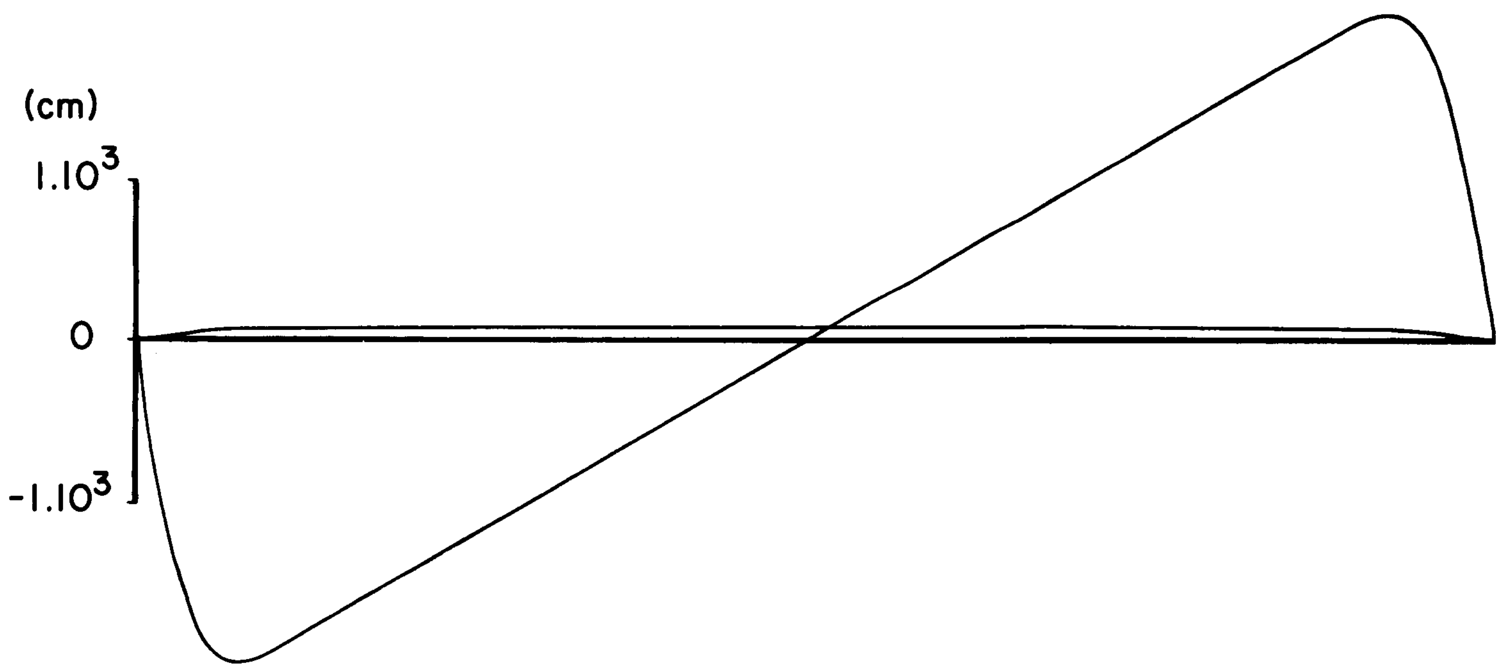

Figure 2. The short-period perturbations of the semimajor axis, caused by the effect of the bulge only (in an enlarged scale).

same order as in the case of the semimajor axis. It is shown in Figure 3, in which the dotted line corresponds again to the perturbation of the element ω without considering the atmospheric bulge. The bulge also causes a secular change in the argument of perigee, which has, in this case, the value of $+2^{\circ}.83 \times 10^{-5}$ during one revolution.

The numerical results were computed mainly to check the whole theory and computations procedure. This analysis was done by a method uncommon in classical celestial mechanics. The implementation of this calculation taxes current computer technology.

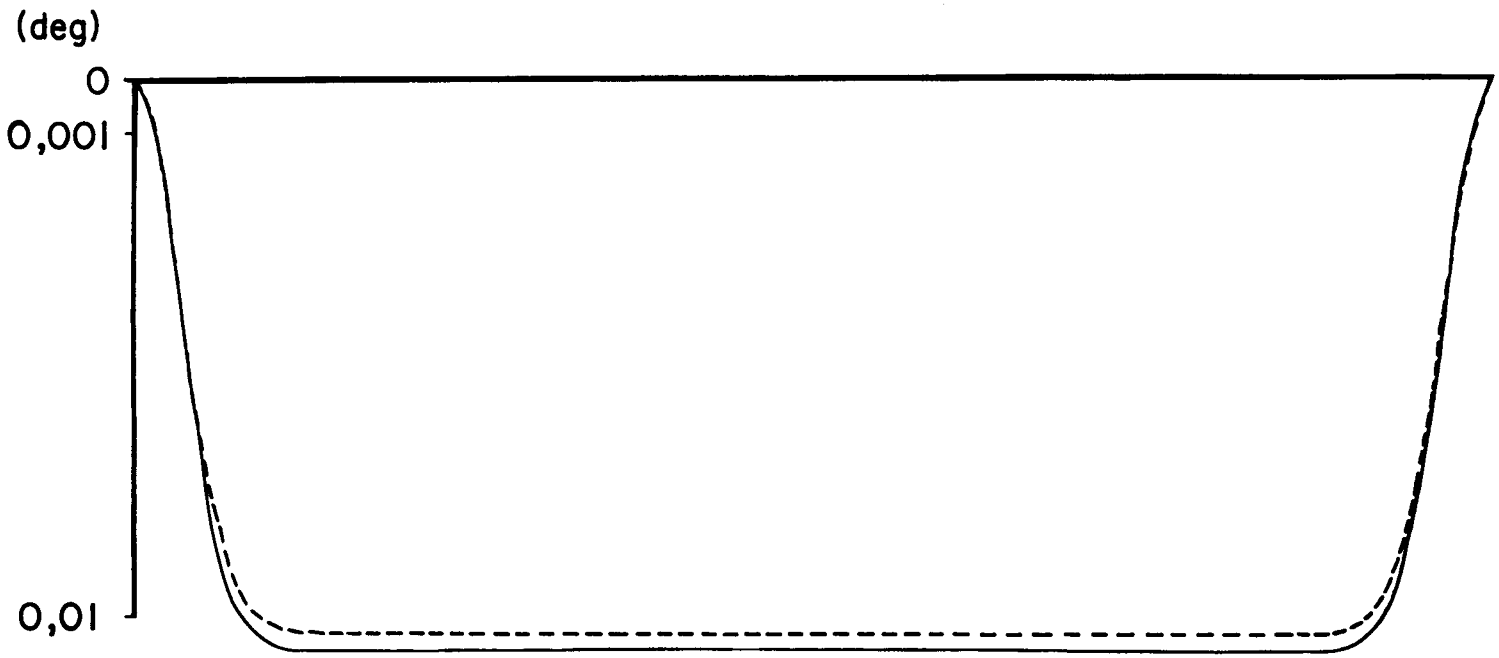

Figure 3. The short-period perturbations of the argument of perigee. The dashed line denotes the perturbations without considering the bulge.

6. ACKNOWLEDGMENTS

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NOTICE

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