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ATTITUDE STABILITY OF A SPINNING PASSIVE ORBITING SATELLITE

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ABSTRACT

The attitude stability of motion of satellites in the neighborhood of equilibrium positions has been studied extensively, however, it is found that these studies have been limited principally to autonomous systems, i.e., systems for which the equations of motion do not exhibit explicit time dependence. The fact remains that a number of important satellite problems are defined by equations of motion containing time-dependent coefficients. Two such examples are: (a) the case in which an unsymmetric satellite moves in a circular orbit and (b) the case in which a symmetric satellite moves in an elliptical orbit. The resulting nonautonomous systems are of a special type in the sense that some of the coefficients appearing in the equations of motion are periodic functions of time. Furthermore, the periodic terms have relatively small amplitudes when compared to the terms with constant coefficients.

There are no general methods available for investigating the stability of motion of multi-degree-of-freedom systems with periodic coefficients such as the type encountered in problems dealing with rotating bodies. The only method used with any degree of success consists of a numerical integration of the equations of motion in conjunction with Floquet's theory. This method is not very satisfactory because, as with any numerical integration, it investigates the stability of the system only at discrete points in the parameter space. This research develops techniques enabling one to investigate stability in

entire regions of the parameter space. Three separate methods of analysis are employed, which together yield the required stability information. The first technique consists of an adaptation of the Liapounov second or "direct" method of analysis into a form that is suitable for use with the periodic systems. The difference between the Hamiltonian function and the Hamiltonian function evaluated at an equilibrium position is shown to be a suitable testing function for use with the stability theorem. This method is used to describe one type of instability boundary and to show the approximate locations of the resonant instability regions. A second method of analysis is developed which is suitable for describing the regions of parametric resonance of multi-degree-of-freedom, linear, periodic systems of the type encountered in rigid body dynamics problems. Because periodic motion can take place on the boundaries of the resonance regions an infinite determinant is written, the value of which must be zero on the boundaries of the instability regions. This determinant is then used to define the boundaries of the regions of instability. A third method of analysis is employed in which asymptotic methods (in the sense of Krilov-Bogoliubov-Mitropolsky) are applied to the multi-degree-of-freedom periodic systems. This technique allows additional regions of instability to be defined and, in contrast with the infinite determinant approach, it is applicable to both linear and nonlinear systems.

The problem of the stability of motion of a rigid spinning satellite which has unequal moments of intertia about axes perpendicular to the spin axis and whose mass center moves in a circular orbit is studied using the techniques described above. The boundaries of the

regions of instability are defined for the motion of the linearized system in the neighborhood of the equilibrium position in which the spin axis is normal to the orbit plane. Unstable regions identified as principal instability regions are in agreement with previous researchers. On the other hand, unstable motion is shown to exist in regions that were presumed to be stable by previous researchers.

The stability of motion of a spinning, rigid symmetric satellite in an elliptic orbit has also been investigated. Consideration is given to the linearized as well as the nonlinear system and the equilibrium position in which the spin axis is normal to the orbit plane is investigated. The analysis of the linearized system can be directly compared to a previous investigation and reasonable agreement is obtained. Even for the linearized system, the present analysis is much more meaningful since it furnishes continuous stability diagrams rather than diagrams consisting of isolated points. In addition to the instability regions predicted by the linearized system, new instability regions associated with the nonlinear terms are obtained; the occurrence of nonlinear "stiffening" of the system tending to limit the amplitude of resonance oscillation is noted. No previous analysis of the complete nonlinear system of equations is known to have been performed.

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SECTION I

INTRODUCTION

1.1   Statement of the Problem

The motion of artificial earth satellites can be described in many cases by the translational motion of the center of mass and the rotational motion of a satellite about its center of mass. The latter motion is referred to as the attitude motion and forms the object of our interest.

Investigations of the attitude motions of space satellites are of current interest since random orientation or uncontrolled tumbling is undesirable in many cases as it may interfere with the mission requirements for communication or observation. The satellite attitude may be controlled by active or passive means. Active systems require attitude sensors and control devices that impose weight and reliability penalties on the satellite. On the other hand, passive attitude control may be applied in some cases and is of interest because it may be achieved with little or no penalty to the satellite. The present study deals with the passive attitude control of a spinning rigid satellite.

The attitude motion of a satellite can be defined with respect to an inertial system of coordinates, however, it is more convenient to define it with respect to an orbiting system of coordinates. In the case of a circular orbit, the orbiting system of coordinates consists of

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two axes in the orbit plane, one tangent and one normal to the orbit, and a third axis perpendicular to the orbit plane. For an elliptic orbit, the two axes in the orbit plane are along the radial and transverse directions. The configuration in which the satellite attitude motion is unaccelerated is called an equilibrium configuration. Of interest is the attitude motion in the neighborhood of an equilibrium configuration and in particular the stability of this motion.

A system described by differential equations with constant coefficients is called autonomous. Spinning satellite problems defined by autonomous systems of differential equations have been investigated extensively. For the most part, the study has been restricted to symmetric satellites with arbitrary spin or to asymmetric satellites with zero spin relative to an orbiting frame of reference whereas the center of the satellite was confined to a circular orbit. The major force acting upon the satellites was due to a radially symmetric inverse square gravitational field. Adequate mathematical techniques for the treatment of autonomous systems are available.

Systems defined by differential equations with time-dependent coefficients are said to be nonautonomous. The investigation of nonautonomous satellite dynamics problems has been very limited due to lack of adequate mathematical techniques. This research has been devoted to the development of mathematical techniques for the stability investigation of the motion of satellites described by nonautonomous systems

of equations, and in particular systems with periodic coefficients.
The methods of analysis have been applied to the problem of a slightly
asymmetric satellite with arbitrary spin confined to a circular orbit
and the problem of a spinning symmetric satellite moving in an elliptic
orbit of low eccentricity.

1.2   Status of Satellite Stability Research

The two major methods of passive stabilization of satellites
are gravity-gradient stabilization and spin stabilization. For a
body in a circular orbit with zero spin relative to an orbiting frame
of reference, the differential-gravity torque tends to align the axis
of minimum moment of inertia with the radial direction from the center
of force to the center of mass of the satellite. A spin-stabilized
body tends to maintain the direction of the spin axis fixed in an
inertial space if no disturbing torques are present. In this case,
differential-gravity torques can be used to impart the spin axis a
steady precession.

Previous work [1] has shown that for rigid bodies the attitude
motion has negligible effect upon the orbital motion of the body center
of mass, provided the satellite dimensions are small relative to the
distance to the center of force. This allows one to reduce the degree
of freedom of the dynamical system by assuming that the orbital motion
is known. This assumption, referred to as orbital constraints, has
been widely used in the study of the attitude motion of spinning satellites.

There are various factors affecting the attitude motion of a satellite. At moderately high altitudes the dominant torque is the differential-gravity torque [2]. At low altitudes the aerodynamic torques may become predominant. Analyses by Beletskii [3] showed that aerodynamic and gravity torques can disturb the motion of a spinning satellite to the extent that an equilibrium may not exist. Using an energy approach Thomson and Reiter [4] and Meirovitch [5] have shown that, for certain satellite configuration, internal energy dissipation due to hysteretic damping can destabilize the spinning motion of a satellite.

Stable equilibrium configurations in the presence of gravity torques have been found for a number of cases involving rigid satellites in a circular orbit. DeBra and Delp [6] investigated the stability of a satellite of unequal moments of inertia possessing zero spin relative to an orbiting frame of reference whereas its mass center was moving in a circular orbit. The attitude stability of a symmetric satellite with arbitrary spin confined to a circular orbit was studied by Thomson [7]. Both works, [6] and [7], used an infinitesimal analysis. The stability of a gravity-gradient stabilized satellite was investigated by Beletskii [8] by means of the Liapounov direct method. Subsequently the stability of motion of a rigid symmetric satellite was analyzed by means of the same method by Pringle [9] and Likins [10].

The stability of a rigid satellite with elastically connected moving parts and possessing zero spin relative to an orbiting frame of reference was studied by Meirovitch [1] and Nelson and Meirovitch [11].

The mathematical model used in Reference [1] as well as References [6] to [11] led to a mathematical formulation consisting of sets of autonomous differential equations.

A number of factors that may have significant influence on the satellite stability renders the system nonautonomous by introducing periodic coefficients in the differential equations of motion. Included in this group are the following:

    i. Rigid, unsymmetric satellite with arbitrary spin moving in a circular orbit.

    ii. Rigid, symmetric satellite with arbitrary spin moving in an elliptic orbit.

    iii. Rigid, symmetric satellite with elastically connected moving parts having arbitrary spin and confined to a circular orbit.

    iv. Rigid satellite in a circular orbit subjected to periodic torques due to solar pressure.

    v. Rigid satellite in a circular orbit subjected to periodic torques due to the nonuniformity of the earth atmosphere.

The first problem was studied by Kane and Shippy [12] and the second one by Kane and Barba [13] using an analysis based on Floquet's

theory. This analysis involves numerical integration of the linearized equations of motion for specific values of the parameters of the problem. Stability is checked only at discrete points in the parameter space and does not furnish continuous stability diagrams.

1.3   Status of Mathematical Methods for Stability Analysis

The problems of attitude motion of satellites lead to systems of coupled, nonlinear differential equations. For the most part, the general solution of these equations is not possible and we shall be interested in the case in which a stability statement suffices. Therefore, the analysis consists of determining equilibrium positions and examining the stability of motion of a system about these equilibrium positions.

The definitions that will be used for equilibrium position and for stability at an equilibrium position will be as follows. Given a system whose essential features are described by n generalized coordinates and n generalized velocities, x_i, $(i=1,2,\ldots,2n)$, an equilibrium position is said to exist at $x_i = c_i$, $(i=1,2,\ldots,2n)$, where c_i are constants if these values satisfy the differential equations of motion. By a suitable coordinate transformation, any equilibrium position of a mechanical system can be translated to the origin, $x_1 = x_2 = \ldots = x_{2n} = 0$, and this will be assumed to be the case in further discussion. An equilibrium position will be defined as stable in the sense of Liapounov [14] if there exist positive numbers ϵ and η and time t_0 such that

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$$\sum_{i=1}^{2n} x_i^2 \le \epsilon \tag{1.1}$$

for all motion subsequent to an initial perturbation from the equilibrium position, where the initial perturbation satisfies

$$\sum_{i=1}^{2n} x_{i0}^2 \le \eta \quad \text{at} \quad t = t_0 \tag{1.2}$$

The majority of solved satellite dynamics problems deal with autonomous systems. Solutions were obtained by both infinitesimal and Liapounov analyses. Linearized analyses were used in the early studies of autonomous systems; fortunately, however, the Liapounov direct method has more recently been adopted for the stability investigation of autonomous systems. The Liapounov direct method allows one to make conclusive statements as to the system stability of motion in many critical cases. One form of the Liapounov stability theorem states that: If there exists a differentiable function $V_L(x_1, x_2, \ldots, x_{2n})$, known as a Liapounov function, that satisfies the conditions

a. $V_L(x_1, x_2, \ldots, x_{2n}) \ge 0$ \hfill (1.3)

in a definable region surrounding the origin where the equal sign applies only at the origin (i.e., V_L is positive definite in the neighborhood of the origin with a relative minimum at the origin).

b. $\dfrac{dV_L}{dt} = \displaystyle\sum_{i=1}^{2n} \dfrac{\partial V_L}{\partial x_i} \dot{x}_i \le 0$ (1.4)

where dV_L/dt is taken along an integral curve then the
equilibrium point $x_i = 0$, $(i=1,2,\ldots,2n)$ is stable. The
equilibrium is said to be asymptotically stable if dV_L/dt
is negative definite* (i.e., dV_L/dt is negative in the neighbor-
hood of the origin and equal to zero at the origin).

The Hamiltonian function has been used extensively as a Liapounov
function for autonomous systems [1], [9], [10], [11].

Periodic systems are a special class of nonautonomous systems in
which the time dependence appears in the form of periodic coefficients.
The differential equations for the problems associated with an n-degree-
of-freedom periodic system may be written in the following form

$$\dot{y}_i = \sum_{j=1}^{2n} b_{ij}(t)y_j + Y_i(y_1,y_2,\ldots,y_{2n},t), \quad (i=1,2,\ldots,2n)$$
(1.5)

where the coefficients $b_{ij}(t)$ are periodic with period T and the
functions Y_i consist of second and higher order terms in the y_i's
multiplied by periodic coefficients with period T .

As with the autonomous systems, the equations are identically
satisfied at the origin so that the origin constitutes an equilibrium

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* This condition is unnecessarily stringent and the above definition is
  valid if dV_L/dt is only negative semi-definite (i.e., it can be zero
  at points other than the origin) provided the motion is coupled [15][16].

position in the neighborhood of which the nonlinear terms can be regarded as small. Hence, it is natural to investigate the linearized equations

$$\{\dot{y}\} = [b(t)]\{y\} \tag{1.6}$$

It can be shown that every linear system with periodic coefficients is reducible by means of a nonsingular transformation with periodic coefficients to a linear system with constant coefficients [17] and that the stability characteristics of the system of equations are unchanged by such transformations. Consequently, it can be shown [18] that a characteristic equation exists for the periodic system, and that the linearized equations yield conclusive information about the character of the equilibrium only when no characteristic number* has a real part that is equal to 0. When one or more of the characteristic numbers has a zero real part we have a critical case in which the nonlinear terms must be considered in order to make conclusive stability statements. Unfortunately, this appears to be the case that must be studied in satellite attitude stability problems.

Theorems have been advanced using the Liapounov second or direct method that are applicable to nonautonomous systems and that

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* Note that the characteristic number discussed here can also be expressed in terms of the natural logarithm of a "characteristic multiplier" as discussed in the next chapter.

would yield conclusive stability information if they could be applied to the periodic systems. One such form of the Liapounov theorem for nonautonomous systems is the following [19]: Given a system characterized by the differential equations

$$\dot{x}_i = f_i(x_j, t) , \qquad (i,j=1,2,\ldots,2n) \tag{1.7}$$

which has an equilibrium position at the origin. The equilibrium is stable if there exists a positive definite function $V(x,t)$ such that its total derivative $\dot{V}$ for the differential system, Eq's. (1.7) is not positive.

To date, no application of the Liapounov method of analysis to periodic satellite dynamics problem is believed to have been successful. Some success has been achieved in the application of a method suggested by Cesari [20] to the linearized equations, using numerical integration of the equations of motion in conjunction with Floquet's theorem for specific values of the parameters of the problems. The limitations of this approach by Kane, et al., [12][13] have been discussed previously.

1.4   Review of the Present Investigation

Three separate methods of analysis were developed in the course of this investigation. Together they yield the desired stability information about periodic systems of the type being studied. The first method consists of an adaptation of the Liapounov direct method into a

form that is suitable for use with periodic systems. A stability theorem is presented, the proof of which is included in Appendix A. The difference between the Hamiltonian function and the Hamiltonian function evaluated at an equilibrium position is shown to be a suitable testing function for use with the stability theorem. The stability theorem may be used for defining stability in the sense of Liapounov. However, an alternate use of this theorem is developed with the aid of Floquet's theorem, by means of which approximate locations of resonance instability regions may be found when the periodic influence on the system is small. In addition, this theorem allows certain stability boundaries to be located that are not readily defined by other methods (Section II).

The other two methods of analysis are developed for the purpose of describing the regions of resonance instability. One method is based upon approximate evaluation of an infinite determinant, while the second is based upon an asymptotic expansion of the equations of motion in terms of a small parameter.

Although the analysis methods were developed principally for application to multi-degree-of-freedom systems of the type encountered in space dynamics problems, they should also be applicable to other types of systems. To illustrate the analysis methods, they have been applied to the Mathieu equation which is a single-degree-of-freedom equation with a periodic coefficient; this analysis is included in Appendix B.

Sections III and IV present the analysis of the stability of motion of spinning rigid satellites under the influence of periodic disturbances.

In Section III the effect of asymmetry of the spinning body about the spin axis is studied for a circular orbit. The equations of motion and energy relations are derived for the system which is linearized about the equilibrium position. At the equilibrium position the spin axis of the satellite is perpendicular to the orbit plane. When the nonlinear terms are neglected, the spin motion becomes uncoupled from other satellite attitude motions and the spin rate of the asymmetric body is found to be an explicit time dependent, periodic motion. This time dependent motion appears in the linearized equations of motion for the spin axis as periodic coefficients. Consequently, a stability analysis using the methods of Section II is performed. The limits of stability are defined and compared with the previous linearized analysis of this case [12].

Section IV presents an analysis of the stability of motion of a rigid, symmetrical, spinning body in an elliptical orbit. Periodic coefficients appear in the equations of motion by virtue of the periodic orbital motion of the center of mass. An equilibrium position is found in which the spin axis is perpendicular to the orbit plane. The stability of this nonlinear system is investigated in the neighborhood of this equilibrium position, using the Liapounov - and asymptotic - types of analyses of

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Section II. Consideration is given to the linearized system as well

as the nonlinear system. The analysis of the linearized system can

be directly compared with a previous investigation, Reference [13],

which checked the stability of the linearized system at isolated points.

The comparison shows reasonable agreement. In addition to the instabil-

ity regions predicted by the linearized system, new instability regions

associated with the nonlinear terms are obtained; the occurrence of

nonlinear "stiffening" of the system tending to limit the amplitude

of resonance oscillation is noted (see p. 171). Thus, the present inves-

tigation not only includes continuous stability diagrams which could not

be obtained by previous methods but also extends the analysis into the

nonlinear regime where no previous stability investigations had been

performed.

SECTION II

METHODS OF STABILITY ANALYSIS OF PERIODIC SYSTEMS

2.1   Liapounov-Type Stability Analysis

The conclusive stability statement that can be made by means of
the Liapounov type of analysis on autonomous systems would be of great
value if it could be made in connection with multi-degree-of-freedom
conservative* systems with periodic coefficients. Such an approach
may also be useful in locating regions of instability in linearized
systems with periodic coefficients. The periodic terms may enter the
equations by virtue of a periodically varying potential function or by
means of assumed periodic behavior of one, or more, coordinate that is
not subject to the stability investigation but appears in the form of
known time-dependent coefficients.

We will assume an arbitrary n-degree-of-freedom system and will
utilize the definitions of equilibrium position and stability as given
in Section I. It should be noted that the equilibrium position may be
defined in terms of a restricted number of coordinates (for example the
attitude stability problem of a spinning satellite may be defined in

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*Conservative is used here in the sense that the external forces are
derivable from a potential function that is independent of the velocities
even though the potential function and the resulting forces may be
time-dependent.

terms of the position of the spin axis as in the case of the constrained system of Section III) and other coordinates such as the orbit parameters or spin angle may appear as time-dependent coefficients.

The Hamiltonian function has been widely used as a testing function in conjunction with the Liapounov stability analyses of autonomous mechanical systems. However, it has not been used in the case of systems with periodic coefficients. To discover why this had been the case, let us consider an unconstrained conservative system with nonrotating coordinates. The usual forms of the Liapounov theorem state [19] that the motion in the neighborhood of an equilibrium position will be stable if a function of the coordinates and time can be found which is positive definite in the neighborhood of the equilibrium and has a negative or zero time derivative. The total energy is a suitable Liapounov function in the case of an unconstrained conservative system with nonrotating coordinates problem, so the Liapounov theorem can be seen to require that the total energy have a relative minimum at the equilibrium position and that energy is either dissipated or unchanged during any small motions near the equilibrium. This latter requirement appears to be too stringent in the case of systems with periodic coefficients, since intuition tells us that a stable equilibrium could exist such that energy could flow in and out of the system during small motions near the equilibrium, as long as the net energy addition after a period of time is not a cumulative effect. Consequently, the following stability theorem is suggested.

2.1.1 Stability Theorem

Given a system described by the differential equations

$$\dot{x}_s = X_s(x_1, x_2, \ldots, x_{2n}, t), \qquad (s = 1, 2, \ldots, 2n) \qquad (2.1)$$

for which an equilibrium position, E, exists at $x_1 = x_2 = \ldots = x_{2n} = 0$, then the perturbed motion about this equilibrium position is said to be stable if a continuous function V can be found such that

a. $V(x_1, x_2, \ldots, n_{2n}, t)$ is positive definite in the neighborhood of E , zero at E , and

b. $$\int_{t_0}^{t} \frac{dV}{dt} \, dt \leq M(x_{10}^2 + x_{20}^2 + \ldots + x_{2n,0}^2) \qquad (2.2)$$

for motion subsequent to $t = t_0$, in which M is a finite, positive constant and $x_{10}, x_{20}, \ldots, x_{2n,0}$ are initial small displacements at $t = t_0$.

Proof of the preceding theorem is given in Appendix A. Similar theorems giving the conditions for instability or asymptotic stability may be developed in the same manner. It should be noted that the preceding stability theorem gives conditions that are sufficient for concluding that a given motion is stable but does not give the necessary conditions. Consequently, stable motions may exist which would not meet the requirements of this theorem.

To make use of the stability theorem it is necessary to select a testing function which can have a relative minimum at an equilibrium position and for which a meaningful value of the integral, Eq. (2.2), can be obtained. The Hamiltonian function will be investigated for this purpose based on physical reasoning. This physical reasoning will be applied to a nonrotating coordinate system before applying it to rotating systems.

2.1.2 <u>Testing</u> <u>Function.</u> <u>Nonrotating</u> <u>Coordinates.</u>

First let us introduce the notation:

KE = kinetic energy

PE = potential energy

L = KE - PE = Lagrangian function

q_i = generalized coordinate

$\dot{q}_i$ = generalized velocity

p_i = generalized momentum

Q_i = generalized conservative force

By definition, the generalized momentum and generalized conservative are related to the Lagrangian by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad , \quad Q_i = \frac{\partial L}{\partial q_i} \quad , (i=1,2,\ldots,n) \tag{2.3}$$

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Furthermore, we have the Lagrange's equations of motion for a conservative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad , \quad \dot{p}_i = Q_i \qquad ,(i=1,2;\ldots,n) \qquad (2.4)$$

which states that the time rate of change of the generalized momentum is equal to the generalized conservative force. The definition of the Hamiltonian is

$$H(q,p,t) = \sum_i p_i \dot{q}_i - L(q,\dot{q},t) \qquad (2.5)$$

where the first of Eq's. (2.3) have been used to eliminate $\dot{q}_i$ from H . Taking the time derivative of both sides of Eq. (2.5) one obtains

$$\frac{dH}{dt} = \sum_i \left\{ \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial t} \right\} = \sum_i \left\{ \dot{q}_i\frac{dp_i}{dt} - \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} - \frac{\partial L}{\partial t} \right\} \qquad (2.6)$$

which gives the following canonical equations

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -Q_i \quad , \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \qquad (2.7)$$

and also

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (2.8)$$

One can also note that all but the last term on the right side of Eq. (2.6) add to zero by use of Lagrange's equations of motion so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$ (2.9)

When a nonrotating coordinate system is used to describe a mechanical system subjected to conservative external forces, the above general relations take on special forms such that the kinetic energy is quadratic in the velocities, the potential energy is a function of only the spatial coordinates, the total energy (KE + PE) is equal to the Hamiltonian, and the generalized force and momentum are equal to the linear force and momentum.

In the absence of explicit time dependence in the Hamiltonian, the total energy is constant. With time dependence the total energy changes in accordance with Eq. (2.9).

Now we can consider the mechanisms through which an instability exists in the neighborhood of an equilibrium position E (the origin). If we consider motion near E in which the energy level is larger by an amount ΔH than the energy level at E , we see that an instability could exist due to exchange of energy between KE and PE in one or more coordinates. But for a mechanical system, KE is positive definite (i.e., KE increases as the p_i depart from zero). So, if the motion is not to diverge from the neighborhood of the equilibrium it is only

necessary that the potential energy increases as q_i increase. More specifically, along any path diverging from E , such as the path

$$\bar{r} = a_1 q_1 \bar{e}_1 + a_2 q_2 \bar{e}_2 + \ldots + a_n q_n \bar{e}_n \qquad (2.10)$$

where $a_1, a_2, \ldots, a_n$ are arbitrary positive constants, we require that

$$\nabla PE \cdot \bar{r} = \nabla H \cdot \bar{r} = -\bar{Q} \cdot \bar{r} > 0 \qquad (2.11)$$

This essentially states that the generalized force Q_i must act towards the equilibrium. This is equivalent to the requirement that PE (and H) have a relative minimum in the neighborhood of the origin. A key element in the above discussion is that if H is time dependent, it is reasonable to require that H be positive definite <u>for all time</u> and, thus, would fulfill the first stability requirement of the proposed theorem.

Another possible mechanism for instability exists, even if H is positive definite in the neighborhood of E . It is possible that the energy of the system will build up over a period of time such that the integral of the time derivative of the difference between the energy of the motion and the energy at the equilibrium position increases without bound

$$I = \int_{t_0}^{t} \frac{d}{dt} (H - H_E) dt \to \infty \quad \text{as} \quad t \to \infty \qquad (2.12)$$

where H_E is the Hamiltonian evaluated at the equilibrium position and it is a function which depends on t only. If this integral increases without bound, then unbounded values of one or more of the q_i or $\dot{q}_i$ would be expected. Conversely, if this integral can be shown to be bounded in accordance with the second requirement of the stability theorem, the motion will be bounded and of arbitrarily small magnitude, depending upon the initial disturbance that is assumed.

No criterion is known to exist for the selection of an optimum testing function for the purpose of a Liapounov type of stability analysis. However, the above arguments give a physical interpretation of the requirements on the Hamiltonian for a stable equilibrium to exist in a nonautonomous system and shows the relationship between these requirements and those of the proposed stability theorem. Consequently, $H - H_E$ appears as a likely choice for use as a testing function in conjunction with the proposed theorem.

2.1.3 Testing Function. Rotating Coordinates.

In this case the Hamiltonian can be shown to be

$$H = \sum_i \dot{q}_i p_i - L = KE^* + U \qquad (2.13)$$

where KE^* is the portion of the kinetic energy expression that is quadratic in the velocities and U is the dynamic potential given by

$$U = PE - \gamma \qquad (2.14)$$

in which γ is the portion of the kinetic energy expression that does not depend on the velocity. The discussion of the preceding section is equally valid for the rotating coordinate system, except that we must use KE* and U instead of KE and PE , respectively. It should be noted that the Hamiltonian is no longer equal to the total energy of the system and the generalized force and momentum are not, in general, equal to the linear force and momentum. Once again the function $V = H - H_E$ appears to be a reasonable testing function for use with the proposed stability theorem.

2.1.4 Application of the Stability Theorem.

In many cases, the parameters of the problem can be specified so as to satisfy the first condition of the stability theorem, namely that the testing function V be positive definite in the neighborhood of the equilibrium position, at all times. This may be done in a direct way by proving that the Hessian matrix associated with V is positive definite for all times [1], or alternately, by means of comparison testing function [21]. The latter consists of assuming that it is possible to find a positive definite function $W(x_1,x_2,...,x_{2n})$ which does not depend explicitly on t and such that

$$V(x_1,x_2,...,x_{2n},t) \geq W(x_1,x_2,...,x_{2n}) \qquad (2.15)$$

When the time dependence of V is periodic one may regard the function $V = c$ as representing a pulsating 2n-dimensional surface. When V is positive definite in the neighborhood of E , the function $V = c$ becomes a 2n-dimensional closed surface which changes in shape and size as a function of time.

In order to check the second stability requirement we must obtain information about the integral

$$I = \int_{t_0}^{t} \frac{dV}{dt}\, dt = \int_{t_0}^{t} \frac{\partial V}{\partial t}\, dt \tag{2.16}$$

In general, V will have the form of a series of terms consisting of a periodic function multiplied by second, or higher, power functions of the generalized coordinates q_i and generalized velocities $\dot{q}_i$. Consequently, we must at least know the form of the solution, q_i and $\dot{q}_i$, in order to determine the behavior of the integral, Eq. (2.16).

For some types of problems it is possible to state that the solution is of the form

$$q_i = \sum_j f_j^{(i)}(t)\, e^{i\gamma_j t} \tag{2.17}$$

where $f_j^{(i)}(t)$ are nonperiodic functions of time (or constants) and γ_j are real numbers.* Equation (2.16) can then be re-expressed in the form of terms such as

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*The i appearing in the exponential is $\sqrt{-1}$, not an index.

$$\int_{t_0}^{t} g_\ell(t)\ e^{i\sigma_\ell t} \cos \omega t\ dt \qquad\qquad (2.18)$$

where ω is the frequency of one of the periodic terms in the Hamiltonian. In the case of undamped systems in which the periodic terms in the Hamiltonian are small (which is certainly the case in many satellite dynamics problems) the $g_\ell(t)$ will be constant or slowly varying functions of time and the integral (2.18) will behave like the product of trigonometric functions and will diverge only if one of the σ_ℓ becomes equal to one of the frequencies in the periodic forcing function. This will be investigated further in the case of each example. We will call this a resonance oscillation.

Estimation of the frequencies of oscillation of the system will depend on the particular system under consideration. In some cases it could be accomplished by means of Floquet's theorem or an asymptotic expansion in terms of a small parameter.

The conclusions which are appropriate for the types of perturbed motion that are under study can be stated as a corollary to the stability theorem:

Corollary - An undamped system subjected to conservative forces and characterized by differential equations and a Hamiltonian function with periodic coefficients will admit stable motion in the neighborhood of an equilibrium position if

  a. the Hamiltonian function has a relative minimum at the

    equilibrium position and

b. no resonance occurs between the motion in the neighborhood

of the equilibrium and the periodic coefficients in the

Hamiltonian.

In some cases, such as the examples treated here, it may be desirable
to apply the stability theorem to a linearized system, in which case
the statement concerning the stability of motion should be regarded
as pertinent to the small motion only. Hence, in this case, stability
will occur in the indicated regions of the parameter space only if the
linear terms dominate the motion.

2.2 <u>Analysis Based on an Infinite Determinant</u>

A method similar to that employed by Bolotin [22] has been
adapted for use in defining the regions of resonance of a linear system.
The present study represents a substantial advance in the use of this
type of analysis, including its application to a system of second order
equations which contain the velocity terms that are typical of gyroscopic
motions. This method has been applied to the problem of the spinning,
unsymmetrical body.

2.2.1 <u>Discussion of the Motion of Linear, Periodic Systems.</u>

According to Floquet's theorem, a system of n second order
linear differential equations with periodic coefficients possess 2n
linearly independent solutions* of the form

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*Discrete characteristic multipliers are assumed.

$$\left\{x^{(j)}\right\} = \left\{f^{(j)}\right\} e^{(t/T)\ell n \; \rho_j} , \; (j=1,2,\ldots,2n) \tag{2.19}$$

where the $f_k^{(j)}$ are periodic functions with period T and the ρ_j are called the characteristic multipliers. Note that

$$\ell n \; \rho = \ell n \left| \rho \right| + i \; \arg \rho \tag{2.20}$$

The motion about the identically zero solution will be stable if no characteristic multiplier has an absolute value that is greater than one. The motion will be asymptotically stable if all characteristic multipliers have absolute values that are less than one and will be unstable if any characteristic multiplier has an absolute value that is greater than one.

The system under consideration is linear with periodic coefficients and has a Hamiltonian function from which the equations of motion may be written in canonical form, i.e.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \; , \; \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad ,(i=1,2,\ldots,n) \tag{2.21}$$

A theorem due to Liapounov [21] states that for such a case the characteristic multipliers occur in reciprocal pairs. Consequently, if ρ_j is a characteristic multiplier, then $1/\rho_j$ is also a characteristic multiplier. Also the characteristic multipliers for the system under consideration occur in complex conjugate pairs.*

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*This has yet to be shown for the system under study.

A pair of particular solutions corresponding to reciprocal roots may be written

$$\left\{x^{(j)}\right\} = \left\{f^{(j)}\right\} e^{(t/T)\ell n\, \rho_j}$$

$$\left\{x^{(k)}\right\} = \left\{f^{(k)}\right\} e^{-(t/T)\ell n\, \rho_j}$$

(2.22)

The region of parameter space in which ρ_j is real and different from ± 1 is clearly the region of unstable motion, since one of the reciprocal characteristic multipliers must be greater than one. Upon further variation of the parameters of the problem, the roots will become complex conjugate pairs

$$\rho_j = a + ib \; , \quad \rho_k = \rho_j^{-1} = a - ib$$

(2.23)

and since $\rho_j \rho_k = 1$ they will have an absolute value equal to one. This then indicates that the region of complex ρ_j is the region of bounded motion. Since the characteristic multipliers are continuous functions of the parameters of the problem, the boundaries of the regions of stability will be given by the cases when pairs of roots, $\rho = 1$ or $\rho = -1$, occur. But we can use Floquet's theorem to show that

$$\left\{x^{(j)}(t+T)\right\} = \left\{f^{(j)}(t+T)\right\} e^{[(t+T)/T]\ell n\, \rho_j} = \rho_j \left\{x^{(j)}(t)\right\}$$

$$= \rho_j \left\{f^{(j)}(t)\right\} e^{(t/T)\ell n\, \rho_j}$$

(2.24)

which, on the boundaries of the regions of instability, gives us

$$\rho_j = 1 \quad , \quad x^{(j)}(t+T) = x^{(j)}(t)$$

$$\tag{2.25}$$

$$\rho_j = -1 \quad , \quad x^{(j)}(t+T) = -x^{(j)}(t)$$

The first of Eq's. (2.25) tells us that a motion which is periodic with period T will be admitted on a boundary where $\rho = 1$. The second of Eq's. (2.25) indicates that a motion that is periodic with period $2T$ will be admitted on a boundary where $\rho = -1$. In addition, any distinct instability region (region of real characteristic multipliers) must be bounded by a single value of ρ (i.e., $\rho = 1$ or $\rho = -1$). This is seen to be true because, in order for the values $\rho = 1$ and $\rho = -1$ to occur on different boundaries of a given instability region, it would be necessary that $\rho_j = 0$ and $1/\rho_j = \infty$ at some location within the instability region, since ρ is a continuous function of the parameters of the system. But this is not possible. This leads to the formulation of a stability theorem.

2.2.2 Stability Theorem for Linear Systems.

Theorem

Periodic solutions with period T or $2T$ are admitted on the boundaries between regions of stability and regions of instability in canonical systems which are described by systems

of linear differential equations with periodic coefficients. Solutions of the same period bound each distinct region of instability.

2.2.3 General Application of the Stability Theorem.

Application of the preceding stability theorem consists of investigations to define the locations in parameter space along which solutions with period T or $2T$ can exist. Separate Fourier expansions with period T and $2T$ may be made such that _for period T_ we have

$$\{x\} = \sum_{n=-\infty}^{\infty} \{a_n\} e^{i\, 2n\pi t/T} = \sum_{n=1}^{\infty} \{b_n\} \sin 2n\pi t/T$$

$$+ \sum_{n=0}^{\infty} \{c_n\} \cos 2n\pi t/T \tag{2.26}$$

and _for period $2T$_ we can write

$$\{x\} = \sum_{n=-\infty}^{\infty} \{a_n\} e^{i\, n\pi t/T} = \sum_{n=1}^{\infty} \{b_n\} \sin n\pi t/T$$

$$+ \sum_{n=0}^{\infty} \{c_n\} \cos n\pi t/T \tag{2.27}$$

Either the exponential or trigonometric form of one of the above solutions may be substituted into the differential equations and the resulting coefficients of equal harmonics may be equated, giving an infinite system of linear equations in terms of an infinite number of coefficients. For a nontrivial solution to exist, the determinant of the coefficients must be zero. Evaluation of the infinite determinant has been possible in the case of Hill's equation [23], and this process has been used further by Mettler [24]. In both cases, the equations being studied are of the second order, and do not include velocity terms. The same techniques do not appear applicable to systems with gyroscopic terms.

In some cases, a reasonable approximation may be achieved by taking only the first few terms of the periodic expansion, Eq. (2.26) or Eq. (2.27). This approach will be taken in the following problem solutions.

2.3   Analysis Based on Asymptotic Expansion in Terms of a Small Parameter

A method of analysis which is similar to that originated by Kryloff and Bogoliuboff [25] and further developed by Bogoliuboff and Mitropolski [26] has been employed. In contrast with these authors, however, the method used in this case involves an expansion of the

equations of motion using assumed simultaneous resonance and nonresonance solutions. As a result, one can define unstable regions of motion of a multi-degree-of-freedom system including those regions in which resonance occurs between periodic terms and the sum or difference of natural frequencies. Systems involving gyroscopic terms have not been discussed in References [25] or [26] and no previous application of this type of asymptotic expansion in the treatment of satellite dynamics stability problems is known.

Perturbation solutions in general involve assumption of the form of the solution of a perturbed system of equations as a power series in a small parameter which appears in the equation of motion. Substitution of this assumed solution into the equation of motion then allows recursive solution for the coefficient of each power of the small parameter, so that the equations of motion are satisfied to any desired accuracy (i.e. any desired power of the small parameter). The absence of secular terms which diverge as $t \to \infty$ and the ability to evaluate the coefficients is frequently accepted as evidence that the assumed form of solution is satisfactory.

2.3.1 Methods of Kryloff, Bogoliuboff, and Mitropolski

The methods employed by Kryloff, Bogoliuboff, and Mitropolski (KBM) are most frequently applied to a single-degree-of-freedom system

with equation of motion of the form

$$\ddot{x} + \omega^2 x = \epsilon \; f(x,\dot{x},t) \tag{2.28}$$

in which ϵ is a small parameter and $f(x,\dot{x},t)$ is an arbitrary periodic function of time and may be either linear or nonlinear in x and $\dot{x}$. In the limit, as ϵ approaches zero, the motion is periodic and of the form

$$x = a \cos (\omega t + \delta) \tag{2.29}$$

where a, ω, and δ are constants. We may look upon the left side of Eq. (2.28) as the unperturbed equation and the terms on the right will be regarded as a perturbation. The assumption is then made that, for small ϵ, the amplitude and phase angle are no longer constant but functions of ϵ and time may be expressed as

$$\frac{da}{dt} = \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \ldots$$

$$\tag{2.30}$$

$$\frac{d\delta}{dt} = \epsilon \delta_1 + \epsilon^2 \delta_2 + \epsilon^3 \delta_3 + \ldots$$

and that the solution contains additional terms of frequency different from that of Eq. (2.29) so as to satisfy the differential equation. The

resulting assumed solution is

$$x = a \cos(\omega t + \delta) + u_1 \epsilon + u_2 \epsilon^2 + \ldots \qquad (2.31)$$

The assumed solution is substituted into the equations of motion, the coefficient of every power of ϵ is set equal to zero, and the resulting equations are solved in succession by harmonic balance. By this process the differential equation is satisfied to any desired power of ϵ.

The KBM procedure has also been applied satisfactorily to multi-degree-of-freedom systems of the form

$$\ddot{x}_i + \sum_{j=1}^{n} c_{ij} x_j = f_i(\epsilon, x_1, \dot{x}_1, x_2, \dot{x}_2, \ldots, x_n, \dot{x}_n, t) ,$$
$$(i = 1, 2, \ldots n) \qquad (2.32)$$

where the c_{ij} are constants and the f_i can be expressed as power series in ϵ in which the time dependence appears as periodic coefficients. However, the assumed forms of solution in this case are not satisfactory for the type of problem under study in the present research.

Extensive use has been made of the asymptotic methods in the study of oscillations since the initial work by Kryloff and Bogoliuboff in 1937 [25]. However, no mathematical foundation was presented for these methods until 1955, when Bogoliuboff and Mitropolski [26] showed that the difference between the true solution and the solution obtained by

asymptotic methods could be made arbitrarily small for systems of nonlinear equations under rather general conditions, and that the properties of the exact solution are given by the properties of the asymptotic solution under much less general conditions.

The type of problems currently under study are not among those for which the mathematical foundations have been shown by Bogoliuboff and Mitropolski. Furthermore, the present research does not seek to provide these mathematical justifications. Instead, the present effort has been concentrated on finding forms of asymptotic solutions which are capable of satisfying the differential equations of the specific problems being studied to any given accuracy in terms of a small parameter.

2.3.2 Application to Rigid Body Dynamics

Some problems of rigid body dynamics may be expressed in the form

$$\ddot{\theta}_1 + \beta_1 \dot{\theta}_2 + \beta_2 \theta_1 = \sum_{n=1}^{\infty} \epsilon^n f_{1n}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t)$$

$$\ddot{\theta}_2 + \beta_3 \dot{\theta}_1 + \beta_4 \theta_2 = \sum_{n=1}^{\infty} \epsilon^n f_{2n}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t)$$

$$(2.33)$$

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where in the limit, as ϵ approaches zero, the solution becomes of the form

$$\theta_1 = a \cos (\omega_1 t + \delta_1) + b \cos (\omega_2 t + \delta_2)$$

$$(2.34)$$

$$\theta_2 = a\lambda_1 \sin (\omega_1 t + \delta_1) + b\lambda_2 \sin (\omega_2 t + \delta_2)$$

in which ω_1 and ω_2 are the natural frequencies of the unperturbed system, a and b are arbitrary amplitudes, δ_1 and δ_2 are arbitrary phase angles, and λ_1 and λ_2 are constants that are obtained in the process of solving the unperturbed equations of motion. We may assume in general that the functions appearing on the right side of Eq's. (2.33) can be expressed in terms of products of periodic functions of period T, and powers of the coordinates and velocities, θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$.

Application of the KBM type of asymptotic solution will depend upon the particular problem being studied and will be applied in different manners to the same problem in order to describe different oscillatory phenomena. Specifically, one can distinguish two significantly different types of motion in each case: resonant and non-resonant. Resonant motion will be shown to take place when ω_1 or ω_2 is sufficiently close to a frequency occuring on the right side

of Eq's (2.33).

To investigate nonresonance motion we may inspect solutions of the form

$$\theta_1 = a_1 \cos(\psi_1 t + \delta_1) + a_2 \cos(\psi_2 t + \delta_2) + \epsilon u_{11} + \epsilon^2 u_{12} + \cdots$$

$$(2.35)$$

$$\theta_2 = a_1 \lambda_1 \sin(\psi_1 t + \delta_1) + a_2 \lambda_2 \sin(\psi_2 t + \delta_2) + \epsilon u_{21} + \epsilon^2 u_{22} + \cdots$$

in which λ_1, λ_2, δ_1, and δ_2 may be expressed as power series in ϵ, the amplitudes a_1 and a_2 are functions of time such as is given by the first of Eq's (2.30), and the frequencies are given by

$$\psi_1 = \omega_1 + \epsilon \Delta_{11} + \epsilon^2 \Delta_{12} + \cdots$$

$$(2.36)$$

$$\psi_2 = \omega_2 + \epsilon \Delta_{21} + \epsilon^2 \Delta_{22} + \cdots$$

Substitution into the differential equations, Eq's (2.33) and solution for the coefficients shows that a_1, a_2, δ_1, and δ_2 are arbitrary constants, in the nonresonant case, and are independent of ϵ. The values if λ_1, λ_2, ψ_1, and ψ_2 are also constants, but are expressed in terms of power series in ϵ. Thus it is seen that in the nonresonant case, the perturbations cause a shift in the frequency of oscillation

and in the relative amplitudes of θ_1 and θ_2 oscillations, and introduce additional terms given by u_{11}, u_{12}, u_{21}, $\ldots$ which include constant amplitude oscillations of frequency

$$\ell \, \omega_1 \pm m \, \omega_2 \pm \frac{2n\pi}{T} \qquad\qquad (2.37)$$

where ℓ, m, and n are integers. Consequently the nonresonance oscillations are bounded, multifrequency oscillations and are of arbitrarily small amplitude depending upon the assumed initial amplitudes given by a_1 and a_2. Therefore, the region of nonresonant oscillation is a region of stable motion.

Resonant oscillation can occur when one of the natural frequencies (for example ω_1) is sufficiently close to a frequency given by Eq. (2.36). Defining the resonant frequency as ω_{res}, a solution of the following form is possible

$$\theta_1 = a \cos (\omega_{res} t + \delta_1) + b \cos (\omega_2 t + \delta_2) + u_{11} \epsilon + u_{12} \epsilon^2 + \ldots$$
$$(2.38)$$

$$\theta_2 = a\lambda_1 \sin (\omega_{res} t + \delta_1) + b\lambda_2 \cos (\omega_2 t + \delta_2) + u_{21} \epsilon$$
$$+ u_{22} \epsilon^2 + \ldots$$

Substitution into the differential equations of motion can be made

treating the amplitude "a" as a function of time and writing

$$\omega_{res} = \omega_1 + \Delta_1 \epsilon + \Delta_2 \epsilon^2 \ldots \qquad (2.39)$$

The result is that the nonresonant part of the oscillation yields the same type of stable oscillatory terms as before. The resonant part of the solution in general has a time dependent amplitude, a, and may be unstable.

Our interest is in describing the size of the unstable resonance region. Consequently we wish to determine the range of the parameters Δ_1, Δ_2, ... of Eq. (2.39) for which the amplitude, a, exists and is time dependent. But in the previous section we found that the boundaries of this resonance region admitted constant amplitude oscillatory motion in the case of linear systems. This must also be the case for nonlinear systems, if the solution is to be an analytic function of time and the parameters of the problem and the solution is not unstable everywhere. Consequently in these cases we may locate the boundaries of the instability regions as those values of Δ_1, Δ_2, etc., for which the equations of motion are satisfied when a is assumed to be a constant. This is the approach assumed in the subsequent problem solutions.

Apparent advantages of the asymptotic method include its applicability to nonlinear systems and its ability to define stability boundaries where the sum or difference of the natural frequencies is in resonance with a parametric excitation.

SECTION III

SPINNING, UNSYMMETRICAL SATELLITE IN A
CIRCULAR ORBIT

The present research is concerned with the problem of stability
of motion of a spinning, unsymmetrical, rigid satellite in a circular
orbit. When the satellite possesses rotational motion relative to an
orbiting frame of reference the problem formulation involves periodic
coefficients.

Previous work [1],[11] has shown that for rigid bodies there is
no coupling between the orbital motion of the center of mass of the body
and the attitude motion of the body about the center of mass. This
assumption, referred to as orbital constraints, will be used in the present
study.

Particular emphasis will be placed upon the case in which the
body is nearly, but not exactly, symmetrical with respect to the spin
axis. It is felt that this is a case of great interest since, for
spin-stabilized satellites, a practical satellite system would be made
nearly symmetrical with respect to the spin axis to minimize the periodic
excitations caused by gravitational torques.

3.1   Coordinate Systems

An orbital frame of reference with its origin at the satellite
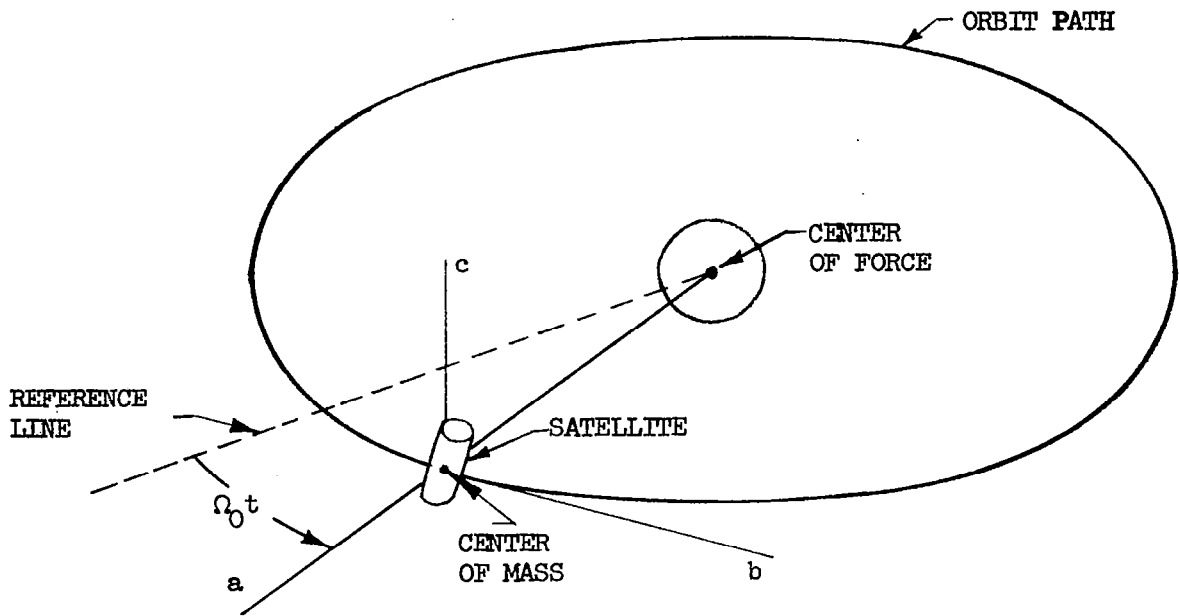center of mass and its orientation as shown in Figure 3.1a is chosen.

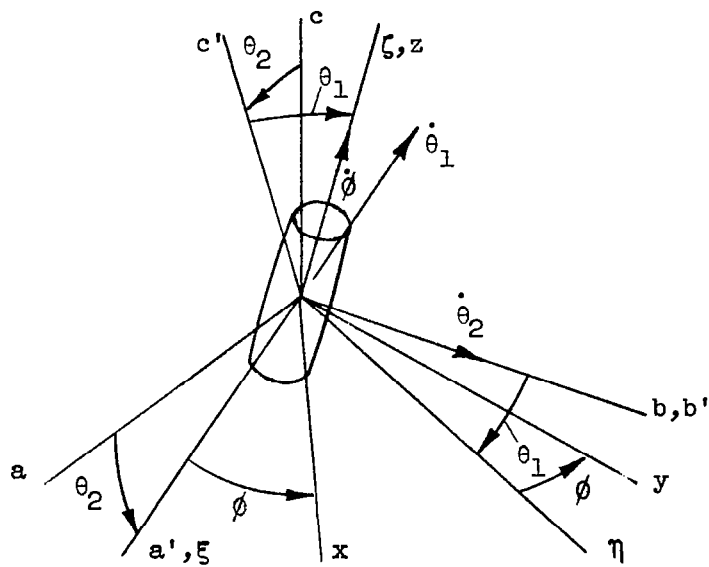FIG. 3.1a

THE SATELLITE AND THE ORBITAL AXES

FIG. 3.1b

COORDINATE SYSTEMS AND ANGULAR VELOCITIES

Axis a is along a radial line from the center of force (center of the earth) to the center of the satellite, axis b along the orbit path, and axis c perpendicular to axes a and b . The orbit angular velocity, denoted by Ω_0 , is related to the constant K , which is the product of the universal gravitational constant times the earth's mass, and the orbit radius R_c by $\Omega_0^2 = K/R_c^3$. Hence a, b, c forms an orbiting frame of reference. The orientation of the satellite relative to the a, b, c reference system is obtained by three successive rotations θ_2, θ_1, and $\emptyset$ as shown in Figure 3.1b.

The z axis is taken as the spin axis and the mass moments of inertia about the axes x, y and z are denoted by A, B and C, respectively.

The direction cosines between the x, y, z axes and the a, b, c axes may be written in terms of the following matrix equation:

$$
\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \ell_{xa} & \ell_{xb} & \ell_{xc} \\ \ell_{ya} & \ell_{yb} & \ell_{yc} \\ \ell_{za} & \ell_{zb} & \ell_{zc} \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}
$$

$$
= \begin{bmatrix} c\theta_2 c\emptyset - s\theta_1 s\theta_2 s\emptyset & c\theta_1 s\emptyset & -s\theta_2 c\emptyset - c\theta_2 s\theta_1 s\emptyset \\ -c\theta_2 s\emptyset - s\theta_1 s\theta_2 c\emptyset & c\theta_1 c\emptyset & s\theta_2 s\emptyset - c\theta_2 s\theta_1 c\emptyset \\ s\theta_2 c\theta_1 & s\theta_1 & c\theta_2 c\theta_1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}
$$

(3.1)

where $c\theta_2 = \cos\theta_2$, $s\theta_1 = \sin\theta_1$, etc. The angular velocities about the x, y, and z axes may be written

$$\Omega_x = \Omega_0 \left(-s\theta_2 c\phi - c\theta_2 s\theta_1 s\phi\right) + \dot{\theta}_2 c\theta_1 s\phi - \dot{\theta}_1 c\phi$$

$$\Omega_y = \Omega_0 \left(s\theta_2 s\phi - c\theta_2 s\theta_1 c\phi\right) + \dot{\theta}_2 c\theta_1 c\phi + \dot{\theta}_1 s\phi \qquad (3.2)$$

$$\Omega_z = \Omega_0 \left(c\theta_2 c\theta_1\right) + \dot{\theta}_2 s\theta_1 + \dot{\phi}$$

The following dimensionless quantities will be introduced:

$$r = C/A\ , \qquad \epsilon = (B - A)/A\ ,$$

$$\alpha = \dot{\phi}/\Omega_0\ , \qquad \alpha_1 = \omega_0/\Omega_0 \qquad\qquad (3.3)$$

where $\dot{\phi}$ is the instantaneous spin rate and ω_0 is the average spin rate. Note that ϵ is a measure of the asymmetry of the body with respect to the spin axis and will generally be a small number as mentioned previously.

3.2   Energy Expressions

The kinetic and potential energy expressions may be written as follows

$$KE = \frac{1}{2} A\ \Omega_x^2 + \frac{1}{2} B\ \Omega_y^2 + \frac{1}{2} C\ \Omega_z^2$$

$$PE = -\frac{3}{4}\frac{K}{R_c^3}\left[(C + B - A)\ell_{xa}^2 + (C + A - B)\ell_{ya}^2 + (A + B - C)\ell_{za}^2\right]$$

$$(3.4)$$

3.3   The Equations of Motion

The Lagrangian function, $L = KE - PE$, may be used in conjunction with the Lagrangian formulation to derive the differential equations of motion. For a conservative system, Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \; , \quad (i=1,2,\ldots,n) \tag{3.5}$$

where n is the degree of freedom of the system. Equations (3.5) lead to

$$\ddot{\theta}_1 + \dot{\theta}_2\Omega_0 c\theta_2 + \dot{\theta}_2^2 s\theta_1 c\theta_1 - \Omega_0(\dot{\theta}_2 c\theta_2 s^2\theta_1 - \dot{\theta}_2 c\theta_2 c^2\theta_1)$$

$$- \Omega_0^2 c^2\theta_2 s\theta_1 c\theta_1 - r[\dot{\phi}\dot{\theta}_2 c\theta_1 + \dot{\theta}_1^2 s\theta_1 c\theta_1 + \Omega_0\dot{\theta}_2 c\theta_2(c^2\theta_1 - s^2\theta_1)$$

$$- \Omega_0\dot{\phi}c\theta_2 s\theta_1 - \Omega_0^2 c^2\theta_2 s\theta_1 c\theta_1] + 3(1-r)\Omega_0^2 s^2\theta_2 s\theta_1 c\theta_1 + \epsilon[\ddot{\theta}_1 s^2\phi$$

$$+ 2\dot{\theta}_1\dot{\phi}s\phi c\phi + \ddot{\theta}_2 c\theta_1 s\phi c\phi - \dot{\theta}_1\dot{\theta}_2 s\theta_1 s\phi c\phi + \dot{\theta}_2\dot{\phi}c\theta_1(c^2\phi - s^2\phi)$$

$$+ \Omega_0\dot{\theta}_2(c\theta_2 s^2\phi + s\theta_2 s\theta_1 s\phi c\phi) + 2\Omega_0\dot{\phi}s\theta_2 s\phi c\phi - \Omega_0\dot{\theta}_1 c\theta_2 c\theta_1 s\phi c\phi$$

$$- \Omega_0\dot{\phi}c\theta_2 s\theta_1(c^2\phi - s^2\phi) + \dot{\theta}_1\dot{\theta}_2 s\theta_1 s\phi c\phi + \dot{\theta}_2^2 s\theta_1 c\theta_1 c^2\phi$$

$$+ \Omega_0\dot{\theta}_2 s\theta_2 s\theta_1 s\phi c\phi - \Omega_0\dot{\theta}_2 c\theta_2 s^2\theta_1 c^2\phi + \Omega_0\dot{\theta}_1 c\theta_2 c\theta_1 s\phi c\phi + \Omega_0\dot{\theta}_2 c\theta_2 c^2\theta_1 c^2\phi$$

$$+ \Omega_0^2(4s\theta_2 c\theta_2 c\theta_1 s\phi c\phi - c^2\theta_2 s\theta_1 c\theta_1 c^2\phi + 3s^2\theta_2 s\theta_1 c\theta_1 c^2\phi)] = 0$$

$$\ddot{\theta}_2 c^2\theta_1 - 2\dot{\theta}_2\dot{\theta}_1 s\theta_1 c\theta_1 - \Omega_0(-\dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1 + \dot{\theta}_1 c\theta_2 c^2\theta_1 - \dot{\theta}_1 c\theta_2 s^2\theta_1 + \dot{\theta}_1 c\theta_2$$

$$+ \dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1) - \Omega_0^2(s\theta_2 c\theta_2 - s\theta_2 c\theta_2 s^2\theta_1) + r[\ddot{\phi}s\theta_1 + \ddot{\ddot{\theta}}_1 c\theta_1$$

$$+ \ddot{\theta}_2 s^2\theta_1 + 2\dot{\theta}_2\dot{\theta}_1 s\theta_1 c\theta_1 + \Omega_0(-\dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1 + \dot{\theta}_1 c\theta_2 c^2\theta_1 - \dot{\theta}_1 c\theta_2 s^2\theta_1$$

$$+ \dot{\phi}s\theta_2 c\theta_1 + \dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1 + \Omega_0 s\theta_2 c\theta_2 c^2\theta_1)] - 3(1-r)\,\Omega_0^2 s\theta_2 c\theta_2 c^2\theta_1$$

$$+ \epsilon[\ddot{\theta}_1 c\theta_1 s\phi c\phi - \dot{\theta}_1^2 s\theta_1 s\phi c\phi + \dot{\theta}_1\dot{\phi}c\theta_1(c^2\phi - s^2\phi) + \ddot{\theta}_2 c^2\theta_1 c^2\phi \qquad (3.6)$$

$$- 2\dot{\theta}_2\dot{\theta}_1 s\theta_1 c\theta_1 c^2\phi - 2\dot{\theta}_2\dot{\phi}c^2\theta_1 s\phi c\phi + \Omega_0(-\dot{\theta}_1 s\theta_1 s\theta_2 s\phi c\phi + \dot{\theta}_2 c\theta_1 c\theta_2 s\phi c\phi$$

$$+ \dot{\phi}c\theta_1 s\theta_2 c^2\phi - \dot{\phi}c\theta_1 s\theta_2 s^2\phi + \dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1 c^2\phi - \dot{\theta}_1 c\theta_2 c^2\theta_1 c^2\phi$$

$$+ \dot{\theta}_1 c\theta_2 s^2\theta_1 c^2\phi + 2\dot{\phi}c\theta_2 s\theta_1 c\theta_1 s\phi c\phi - \dot{\theta}_1 c\theta_2 s^2\phi - \dot{\theta}_2 c\theta_2 c\theta_1 s\phi c\phi$$

$$- \dot{\theta}_1 s\theta_2 s\theta_1 s\phi c\phi - \dot{\theta}_2 s\theta_2 s\theta_1 c\theta_1 c^2\phi) - 4\Omega_0^2(s\theta_2 c\theta_2 s^2\phi + s^2\theta_2 s\theta_1 s\phi c\phi$$

$$- c^2\theta_2 s\theta_1 s\phi c\phi - s\theta_2 c\theta_2 s^2\theta_1 c^2\phi)] = 0$$

$$r[\ddot{\phi} + \ddot{\theta}_2 s\theta_1 + \dot{\theta}_2\dot{\theta}_1 c\theta_1 + \Omega_0(-\dot{\theta}_2 s\theta_2 c\theta_1 - \dot{\theta}_1 c\theta_2 s\theta_1)] - \epsilon[\dot{\theta}_1^2 s\phi c\phi$$

$$+ \dot{\theta}_2\dot{\theta}_1 c\theta_1(c^2\phi - s^2\phi) - \dot{\theta}_2^2 c^2\theta_1 s\phi c\phi + \Omega_0(\dot{\theta}_1 s\theta_2 s\phi c\phi - \dot{\theta}_1 c\theta_2 s\theta_1 c^2\phi$$

$$-\dot{\theta}_2 s\theta_2 c\theta_1 s^2\emptyset + \dot{\theta}_2 c\theta_2 s\theta_1 c\theta_1 s\emptyset c\emptyset + \dot{\theta}_1 s\theta_2 s\emptyset c\emptyset + \dot{\theta}_1 c\theta_2 s\theta_1 s^2\emptyset$$

$$+ \dot{\theta}_2 s\theta_2 c\theta_1 c^2\emptyset + \dot{\theta}_2 c\theta_2 s\theta_1 c\theta_1 s\emptyset c\emptyset) + \Omega_0^2(s^2\theta_2 s\emptyset c\emptyset + s\theta_2 c\theta_2 s\theta_1 s^2\emptyset$$

$$- s\theta_2 c\theta_2 s\theta_1 c^2\emptyset - c^2\theta_2 s^2\theta_1 s\emptyset c\emptyset - 3c^2\theta_2 s\emptyset c\emptyset - 3s\theta_2 c\theta_2 s\theta_1 c^2\emptyset$$

$$+ 3s\theta_2 c\theta_2 s\theta_1 s^2\emptyset + 3s^2\theta_2 s^2\theta_1 s\emptyset c\emptyset)] = 0$$

3.4 Equilibrium Positions. The Linearized Equations of Motion

Inspection of the differential equations of motion, Eq's. (3.6), shows that an equilibrium position exists when $\theta_1 = \dot{\theta}_1 = \theta_2 = \dot{\theta}_2 = \dot{\emptyset}$ $= 0$ and $\emptyset = n\pi/2$. This is the equilibrium position studied by De Bra and Delp [6] in which the unsymmetrical rigid body has a position that is fixed with respect to an orbiting frame of reference. The corresponding equilibrium position was studied by Nelson and Meirovitch [11] for the case of a rigid satellite with elastically connected moving parts. We are now interested in the case in which the satellite has a spinning motion relative to the orbiting frame of reference so that $\emptyset$ and $\dot{\emptyset}$ are not constant. However, one notices that by linearizing Eq's. (3.6) the equation for the coordinate $\emptyset$ becomes uncoupled; hence one can solve for $\emptyset$ independently.

In view of the above conclusions, we wish to define an equilibrium position $\theta_1 = \theta_2 = 0$. We note that θ_1 and θ_2 define the attitude of the spin axis relative to the orbiting frame of reference and $\theta_1 = \theta_2 = 0$ corresponds to the position in which the spin axis is perpendicular to the orbit plane.

The equations of motion, Eq's.(3.6), can be linearized about the position $\theta_1 = \theta_2 = 0$. It will also prove convenient to change the time scale to the nondimensional one defined by

$$\tau = \Omega_0 t \; , \quad \frac{d}{dt} = \Omega_0 \frac{d}{d\tau} \; , \quad \frac{d^2}{dt^2} = \Omega_0^2 \frac{d^2}{d\tau^2} \tag{3.7}$$

so that Eqs. (3.6) can be written as

$$\theta_1'' + \theta_2'[2-r(1+\alpha)] - \theta_1[1-r(1+\alpha)]$$

$$+ \epsilon \left\{ \theta_1'(1 - \tfrac{1}{2} r)(1+\alpha)s2\phi + \theta_2' \left[\tfrac{1}{2} r(1+\alpha) \right. \right.$$

$$+ (1 - \tfrac{1}{2} r)(1+\alpha) \, c2\phi \left. \right] - \theta_1 \left[\tfrac{1}{2} r(1+\alpha) \right.$$

$$+ (1 - \tfrac{1}{2} r)(1+\alpha) \, c2\phi \left. \right] + \theta_2(1 - \tfrac{1}{2} r)(4+\alpha)s2\phi \left. \right\} = 0$$

$$\theta_2'' - \theta_1' \, [2-r(1+\alpha)] - \theta_2[4-r(4+\alpha)] \qquad (3.8)$$

$$+ \, \epsilon \left\{ \theta_1' \left[-\frac{1}{2} r(1+\alpha) + (1 - \frac{1}{2} r)(1+\alpha) \, c2\emptyset \right] \right.$$

$$- \, \theta_2' \, (1 - \frac{1}{2} r)(1+\alpha) \, s2\emptyset + \theta_1 \, (1 - \frac{1}{2} r)(1+\alpha) \, s2\emptyset$$

$$\left. - \, \theta_2 \left[\frac{1}{2} r(4+\alpha) - (1 - \frac{1}{2} r)(4+\alpha) \, c2\emptyset \right] \right\} = 0$$

$$\emptyset'' + \frac{3}{2} \frac{\epsilon}{r} \, s2\emptyset = 0$$

where primes indicate differentiations with respect to τ .

3.5   The Spinning Motion

One notices from the third of Eq's.(3.8) that, for motion in the neighborhood of $\theta_1 = \theta_2 = 0$, the spinning motion is independent of the coordinates θ_1 and θ_2 . One can attempt a solution for $\emptyset$ in terms of a power series in $\epsilon = (B-A)/A$ as follows

$$\emptyset = \omega_0 t + \epsilon \emptyset_1(t) + \epsilon^2 \emptyset_2(t) + \dots \qquad (3.9)$$

from which it follows immediately that

$$\alpha = \alpha_1 + \epsilon \, \emptyset_1'(\tau) + \epsilon^2 \emptyset_2'(\tau) + \dots \qquad (3.10)$$

Substituting the above into the third of Eq's.(3.8) and equating terms of equal powers of ϵ to zero, we obtain a sequence of ordinary differential equations. The sequential solution of these differential equations allows one to write

$$\alpha = \alpha_1 + \epsilon \; \frac{3}{4r\alpha_1} \; c2\alpha_1\tau \; + \epsilon^2 \; \frac{9}{64r^2\alpha_1^3} \; c4\alpha_1\tau$$

$$+ \; \epsilon^3 \left[\frac{27}{1024r^3\alpha_1^5} \; (c6\alpha_1\tau - 5 \; c2\alpha_1\tau) \right] + \ldots$$

$$c2\emptyset = c2\alpha_1\tau + \epsilon \left[\frac{3}{8r\alpha_1^2} \; (c4\alpha_1\tau - 1) \right] \qquad (3.11)$$

$$+ \; \epsilon^2 \left[\frac{9}{256r^2\alpha_1^4} \; (c6\alpha_1\tau - c2\alpha_1\tau) \right] + \ldots$$

$$s2\emptyset = s2\alpha_1\tau + \epsilon \; \frac{3}{8r\alpha_1^2} \; s4\alpha_1\tau$$

$$+ \; \epsilon^2 \left[\frac{9}{256r^2\alpha_1^4} \; (3s6\alpha_1\tau - 5s2\alpha_1\tau) \right] + \ldots$$

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An alternate approach that will also prove useful results from noting that the third of Eq's. (3.8) can be integrated once to yield

$$\dot{\emptyset} = \omega_0 \left[1 + \frac{3}{2} \frac{\epsilon}{r\alpha_1^2} c2\emptyset \right]^{1/2} \tag{3.12}$$

3.6   The Hamiltonian Function

Our interest is in the motion about the point $\theta_1 = \theta_2 = 0$. In this neighborhood, as can be seen from the third of Eq's. (3.8), the coordinate $\emptyset$ can be considered as an explicit function of time. One can conceive of a constrained system which is a system identical with the system under consideration but with the $\emptyset$ coordinate a known function of time, namely the solution of the third of Eq's. (3.8). When θ_1 and θ_2 are not small, the motion of this system is not in accordance with the complete, nonlinear equations which indicates that constraint forces must be added. However, as θ_1 and θ_2 approach zero, the terms coupling the $\emptyset$ motion with the θ_1 and θ_2 motions approach zero and the constraint forces approach zero. As a result of this assumption, one can devise a Hamiltonian function containing θ_1 and θ_2 as variables and $\emptyset$ as an explicit function of time. This Hamiltonian is consistent only with the linearized equations and must be used only when θ_1 and θ_2 are small. The Hamiltonian in question can be written as

$$H = \frac{\partial L}{\partial \theta_1'} \theta_1' + \frac{\partial L}{\partial \theta_2'} \theta_2' - L = \frac{1}{2} A \Omega_0^2 \left\{ \theta_1'^2 (1+\epsilon s^2\emptyset) + \theta_2'^2 (c^2\theta_1 \right.$$

$$+ rs^2\theta_1 + \epsilon c^2\theta_1 c^2\emptyset) + \epsilon \theta_1'\theta_2' c\theta_1 s2\emptyset + [- r(\alpha+c\theta_1 c\theta_2)^2$$

$$- s^2\theta_2 - c^2\theta_2 s^2\theta_1 + 3(r-1)s^2\theta_2 c^2\theta_1] + \epsilon [3(c\theta_2 s\emptyset + s\theta_1 s\theta_2 c\emptyset)^2$$

$$\left. - (s\theta_2 s\emptyset - c\theta_2 s\theta_1 c\emptyset)^2] \right\} \tag{3.13}$$

3.7 Application of the Liapounov Type Analysis to the Spinning, Unsymmetrical Satellite in a Circular Orbit

Application of the proposed theorem to the problem of the spinning, unsymmetrical, rigid satellite moving in a circular orbit will now be undertaken. To this end the assumption is made that $\emptyset$ is an explicit function of t so that one can use the Hamiltonian function as given by Eq. (3.13). The Hamiltonian function evaluated at the equilibrium position, $\theta_1 = \theta_2 = \theta_1' = \theta_2' = 0$, is

$$H_E = \frac{1}{2} A \Omega_0^2 [- r(1+\alpha)^2 + 3\epsilon s^2\emptyset] \tag{3.14}$$

so that the testing function can be written as

$$V = H - H_E = \frac{1}{2} A \Omega_0^2 \left\{ \theta_1'^2 (1+\epsilon s^2\phi) + \theta_2'^2 (c^2\theta_1 + rs^2\theta_1 + \epsilon c^2\theta_1 c^2\phi) \right.$$

$$+ \epsilon \, \theta_1' \theta_2' \, c\theta_1 \, s2\phi + [r(1-c^2\theta_1 c^2\theta_2) + 2r\alpha(1-c\theta_1 c\theta_2)$$

$$- s^2\theta_2 - c^2\theta_2 s^2\theta_1 + 3(r-1)s^2\theta_2 c^2\theta_1] + \epsilon \, [3(c\theta_2 s\phi$$

$$\left. + s\theta_1 s\theta_2 c\phi)^2 - 3s^2\phi - (s\theta_2 s\phi - c\theta_2 s\theta_1 c\phi)^2] \right\} \qquad (3.15)$$

Recalling that $V = KE^* + U$, where KE^* includes the terms that are quadratic in the velocities and is a positive definite function in the neighborhood of the equilibrium position E, the dynamic potential, U, can be written in the form

$$U = \frac{1}{2} A \Omega_0^2 [r(1-c^2\theta_1 c^2\theta_2) + 2r\alpha (1-c\theta_1 c\theta_2) - s^2\theta_2 - c^2\theta_2 s^2\theta_1$$

$$+ 3(r-1)s^2\theta_2 c^2\theta_1 + 3\epsilon(c\theta_2 s\phi + s\theta_1 s\theta_2 c\phi)^2 - 3\epsilon \, s^2\phi$$

$$- \epsilon(s\theta_2 s\phi - c\theta_2 s\theta_1 c\phi)^2] \qquad (3.16)$$

As pointed out in Section II it is sufficient to check U for positive definiteness rather than V.

It is easy to check that

$$\frac{\partial U}{\partial \theta_1} \bigg|_{\theta_1 = \theta_2 = 0} = \frac{\partial U}{\partial \theta_2} \bigg|_{\theta_1 = \theta_2 = 0} = 0 \qquad (3.17)$$

which confirms the existence of the equilibrium at $\theta_1 = \theta_2 = 0$.
To determine the conditions for positive definiteness of U we apply
Sylvester's criterion [21]. According to this criterion we conclude
that U is positive definite if the following conditions are satisfied

$$\frac{\partial^2 U}{\partial \theta_1^2} \bigg|_E = A \, \Omega_0^2 \, [r(1+\alpha) - 1 - \epsilon c^2 \phi] > 0$$

$$\qquad (3.18)$$

$$\frac{\partial^2 U}{\partial \theta_1^2} \frac{\partial^2 U}{\partial \theta_2^2} - \left(\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}\right)^2 \bigg|_E = A^2 \, \Omega_0^4 \, [(r-1+r\alpha) \, (4r-4+r\alpha)$$

$$+ \epsilon(-4r + 4 - 4r\alpha + 3r\alpha c^2\phi) - 12\epsilon^2 s^2\phi c^2\phi] > 0$$

It should be noted that if ϵ is set equal to zero the same stability
criteria is obtained as was obtained by Pringle [9] and Likins [10]
for the symmetrical body. When ϵ is not zero the time dependent
terms are introduced through α and ϕ . For small values of ϵ
the boundary of the region within which U is positive definite can be
determined by neglecting terms in ϵ^2 . Writing the binomial expansion
of Eq. (3.12) and retaining the first two terms only, gives

$$\alpha \cong \alpha_1 + \epsilon \frac{3}{4r\alpha_1} c2\emptyset = \alpha_1 + \epsilon \frac{3}{4r\alpha_1} (2c^2\emptyset - 1) \tag{3.19}$$

so that Eq's. (3.18) become

$$r(1+\alpha_1) - 1 - \epsilon \left[\frac{3}{4\alpha_1} + (1 - \frac{3}{2\alpha_1}) c^2\emptyset \right] > 0$$

$$\tag{3.20}$$

$$(r - 1 + r\alpha_1)(4r - 4 + r\alpha_1) + \epsilon \left[(4 - 4r - 4r\alpha_1 + \frac{15}{4\alpha_1} - \frac{15r}{4\alpha_1}) \right.$$

$$\left. + (3r\alpha_1 - \frac{15}{2\alpha_1} + \frac{15r}{2\alpha_1}) c^2\emptyset \right] > 0$$

The expansion used in Eq. (3.19) is valid only when $3\epsilon/2r\alpha_1 < 1$ and the retention of only the first two terms is justified when $3\epsilon/2r\alpha_1 \ll 1$. Consequently, the above expression will not be used to investigate the region in which $r\alpha_1$ is small. In addition, only the region in which α_1 is positive will be considered and ϵ will be assumed to be sufficiently small to neglect terms in ϵ^2 . Under these restrictions Eq's. (3.20) may be extremized by selecting $c^2\emptyset$ to be zero or one.* It is then found that the requirements for stable motion are satisfied if the following four conditions are fulfilled:

$$r > \frac{4}{\alpha_1+4} + \epsilon \frac{16\alpha_1-5}{4\alpha_1(\alpha_1+4)} \quad \text{for} \quad \alpha_1^2 > \frac{5(1-r)}{2r}$$

$$r > \frac{4}{\alpha_1+4} + \epsilon \frac{5}{4\alpha_1(\alpha_1+4)} \quad \text{for} \quad \alpha_1^2 < \frac{5(1-r)}{2r}$$

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* In doing that the stability requirements are rendered stronger than necessary and stability may occur even though the resulting inequalities are not satisfied.

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$$r > \frac{1}{\alpha_1 + 1} + \epsilon \frac{3}{4\alpha_1(\alpha_1 + 1)} \qquad \text{for} \qquad \alpha_1 < \frac{3}{2} \qquad (3.21)$$

$$r > \frac{1}{\alpha_1 + 1} + \epsilon \frac{4\alpha_1 - 3}{4\alpha_1(\alpha_1 + 1)} \qquad \text{for} \qquad \alpha_1 > \frac{3}{2}$$

Figure 3.2 shows the resulting boundary of the stability region for $\epsilon = 0$ and $\epsilon = 0.1$, where the boundary for $\epsilon = 0$ is identical with that shown by Pringle [9] and Likins [10] for the symmetrical body.

The second part of the stability theorem can now be applied. If the linearized equations are assumed to describe the motion near E, one may use Floquet's theorem [27] to state that the solutions are of the form

$$\theta_1 = \sum_{j=1}^{4} f_j(\tau) \, e^{(u_j + iv_j)\tau}$$

$$(3.22)$$

$$\theta_2 = \sum_{j=1}^{4} g_j(\tau) \, e^{(u_j + iv_j)\tau}$$

in which $f_j(\tau)$ and $g_j(\tau)$ are periodic functions with the same period as the periodic coefficients appearing in the differential equations of motion, which is $2\pi/2\alpha_1$ in this case. Consequently, $f_j(\tau)$ and $g_j(\tau)$ may be expanded in terms of Fourier series so that

$$\epsilon = \frac{B-A}{A} = 0$$
$$= 0.1$$

SPIN RATIO, $\alpha_1 = \omega_0/\Omega_0$

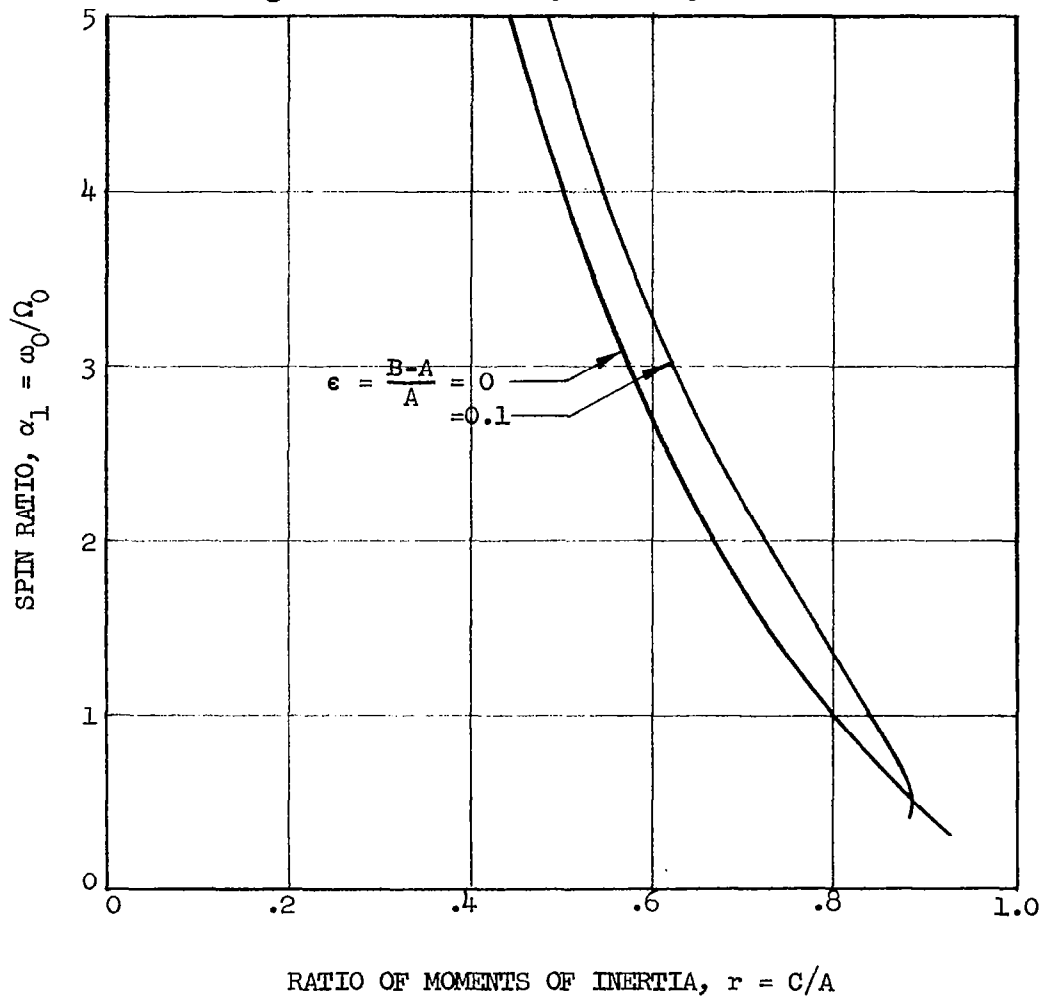RATIO OF MOMENTS OF INERTIA, $r = C/A$

FIGURE 3.2

STABILITY BOUNDARY DICTATED BY POSITIVE
DEFINITE HAMILTONIAN (LINEARIZED SYSTEM)
SPINNING, UNSYMMETRICAL SATELLITE

$$\theta_1 = \sum_{j=1}^{4} \sum_{k=-\infty}^{\infty} f_{jk} \, e^{u_j \tau} \, e^{i(v_j + 2k\alpha_1)\tau}$$

(3.23)

$$\theta_2 = \sum_{j=1}^{4} \sum_{k=-\infty}^{\infty} g_{jk} \, e^{u_j \tau} \, e^{i(v_j + 2k\alpha_1)\tau}$$

In the neighborhood of the equilibrium position, terms in the third and higher powers of the coordinates and velocities will be neglected as small compared with second power terms, so that the partial derivative of the testing function V with respect to time may be written as

$$\frac{\partial V}{\partial \tau} = \frac{\partial(H - H_E)}{\partial \tau} = \frac{1}{2} \epsilon A \Omega_0^2 \left\{ \alpha_1 (\theta_1'^2 - \theta_2'^2) \, s \, 2\alpha_1 \tau + 2\alpha_1 \, \theta_1' \, \theta_2' \, c \, 2\alpha_1 \tau \right.$$

$$\left. - \left[\left(\tfrac{3}{2} - \alpha_1 \right) \theta_1^2 + \left(\tfrac{3}{2} + 4\alpha_1 \right) \theta_2^2 \right] s 2\alpha_1 \tau + 8\alpha_1 \theta_1 \theta_2 \, c 2\alpha_1 \tau \right\} + 0(\epsilon^2)$$

(3.24)

Substituting Eq's. (3.23) into Eq. (3.24) and integrating from τ_0 to τ an expression of the following form is obtained

$$I = \sum_{j=1}^{4} \sum_{\ell=1}^{4} \sum_{k=-\infty}^{\infty} h_{j\ell k} \int_{\tau_0}^{\tau} e^{(u_j + u_\ell)\tau} \, e^{i(v_j + v_\ell + 2k\alpha_1)\tau} \sin(2\alpha_1 \tau + \delta) d\tau$$

(3.25)

Four cases must be investigated.

a. $\quad u_j + u_\ell \leq 0, \quad v_j + v_\ell + 2k\alpha_1 \neq 2\alpha_1$

b. $\quad u_j + u_\ell > 0, \quad v_j + v_\ell + 2k\alpha_1 \neq 2\alpha_1$

$$(3.26)$$

c. $\quad u_j + u_\ell \leq 0, \quad v_j + v_\ell + 2k\alpha_1 = 2\alpha_1$

d. $\quad u_j + u_\ell > 0, \quad v_j + v_\ell + 2k\alpha_1 = 2\alpha_1$

The first two cases are nonresonant cases, so that the value of the integral varies periodically with time. Case a. represents stable motion whereas case b. is clearly impossible since, for large t, the Hamiltonian would oscillate with increasing amplitude. Cases c. and d. correspond to resonant motion. Case c. represents bounded resonant motion and case d. represents divergent motion in which the Hamiltonian tends to increase without bound with time. Whereas one cannot distinguish between cases c. and d. it is possible to say that unstable motion does not occur in the nonresonant case for which the Hamiltonian is positive definite.

As the value of $\epsilon = (B-A)/A$ approaches zero, in the limit, the motion must reduce to

$$\theta_1 = a_{11} e^{i\omega_1 \tau} + a_{12} e^{-i\omega_1 \tau} + a_{13} e^{i\omega_2 \tau} + a_{14} e^{-i\omega_2 \tau}$$

$$(3.27)$$

$$\theta_2 = a_{21} e^{i\omega_1 \tau} + a_{22} e^{-i\omega_1 \tau} + a_{23} e^{i\omega_2 \tau} + a_{24} e^{-i\omega_2 \tau}$$

Where $\pm \omega_1$ and $\pm \omega_2$ are characteristic numbers associated with the equations

$$\theta_1'' + \theta_2' [2-r(1+\alpha_1)] - \theta_1 [1-r(1+\alpha_1)] = 0$$

$$(3.28)$$

$$\theta_2'' - \theta_1' [2-r(1+\alpha_1)] - \theta_2 [4-r(4+\alpha_1)] = 0$$

The characteristic numbers have the values

$$\omega_1 = \sqrt{b + \sqrt{b^2 - c}} \quad , \quad \omega_2 = \sqrt{b - \sqrt{b^2 - c}} \qquad (3.29)$$

in which

$$b = -\frac{1}{2} [1 - r(1-2\alpha_1) - r^2(1+\alpha_1)^2]$$

$$(3.30)$$

$$c = [4 - r(5\alpha_1+8) + r^2(1+\alpha_1)(4+\alpha_1)]$$

A comparison of Eqs. (3.23) and (3.27) shows that

$$\lim_{\epsilon \to 0} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \omega_1 \\ -\omega_1 \\ \omega_2 \\ -\omega_2 \end{Bmatrix} \qquad (3.31)$$

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Consequently when ϵ is small, Eq's. (3.26) lead to the conclusion that resonance must occur near the values

$$\omega_1 = m\alpha_1 \qquad , \quad \omega_2 = m\alpha_1$$

$$(m = \pm 1, \pm 2, \dots) \quad (3.32)$$

$$\omega_1 + \omega_2 = 2m\alpha_1 \quad , \quad \omega_1 - \omega_2 = 2m\alpha_1$$

Figure 3.3 shows the locations in the plane α_1 vs r near which resonance will occur in the linearized system and for small ϵ. It should be noted that the present method gives us the location of instability regions. To describe the width of these regions different methods must be chosen as shown in the following discussion.

3.8 Determination of the Regions of Instability - Infinite Determinant Method

The linearized equations of motion of a spinning, unsymmetrical satellite in a circular orbit may be shown to be:

$$\theta_1'' + \theta_2' \left[-\beta + \epsilon \left(r_1 - \frac{3}{4\alpha_1} \right) \cos 2\alpha_1 \tau \right] + \theta_1' \epsilon r_1 \sin 2\alpha_1 \tau$$

$$+ \theta_2 \epsilon r_2 \sin 2\alpha_1 \tau + \theta_1 \left[\beta + 1 - \epsilon \left(r_1 - \frac{3}{4\alpha_1} \right) \cos 2\alpha_1 \tau \right] + \theta(\epsilon^2) = 0$$

$$\theta_2'' + \theta_1' \left[\beta + \epsilon \left(r_1 + \frac{3}{4\alpha_1} \right) \cos 2\alpha_1 \tau \right] - \theta_2' \epsilon r_1 \sin 2\alpha_1 \tau$$

$$+ \theta_1 \epsilon r_1 \sin 2\alpha_1 \tau + \theta_2 \left[\beta + 3r \left(1 - \frac{\epsilon}{2} \right) - 2 \right. \qquad (3.33)$$

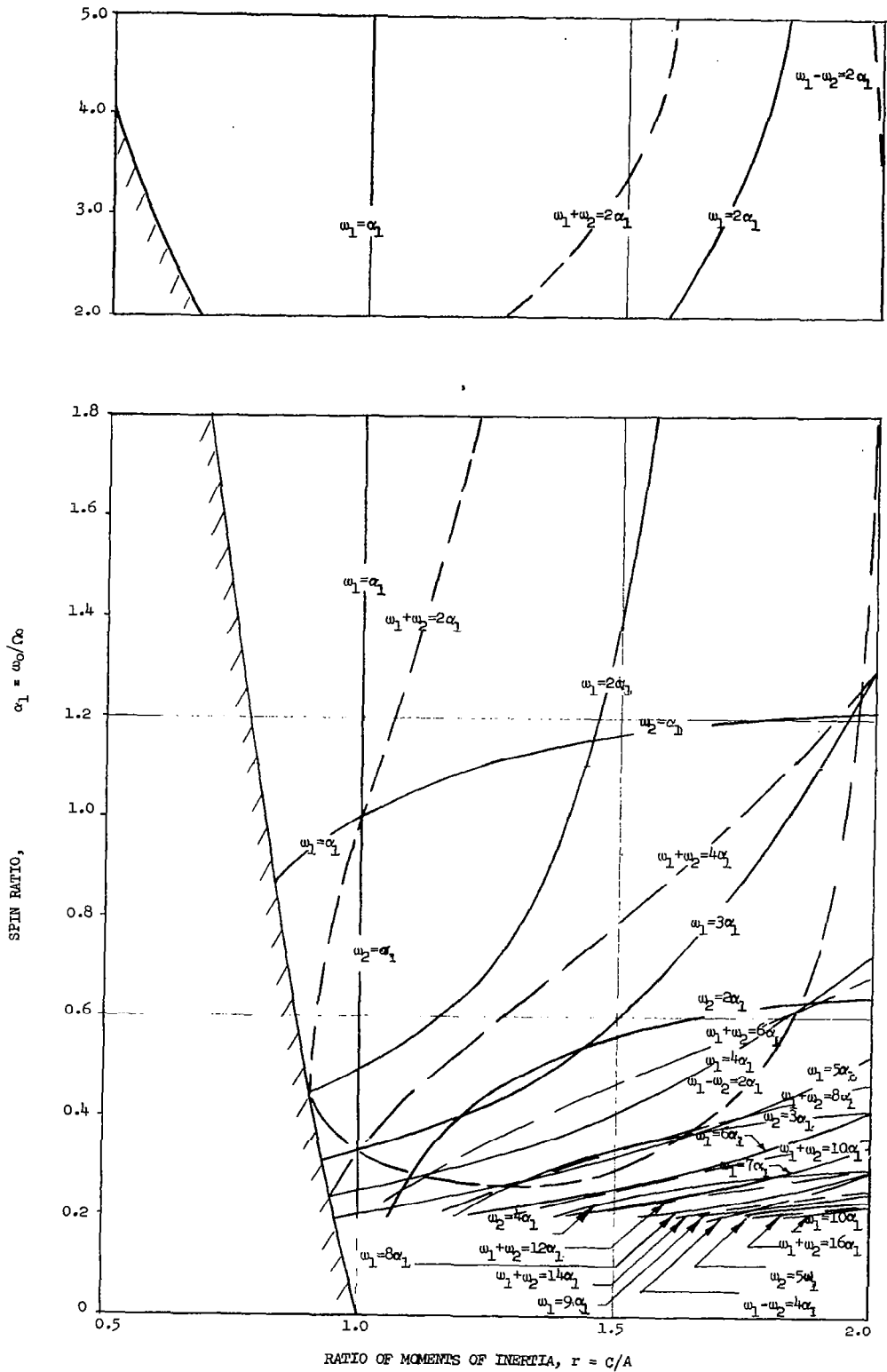$$\left. + \epsilon \left(r_2 + \frac{3}{4\alpha_1} \right) \cos 2\alpha_1 \tau \right] + 0(\epsilon^2) = 0$$

FIGURE 3.3

UNSYMMETRICAL SPINNING SATELLITE

APPROXIMATE LOCATIONS OF RESONANCE REGIONS FOR $\epsilon \ll 1$

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in which

$$\beta = r(\alpha_1 + 1)(1 - \frac{\epsilon}{2}) - 2$$

$$r_1 = (1 - \frac{r}{2})(1 + \alpha_1) \tag{3.34}$$

$$r_2 = (1 - \frac{r}{2})(4 + \alpha_1)$$

and terms in the second and higher powers of ϵ are grouped in the terms $O(\epsilon^2)$ and will be neglected in the first approximation.

The regions of instability are bounded by periodic solutions of period $2T$ and T which can be written in the form

Period 2T

$$\theta_1 = \sum_{n=1,3,5,\ldots}^{\infty} (a_{1n} \sin n\alpha_1 \tau + b_{1n} \cos n\alpha_1 \tau)$$

$$\tag{3.35}$$

$$\theta_2 = \sum_{n=1,3,5,\ldots}^{\infty} (a_{2n} \sin n\alpha_1 \tau + b_{2n} \cos n\alpha_1 \tau)$$

Period T

$$\theta_1 = \sum_{n=0,2,4,\ldots}^{\infty} (a_{1n} \sin n\alpha_1 \tau + b_{1n} \cos n\alpha_1 \tau)$$

$$\tag{3.36}$$

$$\theta_2 = \sum_{n=0,2,4,\ldots}^{\infty} (a_{2n} \sin n\alpha_1 \tau + b_{2n} \cos n\alpha_1 \tau)$$

where advantage has been taken of the specific form of Eq's.(3.33) in eliminating the even n terms in Eq's.(3.35) and the odd n terms in Eq's.(3.36).

The general procedure at this point consists of substituting a finite number of terms of either Eq's.(3.35) or (3.36) into the equations of motion, equating the coefficients of equal harmonics, setting up the determinant of the coefficients of the assumed periodic series, expanding this determinant, and solving the resulting equation for the instability boundaries. This then yields an approximation of the values of the parameters that satisfy the infinite determinant.

Experience with this procedure shows that for the problem under consideration the use of only the $n = 1$ terms of Eq's.(3.35) will define a first approximation for the regions near $\omega_1 = \alpha_1$ and $\omega_2 = \alpha_1$, where ω_1 and ω_2 are the natural frequencies of the unperturbed system. If the $n = 3$ terms are also included, a better approximation of the regions near $\omega_1 = \alpha_1$ and $\omega_2 = \alpha_1$ will be obtained together with a first approximation of the regions near $\omega_1 = 3\alpha_1$ and $\omega_2 = 3\alpha_1$. As more terms are included, the accuracy of the approximation of the lower order parametric resonances is improved and regions at which the natural frequencies are approximately equal to successively higher odd multiples of the average spin frequency, α_1, are defined. When Eq's.(3.36) rather than Eq's.(3.35) are used the even numbered regions are defined in a similar manner.

Inspection of Figure 3.3 of the previous section shows that regions near $\omega_1 = 2\alpha_1$ and $\omega_2 = 2\alpha_1$ are to be expected for a satellite configurations with $1 < r < 1.8$. These regions can be defined in the first approximation by assuming

$$\theta_1 = a_1 + b_1 \sin 2\alpha_1 \tau + c_1 \cos 2\alpha_1 \tau$$

$$(3.37)$$

$$\theta_2 = a_2 + b_2 \sin 2\alpha_1 \tau + c_2 \cos 2\alpha_1 \tau$$

Substituting Eq's.(3.37) into the equations of motion, Eq's.(3.33), and equating the constant terms, the coefficients of $\sin 2\alpha_1 \tau$ and the coefficients of $\cos 2\alpha_1 \tau$ to zero, gives the following six algebraic equations:

$$a_1(\beta+1) + b_2 \epsilon \left[\alpha_1 \left(r_1 - \frac{3}{4\alpha_1} \right) + \frac{1}{2} r_2 \right] + c_1 \epsilon \left[-\alpha_1 r_1 - \frac{1}{2} \left(r_1 - \frac{3}{4\alpha_1} \right) \right] = 0$$

$$a_1 \epsilon \left(r_1 - \frac{3}{4\alpha_1} \right) - b_2 2\alpha_1 \beta + c_1 \left(-4\alpha_1^2 + \beta + 1 \right) = 0$$

$$a_1 \epsilon r_1 + b_2 \left[-4\alpha_1^2 + \beta + 3r \left(1 - \frac{\epsilon}{2} \right) - 2 \right] - c_1 2\alpha_1 \beta = 0$$

$$a_2 \left[\beta + 3r \left(1 - \frac{\epsilon}{2} \right) - 2 \right] + b_1 \epsilon \left[\alpha_1 \left(r_1 + \frac{3}{4\alpha_1} \right) + \frac{1}{2} r_1 \right] \qquad (3.38)$$

$$+ c_2 \epsilon \left[\alpha_1 r_1 + \frac{1}{2} \left(r_2 + \frac{3}{4\alpha_1} \right) \right] = 0$$

$$a_2 \epsilon r_2 + b_1 \left(-4\alpha_1^2 + \beta + 1 \right) + c_2 \epsilon \left[\alpha_1 r_1 + \frac{1}{2} \left(r_2 + \frac{3}{4\alpha_1} \right) \right] = 0$$

$$a_2 \epsilon \left(r_2 + \frac{3}{4\alpha_1} \right) + b_1 2\alpha_1 \beta + c_2 \left[-4\alpha_1^2 + \beta + 3r \left(1 - \frac{\epsilon}{2} \right) - 2 \right] = 0$$

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For nontrivial solutions to exist, the determinant of the coefficients must be zero. In this case the determinant can be expressed as the product of two 3×3 determinants and will be zero if either of the 3×3 determinants is zero.

$$\begin{vmatrix} \beta + 1 & \epsilon\left[\alpha_1\left(r_1 - \frac{3}{4\alpha_1}\right) + \frac{1}{2}r_2\right] & \epsilon\left[-\alpha_1 r_1 - \frac{1}{2}\left(r_1 - \frac{3}{4\alpha_1}\right)\right] \\ \epsilon\left(r_1 - \frac{3}{4\alpha_1}\right) & -2\alpha_1\beta & -4\alpha_1^2 + \beta + 1 \\ \epsilon r_1 & -4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 & -2\alpha_1\beta \end{vmatrix} = 0$$

$$\begin{vmatrix} \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 & \epsilon\left[\alpha_1\left(r_1 + \frac{3}{4\alpha_1}\right) + \frac{1}{2}r_1\right] & \epsilon\left[\alpha_1 + \frac{1}{2}\left(r_2 + \frac{3}{4\alpha_1}\right)\right] \\ \epsilon r_2 & -4\alpha_1^2 + \beta + 1 & 2\alpha_1\beta \\ \epsilon\left(r_2 + \frac{3}{4\alpha_1}\right) & 2\alpha_1\beta & -4\alpha_1^2 + \beta + 3r\left(1 - \frac{\epsilon}{2}\right) - 2 \end{vmatrix} = 0$$

$$(3.39)$$

In the process of determining the boundaries of the regions of instability (which are surfaces in α_1, r, ϵ space) it is convenient to express the value of α_1 in the following terms:

$$2\alpha_1 = \bar{\omega}_i + \Delta_1\epsilon + \Delta_2\epsilon^2 + \dots \qquad (i=1,2) \qquad (3.40)$$

where $\bar{\omega}_i$ is the natural frequency of the system in the absence of the periodic terms in the equations of motion and is given by

$$\bar{\omega}_i = \left\{ \frac{\beta^2 + 2\beta + 3r(1 - \frac{\epsilon}{2}) - 1}{2} \pm \left[\left(\frac{\beta^2 + 2\beta + 3r(1 - \frac{\epsilon}{2}) - 1}{2} \right)^2 \right. \right.$$

$$\left. \left. - (\beta + 1)[\beta + 3r(1 - \frac{\epsilon}{2}) - 2] \right]^{1/2} \right\}^{1/2} \tag{3.41}$$

where $\bar{\omega}_1$ will be associated with the positive sign under the radical and $\bar{\omega}_2$ with the negative sign. The boundaries of the regions of instability can then be expressed in terms of Eq. (3.40). One finds that $\Delta_1 = 0$ and Δ_2 takes on either of the following values

$$\Delta_2 = \frac{1}{(\beta+1)[16\alpha_1^3 + 2\alpha_1(-2\beta-3r+1-\beta^2)]} \left\{ -\left(r_1 - \frac{3}{4\alpha_1} \right)^2 \left(2\alpha_1^2\beta + 2\alpha_1^2 - \frac{1}{2}\beta - \frac{3}{2}r + 1 \right) \right.$$

$$\left. + \left(r_1 - \frac{3}{4\alpha_1} \right) \left[-r_2\alpha_1\beta + r_1\alpha_1 \left(-8\alpha_1^2 + \beta + 3r - 1 \right) \right] - \frac{r_1 r_2}{2} \left(-4\alpha_1^2 + \beta + 1 \right) + 2\alpha_1^2 r_1^2 \beta \right\}$$

or

$$\tag{3.42}$$

$$\Delta_2 = \frac{1}{(\beta+3r-2)[16\alpha_1^3 + 2\alpha_1(-2\beta-3r+1-\beta^2)]} \left\{ \frac{1}{2} \left(r_2 + \frac{3}{4\alpha_1} \right)^2 \left(-4\alpha_1^2 + \beta + 1 \right) \right.$$

$$- \left(r_2 + \frac{3}{4\alpha_1} \right) \left(4\alpha_1^3 r_1 - \alpha_1 r_1 + \alpha_1 r_2\beta + 2\alpha_1^2 r_1 \beta + \frac{3}{2} \alpha_1\beta \right)$$

$$\left. + \alpha_1 r_2 \left(r_1 + \frac{3}{4\alpha_1} \right) \left(-4\alpha_1^2 + \beta + 3r - 2 \right) + \frac{r_1 r_2}{2} \left(-4\alpha_1^2\beta - 4\alpha_1^2 + \beta + 3r - 2 \right) \right\}$$

Figure 3.4 shows the resulting stability boundaries for $r = 1.5$.
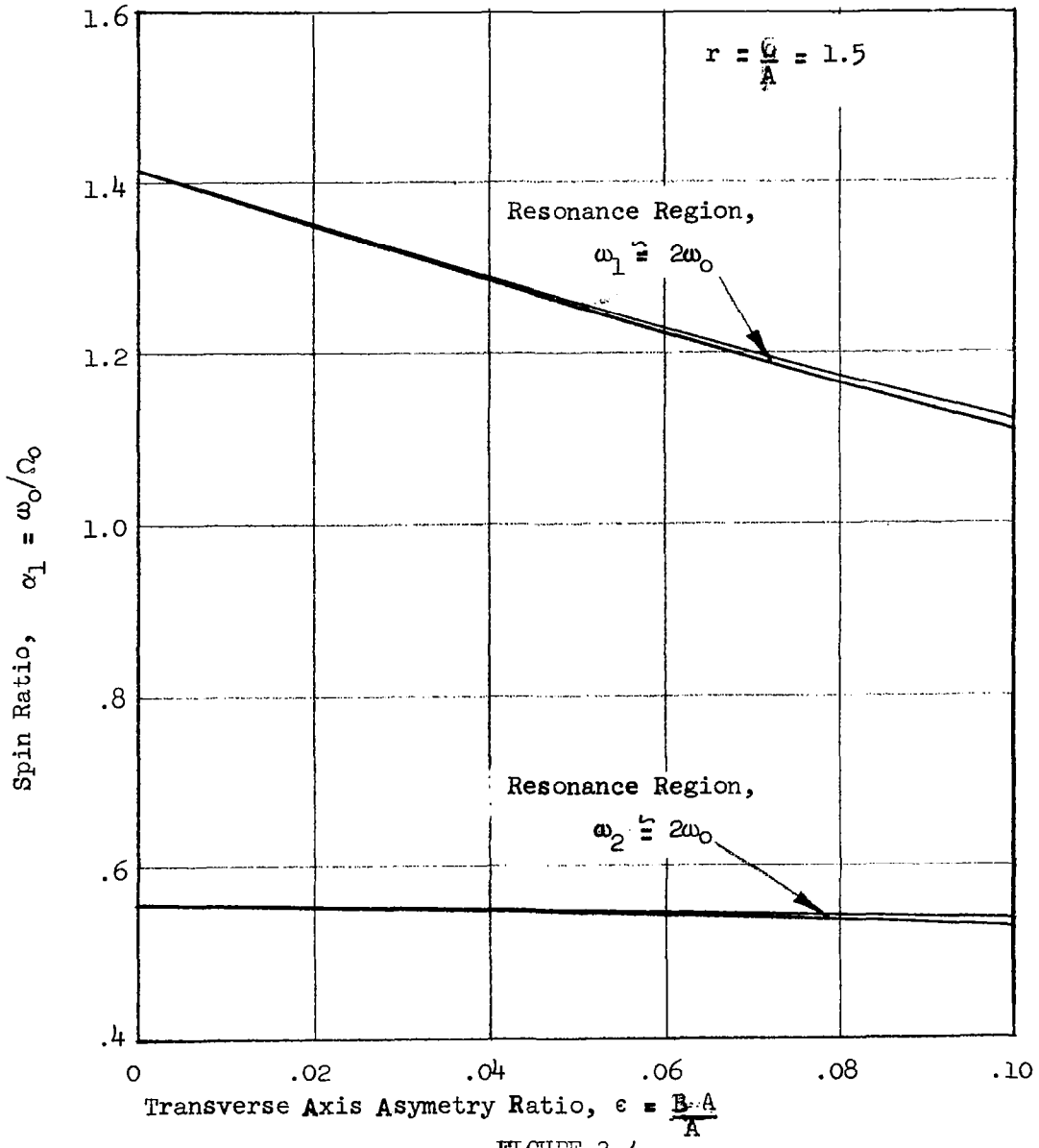
FIGURE 3.4

UNSYMMETRICAL SPINNING SATELLITE
RESONANCE REGIONS FOR OSCILLATIONS
WITH FREQUENCY $2\omega_0$ (LINEARIZED SYSTEM)

One can also investigate the regions of principal instability, which are defined as the regions where α_1 is approximately equal to one of the natural frequencies. Reference to Figure 3.3 shows, however, that for most values of r this type of resonance will not be experienced for $\alpha_1 \cong \omega_1$. To describe this region we will first assume a solution of the form

$$\theta_1 = a_1 \sin \alpha_1 \tau + b_1 \cos \alpha_1 \tau$$

$$(3.43)$$

$$\theta_2 = a_2 \sin \alpha_1 \tau + b_2 \cos \alpha_1 \tau$$

Substituting into the equations of motion, Eq's. (3.33), and equating coefficients of the periodic terms one obtains the following four algebraic equations

$$a_1 \left[-\alpha_1^2 + \frac{\alpha_1 r_1}{2} \epsilon + \beta + 1 + \frac{\epsilon}{2} (r_1 - \frac{3}{4\alpha_1}) \right] + b_2 \left[\alpha_1 \beta + \frac{\alpha_1}{2}(r_1 - \frac{3}{4\alpha_1}) \epsilon \right.$$
$$\left. + \frac{r_2}{2} \epsilon \right] = 0$$

$$a_1 \left[\alpha_1 \beta + \frac{\alpha_1}{2} (r_1 + \frac{3}{4\alpha_1}) \epsilon + \frac{r_1}{2} \epsilon \right] + b_2 \left[-\alpha_1^2 + \frac{\alpha_1 r_1}{2} \epsilon + \beta + 3r(1 - \frac{\epsilon}{2}) \right.$$
$$\left. - 2 + \frac{1}{2} (r_2 + \frac{3}{4\alpha_1}) \epsilon \right] = 0$$

$$a_2 \left[-\alpha_1 \beta + \frac{\alpha_1}{2} (r_1 + \frac{3}{4\alpha_1}) \epsilon + \frac{r_2}{2} \epsilon \right] + b_1 \left[-\alpha_1^2 - \frac{\alpha_1 r_1}{2} \epsilon \right.$$
$$\left. + \beta + 1 - \frac{1}{2} (r_1 - \frac{3}{4\alpha_1}) \epsilon \right] = 0 \qquad (3.44)$$

$$a_2\left[-\alpha_1^2 - \frac{\alpha_1 r_1}{2}\epsilon +\beta+3r(1-\frac{\epsilon}{2}) - 2 - \frac{\epsilon}{2}(r_2 + \frac{3}{4\alpha_1})\right]$$

$$+ b_1\left[-\alpha_1\beta + \frac{\alpha_1}{2}(r_1 + \frac{3}{4\alpha_1})\epsilon + \frac{r_1}{2}\epsilon\right] = 0$$

For nontrivial solutions to exist, the determinant of the coefficients must be zero, which in this case can be expressed as the product of the following two 2×2 determinants.

$$\begin{vmatrix} -\alpha_1^2+\beta+1+\frac{\epsilon}{2}[\alpha_1 r_1+r_1 - \frac{3}{4\alpha_1}] & \alpha_1\beta + \frac{\epsilon}{2}[\alpha_1 r_1+r_2 - \frac{3}{4}] \\ \alpha_1\beta + \frac{\epsilon}{2}[\alpha_1 r_1+r_1 + \frac{3}{4}] & -\alpha_1^2+\beta+3r(1-\frac{\epsilon}{2}) - 2 + \frac{\epsilon}{2}[\alpha_1 r_1+r_2 + \frac{3}{4\alpha_1}] \end{vmatrix} = 0$$

$$\begin{vmatrix} -\alpha_1\beta + \frac{\epsilon}{2}[\alpha_1 r_1+r_2 - \frac{3}{4}] & -\alpha_1^2+\beta+1 - \frac{\epsilon}{2}[\alpha_1 r_1+r_1 - \frac{3}{4\alpha_1}] \\ -\alpha_1^2+\beta+3r(1 - \frac{\epsilon}{2}) - 2 - \frac{\epsilon}{2}[\alpha_1 r_1+r_2 + \frac{3}{4\alpha_1}] & -\alpha_1\beta + \frac{\epsilon}{2}[\alpha_1 r_1+r_1 + \frac{3}{4}] \end{vmatrix} = 0$$

$$(3.45)$$

The definition of α_1 at the boundary of the resonance instability region as given in a fashion similar to Eq. (3.40) will be used.

$$\alpha_1 = \bar{w}_i + \Delta_1\epsilon + \dots \tag{3.46}$$

The natural frequencies of the system are given in Eq. (3.41). After expanding the above determinants Δ_1 can be evaluated on the boundary as follows

$$\Delta_1 = \pm \frac{1}{4\alpha_1(2\alpha_1^2-2\beta-3r+1-\beta^2)}\left[-\alpha_1(\alpha_1+\beta)(2\alpha_1 r_1+r_1+r_2)\right.$$

$$\left.+ (\beta+1)(\alpha_1 r_1+r_2 + \frac{3}{4\alpha_1}) + (\beta+3r-2)(\alpha_1 r_1+r_1 - \frac{3}{4\alpha_1})\right] \qquad (3.47)$$

Figure 3.5 shows the resulting stability boundary for $r = 1.5$.

We can see at this time that the analysis method based on an infinite determinant defines the resonance instability regions for which the natural frequency is approximately equal to a multiple of the frequency expressed in the periodic terms of the equations of motion

$$\omega_i \cong n\alpha_1 \qquad (3.48)$$

In its present form this method does not define the regions of the type

$$\omega_1 \pm \omega_2 \cong 2n\alpha_1 \qquad (3.49)$$

3.9   Determination of the Regions of Instability-Asymptotic Method

With the aid of Eq's. (3.11), the linearized equations of motion of the rigid unsymmetrical spinning body, Eq's. (3.8), may be expressed in the following form:

$$r = \frac{C}{A} = 1.5$$

Spin Ratio , $\alpha_1 = w_0/\Omega_0$

Region of
Instability,
$w_2 \cong \alpha_1$
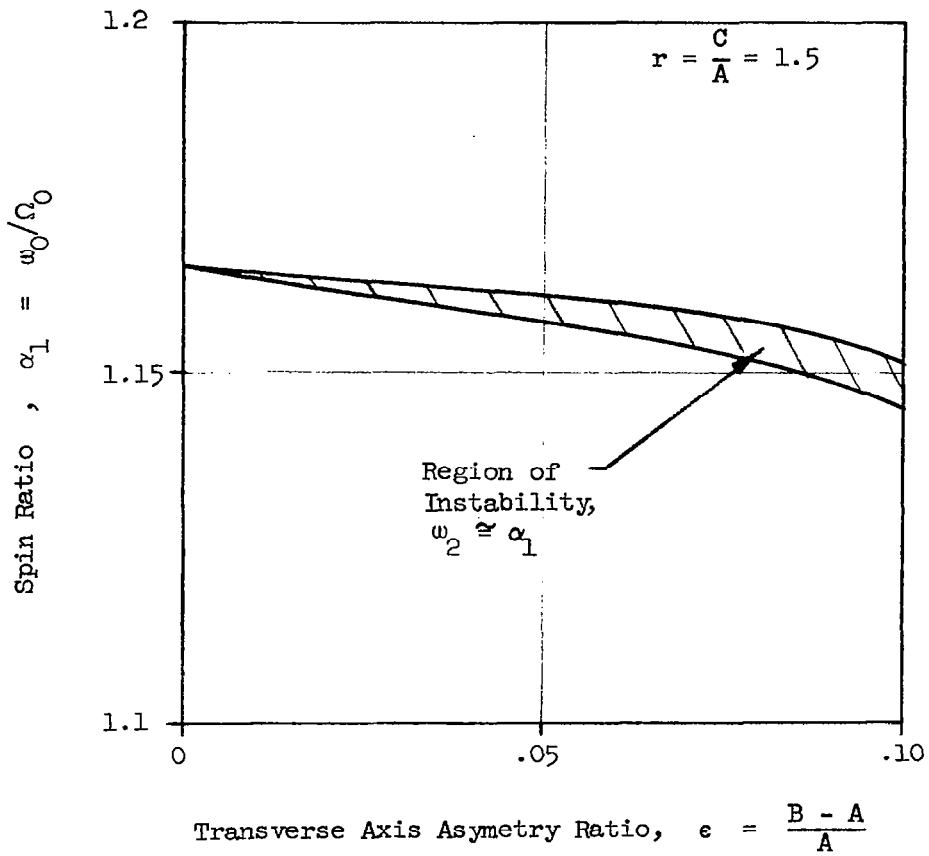
Transverse Axis Asymetry Ratio, $\epsilon = \frac{B - A}{A}$

FIGURE 3.5

UNSYMMETRICAL SPINNING SATELLITE RESONANCE
REGIONS FOR OSCILLATIONS WITH FREQUENCY α_1

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$$\theta_1'' + \beta_1 \theta_2' + \beta_2 \theta_1 = \epsilon\left\{ d_{11}(c2\alpha_1\tau)\ \theta_1 + d_{12}(s2\alpha_1\tau)\ \theta_2 \right.$$

$$+ d_{13}(s2\alpha_1\tau)\ \theta_1' - d_{11}(c2\alpha_1\tau)\ \theta_2'\bigg\} + \epsilon^2\bigg\{(d_{21}\ c2\alpha_1\tau$$

$$+ d_{22}\ c4\alpha_1\tau)\ \theta_1 + d_{23}(s4\alpha_1\tau)\ \theta_2 + d_{24}(s4\alpha_1\tau)\ \theta_1'$$

$$- (d_{21}\ c2\alpha_1\tau + d_{22}\ c4\alpha_1\tau)\ \theta_2'\bigg\} + \epsilon^3\bigg\{(d_{31}\ c2\alpha_1\tau + d_{32}\ c4\alpha_1\tau$$

$$+ d_{33}\ c6\alpha_1\tau)\ \theta_1 + (d_{34}\ s2\alpha_1\tau + d_{35}\ s6\alpha_1\tau)\ \theta_2 + (d_{36}\ s2\alpha_1\tau$$

$$+ d_{37}\ s6\alpha_1\tau)\ \theta_1' - (d_{31}\ c2\alpha_1\tau + d_{32}\ c4\alpha_1\tau + d_{33}\ c6\alpha_1\tau)\ \theta_2'\bigg\} + O(\epsilon^4)$$

$$(3.50)$$

$$\theta_2'' + \beta_3 \theta_1' + \beta_4 \theta_2 = \epsilon\bigg\{e_{11}(s2\alpha_1\tau)\theta_1 + e_{12}(c2\alpha_1\tau)\ \theta_2 + e_{13}(c2\alpha_1\tau\ \theta_1'$$

$$- e_{11}(s2\alpha_1\tau)\ \theta_2'\bigg\} + \epsilon^2\bigg\{e_{21}(s4\alpha_1\tau)\ \theta_1 + (e_{22}\ c2\alpha_1\tau + e_{23}\ c4\alpha_1\tau)\ \theta_2$$

$$+ (e_{22}\ c2\alpha_1\tau + e_{24}\ c4\alpha_1\tau)\ \theta_1' + e_{21}(s4\alpha_1\tau)\ \theta_2'\bigg\} + \epsilon^3\bigg\{(e_{31}\ s2\alpha_1\tau$$

$$+ e_{32}\ s6\alpha_1\tau)\ \theta_1 + (e_{33}\ c2\alpha_1\tau + e_{34}\ c4\alpha_1\tau + e_{35}\ c6\alpha_1\tau)\ \theta_2$$

$$+ (e_{36}\ c2\alpha_1\tau + e_{34}\ c4\alpha_1\tau + e_{37}\ c6\alpha_1\tau)\ \theta_1'$$

$$+ (e_{38} \, s2\alpha_1 \tau + e_{39} \, s6\alpha_1 \tau) \, \theta_2'\}$$

$$+ 0(\epsilon^4)$$

where all terms up to the 3rd power of ϵ have been included and

$$\beta_1 = 2 - r(1-\tfrac{\epsilon}{2})(1+\alpha_1) - \epsilon^2(1-\tfrac{r}{2}) \, \frac{3}{8r \, \alpha_1^2}$$

$$\beta_2 = -1 + r(1-\tfrac{\epsilon}{2})(1+\alpha_1) + \epsilon^2(1-\tfrac{r}{2}) \, \frac{3}{8r \, \alpha_1^2}$$

$$\beta_3 = -2 + r(1-\tfrac{\epsilon}{2})(1+\alpha_1) - \epsilon^2(1-\tfrac{r}{2}) \, \frac{3}{8r \, \alpha_1^2}$$

$$\beta_4 = -4 + r(1-\tfrac{\epsilon}{2})(4+\alpha_1) + \epsilon^2(1-\tfrac{r}{2}) \, \frac{3}{2r \, \alpha_1^2}$$

$$d_{11} = (1-\tfrac{r}{2})(1+\alpha_1) - \frac{3}{4\alpha_1}$$

$$d_{12} = -(1-\tfrac{r}{2})(4+\alpha_1)$$

$$d_{13} = -(1-\tfrac{r}{2})(1+\alpha_1)$$

$$d_{21} = \frac{3}{8\alpha_1}$$

$$d_{22} = - \frac{9}{64r \ \alpha_1^3} + (1-\frac{r}{2}) \ (1+2\alpha_1) \ \frac{3}{8r \ \alpha_1^2}$$

$$d_{23} = - (1-\frac{r}{2}) \ (4+2\alpha_1) \ \frac{3}{8 \ \alpha_1^2 r}$$

$$d_{24} = - (1-\frac{r}{2}) \ (1+2\alpha_1) \ \frac{3}{8 \ \alpha_1^2 r}$$

$$d_{31} = \frac{135}{1024r^2 \ \alpha_1^5} - (1-\frac{r}{2}) \ (1+3\alpha_1) \ \frac{9}{256r^2 \ \alpha_1^4}$$

$$d_{32} = \frac{9}{128r^2 \ \alpha_1^3}$$

(3.51)

$$d_{33} = - \frac{27}{1024r^2 \ \alpha_1^5} + (1-\frac{r}{2}) \ (1+7\alpha_1) \ \frac{9}{256r^2 \ \alpha_1^4}$$

$$d_{34} = (1-\frac{r}{2})(20+3\alpha_1) \ \frac{9}{256r^2 \ \alpha_1^4}$$

$$d_{35} = - (1-\frac{r}{2}) \ (4+3\alpha_1) \ \frac{27}{256r^2 \ \alpha_1^4}$$

$$d_{36} = (1-\frac{r}{2}) \ (5+3\alpha_1) \ \frac{9}{256r^2 \ \alpha_1^4}$$

$$d_{37} = -\left(1-\frac{r}{2}\right)\left(1+3\alpha_1\right)\frac{27}{256r^2\,\alpha_1^4}$$

$$e_{11} = -\left(1-\frac{r}{2}\right)\left(1+\alpha_1\right)$$

$$e_{12} = -\frac{3}{4\alpha_1} - \left(1-\frac{r}{2}\right)\left(4+\alpha_1\right)$$

$$e_{13} = -\frac{3}{4\alpha_1} - \left(1-\frac{r}{2}\right)\left(1+\alpha_1\right)$$

$$e_{21} = -\left(1-\frac{r}{2}\right)\left(1+2\alpha_1\right)\frac{3}{8r\,\alpha_1^2}$$

$$e_{22} = \frac{3}{8\alpha_1}$$

$$e_{23} = -\frac{9}{64r\,\alpha_1^3} - \left(1-\frac{r}{2}\right)\left(4+2\alpha_1\right)\frac{3}{8r\,\alpha_1^2}$$

$$e_{24} = -\frac{9}{64r\,\alpha_1^3} - \left(1-\frac{r}{2}\right)\left(1+2\alpha_1\right)\frac{3}{8r\,\alpha_1^2}$$

$$e_{31} = \left(1-\frac{r}{2}\right)\left(5+3\alpha_1\right)\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{32} = -\left(1-\frac{r}{2}\right)\left(1+3\alpha_1\right)\frac{27}{256r^2\,\alpha_1^4}$$

$$e_{33} = \frac{135}{1024r^2\,\alpha_1^5} + \left(1-\frac{r}{2}\right)\left(4+2\alpha_1\right)\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{34} = \frac{9}{128r\,\alpha_1^3}$$

$$e_{35} = -\frac{27}{1024r^2\,\alpha_1^5} - (1-\frac{r}{2})\,(4+7\alpha_1)\,\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{36} = \frac{27}{1024r^2\,\alpha_1^5} + (1-\frac{r}{2})\,(1+3\alpha_1)\,\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{37} = -\frac{27}{1024r^2\,\alpha_1^5} - (1-\frac{r}{2})\,(1+7\alpha_1)\,\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{38} = (1-\frac{r}{2})\,(5+7\alpha_1)\,\frac{9}{256r^2\,\alpha_1^4}$$

$$e_{39} = -(1-\frac{r}{2})\,(1+7\alpha_1)\,\frac{9}{256r^2\,\alpha_1^4}$$

The left side of each of Eq's. (3.50) may be considered as unperturbed equations, while the periodic perturbations are given on the right sides, and consist of periodic terms multiplied by displacements and velocities. When terms of first order in ϵ are considered, the perturbing terms can be seen to include coefficients of frequency $2\alpha_1$. Terms in ϵ^2 include periodic coefficients of frequency $2\alpha_1$ and $4\alpha_1$; terms in ϵ^3 include periodic coefficients of frequency $2\alpha_1$, $4\alpha_1$, and $6\alpha_1$; and if terms in higher powers of ϵ are considered, additional higher frequencies are included in the perturbations.

Two types of motion will be studied - nonresonant and resonant oscillations. The resonance oscillations will be further distinguished as to resonance frequencies that are approximately equal to natural frequencies and resonance frequencies that are approximately equal to the sum or difference of natural frequencies. The term natural frequency is given to the frequencies of oscillation of the system in the absence of the periodic terms and are given by

$$\bar{\omega}_i = \left\{ \frac{\beta_2 + \beta_4 - \beta_1 \beta_3}{2} + (-1)^{i+1} \left[\left(\frac{\beta_2 + \beta_4 - \beta_1 \beta_3}{2} \right)^2 - \beta_2 \beta_4 \right]^{1/2} \right\}^{1/2},$$

$$(i = 1, 2) \qquad\qquad (3.52)$$

We will designate $\bar{\omega}_1$ as the higher natural frequency and $\bar{\omega}_2$ as the lower one. The unperturbed equations admit solutions of the form

$$\theta_1 = a_1 \, c(\bar{\omega}_1 \, \tau + \delta_1) + a_2 \, c(\bar{\omega}_2 \, \tau + \delta_2)$$

$$(3.53)$$

$$\theta_2 = a_1 \lambda_1 \, s(\bar{\omega}_1 \tau + \delta_1) + a_2 \lambda_2 \, s(\bar{\omega}_2 \tau + \delta_2)$$

in which a_1 and a_2 are arbitrary amplitudes, δ_1 and δ_2 are arbitrary phase angles, and λ_1 and λ_2 are the ratios of amplitudes of θ_1 and θ_2 motion necessary to satisfy the unperturbed equations of motion and are given by

$$\lambda_i = (\bar{\omega}_i^2 - \beta_2)/\beta_1\bar{\omega}_i = \beta_3\bar{\omega}_i/(\bar{\omega}_i^2 - \beta_4), \quad (i = 1,2) \tag{3.54}$$

3.9.1 Nonresonance Oscillation

When the natural frequencies of the unperturbed system are sufficiently different from the frequencies appearing in the perturbation terms on the right-hand side of Eq's. (3.50), the equations of motion can be satisfied by solutions of the form

$$\theta_1 = a_1 \, c(\psi_1\tau+\delta_1) + a_2 \, c(\psi_2\tau+\delta_2) + u_{11}(\tau)\epsilon + u_{12}(\tau)\epsilon^2 + \ldots$$

$$\tag{3.55}$$

$$\theta_2 = a_1(\lambda_1+\epsilon\lambda_{11}+\epsilon^2\lambda_{12}+\ldots) \, s(\psi_1\tau+\delta_1) + a_2(\lambda_2+\epsilon\lambda_{21}+\epsilon^2\lambda_{22}+\ldots) \, s(\psi_2\tau+\delta_2)$$

$$+ u_{21}(\tau)\epsilon + u_{22}(\tau)\epsilon^2 + \ldots$$

in which the frequencies ψ_1 and ψ_2 are almost equal to the natural frequencies and are given by

$$\psi_1 = \bar{\omega}_1 + \Delta_{11}\epsilon + \Delta_{12}\epsilon^2 + \ldots$$

$$\psi_2 = \bar{\omega}_2 + \Delta_{21}\epsilon + \Delta_{22}\epsilon^2 + \dots \tag{3.56}$$

In the above assumed form of solution, a_1 and a_2 are arbitrary small amplitudes and δ_1 and δ_2 are arbitrary phase angles. The ratios of amplituded, given by the λ's, and the difference between the frequency of oscillation and the natural frequency of the unperturbed system, given in terms of the Δ_{ij}, are treated as constants. In a somewhat more general form the amplitude could be treated as a function of time, but formal substitution into the equations of motion and solution for the coefficients of the time-dependent terms in the amplitude shows the latter to be zero for nonresonant oscillation.

In the case of nonresonant oscillation no coupling occurs between the two principal oscillations (one near the frequency $\bar{\omega}_1$ and the other near $\bar{\omega}_2$) so one needs to consider only the case of a single oscillation. Consequently, the solution derived below will yield the solution with frequency of the principle oscillation near $\bar{\omega}_1$ when the coefficients are derived using $\bar{\omega}_1$ and will yield the solution with principle frequency near $\bar{\omega}_2$ when $\bar{\omega}_2$ is used in computing the coefficients. The assumed form of solution is given below and the expansion will be completed through all terms in ϵ^2. Hence, assume

$$\theta_1 = a\ c(\psi\ \tau+\delta) + u_{11}\ \epsilon + u_{12}\epsilon^2$$

$$\theta_2 = a(\lambda + \epsilon\lambda_1 + \epsilon^2\lambda_2)\, s(\psi\, \tau+\delta) + u_{21}\epsilon + u_{22}\epsilon^2$$

$$\psi = \bar{\omega} + \Delta_1\, \epsilon + \Delta_2\, \epsilon^2$$

$$\theta_1' = -a(\bar{\omega} + \epsilon\Delta_1 + \epsilon^2\Delta_2)\, s(\psi\, \tau+\delta) + u_{11}'\, \epsilon + u_{12}'\, \epsilon^2$$

$$\theta_2' = a\left[\bar{\omega}\,\lambda + \epsilon(\bar{\omega}\,\lambda_1 + \Delta_1\,\lambda) + \epsilon^2(\bar{\omega}^2\,\lambda_2 + \Delta_1\lambda_1 + \Delta_2\lambda)\right] c(\psi\, \tau+\delta)$$

$$+ u_{21}'\, \epsilon + u_{22}'\, \epsilon^2 \qquad\qquad (3.57)$$

$$\theta_1'' = -a\left[\bar{\omega}^2 + \epsilon(2\Delta_1\bar{\omega}) + \epsilon^2(\Delta_1^2 + 2\Delta_2\bar{\omega})\right] c(\psi\, \tau+\delta) + u_{11}''\epsilon + u_{12}''\epsilon^2$$

$$\theta_2'' = -a\left[\bar{\omega}^2\,\lambda + \epsilon(2\Delta_1\bar{\omega}\,\lambda + \bar{\omega}^2\,\lambda_1) + \epsilon^2(2\Delta_1\,\bar{\omega}\,\lambda_1 + \bar{\omega}^2\,\lambda_2 + \Delta_1^2\,\lambda\right.$$

$$\left. + 2\Delta_2\,\bar{\omega}\,\lambda)\right] s(\psi\, \tau+\delta) + u_{21}''\, \epsilon + u_{22}''\, \epsilon^2$$

Substituting into Eq's. (3.50) and noting that

$$\bar{\omega}^2 - \lambda\,\beta_1\,\bar{\omega} - \beta_2 = 0$$

$$\qquad\qquad (3.58)$$

$$\lambda\,\bar{\omega}^2 + \beta_3\,\bar{\omega} - \lambda\,\beta_4 = 0$$

we obtain*

---

*Although it appears that the equations are separated according to their order in ϵ, one should recall that u_{ij}' and u_{ij}'' contain terms in ϵ, $\epsilon^2\dots$.

$$\epsilon\left\{a\left[-2\Delta_1\overline{w} + \beta_1(\overline{w}\lambda_1 + \Delta_1\lambda)\right]c(\psi\tau+\delta) + u_{11}'' + \beta_1 u_{21}' + \beta_2 u_{11}\right\}$$

$$+ \epsilon^2\left\{a\left[-(\Delta_1^2 + 2\Delta_2\overline{w}) + (\overline{w}\lambda_2 + \Delta_1\lambda_1 + \Delta_2\lambda)\beta_1\right]c(\psi\tau+\delta) + u_{12}'' + \beta_1 u_{22}'\right.$$

$$\left. + \beta_2 u_{12}\right\} = \epsilon\left\{\frac{a}{2}\,c(2\alpha_1\tau+\psi\tau+\delta)\left[d_{11} - d_{12}\lambda + d_{13}\overline{w} - d_{11}\lambda\overline{w}\right]\right.$$

$$\left. + \frac{a}{2}\,c(2\alpha_1\tau-\psi\tau-\delta)\left[d_{11} + d_{12}\lambda - d_{13}\overline{w} - d_{11}\lambda\overline{w}\right]\right\}$$

$$+ \epsilon^2\left\{c2\alpha_1\tau(d_{11}u_{11} - d_{11}u_{21}') + s2\alpha_1\tau(d_{12}u_{21} + d_{13}u_{11}')\right. \qquad (3.59)$$

$$+ \frac{a}{2}\,c(2\alpha_1\tau+\psi\tau+\delta)\left[-d_{12}\lambda_1 + d_{13}\Delta_1 - d_{11}(\overline{w}\lambda_1 + \Delta_1\lambda) + d_{21}(1-\overline{w}\lambda)\right]$$

$$+ \frac{a}{2}\,c(2\alpha_1\tau-\psi\tau-\delta)\left[d_{12}\lambda_1 - d_{13}\Delta_1 - d_{11}(\overline{w}\lambda_1 + \Delta_1\lambda) + d_{21}(1-\overline{w}\lambda)\right]$$

$$+ \frac{a}{2}\,c(4\alpha_1\tau+\psi\tau+\delta)\left[d_{22} - d_{23}\lambda + d_{24}\overline{w} - d_{22}\overline{w}\lambda\right] + \frac{a}{2}\,c(4\alpha_1\tau-\psi\tau-\delta)\left[d_{22}\right.$$

$$\left.\left. + d_{23}\lambda - d_{24}\overline{w} - d_{22}\overline{w}\lambda\right]\right\}$$

$$\epsilon\left\{a\left[-(2\Delta_1\overline{w}\lambda + \overline{w}^2\lambda_1) - \Delta_1\beta_3 + \lambda_1\beta_4\right]s(\psi\tau+\delta) + u_{21}'' + \beta_3 u_{11}'\right.$$

$$\left. + \beta_4 u_{21}\right\} + \epsilon^2\left\{a\left[-(2\Delta_1\overline{w}\lambda_1 + \overline{w}^2\lambda_2 + \Delta_1^2\lambda) - \Delta_2\beta_3\right.\right.$$

$$\left.\left. + \lambda_2\beta_4\right]s(\psi\tau+\delta)\,u_{22}'' + \beta_3 u_{12}' + \beta_4 u_{22}\right\} = \epsilon\left\{\frac{a}{2}\,s(2\alpha_1\tau\right.$$

$$+\psi\tau+\delta)\left[e_{11} + e_{12}\lambda - e_{13}\overline{\omega} - e_{11}\overline{\omega}\lambda\right] + \frac{a}{2}\,s(2\alpha_1\tau-\psi\tau-\delta)\left[e_{11} - e_{12}\lambda\right.$$

$$\left.\left. + e_{13}\overline{\omega} - e_{11}\lambda\overline{\omega}\right]\right\} + \epsilon^2\left\{s\,2\alpha_1\tau(e_{11}u_{11}-e_{11}u_{21}) + c\,2\alpha_1\tau(e_{12}u_{21}\right.$$

$$+ e_{13}u_{11}) + \frac{a}{2}\,s(2\alpha_1\tau+\psi\tau+\delta)\left[e_{12}\lambda_1 - e_{13}\Delta_1 - e_{11}(\overline{\omega}\lambda_1+\Delta_1\lambda)\right.$$

$$\left. + e_{22}(\lambda-\overline{\omega})\right] + \frac{a}{2}\,s(2\alpha_1\tau-\psi\tau-\delta)\left[-e_{12}\lambda_1 + e_{13}\Delta_1 - e_{11}(\overline{\omega}\lambda_1+\Delta_1\lambda)\right.$$

$$\left. - e_{22}(\lambda-\overline{\omega})\right] + \frac{a}{2}\,s(4\alpha_1\tau+\psi\tau+\delta)\left[e_{21} + e_{23}\lambda - e_{24}\overline{\omega} + e_{21}\overline{\omega}\lambda\right]$$

$$+ \frac{a}{2}\,s(4\alpha_1\tau-\psi\tau-\delta)\left[e_{21} - e_{23}\lambda + e_{24}\overline{\omega} + e_{21}\overline{\omega}\lambda\right]\right\}$$

Equating the coefficients of the first power of ϵ in Eq's. (3.59) and requiring that u_{11} and u_{21} include the terms of frequency other than ψ gives

$$2\Delta_1\overline{\omega} + \beta_1(\overline{\omega}\lambda_1 + \Delta_1\lambda) = 0$$

$$- (2\Delta_1\,\overline{\omega}\lambda+\overline{\omega}^2\lambda_1) - \Delta_1\beta_3 + \lambda_1\beta_4 = 0$$

$$u''_{11} + u'_{21}\beta_1 + u_{11}\beta_2 = \frac{a}{2}\,c(2\alpha_1\tau+\psi\tau+\delta)\left[d_{11} - d_{12}\lambda + d_{13}\overline{\omega} - d_{11}\lambda\overline{\omega}\right]$$

$$+ \frac{a}{2} \, c(2\alpha_1 \tau - \psi \tau - \delta) \left[d_{11} + d_{12}\lambda - d_{13}\bar{\omega} - d_{11}\lambda\bar{\omega} \right] \qquad (3.60)$$

$$u_{21}'' + u_{11}'\beta_3 + u_{21}\beta_4 = \frac{a}{2} \, s(2\alpha_1\tau + \psi\tau + \delta) \left[e_{11} + e_{12}\lambda - e_{13}\bar{\omega} - e_{11}\bar{\omega}\lambda \right]$$

$$+ \frac{a}{2} \, s(2\alpha_1\tau - \psi\tau - \delta) \left[e_{11} - e_{12}\lambda + e_{13}\bar{\omega} - e_{11}\lambda\bar{\omega} \right]$$

The first two of Eq's. (3.60) are satisfied only if

$$\Delta_1 = \lambda_1 = 0 \qquad (3.61)$$

The second two of Eq's (3.60) are satisfied, ignoring higher order terms in ϵ arising in the total derivative terms of u_{11} and u_{22}, when

$$u_{11} = f_{11} \, a \, c(2\alpha_1\tau + \psi\tau + \delta) + f_{12} \, a \, c(2\alpha_1\tau - \psi\tau - \delta)$$

$$(3.62)$$

$$u_{12} = g_{11} \, a \, s(2\alpha_1\tau + \psi\tau + \delta) + g_{12} \, a \, s(2\alpha_1\tau - \psi\tau - \delta)$$

in which

$$f_{11} = \frac{1}{2} \, \frac{1}{\left[\beta_2 - (2\alpha_1 + \bar{\omega})^2\right]\left[\beta_4 - (2\alpha_1 + \bar{\omega})^2\right] + \beta_1\beta_3(2\alpha_1 + \bar{\omega})^2} \left\{ d_{11}(1 - \lambda\bar{\omega}) \right.$$

$$- d_{12}\lambda + d_{13}\bar{\omega} \right] \left[\beta_4 - (2\alpha_1 + \bar{\omega})^2\right] - \left[e_{11}(1 - \lambda\bar{\omega}) + e_{12}\lambda \right.$$

$$- e_{13}\overline{w}\Big]\Big[\beta_1(2\alpha_1+\overline{w})\Big]\Big\}$$

$$g_{11} = \frac{1}{2}\,\frac{1}{\Big[\beta_2 - (2\alpha_1+\overline{w})^2\Big]\Big[\beta_4-(2\alpha_1+\overline{w})^2\Big] + \beta_1\beta_3(2\alpha_1+\overline{w})^2}\,\Big\{\Big[d_{11}(1-\lambda\overline{w})$$

$$- d_{12}\lambda + d_{13}\overline{w}\Big]\Big[\beta_3(2\alpha_1+\overline{w})\Big] + \Big[e_{11}(1-\lambda\overline{w}) + e_{12}\lambda - e_{13}\overline{w}\Big]\Big[\beta_2$$

$$- (2\alpha_1+\overline{w})^2\Big]$$

$$(3.63)$$

$$f_{12} = \frac{1}{2}\,\frac{1}{\Big[\beta_2 - (2\alpha_1-\overline{w})^2\Big]\Big[\beta_4-(2\alpha_1-\overline{w})^2\Big] + \beta_1\beta_3(2\alpha_1-\overline{w})^2}\,\Big\{\Big[d_{11}(1-\lambda\overline{w})$$

$$+ d_{12}\lambda - d_{13}\overline{w}\Big]\Big[\beta_4 - (2\alpha_1-\overline{w})^2\Big] - \Big[e_{11}(1-\overline{w}\lambda) - e_{12}\lambda$$

$$+ e_{13}\overline{w}\Big]\Big[\beta_1(2\alpha_1-\overline{w})\Big]\Big\}$$

$$g_{12} = \frac{1}{2}\,\frac{1}{\Big[\beta_2-(2\alpha_1-\overline{w})^2\Big]\Big[\beta_4-(2\alpha_1-\overline{w})^2\Big] + \beta_1\beta_3(2\alpha_1-\overline{w})^2}\,\Big\{\Big[d_{11}(1-\lambda\overline{w})$$

$$+ d_{12}\lambda - d_{13}\overline{w}\Big]\Big[\beta_3(2\alpha_1-\overline{w})\Big] + \Big[e_{11}(1-\overline{w}\lambda) - e_{12}\lambda + e_{13}\overline{w}\Big]\Big[\beta_2-(2\alpha_1-\overline{w})^2\Big]\Big\}$$

Next the coefficients of ϵ^2 terms in Eq's. (3.59) can be equated.[*]

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[*]Note that, in general, terms arising from the derivatives of u_{11} and u_{22} must be included.

After substitution of Eq's (3.61) through (3.63) and applying harmonic balance the result obtained is given by the following four relations

$$\Delta_2(\lambda\beta_1 - 2\overline{\omega}) + \lambda_2\overline{\omega}\beta_1 = \frac{1}{2}\left\{ d_{11}(f_{11}+f_{12}) + d_{12}(g_{11}+g_{12}) \right.$$

$$\left. -d_{13}\left[f_{11}(2\alpha_1+\overline{\omega}) + f_{12}(2\alpha_1-\overline{\omega})\right] - d_{11}\left[g_{11}(2\alpha_1+\overline{\omega}) + g_{12}(2\alpha_1-\overline{\omega})\right]\right\}$$

$$\Delta_2(-2\overline{\omega}\lambda-\beta_3) + \lambda_2(\beta_4-\overline{\omega}^2) = \frac{1}{2}\left\{ e_{11}(-f_{11}+f_{12}) + e_{12}(g_{11}-g_{12}) \right.$$

$$\left. -e_{13}\left[f_{11}(2\alpha_1+\overline{\omega}) - f_{12}(2\alpha_1-\overline{\omega})\right] - d_{11}\left[-g_{11}(2\alpha_1-\overline{\omega}) + g_{12}(2\alpha_1-\overline{\omega})\right]\right.$$

$$(3.64)$$

$$u_{12}'' + \beta_1 u_{22}' + \beta_2 u_{12} = \frac{a}{2}\,c(2\alpha_1\tau+\psi\tau+\delta)\left[d_{21}(1-\overline{\omega}\lambda)\right] + \frac{a}{2}\,c(2\alpha_1\tau$$

$$-\psi\tau-\delta)\left[d_{21}(1-\overline{\omega}\lambda)\right] + \frac{a}{2}\,c(4\alpha_1\tau+\psi\tau+\delta)\left[d_{22}(1-\overline{\omega}\lambda)-d_{23}\lambda+d_{24}\overline{\omega}+d_{11}f_{11}\right.$$

$$- d_{12}g_{11} + d_{13}(2\alpha_1+\overline{\omega})\,f_{11} - d_{11}(2\alpha_1+\overline{\omega})\,g_{11}\right] + \frac{a}{2}\,c(4\alpha_1\tau$$

$$-\psi\tau-\delta)\left[d_{22}(1-\overline{\omega}\lambda) + d_{23}\lambda - d_{24}\overline{\omega} + d_{11}f_{12} - d_{12}g_{12}\right.$$

$$+ d_{13}(2\alpha_1-\overline{w}) \, f_{12} - d_{11}(2\alpha_1-\overline{w}) \, g_{12} \Big] \qquad\qquad (3.65)$$

$$u''_{22} + u'_{12}\beta_3 + u_{22}\beta_4 = \frac{a}{2} \, s(2\alpha_1\tau+\psi\tau+\delta) \left[e_{22}(\lambda-\overline{w}) \right]$$

$$+ \frac{a}{2} \, s(2\alpha_1\tau-\psi\tau-\delta) \left[e_{22}(\overline{w}-\lambda) \right] + \frac{a}{2} \, s(4\alpha_1\tau-\psi\tau-\delta) \left[e_{21}(1+\overline{w}\lambda) \right.$$

$$+ e_{23}\lambda - e_{24}\overline{w} - e_{11}f_{11} - e_{12}g_{11} + e_{13}(2\alpha_1+\overline{w}) \, f_{11} + e_{11}(2\alpha_1$$

$$+ \overline{w}) \, g_{11} \Big] + \frac{a}{2} \, s(4\alpha_1\tau-\psi\tau-\delta) \left[e_{21}(1+\overline{w}\lambda) - e_{23}\lambda + e_{24}\overline{w} - e_{11}f_{12} \right.$$

$$- e_{12}g_{12} + e_{13}(2\alpha_1-\overline{w}) \, f_{12} + e_{11}(2\alpha_1-\overline{w})g_{12} \Big]$$

Equations (3.64) give the following values for Δ_2 and λ_2

$$\Delta_2 = \frac{1}{2} \; \frac{1}{(\lambda\beta_1-2\overline{w})(\beta_4-\overline{w}^2) + (2\overline{w}\lambda+\beta_3) \, \beta_1\overline{w}} \left\{ \Big\langle d_{11}(f_{11}+f_{12}) \right.$$

$$+ d_{12}(g_{11}+g_{12}) - d_{13}\Big[f_{11}(2\alpha_1+\overline{w}) + f_{12}(2\alpha_1-\overline{w}) \Big] - d_{11}\Big[g_{11}(2\alpha_1+\overline{w})$$

$$+ g_{12}(2\alpha_1-\overline{w}) \Big] \Big\rangle (\beta_4-\overline{w}^2) - \Big\langle e_{11}(-f_{11}+f_{12}) + e_{12}(g_{11}-g_{12}) \; \cdot$$

$$- e_{13}\Big[f_{11}(2\alpha_1+\overline{w}) - f_{12}(2\alpha_1-\overline{w}) \Big] - e_{11}\Big[-g_{11}(2\alpha_1+\overline{w}) \qquad (3.66)$$

$$+ g_{12}(2\alpha_1 - \overline{\omega})\Big]\Big\rangle \overline{\omega}\beta_1\Big\}$$

$$\lambda_2 = \frac{1}{2} \; \frac{1}{(\lambda\beta_1 - 2\overline{\omega})(\beta_4 - \overline{\omega}^2) + (2\overline{\omega}\lambda + \beta_3)\;\beta_1\overline{\omega}} \Big\{\Big\langle d_{11}(f_{11} + f_{12})$$

$$+ d_{12}(g_{11} + g_{12}) - d_{13}\Big[f_{11}(2\alpha_1 + \overline{\omega}) + f_{12}(2\alpha_1 - \overline{\omega})\Big] - d_{11}\Big[g_{11}(2\alpha_1$$

$$+ \overline{\omega}) + g_{12}(2\alpha_1 - \overline{\omega})\Big]\Big\rangle(2\overline{\omega}\lambda + \beta_3) + \Big\langle e_{11}(-f_{11} + f_{12}) + e_{12}(g_{11} - g_{12})$$

$$- e_{13}\Big[f_{11}(2\alpha_1 + \overline{\omega}) - f_{12}(2\alpha_1 - \overline{\omega})\Big] - e_{11}\Big[-g_{11}(2\alpha_1 + \overline{\omega})$$

$$+ g_{12}(2\alpha_1 - \overline{\omega})\Big]\Big\rangle(\lambda\beta_1 - 2\overline{\omega})\Big\}$$

and from Eq's. (3.65) one can obtain u_{12} and u_{22}

$$u_{12} = f_{21}\; a\; c(2\alpha_1\tau + \psi\tau + \delta) + f_{12}\; a\; c(2\alpha_1\tau - \psi\tau - \delta)$$

$$+ f_{23}\; a\; c(4\alpha_1\tau + \psi\tau + \delta) + f_{24}\; a\; c(4\alpha_1\tau - \psi\tau - \delta)$$

$$(3.67)$$

$$u_{22} = g_{21}\; a\; s(2\alpha_1\tau + \psi_1\tau + \delta) + g_{22}\; a\; s(2\alpha_1\tau - \psi\tau - \delta)$$

$$+ g_{23} \, a \, s(4\alpha_1 \tau + \psi \tau + \delta) + g_{24} \, a \, s(4\alpha_1 \tau - \psi \tau - \delta)$$

in which

$$f_{21} = \frac{1}{2} \; \frac{d_{21}(1-\overline{\omega}\lambda)\left[\beta_4 - (2\alpha_1 + \overline{\omega})^2\right] - e_{22}(\lambda - \overline{\omega})\left[\beta_1(2\alpha_1 + \overline{\omega})\right]}{\left[\beta_2 - (2\alpha_1 + \overline{\omega})^2\right]\left[\beta_4 - (2\alpha_1 + \overline{\omega})^2\right] - \beta_1\beta_3(2\alpha_1 + \overline{\omega})^2}$$

$$g_{21} = \frac{1}{2} \; \frac{d_{21}(1-\overline{\omega}\lambda)\left[\beta_3(2\alpha_1 + \overline{\omega})\right] + e_{22}(\lambda - \overline{\omega})\left[\beta_2 - (2\alpha_1 + \overline{\omega})^2\right]}{\left[\beta_2 - (2\alpha_1 + \overline{\omega})^2\right]\left[\beta_4 - (2\alpha_1 + \overline{\omega})^2\right] - \beta_1\beta_3(2\alpha_1 + \overline{\omega})^2}$$

$$f_{22} = \frac{1}{2} \; \frac{d_{21}(1-\lambda\overline{\omega})\left[\beta_4 - (2\alpha_1 - \overline{\omega})^2\right] + e_{22}(\lambda - \overline{\omega})\left[\beta_1(2\alpha_1 - \overline{\omega})\right]}{\left[\beta_2 - (2\alpha_1 - \overline{\omega})^2\right]\left[\beta_4 - (2\alpha_1 - \overline{\omega})^2\right] - \beta_1\beta_3(2\alpha_1 - \overline{\omega})^2}$$

$$g_{22} = \frac{1}{2} \; \frac{d_{21}(1-\lambda\omega)\left[\beta_3(2\alpha_1 - \overline{\omega})\right] - e_{22}(\lambda - \overline{\omega})\left[\beta_2 - (2\alpha_1 - \overline{\omega})^2\right]}{\left[\beta_2 - (2\alpha_1 - \overline{\omega})^2\right]\left[\beta_4 - (2\alpha_1 - \overline{\omega})^2\right] - \beta_1\beta_3(2\alpha_1 - \overline{\omega})^2}$$

$$f_{23} = \frac{1}{2} \cdot \frac{1}{\left[\beta_2 - (4\alpha_1 + \bar{w})^2\right]\left[\beta_4 - (4\alpha_1 + \bar{w})^2\right] - \beta_1\beta_3(4\alpha_1 + w)^2} \left\{\left[d_{22}(1\right.\right.$$

$$\left. -\bar{w}\lambda) - d_{23}\lambda + d_{24}\bar{w} + d_{11}f_{11} - d_{12}g_{11} + d_{13}(2\alpha_1 + \bar{w})\, f_{11}\right.$$

$$\left. - d_{11}(2\alpha_1 + \bar{w})\, g_{11}\right]\left[\beta_4 - (4\alpha_1 + \bar{w})^2\right] - \left[e_{21}(1 + \bar{w}\lambda) + e_{23}\lambda\right.$$

$$\left. - e_{24}\bar{w} - e_{11}f_{11} - e_{12}g_{11} + e_{13}(2\alpha_1 + \bar{w})\, f_{11} + e_{11}(2\alpha_1\right.$$

$$\left. \left. + \bar{w})\, g_{11}\right]\left[\beta_1(4\alpha_1 + \bar{w})\right]\right\}$$

$$(3.68)$$

$$g_{23} = \frac{1}{2} \cdot \frac{1}{\left[\beta_2 - (4\alpha_1 + \bar{w})^2\right]\left[\beta_4 - (4\alpha_1 + \bar{w})^2\right] - \beta_1\beta_3(4\alpha_1 + \bar{w})^2} \left\{\left[d_{22}(1\right.\right.$$

$$\left. -\bar{w}\lambda) - d_{23}\lambda + d_{24}\bar{w} + d_{11}f_{11} - d_{12}g_{11} + d_{13}(2\alpha_1 + \bar{w})f_{11}\right.$$

$$\left. - d_{11}(2\alpha_1 + \bar{w})g_{11}\right]\left[\beta_3(4\alpha_1 + \bar{w})\right] + \left[e_{21}(1 + \bar{w}\lambda) + e_{23}\lambda - e_{24}\bar{w}\right.$$

$$\left. \left. - e_{11}f_{11} - e_{12}g_{11} + e_{13}(2\alpha_1 + \bar{w})\, f_{11} + e_{11}(2\alpha_1 + \bar{w})\, g_{11}\right]\left[\beta_2\right.\right.$$

$$- (4\alpha_1 + \overline{w})^2 \Big] \Big\}$$

$$f_{24} = \frac{1}{2} \ \frac{1}{\Big[\beta_2 - (4\alpha_1 - \overline{w})^2\Big]\Big[\beta_4 - (4\alpha_1 - \overline{w})^2\Big] - \beta_1\beta_3(4\alpha_1 - w)^2} \ \Big\{ \ \Big[d_{22}(1$$

$$-w\lambda) + d_{23}\lambda - d_{24}\overline{w} + d_{11}f_{12} - d_{12}g_{12} + d_{13}(2\alpha_1 - \overline{w}) \ f_{12}$$

$$- d_{11}(2\alpha_1 - \overline{w}) \ g_{12}\Big]\Big[\beta_4 - (4\alpha_1 - \overline{w})^2\Big] - \Big[e_{21}(1 + \overline{w}\lambda) - e_{23}\lambda$$

$$+ e_{24}\overline{w} - e_{11}f_{12} - e_{12}g_{12} + e_{13}(2\alpha_1 - \overline{w}) \ f_{12} + e_{11}(2\alpha_1$$

$$-w) \ g_{12}\Big] \ \Big[\beta_1(4\alpha_1 - w)\Big]\Big\}$$

$$g_{24} = \frac{1}{2} \ \frac{1}{\Big[\beta_2 - (4\alpha_1 + \overline{w})^2\Big]\Big[\beta_4 - (4\alpha_1 - \overline{w})^2\Big] - \beta_1\beta_3(4\alpha_1 - \overline{w})^2} \ \Big\{ \Big[d_{22}(1$$

$$-\overline{w}\lambda) + d_{23}\lambda - d_{24}\overline{w} + d_{11}f_{12} - d_{12}g_{12} + d_{13}(2\alpha_1 - \overline{w}) \ f_{12}$$

$$- d_{11}(2\alpha_1 - \overline{w}) \ g_{12}\Big]\Big[\beta_3(4\alpha_1 - \overline{w})\Big] \ + \Big[e_{21} \ (1 + \overline{w}\lambda) - e_{23}\lambda + e_{24}\overline{w}$$

$$- e_{11}f_{12} - e_{12}g_{12} + e_{13}(2\alpha_1 - \overline{w}) \ f_{12} + e_{11}(2\alpha_1 - \overline{w})g_{12}\Big]\Big[\beta_2$$

$$- (4\alpha_1-\omega)^2 \Big]\Big\}$$

Consequently the nonresonant solution of Eq's. (3.50), showing terms through the second power of ϵ, can be written as

$$\theta_1 = a\Big\{c(\psi\tau+\delta) + (f_{11}\epsilon+f_{21}\epsilon^2)\ c(2\alpha_1\tau+\psi\tau+\delta) + (f_{12}\epsilon$$

$$+f_{22}\epsilon^2\ c(2\alpha_1\tau-\psi\tau-\delta) + f_{23}\epsilon^2\ c(4\alpha_1\tau+\psi\tau+\delta) + f_{24}\epsilon^2\ c(4\alpha_1\tau-\psi\tau-\delta)$$

$$+ O(\epsilon^3)\Big\}$$

$$(3.69)$$

$$\theta_2 = a\Big\{(\lambda+\epsilon^2\lambda_2)\ s(\psi\tau+\delta) + (g_{11}\epsilon+g_{21}\epsilon^2)\ s(2\alpha_1\tau+\psi\tau+\delta) + (g_{12}\epsilon$$

$$+ g_{22}\epsilon^2)\ s(2\alpha_1\tau-\psi\tau-\delta) + g_{23}\epsilon^2\ s(4\alpha_1\tau+\psi\tau+\delta) + g_{24}\epsilon^2\ s(4\alpha_1\tau$$

$$- \psi\tau-\delta) + O(\epsilon^3)\Big\}$$

$$\psi = \bar{\bar{\omega}} + \Delta_2\epsilon^2 + O(\epsilon^3)$$

Note that the phase angle δ is completely arbitrary in the nonresonant solution. It can be seen that the frequency of the principal oscillation is slightly different from the natural frequency of the unperturbed system the difference being a term in the second (and higher) power of the small parameter, ϵ. Additional oscillatory terms appear, but with smaller amplitudes. Terms of frequency $2\alpha_1 \pm \psi$ appear with a coefficient in the first and higher powers of ϵ, terms of frequency $4\alpha_1 \pm \psi$ appear with a coefficient in the second and higher power of ϵ, and it is apparent from the recursive process used to determine the oscillatory terms that terms of frequency $2n\alpha_1 \pm \psi$ appear with a coefficient in the nth and higher powers of ϵ. It is instructive to look at the typical size of the coefficients appearing in the nonresonant solution, Eq's. (3.69). Taking as an example the nonresonant solution with principal frequency near the first natural frequency and with the following parameters:

$$r = 1.5, \qquad \alpha_1 = 1.0, \qquad \epsilon = 0.1 \qquad (3.70)$$

One obtains

$$\bar{\omega} = \bar{\omega}_1 = 2.1$$

$$(3.71)$$

$$\lambda = -1.433$$

and the solution is

$$\psi = 1.00073\bar{\omega}$$

$$\theta_1 = a\left\{c(\psi\tau+\delta) + .0163\ c(2\alpha_1\tau+\psi\tau+\delta) + .0586\ c(2\alpha_1\tau-\psi\tau-\delta)\right.$$

$$\left. + .00004\ c(4\alpha_1\tau+\psi\tau+\delta) - .0062\ c(4\alpha_1\tau-\psi\tau-\delta) + 0(\varepsilon^3)\right\}$$

$$\theta_2 = a\left\{-1.435\ s(\psi\tau+\delta) - .0164\ s(2\alpha_1\tau+\psi\tau+\delta) - .1203\ s(2\alpha_1\tau-\psi\tau-\delta)\right.$$

$$\left. - .00003\ s(4\alpha_1\tau+\psi\tau+\delta) + .0044\ s(4\alpha_1\tau-\psi\tau-\delta) + 0(\varepsilon^3)\right\}$$

$$(3.72)$$

In spite of the fact that a relatively large value of ε has been chosen in the example, (10 percent difference in the transverse moments of inertia of the rigid body) the frequency of the principal oscillation is only slightly different from the natural frequency of the unperturbed system, and the relative amplitudes of the oscillations at other frequencies are seen to be small. Thus the coefficients of higher frequency terms are expected to be extremely small, since they appear only in terms with coefficients including ε to a high power as a factor.

3.9.2 Resonance Oscillations

In the analysis of the previous section, the form of the solution would have been entirely different if the frequency of oscillation, ψ_1, had been equal to one of the frequencies appearing on the right side of the perturbed equations. The phase angle between the resonant motion and the periodic coefficients becomes important in this case. Also, the amplitude of the motion must, in general, be a function of time. Consequently, we may investigate a solution of the form

$$\theta_1 = (b + b_1 \tau \epsilon + b_2 \tau \epsilon^2 + \ldots)\ c(\Phi \tau + \delta) + u_{11}(\tau)\ \epsilon + u_{12}(\tau)\ \epsilon^2 + \ldots$$

$$(3.73)$$

$$\theta_2 = (b + b_1 \tau \epsilon + b_2 \tau \epsilon^2 + \ldots)\ (\lambda + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \ldots)\ s(\Phi \tau + \delta + \delta_1 \epsilon + \delta_2 \epsilon^2 + \ldots)$$

$$+ u_{21}(\tau)\ \epsilon + u_{22}(\tau)\ \epsilon^2 + \ldots$$

In the interest of brevity this analysis is shown for the first or "principal" region of instability for which terms only through the first power of ϵ are required. This occurs when a natural frequency is nearly equal to α_1, so that the frequency of oscillation is

$$\Phi = \alpha_1 = \overline{\omega} + \Delta_1 \epsilon + \Delta_2 \epsilon^2 + \ldots \qquad (3.74)$$

Introducing the assumed solution into Eq's. (3.50) and again using Eq's. (3.58) one obtains

$$b\ c(\alpha_1 \tau + \delta)\left[-2\Delta_1\overline{\omega} + \beta_1\lambda_1\overline{\omega} + \beta_1\Delta_1\lambda\right] + s(\alpha_1\tau+\delta)\left[b_1(-2\overline{\omega}+\beta_1\lambda)\right.$$

$$\left. - b\ \delta_1\lambda\overline{\omega}\beta_1\right] + u_{11}'' + u_{21}'\beta_1 + u_{11}\beta_2 = \frac{b}{2}\ c(3\alpha_1\tau+\delta)\left[d_{11}(1-\lambda\overline{\omega})\right.$$

$$\left. - d_{12}\lambda + d_{13}\overline{\omega}\right] + \frac{b}{2}\ c(\alpha_1\tau-\delta)\left[d_{11}(1-\lambda\overline{\omega}) + d_{12}\lambda - d_{13}\overline{\omega}\right]$$

$$(3.75)$$

$$b\ s(\alpha_1\tau+\delta)\left[-2\overline{\omega}\Delta_1\lambda - \omega^2\lambda_1 - \beta_3\Delta_1 + \beta_4\lambda_1\right] + c(\alpha_1\tau+\delta)\left[b_1(2\lambda\overline{\omega}+\beta_3)\right.$$

$$\left. + b(\beta_4-\overline{\omega}^2)\ \delta_1\lambda\right] + u_{21}'' + u_{11}'\beta_3 + u_{21}\beta_4 = \frac{b}{2}\ s(3\alpha_1\tau+\delta)\left[e_{11}(1-\lambda\overline{\omega})\right.$$

$$\left. + e_{12}\ \lambda - e_{13}\overline{\omega}\right] + \frac{b}{2}\ s(\alpha_1\tau-\delta)\left[e_{11}(1-\lambda\overline{\omega}) - e_{12}\lambda + e_{13}\overline{\omega}\right]$$

The quantities u_{11} and u_{21} can be selected so as to include the non-resonant oscillation of frequency $3\alpha_1$ and to include no resonant motion of frequency α_1. The coefficients of $\cos \alpha_1\tau$ and $\sin \alpha_1\tau$

(the resonant terms) can be equated to zero giving the following four equations.

$$\left[\Delta_1(\beta_1\lambda - 2\overline{\omega}) + \lambda_1\beta_1\overline{\omega}\right]c\delta + \left[\frac{b_1}{b}(\beta_1\lambda - 2\overline{\omega}) - \delta_1\lambda\overline{\omega}\beta_1\right]s\delta = \frac{1}{2}\left[d_n\right]c\delta$$

$$\left[\Delta_1(\beta_1\lambda - 2\overline{\omega}) + \lambda_1\beta_1\overline{\omega}\right]s\delta + \left[\frac{b_1}{b}(\beta_1\lambda - 2\overline{\omega}) - \delta_1\lambda\overline{\omega}\beta_1\right]c\delta = \frac{1}{2}\left[d_n\right]s\delta$$

$$(3.76)$$

$$\left[\Delta_1(-\beta_3 - 2\overline{\omega}\lambda) + \lambda_1(\beta_4 - \overline{\omega}^2)\right]s\delta + \left[\frac{b_1}{b}(2\lambda\overline{\omega} + \beta_3) + \delta(\beta_4 - \overline{\omega}^2)\lambda\right]c\delta$$

$$= -\frac{1}{2}\left[e_n\right]s\delta$$

$$\left[\Delta_1(-\beta_3 - 2\overline{\omega}\lambda) + \lambda_1(\beta_4 - \overline{\omega}^2)\right]c\delta - \left[\frac{b_1}{b}(2\lambda\overline{\omega} + \beta_3) + \delta_1(\beta_4 - \overline{\omega}^2)\lambda\right]s\delta$$

$$= \frac{1}{2}\left[e_n\right]c\delta$$

in which

$$d_n = d_{11}(1 - \lambda\overline{\omega}) + d_{12}\lambda - d_{13}\overline{\omega}$$

$$(3.77)$$

$$e_n = e_{11}(1-\lambda\overline{\omega}) - e_{12}\lambda + e_{13}\overline{\omega}$$

For any value of the phase angle δ there must be only one solution to the four simultaneous linear algebraic equations, Eq's. (3.76), in terms of the parameters Δ_1, λ_1, δ_1, and b_1/b.

Our interest is principally in describing the width of the possible unstable resonance region, given by Δ_1, and in noting the value of the coefficient b_1. Equations (3.76) can be solved to give

$$\Delta_1 = \frac{1}{2} \, c2\delta \, \frac{d_n(\beta_4 - \overline{\omega}^2) - e_n\beta_1\overline{\omega}}{(\beta_1\lambda - 2\overline{\omega})(\beta_4 - \overline{\omega}^2) + \beta_1\overline{\omega}(2\overline{\omega}\lambda + \beta_3)}$$

$$\tag{3.78}$$

$$\frac{b_1}{b} = \frac{1}{2} \, s2\delta \, \frac{d_n(\beta_4 - \overline{\omega}^2) - e_n\beta_1\lambda\overline{\omega}}{(\beta_1\lambda - 2\overline{\omega})(\beta_4 - \overline{\omega}^2) + \beta_1\lambda\overline{\omega}(2\overline{\omega}\lambda + \beta_3)}$$

When b_1 is not zero the amplitude of the oscillation will become unbounded for large τ, as may be seen by inspection of Eq's. (3.73). When b_1 is zero the motion is periodic.

The analysis of the previous section showed that bounded oscillatory motion can occur on the boundaries of the resonance regions. Inspection of the second of Eq's. (3.78) shows that this occurs at $\delta = n\pi/2$, ($n = 0,1,2 \ldots$). Furthermore, inspection of the

first of Eq's (3.78) shows that Δ_1 reaches its largest absolute value when $\delta = n\pi/2$, $(n = 0,1,2 \ldots)$.

As a numerical example, the case of the first resonance region, for which $\bar{\omega}_2 = \alpha_1$ and $r = 1.5$, can be selected. This case was investigated by the previous method. Selecting a value of $\epsilon = 0.1$ we find that

$$\Delta_1 = .037 \cos 2\delta$$

$$\tag{3.79}$$

$$b_1/b = .0425 \sin 2\delta$$

Inspection of Eq's. (3.76) shows, as expected, that bounded oscillatory motion ($b_1/b = 0$) occurs on the boundary of the resonance region, since Δ_1 takes on its maximum absolute values when $\delta = n\pi/2$, $(n = 0,1, \ldots)$. This same approach could be repeated for other resonance regions for which $\bar{\omega}_1$ or $\bar{\omega}_2$ was approximately equal to $n\gamma_1$, $(n = 1,2, \ldots)$.

3.9.3 Boundaries of the Resonance Regions

The method of the previous section could be used to describe the resonance regions, but a less time consuming procedure can be used since it is only necessary to describe the boundaries of the

unstable resonance regions. For this purpose, one can look for resonance solutions to the differential equations, Eq's. (3.50), for which the time dependence of the amplitude (see Eq's. (3.73)) is zero. Also, experience with this analysis shows that the equations of motion can be satisfied in this case without the assumed phase shift between the θ_1 and θ_2 motion, as given by the terms δ_1, δ_2, ..., so that the assumed resonance solution on the boundary of the region of instability is

$$\theta_1 = b \, c(\Phi \tau + \delta_j) + u_{11}(\tau)\epsilon + u_{12}(\tau)\epsilon^2 + \ldots$$

(3.80)*

$$\theta_2 = b(\lambda_j + \epsilon\lambda_{j1} + \epsilon\lambda_{j2} + \ldots) \, s(\Phi \tau + \delta_j) + u_{21}(\tau)\epsilon + u_{22}(\tau)\epsilon^2 + \ldots$$

in which Φ is the frequency of the resonance oscillation and can be expressed in the form

$$\Phi = \overline{\omega}_j + \Delta_1 \epsilon + \Delta_2 \epsilon^2 + \ldots \qquad (3.81)$$

and $j = 1$ or 2 depending upon whether the frequency of the resonance oscillation is approximately equal to $\overline{\omega}_1$ or $\overline{\omega}_2$.

Simultaneous resonant and nonresonant oscillations may take place, with the frequency of the principal oscillation of the resonant

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*The phase angle δ_j, $(j = 1,2)$, should not be confused with the higher order phase angles in the second of Eq's. (3.73).

motion near one natural frequency and the frequency of the principal

oscillation of the nonresonant motion near the other natural frequency

Consequently we will assume a complete solution that includes both

resonant and nonresonant solutions. In this case the nonresonant

solution proves to be identical with that obtained earlier. There-

fore the complete solution includes the resonance solution given by

Eq's. (3.80) and (3.81) plus the nonresonance solution given by Eq's.

(3.69). In the following analysis the subscript i designates the

natural frequency near which the nonresonant oscillation takes place

and the subscript j designates the natural frequency near which

the resonant oscillation takes place.

The assumed solution can be substituted into the equations

of motion, Eq's (3.50) and the coefficients of every power of ϵ

can be set equal to zero. Setting the coefficients of ϵ equal to

zero[*] yields

$$b\ c(\Phi\tau+\delta_j)\left[\Delta_1(\beta_1\lambda_j-\overline{\omega}_j) + \lambda_{j1}\beta_1\overline{\omega}_j\right] + u_{11}'' + \beta_1 u_{21}' + \beta_2 u_{11}$$

$$= \frac{a}{2}\ c(2\alpha_1\tau+\psi\tau+\delta_i)\left[d_{11}(1-\overline{\omega}_i\lambda_i) - d_{12}\lambda_i + d_{13}\overline{\omega}_i\right]$$

---

[*]As before, terms in ϵ, ϵ^2, etc. arising in u_{ij}' and u_{ij}'' are carried forward to the higher order approximations.

$$+ \frac{a}{2} c(2\alpha_1\tau - \psi\tau - \delta_i) \left[d_{11}(1 - \bar{w}_i\lambda_i) + d_{12}\lambda_i - d_{13}\bar{w}_i \right]$$

$$+ \frac{b}{2} c(2\alpha_1\tau + \Phi\tau + \delta_j) \left[d_{11}(1 - \bar{w}_j\lambda_j) - d_{12}\lambda_j + d_{13}\bar{w}_j \right]$$

$$+ \frac{b}{2} c(2\alpha_1\tau - \Phi\tau - \delta_j) \left[d_{11}(1 - \bar{w}_j\lambda_j) + d_{12}\lambda_j - d_{13}\bar{w}_j \right]$$

$$(3.82)$$

$$b \, s(\Phi\tau + \delta_j) \left[\Delta_1(-\beta_3 - 2\bar{w}_j\lambda_j) + \lambda_{j1}(\beta_4 - \bar{w}_j{}^2) \right] + u_{21}'' + \beta_3 u_{11}' + \beta_4 u_{21}$$

$$= \frac{a}{2} s(2\alpha_1\tau + \psi\tau + \delta_i) \left[e_{11}(1 - \bar{w}_i\lambda_i) + e_{12}\lambda_i - e_{13}\bar{w}_i \right] + \frac{a}{2} s(2\alpha_1\tau$$

$$- \psi\tau - \delta_i) \left[e_{11}(1 - \bar{w}_i\lambda_i) - e_{12}\lambda_i + e_{13}\bar{w}_i \right] + \frac{b}{2} s(2\alpha_1\tau + \Phi\tau$$

$$+ \delta_j) \left[e_{11}(1 - \bar{w}_j\lambda_j) + e_{12}\lambda_j - e_{13}\bar{w}_j \right] + \frac{b}{2} s(2\alpha_1\tau - \Phi\tau - \delta_j) \left[e_{11}(1 \right.$$

$$\left. - \bar{w}_j\lambda_j) - e_{12}\lambda_j + e_{13}\bar{w}_j \right]$$

Resonance terms are of frequency Φ and will occur on the right side of the above equations when any of the following conditions exists:

$$\Phi = 2\alpha_1 \pm \psi$$

$$(3.83)$$

$$\Phi = 2\alpha_1 \pm \Phi$$

The stability boundary given by the positive sign in the second of Eq's. (3.83) corresponds to the zero spin case, $\alpha_1 = 0$, for which the present equations do not give a valid representation. Also we will consider only positive values of α_1. Recalling that ψ and Φ are approximately equal to $\bar{\omega}_1$ and $\bar{\omega}_2$, the resonances that result from Eq's. (3.83) occur at

$$\bar{\omega}_1 \cong \alpha_1$$

$$\bar{\omega}_2 \cong \alpha_1$$

$$(3.84)$$

$$\bar{\omega}_1 + \bar{\omega}_2 \cong 2\alpha_1$$

$$\bar{\omega}_1 - \bar{\omega}_2 \cong 2\alpha_1$$

The first of these resonances were described in detail in the previous section.

The width of the resonance region for which $\bar{\omega}_1 + \bar{\omega}_2 \cong 2\alpha_1$ can be defined by assuming $i = 1$ and $j = 2$ so that

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$$\psi = \overline{\omega}_1$$

$$\Phi = \overline{\omega}_2 + \Delta_1 \epsilon = 2\alpha_1 - \psi \qquad (3.85)$$

in which terms in ϵ to the second and higher powers are ignored[*]. The boundaries of the resonance region are then given by the spin rate satisfying

$$\alpha_1 = \frac{1}{2}(\overline{\omega}_1 + \overline{\omega}_2 + \epsilon \Delta_1) \qquad (3.86)$$

Substituting into Eq's (3.82) and equating the coefficients of the resonant terms one obtains

$$b \ c(2\alpha_1 \tau - \psi \tau + \delta_2)\left[\Delta_1(\beta_1 \lambda_2 - 2\overline{\omega}_2) + \lambda_{21}\beta_1\overline{\omega}_2\right] = \frac{a}{2} \ c(2\alpha_1 \tau - \psi \tau$$

$$-\delta_1)\left[d_{11}(1 - \overline{\omega}_1 \lambda_1) + d_{12}\lambda_1 - d_{13}\overline{\omega}_1\right] \qquad (3.87)$$

$$b \ s(2\alpha_1 \tau - \psi \tau + \delta_2)\left[\Delta_1(-\beta_3 - 2\overline{\omega}_2 \lambda_2) + \lambda_{21}(\beta_4 - \overline{\omega}_2^2)\right] = \frac{a}{2} \ s(2\alpha_1 \tau - \psi \tau$$

---

[*]A resonance with frequency of its principal oscillation near $\overline{\omega}_1$ (i=2, j=1) is yet to be determined.

$$-\delta_1\Big)\Big[e_{11}(1-\overline{\omega}_1\lambda_1) - e_{12}\lambda_1 + e_{13}\overline{\omega}_1\Big]$$

These equations can be satisfied when

$$\delta_1 - \delta_2 = n\pi \qquad (n = 0, 1, 2, \ldots) \tag{3.88}$$

Consequently, the boundaries of the region of instability, given by Δ_1, are

$$\Delta_1 = \pm\;\frac{a}{2b}\;\frac{(\beta_4-\overline{\omega}_2^2)\Big[d_{11}(1-\overline{\omega}_1\lambda_1)+d_{12}\lambda_1-d_{13}\overline{\omega}_1\Big]-\beta_1\overline{\omega}_2\Big[e_{11}(1-\overline{\omega}_1\lambda_1)-e_{12}\lambda_1+e_{13}\overline{\omega}_1\Big]}{(\beta_1\lambda_2-\overline{\omega}_2)(\beta_4-\overline{\omega}_2^2) + (\beta_3+2\overline{\omega}_2\lambda_2)\;\beta_1\overline{\omega}_2} \tag{3.89}$$

Inspection of the above relation shows that the width of the resonance region is a function of the relative amplitude of motion of the nonresonant and the resonant oscillation, a/b. Consequently, unique graphs of the width of this type of resonance region, such as were given in Figures 3.4 and 3.5, cannot be given. In fact, if the nonresonant oscillation is zero the instability region disappears. Nonetheless, this is a true instability region, since an infinitesimal initial perturbation from the equilibrium would excite both the resonant and nonresonant modes, and unbounded resonant motion would occur for values of spin rate falling between the boundaries given by Eq's. (3.86) and (3.89). If a higher order approximation of the

location of the stability boundary is desired, the boundary may
be described by further terms in the expansion in terms of powers
of ϵ such that

$$\Phi = \bar{\omega}_2 + \Delta_1 \epsilon + \Delta_2 \epsilon^2 + \Delta_3 \epsilon^3 + \ldots$$

$$\tag{3.90}$$

$$\alpha_1 = \frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2 + \epsilon\Delta_1 + \epsilon^2\Delta_2 + \epsilon^3\Delta_3 + \ldots)$$

by expansion of the equations of motion to higher powers of ϵ
and determining the constants to satisfy the equations for each
successive power of ϵ.

When the spin rate is such that no resonance occurs as given
by Eq's. (3.83) then

$$\Delta_1 = \lambda_{j1} = 0 \tag{3.91}$$

and

$$u_{11} = f_{11}^{(i)}a \; c(2\alpha_1\tau + \psi\tau + \delta_i) + f_{12}^{(i)}a \; c(2\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ f_{11}^{(j)}b \; c(2\alpha_1\tau + \Phi\tau + \delta_j) + f_{12}^{(j)}b \; c(2\alpha_1\tau - \Phi\tau - \delta_j)$$

$$\tag{3.92}$$

$$u_{12} = g_{11}^{(i)} a \; s(2\alpha_1 \tau + \psi\tau + \delta_i) + g_{12}^{(i)} a \; s(2\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ g_{11}^{(j)} b \; s(2\alpha_1\tau + \Phi\tau + \delta_j) + g_{12}^{(j)} b \; s(2\alpha_1\tau - \Phi\tau - \delta_j)$$

where the coefficients f and g are given by Eq's. (3.62). The superscripts i and j may take on the values 1 or 2 and designate the natural frequency, $\overline{\omega}_1$ or $\overline{\omega}_2$, that must be used in calculation of the coefficients f and g.

When no resonance takes place in the ϵ order terms, the substitution of the assumed resonance plus nonresonance solutions into the expansion of the equations of motion results in the following coefficient of ϵ^2.

$$b \; c(\Phi\tau + \delta_j)\left[\Delta_2(-2\overline{\omega}_j + \lambda_j\beta_1) + \lambda_{j2}\overline{\omega}_j\beta_1\right] + u_{12}'' + \beta_1 u_{22}' + \beta_2 u_{12}$$

$$= \frac{b}{2} \; c(\Phi\tau + \delta_j) \; h_{j1} + \frac{a}{2} \; c(2\alpha_1\tau + \psi\tau + \delta_i) \; h_{i2} + \frac{b}{2} \; c(2\alpha_1\tau + \Phi\tau + \delta_j) \; h_{j2}$$

$$+ \frac{a}{2} \; c(2\alpha_1\tau - \psi\tau - \delta_i) \; h_{i3} + \frac{b}{2} \; c(2\alpha_1\tau - \Phi\tau - \delta_j) \; h_{j3} + \frac{a}{2} \; c(4\alpha_1\tau$$

$$+\psi\tau + \delta_i) \; h_{i4} + \frac{b}{2} \; c(4\alpha_1\tau + \Phi\tau + \delta_j) \; h_{j4} + \frac{a}{2} \; c(4\alpha_1\tau - \psi\tau - \delta_i) \; h_{i5}$$

$$+ \frac{b}{2} \; c(4\alpha_1\tau - \Phi\tau - \delta_j) \; h_{j5} \tag{3.93}$$

$$b\ s(\Phi_T+\delta_j)\left[\Delta_2(-\beta_3-2\bar{\omega}_j\lambda_j) + \lambda_{j2}(\beta_4-\bar{\omega}_j^{\ 2})\right] + u_{22}'' + u_{12}'\beta_3 + u_{22}\beta_4$$

$$= \frac{b}{2}\ s(\Phi_T+\delta_j)\ \ell_{j1} + \frac{a}{2}\ s(2\alpha_1\tau+\psi\tau+\delta_i)\ \ell_{i2} + \frac{b}{2}\ s(2\alpha_1\tau+\Phi_T$$

$$+ \delta_j)\ \ell_{j2} + \frac{a}{2}\ s(2\alpha_1\tau-\psi\tau-\delta_i)\ \ell_{i3} + \frac{b}{2}\ s(2\alpha_1\tau-\Phi_T-\delta_j)\ \ell_{j3}$$

$$+ \frac{a}{2}\ s(4\alpha_1\tau+\psi\tau+\delta_i)\ \ell_{i4} + \frac{b}{2}\ s(4\alpha_1\tau+\Phi_T+\delta_j)\ \ell_{j4} + \frac{a}{2}\ s(4\alpha_1\tau$$

$$-\psi\tau-\delta_i)\ \ell_{i5} + \frac{b}{2}\ s(4\alpha_1\tau-\Phi_T-\delta_j)\ \ell_{j5}$$

in which

$$h_{k1} = d_{11}(f_{11}+f_{12}) + d_{12}(g_{11}+g_{12}) - d_{13}\left[f_{11}(2\alpha_1+\bar{\omega}_k) + f_{12}(2\alpha_1\right.$$

$$\left.-\bar{\omega}_k)\right] + d_{11}\left[g_{11}(2\alpha_1+\bar{\omega}_k) + g_{12}(2\alpha_1-\bar{\omega}_k)\right]$$

$$h_{k2} = h_{k3} = d_{21}(1-\bar{\omega}_k\lambda_k)$$

$$h_{k4} = d_{22}(1-\bar{\omega}_k\lambda_k) - d_{23}\lambda_k + d_{24}\bar{\omega}_k + d_{11}f_{11} - d_{12}g_{11} + d_{13}(2\alpha_1$$

$$+\bar{\omega}_k)\ f_{11} - d_{11}(2\alpha_1+\bar{\omega}_k)\ g_{11}$$

$$h_{k5} = d_{22}(1-\bar{\omega}_k\lambda_k) + d_{23}\lambda_k - d_{24}\bar{\omega}_k + d_{11}f_{12} - d_{12}g_{12} + d_{13}(2\alpha_1$$

$$-\bar{\omega}_k) \; f_{12} - d_{11}(2\alpha_1-\bar{\omega}_k) \; g_{12} \qquad\qquad (3.94)$$

$$\ell_{k1} = e_{11}(-f_{11}+f_{12}) + e_{12}(g_{11}-g_{12}) - e_{13}\Big[f_{11}(2\alpha_1+\bar{\omega}_k) - f_{12}(2\alpha_1$$

$$-\bar{\omega}_k)\Big] - e_{11}\Big[-g_{11}(2\alpha_1+\bar{\omega}_k) + g_{12}(2\alpha_1-\bar{\omega}_k)\Big]$$

$$\ell_{k2} = -\ell_{k3} = e_{22}(\lambda_k-\bar{\omega}_k)$$

$$\ell_{k4} = e_{21}(1+\bar{\omega}_k\lambda_k) + e_{23}\lambda_k - e_{24}\bar{\omega}_k - e_{11}f_{11} - e_{12}g_{11} + e_{13}(2\alpha_1$$

$$+\bar{\omega}_k) \; f_{11} + e_{11}(2\alpha_1+\bar{\omega}_k) \; g_{11}$$

$$\ell_{k5} = e_{21}(1+\bar{\omega}_k\lambda_k) - e_{23}\lambda_k + e_{24}\bar{\omega}_k - e_{11}f_{12} - e_{12}g_{12} + e_{13}(2\alpha_1$$

$$+\bar{\omega}_k) \; f_{12} + e_{11}(2\alpha_1+\bar{\omega}_k) \; g_{12}$$

Resonant motion will occur in the ϵ^2 expansion (under the assumed condition of no resonance in the ϵ expansion) when Φ is equal to one of the frequencies appearing for the first time on

the right side of Eq's. (3.93). This includes

$$\Phi = 4\alpha_1 \pm \Psi$$

$$(3.95)$$

$$\Phi = 4\alpha_1 \pm \Phi$$

Again eliminating cases corresponding to zero or negative values of α_1, these resonances are seen to correspond to

$$\overline{w}_1 \cong 2\alpha_1$$

$$\overline{w}_2 \cong 2\alpha_1$$

$$(3.96)$$

$$\overline{w}_1 + \overline{w}_2 \cong 4\alpha_1$$

$$\overline{w}_1 - \overline{w}_2 \cong 4\alpha_1$$

The first two of these cases were investigated by the method of the previous section.

The resonance region for which $\overline{w}_1 + \overline{w}_2 \cong 4\alpha_1$ can be described by assuming

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$$\psi \cong \tilde{\bar{\omega}}_1 \qquad \text{(see the third of Eq's (3.69))}$$

(3.97)

$$\Phi = \bar{\omega}_2 + \Delta_{21}\epsilon^2 = 4\alpha_1 - \psi$$

Equating the resonant terms of Eq's. (3.93) one obtains

$$b\ c(4\alpha_1\tau - \psi\tau + \delta_2)\left[\Delta_2(-2\bar{\omega}_2 + \lambda_2\beta_1) + \lambda_{22}\bar{\omega}_2\beta_1\right]$$

$$= \frac{b}{2}\ c(4\alpha_1\tau - \psi\tau + \delta_2)\ h_{21} + \frac{a}{2}\ c(4\alpha_1\tau - \psi\tau - \delta_1)\ h_{15}$$

(3.98)

$$b\ s(4\alpha_1\tau - \psi\tau + \delta_2)\left[\Delta_2(-\beta_3 - 2\bar{\omega}_2\lambda_2) + \lambda_{22}(\beta_4 - \bar{\omega}_2^2)\right]$$

$$= \frac{b}{2}\ s(4\alpha_1\tau - \psi\tau + \delta_2)\ \ell_{21} + \frac{a}{2}\ s(4\alpha_1\tau - \psi\tau - \delta_1)\ \ell_{15}$$

These equations are satisfied when $\delta_1 - \delta_2 = n\pi$, $(n = 0, 1, 2, \ldots)$.
The equations can be solved to give

$$\Delta_2 = \frac{1}{2}\ \frac{1}{(\beta_4 - \bar{\omega}_2^2)(-2\bar{\omega}_2 + \lambda_2\beta_1) + (\beta_3 + 2\bar{\omega}_2\lambda_2)\ \bar{\omega}_2\beta_1}\ \left\{h_{21}(\beta_4 - \bar{\omega}_2^2)\right.$$

(3.99)

$$- \ell_{21} \overline{\omega}_2 \beta_1 \pm \frac{a}{b} \left[h_{15} (\beta_4 - \overline{\omega}_2^2) - \ell_{15} \overline{\omega}_2 \beta_1 \right\}$$

The corresponding values of the spin rate at the boundaries of the instability region are obtained using

$$\alpha_1 = \frac{1}{4} \left[\Phi + \psi \right] = \frac{1}{4} \left[\overline{\omega}_1 + \overline{\omega}_2 + \epsilon^2 \Delta_2 \Big|_{\text{[Per Eq. (3.99)]}} \right.$$

(3.100)

$$\left. + \epsilon^2 \Delta_2 \Big|_{\text{[Per Eq. (3.99)]}} + O(\epsilon^3) \right]$$

Note that the nonresonant frequency correction given by Eq. (3.66) is included, and is calculated for nonresonant oscillation with frequency near $\overline{\omega}_1$.

Inspection of Eq. (3.99) shows that the width of this resonance region is again a function of the relative amplitude of nonresonant to resonant motion, a/b, and the previous remarks are again applicable.

When the spin rate is such that no resonance occurs as given by either Eq's. (3.84) or (3.96), Δ_2 and λ_{j2} are given by Eq's. (3.66) and the nonresonance terms are

$$u_{12} = f_{21}^{(i)}a \; c(2\alpha_1\tau + \psi\tau + \delta_i) + f_{22}^{(i)}a \; c(2\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ f_{23}^{(i)}a \; c(4\alpha_1\tau + \psi\tau + \delta_i) + f_{24}^{(i)}a \; c(4\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ f_{21}^{(j)}b \; c(2\alpha_1\tau + \Phi\tau + \delta_j) + f_{22}^{(j)}b \; c(2\alpha_1\tau - \Phi\tau - \delta_j)$$

$$+ f_{23}^{(j)}b \; c(4\alpha_1\tau + \Phi\tau + \delta_j) + f_{24}^{(j)}b \; c(4\alpha_1\tau - \Phi\tau - \delta_j)$$

$$\tag{3.101}$$

$$u_{22} = g_{21}^{(i)}a \; s(2\alpha_1\tau + \psi\tau + \delta_i) + g_{22}^{(i)}a \; s(2\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ g_{23}^{(i)}a \; s(4\alpha_1\tau + \psi\tau + \delta_i) + g_{24}^{(i)}a \; s(4\alpha_1\tau - \psi\tau - \delta_i)$$

$$+ g_{21}^{(j)}b \; s(2\alpha_1\tau + \Phi\tau + \delta_j) + g_{22}^{(j)}b \; s(2\alpha_1\tau - \Phi\tau - \delta_j)$$

$$+ g_{23}^{(j)}b \; s(4\alpha_1\tau + \Phi\tau + \delta_j) + g_{24}^{(j)}b \; s(4\alpha_1\tau - \Phi\tau - \delta_j)$$

where the coefficients in the above expressions are given by Eq's.
(3.68) and the superscripts i and j denote the natural frequency
that should be used in calculating each coefficient.

When no resonance takes place in the ϵ or ϵ^2 order terms,
the substitution of the assumed resonance plus nonresonance solutions
into the asymptotic expansion of the equations of motion results in

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the following coefficient of e^3.

$$b\left[\Delta_3(\beta_1\lambda_j - 2\overline{\omega}_j) + \lambda_{j3}\beta_1\overline{\omega}_j\right] c(\Phi\tau+\delta_j) + u''_{13} + \beta_1 u'_{23} + \beta_2 u_{13}$$

$$= \frac{b}{2} c(\Phi\tau+\delta_j) m_{j1} + \frac{a}{2} c(2\alpha_1\tau+\psi\tau+\delta_i) m_{i2}$$

$$+ \frac{b}{2} c(2\alpha_1\tau+\Phi\tau+\delta_j) m_{j2} + \frac{a}{2} c(2\alpha_1\tau-\psi\tau-\delta_i) m_{i3}$$

$$+ \frac{b}{2} c(2\alpha_1\tau-\Phi\tau-\delta_j) m_{j3} + \frac{a}{2} c(4\alpha_1\tau+\psi\tau+\delta_i) m_{i4}$$

$$+ \frac{b}{2} c(4\alpha_1\tau+\Phi\tau+\delta_j) m_{j4} + \frac{a}{2} c(4\alpha_1\tau-\psi\tau-\delta_i) m_{i5}$$

$$+ \frac{b}{2} c(4\alpha_1\tau-\Phi\tau-\delta_j) m_{j5} + \frac{a}{2} c(6\alpha_1\tau+\psi\tau+\delta_i) m_{i6}$$

$$+ \frac{b}{2} c(6\alpha_1\tau+\Phi\tau+\delta_j) m_{j6} + \frac{a}{2} c(6\alpha_1\tau-\psi\tau-\delta_i) m_{i7}$$

$$+ \frac{b}{2} c(6\alpha_1\tau-\Phi\tau-\delta_j) m_{j7}$$

<div align="right">(3.102)</div>

$$b\left[\Delta_3(-2\lambda_j\overline{\omega}_j-\beta_3) + \lambda_{j3}(\beta_4-\overline{\omega}_j^2)\right] s(\Phi\tau+\delta_j) + u''_{23} + \beta_3 u'_{13} + \beta_4 u_{23}$$

$$= \frac{b}{2} \, s(\Phi_T + \delta_j) \, n_{j1} + \frac{a}{2} \, s(2\sigma_1 \tau + \psi \tau + \delta_i) \, n_{12}$$

$$+ \frac{b}{2} \, s(2\sigma_1 \tau + \Phi \tau + \delta_j) \, n_{j2} + \dots$$

In the above equations, the coefficients m and n are functions of the parameters of the problem and are not given here.

Resonant motion will occur in the ϵ^3 expansion (under the assumed condition of no resonance in the ϵ or ϵ^2 expansions) when Φ is equal to one of the frequencies appearing for the first time on the right side of Eq's. (3.102). This includes

$$\Phi = 6\sigma_1 \pm \psi$$

$$(3.103)$$

$$\Phi = 6\sigma_1 \pm \Phi$$

Again eliminating the cases corresponding to zero or negative average spin rate, σ_1, these resonances can be seen to correspond to

$$\bar{\omega}_1 \cong 3\sigma_1$$

$$\bar{\omega}_2 \cong 3\sigma_1$$

$$(3.104)$$

$$\bar{\omega}_1 + \bar{\omega}_2 \cong 6\alpha_1$$

$$\bar{\omega}_1 - \bar{\omega}_2 \cong 6\alpha_1$$

The stability boundaries for the first two cases, using Eq's. (3.66), (3.81), (3.91), and (3.103) are given by

$$\alpha_1 = \frac{1}{3}\left[\bar{\omega}_1 + \epsilon^2 \Delta_2 + \epsilon^3 \Delta_3 + \dots\right] \qquad (3.105)$$

where only Δ_3 takes on different values on the two boundaries of the instability region, so that the width of the region is measured in terms of ϵ^3. A similar form of solution for the instability regions described by the third and fourth of Eq's. (3.104) occurs and the width of the instability region is proportional to the third power of the small parameter ϵ.

It is evident that the asymptotic expansion process, when carried to the nth power of ϵ, will describe additional instability regions with frequencies.

$$\bar{\omega}_i \cong n\alpha_1 \quad , \quad (i = 1,2)$$

$$(3.106)$$

$$\bar{\omega}_1 \pm \bar{\omega}_2 \cong 2n\alpha_1$$

and that the width of the resonance instability region will be proportional to ϵ^n.

3.10 Comparison with Previous Analysis

The previous sections established the approximate location of the instability regions of the unsymmetrical spinning rigid body (see Figure 3.3) and included analyses to define some of the instability regions in greater detail.

The only previous analysis known to have been performed on this problem is that of Kane and Shippy [12] in which a method due to Cesari [20] is used to check the stability of motion of the unsymmetrical spinning body for specific spin rates and ratios of moments of inertia of the body. This previous analysis states that approximately 230 points were checked, and on the basis of these points, stability regions were defined. Two of the stability charts of Reference [12] are reproduced on Figure 3.6, corresponding to average spin rates given by $\alpha_1 = 1$ and $\alpha_1 = 5$. On Figure 3.6 the nondimensional parameters K_1 and K_2 are given by

$$K_1 = \frac{B-C}{A} = 1 + \epsilon - r$$

(3.107)
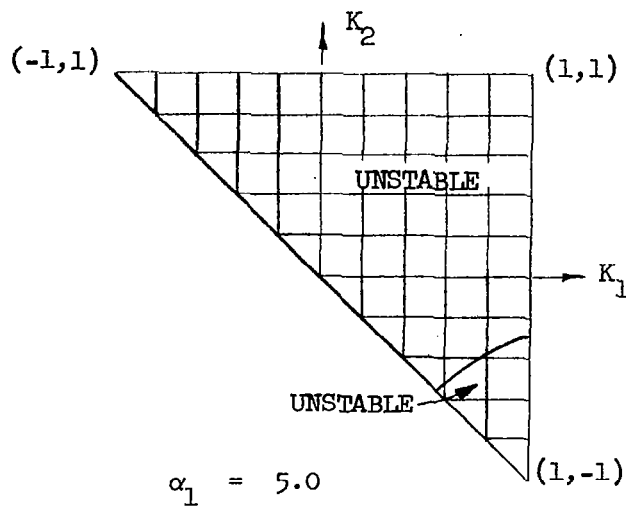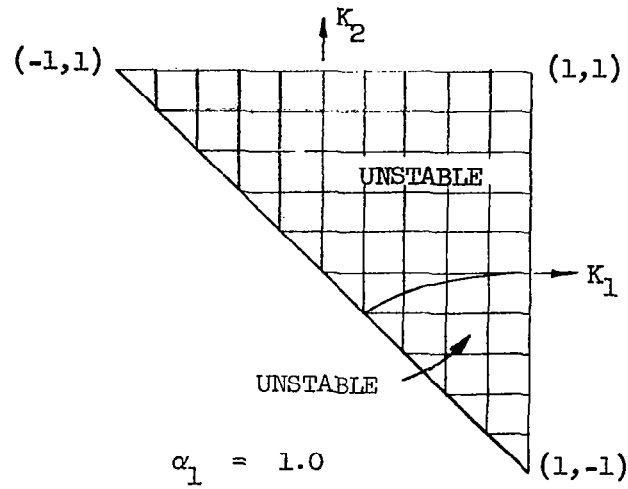
$$K_2 = \frac{C-A}{B} = \frac{r-1}{1+\epsilon}$$

FIGURE 3.6

STABILITY CHARTS OF REFERENCE [12]

Figure 3.7 shows a replotting of these stability charts of Reference [10] in terms of the nondimensional parameters used in the present research -- r and ϵ. The stability boundary on the lower portion of Figure 3.7 can be seen to be a function of α_1, the average spin rate, and this boundary was shown in Reference [12] for $\alpha_1 = 1$ and $\alpha_1 = 5$. Superimposed are the stability boundaries predicted by the requirement that the Hamiltonian be positive definite, as given by Eq's. (3.21). Comparison of the boundaries predicted by Reference [12] with those of the present research indicates that the requirement that the Hamiltonian be positive definite at all times may exclude small regions of parameter space in which stable motion is actually possible. This is not unexpected since the requirement that the Hamiltonian be positive definite was seen to be a sufficient condition for stability to exist, but not a necessary requirement.

By comparison with Figure 3.3 it can immediately be seen that additional regions of instability not shown on Figure 3.7 are predicted in the present research. Specifically, additional regions of instability are expected in the case of $\epsilon << 1$ for an average spin rate of one, $\alpha_1 = 1$, with moment of inertia ratios, r, of approximately 1.39, 1.73, 1.83, and 1.94 and for $\alpha_1 = 5$ with moment of inertia ratios of approximately 1.64, 1.82, and 1.99.

Figure 3.8 shows the estimated regions of instability for $\alpha_1 = 5$ and small ϵ, as defined by the methods of the present
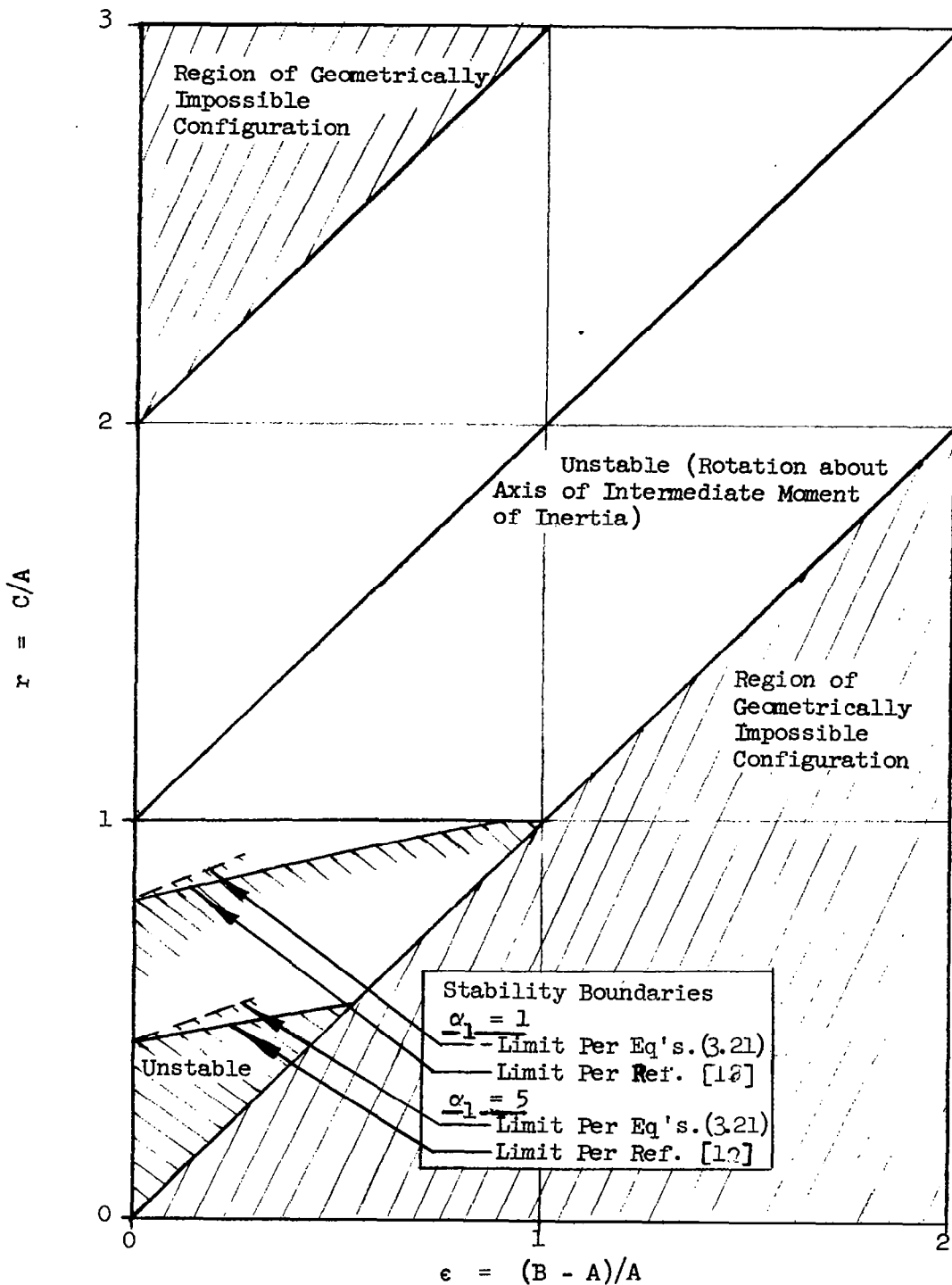
FIGURE 3.7

STABILITY CHARTS OF REFERENCE 12 (IN TERMS OF r AND ε)
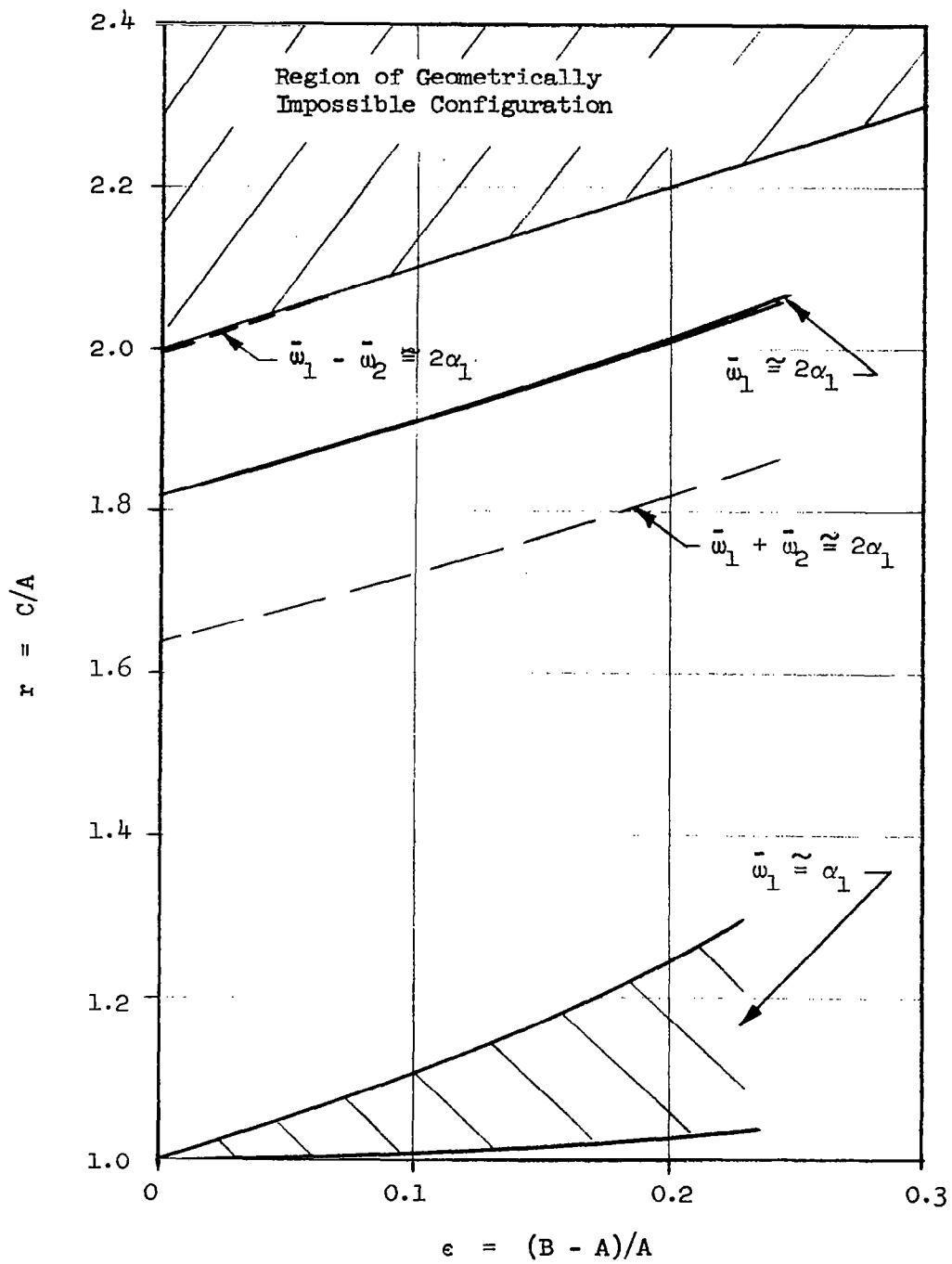AND COMPARISON WITH STABILITY BOUNDARIES OF EQUATIONS (3.21)

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FIGURE 3.8

UNSYMMETRICAL SPINNING SATELLITE RESONANCE

INSTABILITY REGIONS FOR $\alpha_1 = 5$

research. This figure shows the locations of four regions of resonance instability, corresponding to

$$\bar{\omega}_1 \simeq \alpha_1$$

$$\bar{\omega}_2 \simeq 2\alpha_1$$

$$\bar{\omega}_1 + \bar{\omega}_2 \simeq 2\alpha_1$$ (3.108)

$$\bar{\omega}_1 - \bar{\omega}_2 \simeq 2\alpha_1$$

The first of these regions, referred to as the region of principal instability, corresponds to the instability region shown on Figure 3.7, and physically corresponds to the region in parameter space in which the body is spinning about an axis of intermediate moment of inertia. The region shown in Figure 3.8 is the first approximation given by Eq. (3.47), and can be seen to agree closely with Reference [12] for small ϵ. Improved agreement at larger values of ϵ would be expected if a higher order approximation were used, since the width of the region would then be corrected by terms proportional to ϵ^2 and higher powers of ϵ.

The first approximation of the region of resonance instability for $\bar{\omega}_1 \simeq 2\alpha_1$ is shown in Figure 3.8, using Eq's (3.42). As expected,

the width of the region is approximately proportional to ϵ^2 and consequently is very narrow for small ϵ.

The locations of instability regions for which $\overline{w}_1 \pm \overline{w}_2 \simeq 2\alpha_1$ are shown as dotted lines. As discussed in the previous section, the width of these instability regions is a function of the type of initial disturbance to which the body is subjected.

Figure 3.9 presents a similar stability plot for $\alpha_1 = 1$. The remarks relating to Figure 3.8 are again applicable and in addition, several added types of instability region are found to exist. Three regions are seen to be superimposed near $r = 1.0$, including a region of principal instability corresponding to $\overline{w}_2 \simeq \alpha_1$ as well as one for which $\overline{w}_1 \simeq \alpha_1$.

A region corresponding to $\overline{w}_1 + \overline{w}_2 \simeq 4\alpha_1$ is seen to appear in Figure 3.9. Its width was found to be a function of the initial disturbance as previously discussed and as is in general the case for instabilities near $\overline{w}_1 \pm \overline{w}_2 \simeq 2n\alpha_1$. An additional instability region is also seen on Figure 3.9 for which $\overline{w}_1 \simeq 3\alpha_1$. The width of this region has not been defined, but it is expected to be very narrow as its width was previously shown to be proportional to ϵ^3.
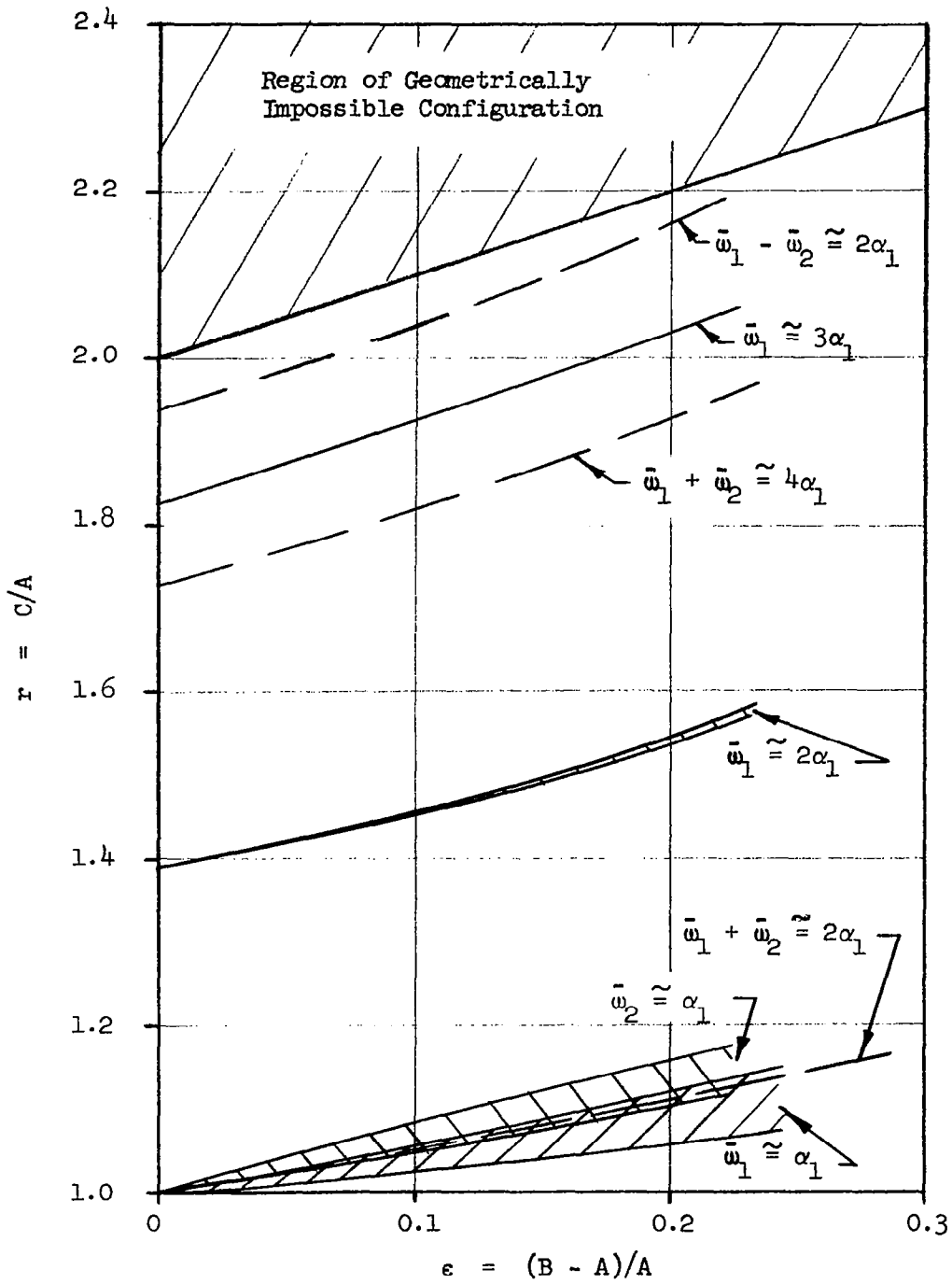
FIGURE 3.9

UNSYMMETRICAL SPINNING SATELLITE RESONANCE

INSTABILITY REGIONS FOR $\alpha_1 = 1$

SECTION IV

SPINNING SYMMETRICAL SATELLITE IN AN ELLIPTIC ORBIT

A circular orbit is often assumed in satellite stability analyses due to the inherent simplicity of mathematical form. On occasions the orbits are elliptic, obtained intentionally or unintentionally. When elliptical orbital motion is considered, the equations of motion describing the attitude motions of a rigid body are found to include periodic coefficients.

A rigid spinning symmetrical body in an elliptical orbit is found to have an equilibrium position in which the spin axis is perpendicular to the orbit plane. A previous analysis of this problem [13] dealt with the attitude stability of motion about the equilibrium position, but the method of analysis that was employed was restricted to the linearized system and included only checks of the stability for specific values of the parameters of the problem. In fact it is the same method as used in Reference 12. Beletskii [28] used a KBM method to study the case of a nonsymmetric satellite in an elliptic orbit. His case is different than the case discussed in this section in the sense that Beletskii's analysis is restricted to one-dimensional libration in the plane of the orbit about a configuration of no spin relative to an orbiting system of coordinates.

The analysis presented in this section gives an analytical solution for the stability characteristics of both the nonlinear and linear systems

in the neighborhood of the equilibrium position, for relatively small orbit eccentricity. An analysis based on the methods discussed in Section III will be employed.

4.1 Orbital Motion

The orbit path of the center of mass of a rigid body traveling under the influence of an inverse square gravitational field is known to be a conic section. The gravitational attractive force is assumed equal to K/R^2 where K is the product of the earth's mass times the universal gravitational constant and R is the radial distance between the earth's center of mass and the satellite center of mass. The assumption of a uniform inverse square gravitational field is an idealization equivalent to the assumption that the earth's density is a function of radius alone. In many satellite systems the desired conic section is an ellipse, implying periodic orbital motion.

According to Moulton [29] the elliptical orbit parameters can be expressed as periodic functions of time consisting of series of terms in increasing powers of the orbit eccentricity, ϵ. These relations are

$$R = R_s \left\{ 1 - (\cos \tau)\epsilon + \frac{1}{2} (1 - \cos 2\tau)\epsilon^2 \right.$$

$$\left. + \frac{3}{8} (\cos \tau - \cos 3\tau)\epsilon^3 + \frac{1}{3} (\cos 2\tau - \cos 4\tau)\epsilon^4 + \ldots \right\}$$

$$\tag{4.1}$$

$$\Theta = \tau + 2(\sin \tau)\epsilon + \frac{5}{4} (\sin 2\tau)\epsilon^2 + \frac{1}{4} (- \sin \tau$$

$$+ \frac{13}{3} \sin 3\tau)\epsilon^3 + \frac{1}{24} (-11 \sin 2\tau + \frac{103}{4} \sin 4\tau)\epsilon^4 + \ldots$$

in which R_s is the orbit semimajor axis, Θ is the angular position of the satellite measured from a radial line through the perigee and τ is the mean anomaly which is a nondimensional quantity related to the time, t, and the orbital period, T, by

$$\tau = 2\pi \, t/T \qquad\qquad (4.2)$$

The orbit period is [30]

$$T = 2\pi \, K^{-1/2} \, R_s^{3/2} \qquad\qquad (4.3)$$

The series expansions for R and Θ in powers of the eccentricity ϵ, Eq's. (4.1), have been shown to converge for value of ϵ up to 0.667 [29]. This region of convergence is more than adequate for the present study, since the eccentricity of earth satellite orbits seldom exceeds 0.1, and the principal interest is in nearly circular orbits possessing even smaller eccentricity, so that one can justify treating ϵ as a small parameter.

The influence of the attitude motion on the orbital motion of the satellite is negligible when the satellite dimensions are small compared with the distance from the center of force to the satellite center of mass.

4.2   Coordinate System

An orbital frame of reference with its origin at the satellite center of mass and its orientation as shown in Figure 3.1a is chosen.

Axis a is along a radial line from the center of force (center of the earth) to the center of the satellite, axis c is perpendicular to the orbit plane, and axis b is perpendicular to axes a and c so as to form a right hand system a,b,c. (Note that axis b is not always directed along the orbit path as it was in the case of a circular orbit.) Hence, a, b, c forms an orbiting frame of reference in which a and b are the radial and transverse coordinates of planar motion. The orientation of the satellite relative to the a, b, c reference system is obtained by three successive rotations θ_2, θ_1, and φ as shown on Figure 3.1b.

The z axis is taken as the symmetry axis and the mass moment of inertia about this axis denoted by C whereas the mass moment of inertia about axes perpendicular to z is denoted by A. The analysis of the symmetric case is simplified by using the ξ, η, ζ, axes. The angular velocity components of the body along these axes are

$$\Omega_\xi = -\dot{\Theta}\ s\theta_2 - \dot{\theta}_1 = \frac{2\pi}{T}[-\Theta'\ s\theta_2 - \theta_1']$$

$$\Omega_\eta = -\dot{\Theta}\ c\theta_2\ s\theta_1 + \dot{\theta}_2\ c\theta_1 = \frac{2\pi}{T}[-\Theta'\ c\theta_2\ s\theta_1 + \theta_2'\ c\theta_1]$$

$$\Omega_\zeta = \Omega_z = \dot{\Theta}\ c\theta_2\ c\theta_1 + \dot{\theta}_2\ s\theta_1 + \dot{\varphi} = \frac{2\pi}{T}[\Theta'\ c\theta_2\ c\theta_1 +$$

$$+ \theta_2'\ s\theta_1 + \varphi']$$

$$(4.4)$$

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where the dot designates differentiation with respect to time and the prime indicates differentiation with respect to τ .

The position of the satellite symmetry axis is described in terms of the angles θ_1 and θ_2. All possible positions are described by using the ranges $-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$ and $-\pi < \theta_2 \leq \pi$. Analogy may be made to the description of points on the surface of the earth, with θ_2 the longitude and θ_1 the latitude. At the points $\theta_1 = \frac{\pi}{2}$ (corresponding to the poles) the coordinate system is singular in the sense that partial derivatives with respect to θ_2 are not defined.

4.3 Energy Expressions

The kinetic energy associated with attitude motion of a symmetric satellite can be written in terms of the angular velocity components along the ξ, η, ζ axes

$$K E = \frac{1}{2} A (\Omega_\xi^2 + \Omega_\eta^2) + \frac{1}{2} C \Omega_\zeta^2$$

$$= \frac{2\pi^2}{T^2} \left\{ A \left[(\theta_1' + \Theta' s\theta_2)^2 + (\theta_2' c\theta_1 - \Theta' c\theta_2 s\theta_1)^2 \right] \right.$$

$$\left. + C (\varphi' + \theta_2' s\theta_1 + \Theta' c\theta_2 c\theta_1)^2 \right\} \quad (4.5)$$

The gravitational potential energy associated with attitude motions of a symmetric satellite reduces to

$$P E = -\frac{3}{2} \frac{K}{R^3} (A - C) s^2\theta_2 c^2\theta_1 \quad (4.6)$$

Note that the translational kinetic energy of the center of mass and the potential energy due to altitude have been omitted. This again implies uncoupling of the attitude motion and the orbital motion and will be referred to as orbital constraints.

4.4   Equations of Motion

We shall seek to derive the equation of motion for a symmetric satellite by means of Lagrange's equations. To this end we write the Lagrangian

$$
L = KE - PE = \frac{2\pi^2}{T^2} \left\{ A \left[(\theta_1' + \Theta' s\theta_2)^2 \right.\right.
$$

$$
\left. + (\theta_2' c\theta_1 - \Theta' c\theta_2 s\theta_1)^2 \right] + C (\varphi' + \theta_2' s\theta_1 \tag{4.7}
$$

$$
\left. + \Theta' c\theta_2 c\theta_1)^2 \right\} + \frac{3K}{2R^3} (A - C) s^2\theta_2 c^2\theta_1
$$

An inspection of Eq. (4.7) reveals that the generalized coordinate φ does not appear in the Lagrangian. Furthermore it will be assumed that there are no nonconservative forces present so that φ is a cyclic or ignorable coordinate. Furthermore, R and Θ are known functions of the mean anomaly τ . Consequently, Lagrange's equations of motion are

$$
\frac{d}{d\tau} \left(\frac{\partial L}{\partial \theta_1'} \right) - \frac{\partial L}{\partial \theta_1} = 0
$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \theta_2'}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \varphi'}\right) = 0 \tag{4.8}$$

The third of Eq's. (4.8) can be integrated immediately to obtain the generalized momentum about the spin axis, p_φ , which is an integral of the system and should be regarded as a constant quantity

$$p_\varphi = \frac{T}{2\pi}\frac{\partial L}{\partial \varphi'} = \frac{2\pi C}{T}(\varphi' + \theta_2' s\theta_1 + \Theta' c\theta_2 c\theta_1) = \text{const.} \tag{4.9}$$

This integral can be used to reduce the degree of freedom of the system by one, using a procedure known as the ignoration of coordinates introduced by Routh [1], [31]. According to this process the Routhian of the system can be devised in the form

$$\mathcal{R} = L - \frac{\partial L}{\partial \varphi'}\varphi' = L - \frac{2\pi}{T}p_\varphi\varphi'$$

$$= \frac{2\pi^2}{T^2} A\left\{(\theta_1' + \Theta' s\theta_2)^2 + (\theta_2' c\theta_1 - \Theta' c\theta_2 s\theta_1)^2\right.$$

$$- \ell^2 r + 2\ell r\theta_2' s\theta_1 + 2\ell r\Theta' c\theta_2 c\theta_1$$

$$+ \frac{3KT^2}{4\pi^2 R^3}(1-r)s^2\theta_2 c^2\theta_1\left.\right\} \tag{4.10}$$

in which

$$r = \frac{C}{A} \qquad , \qquad \ell = \frac{p_\omega T}{2\pi C} \qquad (4.11)$$

and where Eq. (4.9) has been used to eliminate φ'.

The equations of motion become

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{R}}{\partial \theta_1'}\right) - \frac{\partial \mathcal{R}}{\partial \theta_1} = 0$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{R}}{\partial \theta_2'}\right) - \frac{\partial \mathcal{R}}{\partial \theta_2} = 0 \qquad (4.12)$$

which can be written

$$\theta_1'' + \theta_2'(2 - \ell r) + \theta_1(\ell r - 1 - 2\epsilon^2)$$

$$= \theta_2'\left[2(1 - c\theta_2 \ c^2\theta_1) - \theta_2' \ s\theta_1 \ c\theta_1 - \ell r(1 - c\theta_1)\right]$$

$$+ \theta_1\left[\ell r(1 - c\theta_2 \ \frac{s\theta_1}{\theta_1}) - (1 - c^2\theta_2 \ c\theta_1 \ \frac{s\theta_1}{\theta_1})(1 + 2\epsilon^2)\right.$$

$$- (3 + \frac{9}{2} \ \epsilon^2)(1 - r) \ s^2\theta_2 \ c\theta_1 \ \frac{s\theta_1}{\theta_1}\Big]$$

$$+ \epsilon\left\{ c\tau\left[-4\theta_2' \ c\theta_2 \ c^2\theta_1 - 2 \ell r \ c\theta_2 \ s\theta_1 + 4 c^2\theta_2 \ s\theta_1 \ c\theta_1\right.\right.$$

$$- 9(1 - r) \ s^2\theta_2 \ s\theta_1 \ c\theta_1\Big] + s\tau\left[2 \ s\theta_2\right]\Big\}$$

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$$+ \epsilon^2 \left\{ c2\tau \left[- 5\theta_2' \, c\theta_2 \, c^2\theta_1 \; - \; \frac{5}{2} \, \ell \, r \, c\theta_2 \, s\theta_1 \right.\right.$$

$$\left. + \; 7 \, c^2\theta_2 \, s\theta_1 \, c\theta_1 \; - \; \frac{27}{2} \, (1 - r) \, s^2\theta_2 \, s\theta_1 \, c\theta_1 \right]$$

$$\left. + \; s2\tau \left[5 \, s\theta_2 \right] \right\} \; + \; 0 \, (\epsilon^3) \tag{4.13}$$

$$\theta_2'' \; + \; \theta_1' \, (\ell \, r \, - \, 2) \; + \; \theta_2 \left[\ell \, r \; + \; 3r \; - \; 4 \; + \; \epsilon^2 \left(\frac{9}{2} \, r \; - \; \frac{13}{2} \right) \right]$$

$$= \; \theta_2'' \left[1 \; - \; c^2\theta_1 \right] \; + \; \theta_1' \left[2 \, s\theta_1 \, c\theta_1 \, \theta_2' \; - \; 2(1 \; - \; c\theta_2 \, c^2\theta_1) \right.$$

$$\left. + \; \ell \, r(1 \; - \; c\theta_1) \right] + \theta_2 \left[\ell \, r(1 \; - \; c\theta_1 \, \frac{s\theta_2}{\theta_2}) \right.$$

$$\left. + \; (3r \; - \; 4 \; + \; \frac{9}{2} \, \epsilon^2 \, r \; - \; \frac{13}{2} \, \epsilon^2) \, (1 \; - \; c\theta_2 \, c^2\theta_1 \, \frac{s\theta_2}{\theta_2}) \right]$$

$$+ \; \epsilon \left\{ s\tau \left[- \; 2c\theta_2 \, s\theta_1 \, c\theta_1 \right] \; + \; c\tau \left[4c\theta_2 \, c^2\theta_1 \, \theta_1' \right.\right.$$

$$\left.\left. - \; 2 \, \ell \, r \, s\theta_2 \, c\theta_1 \; + \; (13 \; - \; 9r) \, s\theta_2 \, c\theta_2 \, c^2\theta_1 \right] \right\}$$

$$+ \; \epsilon^2 \left\{ s2\tau \left[- \, 5 \, c\theta_2 \, s\theta_1 \, c\theta_1 \right] \; + \; c2\tau \left[5 \, c\theta_2 \, c^2\theta_1 \, \theta_1' \right.\right.$$

$$\left.\left. - \; \frac{5}{2} \, \ell \, r \, s\theta_2 \, c\theta_1 \; + \; \frac{1}{2} \, (41 \; - \; 27r) \, s\theta_2 \, c\theta_2 \, c^2\theta_1 \right] \right\} \; + \; 0(\epsilon^3)$$

The time dependent orbital motion given by Eq's. (4.1) has been included

in the above equations and all terms through the second power of the orbit eccentricity have been included. Note that linear terms in the variables and their derivatives have been added and subtracted in the above forms of the equations of motion so as to render them in a form that will be useful in the subsequent analysis.

4.5 Dynamic Equilibrium Positions

Inspection of the equations of motion, Eq's. (4.13), shows that they are satisfied at the position

$$\theta_1 = \theta_2 = \theta_1' = \theta_2' = \theta_1'' = \theta_2'' = 0 \qquad (4.14)$$

This then is a dynamic equilibrium position at which the spin axis of the symmetrical satellite is perpendicular to the orbit plane.

A number of useful observations can be made regarding the form of the equations of motion, Eq's. (4.13), in the neighborhood of the equilibrium position. The terms grouped on the left are products of constants times terms in the first power of the displacements and their derivatives. The terms on the right are grouped in terms with constant coefficients and terms with time-dependent coefficients. The terms with constant coefficients can be seen to be very small in the neighborhood of the equilibrium position, their magnitudes being proportional to the third and higher powers of the displacements, θ_1 and θ_2, and their derivatives.

The time-dependent terms in Eq's. (4.13) are all multiplied by powers of ϵ. The coefficients of ϵ to the first power consist of

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products of the displacements and velocities to the first and higher powers multiplied by terms harmonic in time and of frequency 1. The coefficients of ϵ^2 on the other hand possess harmonics of frequency 2, etc.

If the equations of motion are linearized in the neighborhood of the equilibrium positon (i.e. powers higher than the first in the displacements and their derivatives are discarded), they take the form

$$\theta_1'' + \theta_2' (2 - \ell r) + \theta_1 (\ell r - 1 - 2\epsilon^2)$$

$$= \epsilon \left\{ c\tau \left[-4\theta_2' - \theta_1 (2\ell r - 4) \right] + s\tau \left[2\theta_2 \right] \right\}$$

$$+ \epsilon^2 \left\{ c2\tau \left[-5\theta_2' + \theta_1 (7 - \frac{5}{2}\ell r) \right] \; s2\tau \left[5\theta_2 \right] \right\} + O(\epsilon^3)$$

(4.15

$$\theta_2'' + \theta_1' (\ell r - 2) + \theta_2 \left[\ell r + 3r - 4 + \epsilon^2 (\frac{9}{2} r - \frac{13}{2}) \right]$$

$$= \epsilon \left\{ c\tau \left[4\theta_1' + \theta_2 (13 - 9r - 2\ell r) \right] - s\tau \left[2\theta_1 \right] \right\}$$

$$+ \epsilon^2 \left\{ c2\tau \left[5\theta_1' + \theta_2 (\frac{41}{2} - \frac{27}{2} r - \frac{5}{2}\ell r) \right] \right.$$

$$\left. - s2\tau \left[5\theta_1 \right] \right\} + O(\epsilon^3)$$

4.6   Liapounov-Type Analysis

Application of the Liapounov method to determine the stability
of motion of the symmetrical, rigid body in an elliptical orbit in the
neighborhood of the equilibrium position will now be undertaken. The difference
between the Hamiltonian function, H, and the Hamiltonian function evaluated
at the equilibrium position, H_E , will be used as a testing function.

The Hamiltonian function has been shown [1] to be related to the
Routhian by

$$H = \frac{\partial \mathcal{R}}{\partial \theta_1'} \theta_1' + \frac{\partial \mathcal{R}}{\partial \theta_2'} \theta_2' - \mathcal{R}$$

$$= \frac{2\pi^2 A}{T^2} \left[\theta_1'^2 + \theta_2'^2 c^2\theta_1 - \Theta'^2 (s^2\theta_2 + c^2\theta_2 s^2\theta_1) \right.$$

$$\left. + \ell^2 r - 2 \ell r \Theta' c\theta_2 c\theta_1 - 3 \frac{KT^2}{4\pi^2} \frac{1}{R^3} (1 - r) s^2\theta_2 c^2\theta_1 \right]$$

$$(4.16)$$

so that

$$H_E = \frac{2\pi^2 A}{T^2} (\ell^2 r - 2 \ell r \Theta') \qquad (4.17)$$

Introducing the time dependent motion, we obtain the testing function

$$V = H - H_E = \frac{2\pi^2 A}{T^2} \left\{ \theta_1'^2 + \theta_2'^2 c^2\theta_1 \right.$$

$$- (s^2\theta_2 + c^2\theta_2 s^2\theta_1) \left[1 + (4 c\tau) \epsilon + (2 + 7 c2\tau)\epsilon^2 \right.$$

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$$+ \frac{1}{2} (9\,c\tau + 23\,c3\tau)\epsilon^3 + \ldots \Big] + 2\,\ell\,r\,(1$$

$$- c\theta_2\,c\theta_1) \Big[1 + (2\,c\tau)\,\epsilon + \frac{5}{2}(c2\tau)\,\epsilon^2 + \frac{1}{4}(- c\tau$$

$$+ 13\,c3\tau)\epsilon^3 + \ldots \Big] - 3(1 - r)\,s^2\theta_2\,c^2\theta_1 \Big[1$$

$$+ (3\,c\tau)\,\epsilon + \frac{3}{2}(1 + 3\,c2\tau)\epsilon^2 + \frac{1}{8}(27\,c\tau$$

$$+ 53\,c3\tau)\epsilon^3 + \ldots \Big] \Big\} \tag{4.18}$$

As before we will define K E* as the portion of the Hamiltonian which is quadratic in the velocities. The remaining terms in the Hamiltonian, defined as the dynamic potential U, are

$$U = \frac{2\pi^2 A}{T^2} \Big\{ - (s^2\theta_2 + c^2\theta_2\,s^2\theta_1) \Big[1 + (4\,c\tau)\,\epsilon$$

$$+ (2 + 7\,c2\tau)\epsilon^2 + \frac{1}{2}(9\,c\tau + 23\,c3\tau)\epsilon^3 + \ldots \Big]$$

$$+ 2\,\ell\,r(1 - c\theta_2\,c\theta_1) \Big[1 + (2\,c\tau)\,\epsilon + \frac{5}{2}\,c2\tau\,\epsilon^2$$

$$+ \frac{1}{4}(- c\tau + 13\,c3\tau)\epsilon^3 + \ldots \Big] - 3(1 - r)\,s^2\theta_2\,c^2\theta_1 \Big[1$$

$$+ (3\,c\tau)\,\epsilon + \frac{3}{2}(1 + 3\,c2\tau)\epsilon^2 + \frac{1}{8}(27\,c\tau$$

$$+ 53\,c3\tau)\epsilon^3 + \ldots \Big] \Big\} \tag{4.19}$$

It is evident that KE^* is positive definite in the neighborhood of the equilibrium position, so it is only necessary to check the positve definiteness of U to determine the positive definiteness of V. Again applying Sylvester's criterion we conclude that V is positve definite if

$$\left. \frac{\partial^2 U}{\partial \theta_1^2} \right|_E > 0$$

$$\left. \frac{\partial^2 U}{\partial \theta_1^2} \frac{\partial^2 U}{\partial \theta_2^2} - \left(\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2} \right)^2 \right|_E > 0 \tag{4.20}$$

which gives the following requirements

$$\ell r - 1 + \epsilon \ c_T(2 \ell r - 4) + \epsilon^2\left[- 2 + c2_T \ (\tfrac{5}{2} \ell r - 7)\right] + O(\epsilon^3) > 0 \tag{4.21}$$

$$\ell r + 3r - 4 + \epsilon \ c_T(2 \ell r + 9r - 13) + \epsilon^2\left[\tfrac{9}{2} r - \tfrac{13}{2} + c2_T(\tfrac{5}{2} \ell r + \tfrac{27}{2} r - \tfrac{41}{2})\right] + O(\epsilon^3) > 0$$

For small values of the orbit eccentricity ϵ we can determine the boundary of the region within which V is positive definite by neglecting terms in ϵ^2. For small ϵ the corresponding approximate inequalities

$$\ell\, r \;>\; 1 \;+\; 2\epsilon\, c_T$$

$$(4.22)$$

$$\ell\, r \;>\; 4 \;-\; 3r \;+\; \epsilon\, c_T\, (5 \;-\; 3r)$$

resulting in three inequalities that must be satisfied for V to be positive definite for all positive values of ϵ . These inequalities have the form

$$\ell \;>\; \frac{1}{r}\,(1 \;+\; 2\epsilon)$$

$$(4.23)$$

$$\ell \;>\; \frac{4 \;-\; 3r}{r} \;\pm\; \epsilon\left(\frac{5}{r} \;-\; 3\right) \begin{cases} + \text{ for } \frac{5}{r} > 3 \\ - \text{ for } \frac{5}{r} < 3 \end{cases}$$

The resulting stability boundary is shown on Figure 4.1. In the neighborhood of the equilibrium and in the limit as ϵ approaches 0, ℓ approaches $\alpha_1 + 1$ where α_1 is the ratio of the average spin rate to the average orbit angular rate as used in the previous chapter. Consequently it can be seen that in this limiting case the stability boundary is identical with that given in Section III and in References 9 and 10.

The second part of the Liapounov stability theorem states that, given a positive definite Liapounov function, the motion will be stable if the following inequality can be satisfied

$$\int_{t_0}^{t} \frac{dV}{dt}\, dt \;=\; \int_{t_0}^{t} \frac{\partial(H \;-\; H_E)}{\partial t}\, dt \;=\; \frac{2\pi^2 A}{T^2} \int_{t_0}^{t} \left\{\, (s^2\theta_2 \right.$$
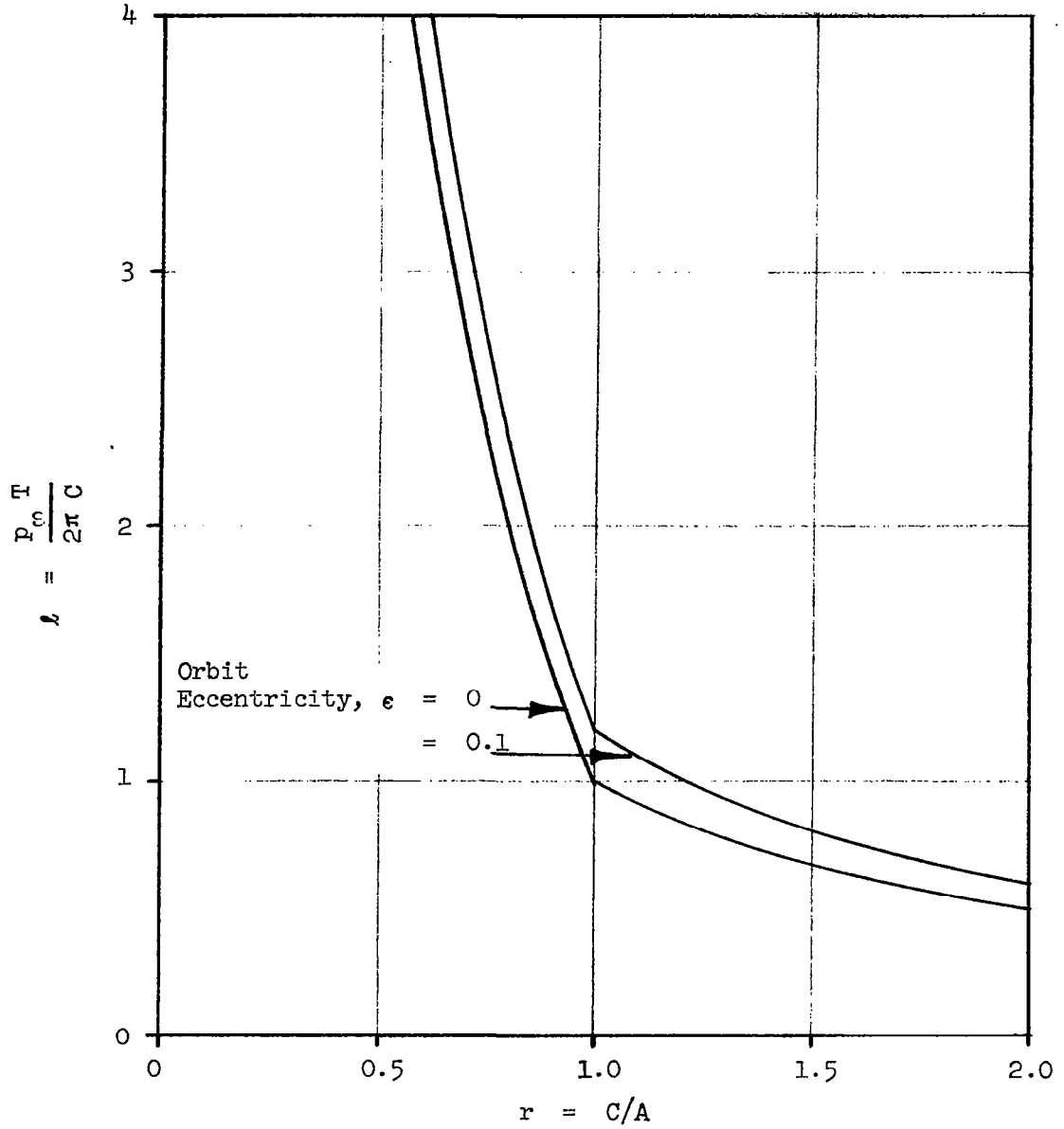
FIGURE 4.1

STABILITY BOUNDARY DICTATED BY POSITIVE DEFINITE HAMILTONIAN
SYMMETRIC, SPINNING SATELLITE IN ELLIPTICAL ORBIT

$$+ c^2\theta_2 \, s^2\theta_1) \left[(4 \, s_\tau) \, \epsilon \; + (14 \, s2\tau)\epsilon^2 \; + \frac{1}{2} (9 \, s_\tau\right.$$

$$\left. + 69 \, s3\tau)\epsilon^3 \; + \; \dots \right] \; - \; 2 \, \ell \, r \, (1 \; - \; c\theta_2 \, c\theta_1) \left[(2 \, s_\tau) \, \epsilon\right.$$

$$\left. + (5 \, s2\tau) \, \epsilon^2 \; + \frac{1}{4} (- \, s_\tau \; + \; 39 \, s3\tau)\epsilon^3 \; + \; \dots \right]$$

$$+ \; 3(1 \; - \; r) \, s^2\theta_2 \, c^2\theta_1 \left[(3 \, s\tau) \, \epsilon \; + (9 \, s2\tau)\epsilon^2\right.$$

$$\left. + \frac{1}{8} (27 \, s\tau \; + \; 159 \, s3\tau)\epsilon^3 \; + \; \dots \right] \bigg\} \; dt \; < \; M(\theta_1^2$$

$$+ \; \theta_2^2 \; + \; \theta_1'^2 \; + \; \theta_2'^2 \;) \bigg|_{t \; = \; t_0} \tag{4.24}$$

In order to investigate this integral, one must know the form of the
solutions, θ_1 and θ_2, and in particular the frequency components if
the solutions are periodic. If the nonlinear terms are neglected in
the equations of motion and a correspondingly reduced Hamiltonian function
is used, one can apply Floquet's theorem as was done in Section III and
in Appendix B. When this is done and following an analogous procedure,
it can be shown that, for ϵ small, the instabilities occur when

$$\omega_1 \; \cong \; \frac{1}{2} \, n$$

$$\omega_2 \; \cong \; \frac{1}{2} \, n \qquad\qquad (n=1,2,\dots) \tag{4.25}$$

$$\omega_1 \; \pm \; \omega_2 \; \cong \; n$$

where ω_1 and ω_2 are the natural frequencies[*] of the linearized system for $\epsilon = 0$

$$\omega_1 = \sqrt{b + \sqrt{b^2 - c}} \qquad\qquad \omega_2 = \sqrt{b - \sqrt{b^2 - c}} \qquad (4.26)$$

and

$$b = \frac{1}{2} (- \ell^2 r^2 + 6 \ell r + 3r - 9) \tag{4.27}$$

$$c = (\ell r - 1)(\ell r + 3r - 4)$$

When the complete nonlinear equations are taken into consideration, Floquet's theorem is no longer applicable and other means of determining the form of the solution must be employed. In the next section, the non-linear equations are analyzed using an asymptotic expansion and it is found that the solutions, θ_1 and θ_2, can be written

$$\theta_1 = \sum_{\ell, m, n = 0}^{\infty} a_{\ell mn} \cos\left[(\ell + m \psi_1 + n \psi_2)\tau + \delta_{\ell mn} \right]$$

$$(4.28)$$

$$\theta_2 = \sum_{\ell, m, n = 0}^{\infty} b_{\ell mn} \sin\left[(\ell + m \psi_1 + n \psi_2)\tau + \delta_{\ell mn} \right]$$

---

[*]The true natural frequencies are obtained by multiplying ω_1 and ω_2 by $\frac{2\pi}{T}$.

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where the $a_{\ell mn}$ are constants or slowly varying functions of time. Also ℓ, m, and n are not all zero simultaneously in the region where the Hamiltonian is positive definite, and the sum of m plus n is equal to an odd integer. Our current interest is in the case of small oscillations about the equilibrium position and small orbit eccentricity, and under these conditions

$$\psi_1 \cong \omega_1 \qquad , \qquad \psi_2 \cong \omega_2 \qquad\qquad (4.29)$$

Substituting Eq's. (4.28) into Eq. (4.24) it can be seen that the integrand includes terms that are composed of θ_1 raised to an even power times a trigonometric function with frequency equal to an integer. The value of the integral will be bounded when no resonance takes place and may be bounded or unbounded when a resonance takes place as in the previous cases. For small amplitudes of oscillation and small ϵ, the form of the integrand of Eq. (4.24) is such that resonance can be seen to occur in the neighborhood of positions where

$$\omega_1 = \frac{p}{2q} , \qquad (p,q=1,2,\ldots)$$

$$(4.30)$$

$$\omega_2 = \frac{p}{2q} , \qquad (p,q,=1,2,\ldots)$$

$$\pm m\omega_1 \pm n\omega_2 = \frac{p}{q} , \quad (m,n,p,q=1,2,\ldots)$$

Inspection of the above equations shows that the nonlinear system theoretically has resonance regions in the neighborhood of any region of parameter space. This has previously been noted by Bogoliuboff and Mitropolski under similar circumstances:* "Because p and q may take up all possible integral values, the set $\{p/q\}$ is compact and hence the ratio p/q , with proper choice of the numbers p and q , may approach any given number."

As a result of consideration of the nonlinear terms, additional resonance regions termed "nonlinear resonance regions", that did not occur in Eq's. (4.25) for the linearized system are found in Eq's. (4.30) for the nonlinear system. We will classify the resonance regions given by Eq's. (4.25) as "linear resonance regions". When oscillations near the equilibrium position are considered, the nonlinear terms with the largest magnitudes are third power products of the displacements and their derivatives. But even these terms are small since they include coefficients that have the amplitude of oscillation raised to the third power as a factor, and this amplitude can be made arbitrarily small by limiting consideration to sufficiently small amplitudes of motion about the equilibrium. Also, it is shown in the next section that when the nonlinear terms are considered, the amplitude of motion affects the frequency of oscillation of the system. Consequently it is found that a system which experiences one of these "nonlinear type resonances" when oscillating with one amplitude will increase its amplitude of oscillation,

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* N.N. Bogoliubov and Y.A. Mitropolski, <u>Asymptotic Methods in the Theory of Non-Linear Oscillations</u> (Delhi, India: Hindustan Publishing Company, 1961), p. 198.

causing a shift in the system natural frequency until a condition of bounded periodic motion is reached.

As a result of the above discussion, our greatest interest will be placed in the resonance regions dictated by the linearized system. The approximate locations of these resonance regions when ϵ is small are given in Figure 4.2.

4.7   Asymptotic Analysis

Inspection of the complete, nonlinear equations of motion, as given in Eq's. (4.13), shows them to be arranged so that the linear terms with constant coefficients appear on the left.

Terms appearing on the right side of Eq's. (4.13) are grouped into (a) the nonlinear terms with constant coefficients and (b) the terms with time-dependent (periodic) coefficients. The terms with periodic coefficients are all multiplied by the small parameter ϵ to the first or higher power, so that an asymptotic analysis similar to that used in the previous section appears to offer promise of furnishing a meaningful solution. However, the nonlinear terms with constant coefficients present a somewhat different problem. Consequently, the analysis is presented first for the linear system with constant coefficients, next for the nonlinear system with constant coefficients and finally for the complete nonlinear periodic system. This last analysis is then specialized to the linear, periodic system.

$$\ell = \frac{p_\phi T}{2\pi C}$$
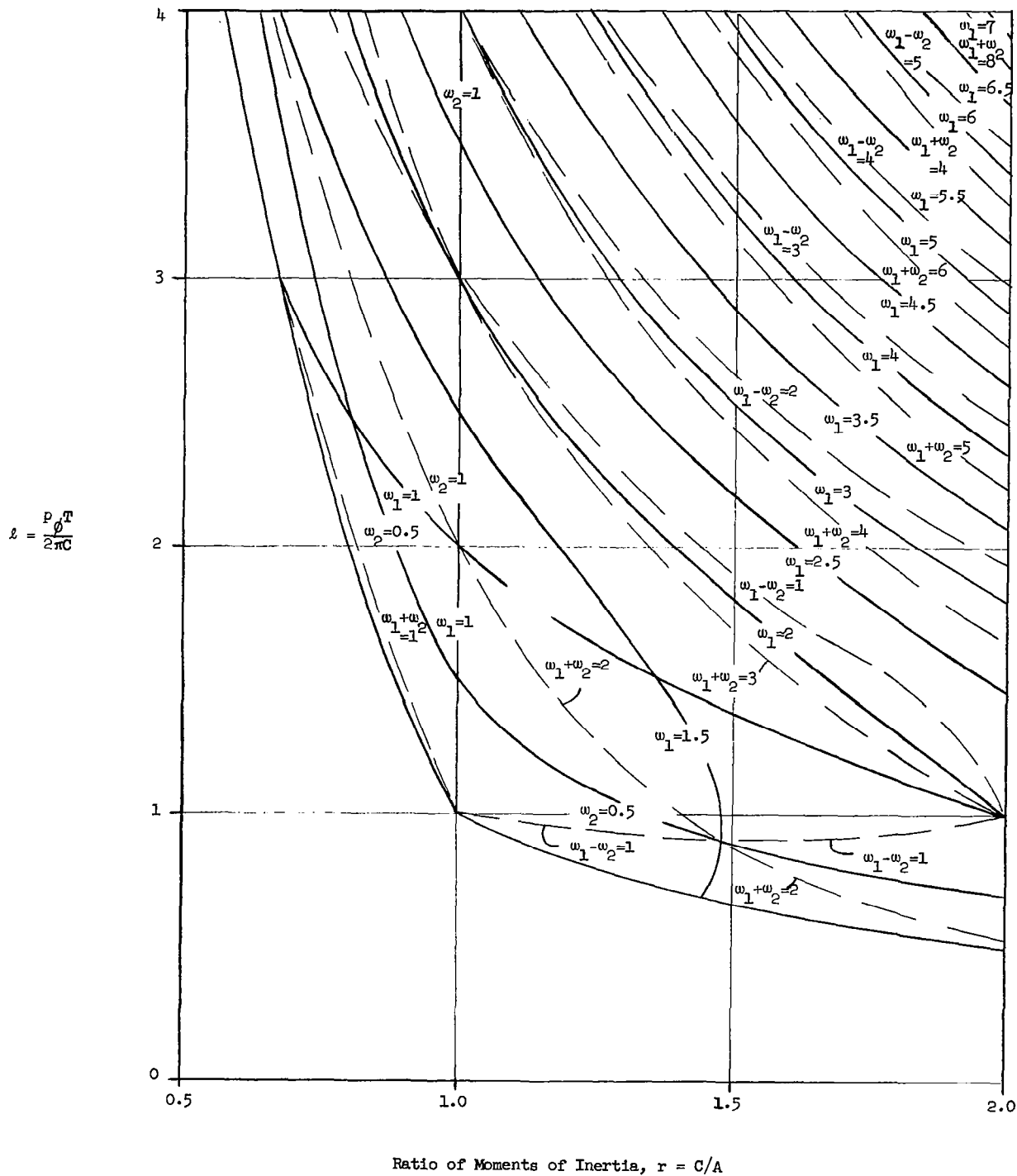
Ratio of Moments of Inertia, $r = C/A$

FIGURE 4.2

SYMMETRIC SPINNING SATELLITE IN AN ELLIPTICAL ORBIT
APPROXIMATE LOCATIONS OF RESONANCE INSTABILITY
REGIONS FOR THE LINEARIZED SYSTEM AND $\epsilon \ll 1$

4.7.1 Underlined: Unperturbed,* Linearized System

Designating the equations of motion with the right side terms set equal to zero as the unperturbed linearized equations we obtain

$$\theta_1'' + \beta_1 \theta_2' + \beta_2 \theta_1 = 0$$

$$\theta_2'' + \beta_3 \theta_1' + \beta_4 \theta_2 = 0 \tag{4.31}$$

in which

$$\beta_1 = 2 - \ell r$$

$$\beta_2 = \ell r - 1 - 2\epsilon^2$$

$$\beta_3 = \ell r - 2 \tag{4.32}**$$

$$\beta_4 = \ell r + 3r - 4 + \epsilon^2 \left(\frac{9}{2} r - \frac{13}{2} \right)$$

The natural frequencies of the unperturbed linearized system are given by

---

*Note that unperturbed system should be regarded in the sense of a constant-coefficients system.

**Different symbols have been used for β_1 and β_3. This is consistent with the notation of the previous chapter.

$$\bar{w}_i = \left\{ \frac{\beta_2 + \beta_4 - \beta_1\beta_3}{2} \right. \tag{4.33}*$$

$$\left. + (-1)^{i+1} \left[\left(\frac{\beta_2 + \beta_4 - \beta_1\beta_3}{2} \right)^2 - \beta_2\beta_4 \right]^{1/2} \right\}^{1/2}, \quad (i=1,2)$$

Also, the ratio of the amplitude of θ_2 motion to the amplitude of θ_1 motion satisfying the unperturbed linearized system is

$$\lambda_i = \frac{\bar{w}_i^2 - \beta_2}{\beta_1\bar{w}_i} = \frac{\beta_3\bar{w}_i}{\beta_4 - \bar{w}_i^2}, \quad (i=1,2) \tag{4.34}$$

4.7.2 Unperturbed, Nonlinear System

Designating the equations of motion with the periodic terms set equal to zero as the unperturbed, nonlinear equations, Eq's. (4.13) lead to

$$\theta_1'' + \beta_1\theta_2' + \beta_2\theta_1 = \sigma \left\{ \theta_2' \left[\theta_2^2 + \theta_1^2 (2 - \frac{1}{2} \ell r) \right. \right.$$

$$- \theta_1\theta_2' - \frac{1}{12} \theta_2^4 + \theta_1^4 (\frac{1}{24} \ell r - \frac{2}{3}) - \theta_1^2 \theta_2^2$$

$$+ \frac{2}{3} \theta_1^3 \theta_2' \left. \right] + \theta_1 \left[\theta_2^2 (\frac{1}{2} \ell r - 4 + 3r + \frac{9}{2} \epsilon^2 r \right.$$

$$- \frac{5}{2} \epsilon^2) + \theta_1^2 (\frac{1}{6} \ell r - \frac{2}{3} - \frac{4}{3} \epsilon^2)$$

$$+ \theta_2^4 (- \frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{6} \epsilon^2 - \frac{3}{2} \epsilon^2 r)$$

---

*The symbol $\bar{w}_i$ differs from w_i in that autonomous linear terms in ϵ^2 have been included in Eq's. (4.32).

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$$+ \theta_1^4 \left(- \frac{1}{120} \ell r + \frac{2}{15} + \frac{4}{15} \epsilon^2\right) + \theta_1^2 \theta_2^2 \left(- \frac{1}{12} \ell r\right.$$

$$\left.\left. + \frac{8}{3} - 2r + \frac{13}{3} \epsilon^2 - 3 \epsilon^2 r\right)\right] + \ldots \Bigg\}$$

$$(4.35)$$

$$\theta_2'' + \beta_3 \theta_1' + \beta_4 \theta_2 = \sigma \Bigg\{ \theta_2'' \left(\theta_1^2 - \frac{1}{3} \theta_1^4\right) + \theta_1' \left[2\theta_1 \theta_2'\right.$$

$$- \theta_2^2 + \left(\frac{\ell r}{2} - 2\right)\theta_1^2 - \frac{4}{3} \theta_1^3 \theta_2' + \theta_1^2 \theta_2^2 + \frac{1}{12} \theta_2^4$$

$$\left. + \left(\frac{2}{3} - \frac{1}{24} \ell r\right) \theta_1^4 \right] + \theta_2 \left[\theta_2^2 \left(\frac{1}{6} \ell r - \frac{8}{3} + 2r\right.\right.$$

$$\left. - \frac{13}{3} \epsilon^2 + 3 r \epsilon^2\right) + \theta_1^2 \left(\frac{1}{2} \ell r - 4 + 3r - \frac{13}{2} \epsilon^2\right.$$

$$\left. + \frac{9}{2} \epsilon^2 r\right) + \theta_2^4 \left(- \frac{1}{120} \ell r + \frac{8}{15} - \frac{2}{5} r + \frac{13}{15} \epsilon^2\right.$$

$$\left. - \frac{3}{5} \epsilon^2 r\right) + \theta_1^4 \left(- \frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{6} \epsilon^2\right.$$

$$\left. - \frac{3}{2} \epsilon^2 r\right) + \theta_1^2 \theta_2^2 \left(- \frac{1}{12} \ell r + \frac{8}{3} - 2r + \frac{13}{3} \epsilon^2\right.$$

$$\left.\left. - 3 \epsilon^2 r\right) \right] + \ldots \Bigg\}$$

In the above equations, the sine and cosine terms have been expanded in series form and all terms through the 5th power of the coordinates and their derivatives have been retained. Terms in the 7th, 9th, and higher odd powers of products of the displacements and their derivatives have

been neglected as they are small in the neighborhood of the equilibrium position. However their influence on the resulting solution will be discussed at the end of this section. As before, only terms in ϵ to the second power have been included. In addition, the left side of Eq's. (4.35) has been multiplied by a factor, σ.

The factor σ included in Eq's. (4.35) can be considered to be a parameter of the problem, so that the solution will be an analytic function of σ and will be the desired solution when $\sigma = 1$. Following a procedure suggested by Kryloff and Bogoliuboff [25] and used by Cunningham [32], an expansion will be made as though σ was a small parameter. However, in this case, σ is not a small parameter so that the expansion can be expected to converge to the correct solution only if it can be shown that the coefficients of successively higher powers of the parameter σ become extremely small. To this end we can consider an assumed solution for which the frequency of the principal oscillation is nearly equal to a natural frequency of the unperturbed, linearized system.

$$\theta_1 = a \cos \Phi + u_{11}\sigma + u_{12}\sigma^2 + \ldots$$

$$\theta_2 = a(\lambda_i + \sigma\lambda_{i1} + \sigma^2\lambda_{i2} + \ldots) \sin \Phi$$

$$+ u_{21}\sigma + u_{22}\sigma^2 + \ldots \qquad (4.36)$$

$$\Phi = (\bar{\omega}_i + \Delta_{i1}\sigma + \Delta_{i2}\sigma^2 + \dots)\tau + \delta$$

where λ_i and $\bar{\omega}_i$ are obtained from Eq's. (4.34) and (4.35), and u_{11}, u_{22}, etc. are selected to include all terms of frequency other than Φ. The subscript i will be dropped throughout the remainder of the analysis of the nonlinear, unperturbed equations. Substituting the assumed solution into the equations of motion, and retaining all terms through the second power in σ,* gives

$$a\left\{-\bar{\omega}^2 + \lambda\omega\beta_1 + \beta_2\right\}c\Phi + \sigma\left\{a\left[-2\bar{\omega}\Delta_1 + \beta_1(\lambda\Delta_1\right.\right.$$

$$\left.\left. + \lambda_1\bar{\omega})\right]c\Phi + u_{11}'' + \beta_1 u_{21}' + \beta_2 u_{11}\right\}$$

$$+ \sigma^2\left\{a\left[-\Delta_1^2 - 2\bar{\omega}\Delta_2 + \beta_1(\lambda\Delta_2 + \lambda_1\Delta_1\right.\right.$$

$$\left.\left. + \lambda_2\bar{\omega})\right]c\Phi + u_{12}'' + \beta_1 u_{22}' + \beta_2 u_{12}\right\} + \sigma^3\left\{\dots\right\}$$

$$+ \dots = \sigma\left\{a^3\left[d_{11}c\Phi + d_{12}c3\Phi\right] + a^5\left[d_{13}c\Phi\right.\right.$$

$$\left.\left. + d_{14}c3\Phi + d_{15}c5\Phi\right]\right\} + \sigma^2\left\{a^3\left[\Delta_1(d_{21}c\Phi + d_{22}c3\Phi)\right.\right.$$

$$+ \lambda_1(d_{23}c\Phi + d_{24}c3\Phi) + \frac{u_{21}'}{a}(d_{25} + d_{26}c2\Phi)$$

---

*Although it appears that the equations are separated according to their order in σ, one should recall that u_{ij}' and u_{ij}'' contain terms in σ, σ^2, ...

$$+ \frac{u_{21}}{a} d_{27} s2\Phi + \frac{u_{11}}{a} (d_{28} + d_{29} c2\Phi) \Big] + a^5 \Big[\ldots \Big] \Big\}$$

$$+ \sigma^3 \Big\{ \ldots \Big\} + \ldots \qquad\qquad (4.37)$$

$$a \Big\{ - \lambda_0 \bar{w}^2 - \beta_3 \bar{w} + \beta_4 \lambda \Big\} s\Phi + \sigma \Big\{ a \Big[-2\bar{w}\Delta_1 \lambda - \bar{w}^2 \lambda_1$$

$$- \beta_3 \Delta_1 + \beta_4 \lambda_1 \Big] s\Phi + u''_{21} + \beta_3 u'_{11} + \beta_4 u_{21} \Big\}$$

$$+ \sigma^2 \Big\{ a \Big[- \lambda \Delta_1^2 - 2\lambda_1 \bar{w} \Delta_2 - \bar{w}^2 \lambda_2 - \beta_3 \Delta_2$$

$$+ \beta_4 \lambda_2 \Big] s\Phi + u''_{22} + \beta_3 u'_{12} + \beta_4 u_{22} \Big\} + \sigma^3 \Big\{ \ldots \Big\}$$

$$+ \ldots = \sigma \Big\{ a^3 \Big[e_{11} s\Phi + e_{12} s3\Phi \Big] + a^5 \Big[e_{13} s\Phi$$

$$+ e_{14} s3\Phi + e_{15} s5\Phi \Big] \Big\} + \sigma^2 \Big\{ a^3 \Big[\Delta_1 (e_{21} s\Phi + e_{22} s3\Phi)$$

$$+ \lambda_1 (e_{23} s\Phi + e_{24} s3\Phi) + \frac{u''_{21}}{a} \Big(\frac{1}{2} - \frac{1}{2} c2\Phi \Big) + \frac{u'_{21}}{a} (- \bar{w} s2\Phi)$$

$$+ \frac{u'_{11}}{a} (e_{25} + e_{26} c2\Phi) + \frac{u_{21}}{a} (e_{27} + e_{28} c2\Phi) + \frac{u_{11}}{a} (e_{29} s2\Phi) \Big]$$

$$+ a^5 \Big[\ldots \Big] \Big\} + \sigma^3 \Big\{ \ldots \Big\} + \ldots$$

where the constants introduced in the above equations are

$$d_{11} = \frac{1}{4} \Big[\lambda^3 \bar{w} + \lambda^2 \Big(\frac{1}{2} \ell r - 4 + 3r + \frac{9}{2} \epsilon^2 r$$

$$- \frac{5}{2} \epsilon^2) + 3\lambda\bar{\omega} (2 - \frac{1}{2} \ell r) - 3\lambda^2\bar{\omega}^2 + 3(\frac{1}{6} \ell r$$

$$- \frac{2}{3} - \frac{4}{3} \epsilon^2) \Big]$$

$$d_{12} = \frac{1}{4} \Big[- \lambda^3\bar{\omega} - \lambda^2(2 \ell r - 4 + 3r + \frac{9}{2} \epsilon^2 r - \frac{5}{2} \epsilon^2)$$

$$+ \lambda\bar{\omega} (2 - \frac{1}{2} \ell r) - \lambda^2\bar{\omega}^2 + (\frac{1}{6} \ell r - \frac{2}{3} - \frac{4}{3} \epsilon^2) \Big]$$

$$d_{13} = \frac{1}{8} \Big[- \frac{1}{12} \lambda^5\bar{\omega} + 5\lambda\bar{\omega} (\frac{1}{24} \ell r - \frac{2}{3}) - \lambda^3\bar{\omega} + \frac{10}{3} \lambda^2\bar{\omega}^2$$

$$+ \lambda^4(- \frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{2} \epsilon^2 - \frac{3}{2} \epsilon^2 r) + (- \frac{1}{24} \ell r$$

$$+ \frac{2}{3} + \frac{4}{3} \epsilon^2) + \lambda^2(- \frac{1}{12} \ell r + \frac{8}{3} - 2r + \frac{13}{3} \epsilon^2 - 3\epsilon^2 r) \Big]$$

$$d_{14} = \frac{1}{16} \Big[\frac{1}{4} \lambda^5\bar{\omega} + 5\lambda\bar{\omega} (\frac{1}{24} \ell r - \frac{2}{3}) + \lambda^3\bar{\omega} + \frac{10}{3} \lambda^2\bar{\omega}^2$$

$$- \lambda^4 (- \frac{1}{8} \ell r + 4 - 3r + \frac{13}{2} \epsilon^2 - \frac{9}{2} \epsilon^2 r) - (\frac{1}{24} \ell r$$

$$+ \frac{2}{3} + \frac{4}{3} \epsilon^2) - \lambda^2(- \frac{1}{12} \ell r + \frac{8}{3} - 2r + \frac{13}{3} \epsilon^2 - 3\epsilon^2 r) \Big]$$

$$d_{15} = \frac{1}{16} \Big[- \frac{1}{12} \lambda^5\bar{\omega} + \lambda\bar{\omega} (\frac{1}{24} \ell r - \frac{2}{3}) + \lambda^3\bar{\omega} + \frac{2}{3} \lambda^2\bar{\omega}^2$$

$$+ \lambda^4 (- \frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{6} \epsilon^2 - \frac{3}{2} \epsilon^2 r) + (- \frac{1}{120} \ell r$$

$$+ \frac{2}{15} + \frac{4}{15} e^2) - \lambda^2 (- \frac{1}{12} \ell\, r + \frac{8}{3} - 2r + \frac{13}{3} e^2$$

$$- 3e^2 r) \Big]$$

$$d_{21} = \frac{1}{4} \Big[\lambda^3 + 3\lambda (2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w}) \Big]$$

$$d_{22} = \frac{1}{4} \Big[- \lambda^3 + \lambda (2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w})$$

$$d_{23} = \frac{1}{4} \Big[3\lambda^2 \bar{w} + 3\bar{w} (2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w}) + 2\lambda(\frac{1}{2} \ell\, r$$

$$- 4 + 3r + \frac{9}{2} e^2 r - \frac{5}{2} e^2) \Big]$$

$$d_{24} = \frac{1}{4} \Big[- 3\lambda^2 \bar{w} + \bar{w}(2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w}) - 2\lambda(\frac{1}{2} \ell\, r - 4$$

$$+ 3r + \frac{9}{2} e^2 r - \frac{5}{2} e^2) \Big]$$

$$d_{25} = \frac{1}{2} \Big[\lambda^2 + 2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w} \Big]$$

$$d_{26} = \frac{1}{2} \Big[- \lambda^2 + 2 - \frac{1}{2} \ell\, r - 2\lambda\bar{w} \Big]$$

$$d_{27} = \lambda^2 \bar{w} + \lambda (\frac{1}{2} \ell\, r - 4 + 3r + \frac{9}{2} e^2 r - \frac{5}{2} e^2)$$

$$d_{28} = \frac{1}{2} \Big[2\lambda\bar{w} (2 - \frac{1}{2} \ell\, r - \frac{1}{2} \lambda\bar{w}) + \lambda^2 (\frac{1}{2} \ell\, r - 4$$

$$+ 3r + \frac{9}{2} e^2 r - \frac{5}{2} e^2) + (\frac{1}{2} \ell r - 2 - 4 e^2) \Big]$$

$$d_{29} = \frac{1}{2} \Big[2\lambda\bar{w}(2 - \frac{1}{2} \ell r - \frac{1}{2} \lambda\bar{w}) - \lambda^2 (\frac{1}{2} \ell r - 4$$

$$+ 3r + \frac{9}{2} e^2 r - \frac{5}{2} e^2) + (\frac{1}{2} \ell r - 2 - 4 e^2) \Big]$$

$$e_{11} = \frac{1}{4} \Big[-\lambda\bar{w}^2 + 3\lambda^2\bar{w} - \bar{w}(\frac{1}{2} \ell r - 2 + 2\lambda\bar{w})$$

$$+ \lambda^3(\frac{1}{2} \ell r - 8 + 6r - 13 e^2 + 9 r e^2) + \lambda (\frac{1}{2} \ell r$$

$$- 4 + 3r - \frac{13}{2} e^2 + \frac{9}{2} e^2 r) \Big]$$

$$e_{12} = \frac{1}{4} \Big[-\lambda\bar{w}^2 - \lambda^2\bar{w} - \bar{w}(\frac{1}{2} \ell r - 2 + 2\lambda\bar{w})$$

$$- \lambda^3(\frac{1}{6} \ell r - \frac{8}{3} + 2r - \frac{13}{3} e^2 + 3 r e^2) + \lambda (\frac{1}{2} \ell r$$

$$- 4 + 3r - \frac{13}{2} e^2 + \frac{9}{2} e^2 r) \Big]$$

$$e_{13} = \frac{1}{8} \Big[\frac{1}{3} \lambda\bar{w}^2 - \frac{5}{12} \bar{w}\lambda^4 - \bar{w}(\frac{2}{3} - \frac{1}{24} \ell r - \frac{4}{3} \lambda\bar{w})$$

$$- \bar{w}\lambda^2 + \lambda^5(-\frac{1}{24} \ell r + \frac{40}{3} - 2r + \frac{13}{3} e^2 - 3 e^2 r)$$

$$+ \lambda(-\frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{6} e^2 - \frac{3}{2} e^2 r) + \lambda^3 (-\frac{1}{12} \ell r$$

$$+ \frac{8}{3} - 2r + \frac{13}{3} e^2 - 3e^2 r) \Big]$$

$$e_{14} = \frac{1}{16} \Big[\frac{5}{3} \lambda \bar{\omega}^2 + \frac{5}{12} \bar{\omega}\lambda^4 - 3\bar{\omega} \left(\frac{2}{3} - \frac{1}{24} \ell r - \frac{4}{3} \lambda\bar{\omega} \right)$$

$$- \bar{\omega}\lambda^2 - \lambda^5 \left(-\frac{1}{24} \ell r + \frac{40}{3} - 2r + \frac{13}{3} e^2 - 3e^2 r \right)$$

$$+ \lambda \left(-\frac{1}{8} \ell r + 4 - 3r + \frac{13}{2} e^2 - \frac{9}{2} e^2 r \right)$$

$$+ \lambda^3 \left(-\frac{1}{12} \ell r + \frac{8}{3} - 2r + \frac{13}{3} e^2 - 3e^2 r \right) \Big]$$

$$e_{15} = \frac{1}{16} \Big[\frac{1}{3} \lambda\bar{\omega}^2 - \frac{1}{12} \bar{\omega}\lambda^4 - \bar{\omega} \left(\frac{2}{3} - \frac{1}{24} \ell r - \frac{4}{3} \lambda\bar{\omega} \right)$$

$$+ \bar{\omega}\lambda^2 + \lambda^5 \left(-\frac{1}{120} \ell r + \frac{8}{3} - \frac{2}{5} r + \frac{13}{15} e^2 - \frac{3}{5} e^2 r \right)$$

$$+ \lambda \left(-\frac{1}{24} \ell r + \frac{4}{3} - r + \frac{13}{6} e^2 - \frac{3}{2} e^2 r \right) - \lambda^3 \left(-\frac{1}{12} \ell r \right.$$

$$+ \frac{8}{3} - 2r + \frac{13}{3} e^2 - 3e^2 r) \Big]$$

$$e_{21} = \frac{1}{4} \Big[-5\lambda\bar{\omega} + 3\lambda^2 - \frac{\ell r}{2} + 2) \Big]$$

$$e_{22} = \frac{1}{4} \Big[-5\lambda\bar{\omega} - \lambda^2 - \frac{\ell r}{2} + 2 \Big]$$

$$e_{23} = \frac{1}{4} \Big[6\bar{\omega}\lambda - 3\bar{\omega}^2 + \left(\frac{1}{2} \ell r - 4 + 3r - \frac{13}{2} e^2 \right.$$

$$+ \frac{9}{2} \epsilon^2 r) + 3\lambda^2(\frac{1}{2} \ell r - 8 + 6r - 13\epsilon^2 + 9\epsilon^2 r) \Big]$$

$$e_{24} = \frac{1}{4} \Big[- 2\bar{\omega}\lambda - 3\bar{\omega}^2 + (\frac{1}{2} \ell r - 4 + 3r - \frac{13}{2} \epsilon^2$$

$$+ \frac{9}{2} \epsilon^2 r) - \lambda^2(\frac{1}{2} \ell r - 8 + 6r - 13\epsilon^2 + 9\epsilon^2 r) \Big]$$

$$e_{25} = \frac{1}{2} \Big[2\lambda\bar{\omega} + \frac{\ell r}{2} - 2 - \lambda^2 \Big]$$

$$e_{26} = \frac{1}{2} \Big[2\lambda\bar{\omega} + \frac{\ell r}{2} - 2 + \lambda^2 \Big]$$

$$e_{27} = \frac{1}{2} \Big[2\lambda\bar{\omega} + (\frac{1}{2} \ell r - 4 + 3r - \frac{13}{2} \epsilon^2 + \frac{9}{2} \epsilon^2 r)$$

$$+ \lambda^2(\frac{1}{2} \ell r - 8 + 6r - 13\epsilon^2 + 9\epsilon^2 r) \Big]$$

$$e_{28} = \frac{1}{2} \Big[2\lambda\bar{\omega} + (\frac{1}{2} \ell r - 4 + 3r - \frac{13}{2} \epsilon^2 + \frac{9}{2} \epsilon^2 r)$$

$$- \lambda^2(\frac{1}{2} \ell r - 8 + 6r - 13\epsilon^2 + 9\epsilon^2 r) \Big]$$

$$e_{29} = 2 - \frac{\ell r}{2} - 2\lambda\bar{\omega}^2 + \lambda (\frac{1}{2} \ell r - 4 + 3r$$

$$- \frac{13}{2} \epsilon^2 + \frac{9}{2} \epsilon^2 r) \tag{4.38}$$

The first term in each of Eq's. (4.37) is zero by virtue of
Eq. (4.34). By equating the coefficients of σ and applying harmonic

balance, the following four equations are obtained

$$\Delta_1(\beta_1\lambda - 2\bar{\omega}) + \lambda_1(\beta_1\bar{\omega}) = a^2 d_{11} + a^4 d_{13}$$

$$\Delta_1(-\beta_3 - 2\bar{\omega}\lambda) + \lambda_1(\beta_4 - \bar{\omega}^2) = a^2 e_{11} + a^4 e_{13}$$

$$(4.39)$$

$$u''_{11} + \beta_1 u'_{21} + \beta_2 u_{11} = a^3(d_{12} + a^2 d_{14}) c3\bar{\Phi} + a^5 d_{15} c5\bar{\Phi}$$

$$u''_{21} + \beta_3 u'_{11} + \beta_4 u_{21} = a^3(e_{12} + a^2 e_{14}) s3\bar{\Phi} + a^5 e_{15} s5\bar{\Phi}$$

From the first two of these equations we obtain

$$\Delta_1 = a^2 \frac{(d_{11} + a^2 d_{13})(\beta_4 - \bar{\omega}^2) - (e_{11} + a^2 e_{13}) \beta_1\bar{\omega}}{(\beta_1\lambda - 2\bar{\omega})(\beta_4 - \bar{\omega}^2) + \beta_1\bar{\omega}(\beta_3 + 2\lambda\bar{\omega})}$$

$$(4.40)$$

$$\lambda_1 = a^2 \frac{(d_{11} + a^2 d_{13})(\beta_3 + 2\lambda\bar{\omega}) + (e_{11} + a^2 e_{13})(\beta_1\lambda - 2\bar{\omega})}{(\beta_1\lambda - 2\bar{\omega})(\beta_4 - \bar{\omega}^2) + \beta_1\bar{\omega}(\beta_3 + 2\lambda\bar{\omega})}$$

The second two of Eq's. (4.39) are satisfied, neglecting terms in ϵ arising from the derivatives of u_{11} and u_{21}, by

$$u_{11} = A_1 c3\bar{\Phi} + A_2 c5\bar{\Phi}$$

$$(4.41)$$

$$u_{21} = A_3 s3\bar{\Phi} + A_4 s5\bar{\Phi}$$

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in which the coefficients are

$$A_1 = a^3 \frac{(\beta_4 - 9\bar{\omega}^2)(d_{12} + a^2 d_{14}) - 3\bar{\omega}\beta_1(e_{12} + a^2 e_{14})}{(\beta_4 - 9\bar{\omega}^2)(\beta_2 - 9\bar{\omega}^2) + 9\bar{\omega}^2 \beta_1 \beta_3}$$

$$A_2 = a^5 \frac{(\beta_4 - 25\bar{\omega}^2) d_{15} - 5\bar{\omega}\beta_1 e_{15}}{(\beta_4 - 25\bar{\omega}^2)(\beta_2 - 25\bar{\omega}^2) + 25\bar{\omega}^2 \beta_1 \beta_3}$$

$$\hspace{11cm}(4.42)$$

$$A_3 = a^3 \frac{3\bar{\omega}\beta_3(d_{12} + a^2 d_{14}) + (\beta_2 - 9\bar{\omega}^2)(e_{12} + a^2 e_{14})}{(\beta_4 - 9\bar{\omega}^2)(\beta_2 - 9\bar{\omega}^2) + 9\bar{\omega}^2 \beta_1 \beta_3}$$

$$A_4 = a^5 \frac{5\bar{\omega}\beta_3 d_{15} + (\beta_2 - 25\bar{\omega}^2) e_{15}}{(\beta_4 - 25\bar{\omega}^2)(\beta_2 - 25\bar{\omega}^2) + 25\bar{\omega}^2 \beta_1 \beta_3}$$

We see that the presence of the nonlinear terms, in the first approx-imation, introduces a correction to the frequency of the principle oscilation, Δ_1, in which the amplitude squared appears as a factor. Similarly λ_1 includes a^2 as a factor and the additional harmonic terms given in u_{11} and u_{21} have amplitudes in which a^3 appears as a factor. Consequently for small motion about the equilibrium position, these terms are very small.

Equating the terms of Eq's. (4.37) that are proportional to σ^{2*}, introducing the results of the first approximation, and applying harmonic balance we obtain the following four equations

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*Terms arising from the derivatives of u_{11} and u_{21} have been included.

$$\Delta_2(\beta_1\lambda_0 - 2\overline{\omega}) + \lambda_2(\beta_1\overline{\omega}) = \Delta_1^2 - \beta_1\lambda_1\Delta_1 + a^2\left[\Delta_1 d_{21}\right.$$

$$\left. + \lambda_1 d_{23} + \frac{3}{2}\frac{A_3}{a}\overline{\omega} d_{26} + \frac{1}{2}\frac{A_3}{a} d_{27} + \frac{1}{2}\frac{A_1}{a} d_{29}\right]$$

$$+ a^4\left[\ldots\right]$$

$$\Delta_2(-\beta_3 - 2\lambda\overline{\omega}) + \lambda_2(\beta_4 - \overline{\omega}^2) = \lambda\Delta_1^2 + 2\overline{\omega}\lambda_1\Delta_1 + a^2\left[\Delta_1 e_{21}\right.$$

$$\left. + \lambda_1 e_{23} + \frac{9}{4}\frac{\overline{\omega}^2}{a}A_3 + \frac{3}{2}\frac{\overline{\omega}^2}{a}A_3 - \frac{3}{2}\frac{\overline{\omega}}{a}A_1 e_{26}\right.$$

$$\left. + \frac{1}{2a}A_3 e_{28} - \frac{1}{2a}A_1 e_{29}\right] + a^4\left[\ldots\right]$$

$$u''_{12} + \beta_1 u'_{22} + \beta_2 u_{12} = a^3\left\{c3\overline{\phi}\left[\Delta_1(d_{22} + 18\overline{\omega} A_1 - 3\beta_1 A_3)\right.\right.$$

$$\left. + \lambda_1 d_{24} + \frac{3\overline{\omega}}{2}A_3 d_{25} + \frac{5}{2}\frac{\overline{\omega}}{a}A_4 d_{26} + \frac{1}{2a}A_4 d_{27}\right.$$

$$\left. + \frac{1}{a}A_1 d_{28} + \frac{1}{2a}A_2 d_{29}\right] + c5\overline{\phi}\left[\Delta_1(50 A_2\overline{\omega} - 5\beta_1 A_4)\right.$$

$$\left. + 5\frac{\overline{\omega}}{a}A_4 d_{25} + \frac{3}{2}\frac{\overline{\omega}}{a}A_3 d_{26} - \frac{1}{2a}A_3 d_{27} + \frac{1}{a}A_2 d_{28}\right.$$

$$\left. + \frac{1}{2a}A_1 d_{29}\right] + c7\overline{\phi}\left[\frac{5}{2}\frac{\overline{\omega}}{a}A_4 d_{26} - \frac{1}{2a}A_4 d_{27}\right.$$

$$\left.\left. + \frac{1}{2a}A_2 d_{29}\right]\right\} + a^4\left\{\ldots\right\}$$

$$u''_{22} + \beta_3 u'_{12} + \beta_4 u_{22} = a^2 \left\{ s3\tilde{\phi} \left[\Delta_1 (e_{22} + 18A_3\bar{\omega} \right. \right.$$

$$+ 3\beta_3 A_1) + \lambda_1 e_{24} - \frac{9}{2} \frac{\bar{\omega}^2}{a} A_3 + \frac{25}{4} \frac{\bar{\omega}^2}{a} A_4$$

$$+ \frac{5}{2} \frac{\bar{\omega}^2}{a} A_4 - 3 \frac{\bar{\omega}}{a} A_1 e_{25} - \frac{5}{2} \frac{\bar{\omega}}{a} A_2 e_{26} + \frac{1}{a} A_3 e_{27}$$

$$+ \frac{1}{2a} A_4 e_{28} - \frac{1}{2a} A_2 e_{29} \left] + s5\tilde{\phi} \left[\Delta_1 (50A_4\bar{\omega} + 5\beta_3 A_2) \right. \right.$$

$$- \frac{25}{2a} A_4 + \frac{9}{4} \frac{\bar{\omega}^2}{a} A_3 - \frac{3}{2} \frac{\bar{\omega}^2}{a} A_3 - 5 \frac{\bar{\omega}}{a} A_2 e_{25}$$

$$- \frac{3}{2} \frac{\bar{\omega}}{a} A_1 e_{26} + \frac{1}{a} A_4 e_{27} + \frac{1}{2a} A_3 e_{28} + \frac{1}{2a} A_1 e_{29} \right]$$

$$+ s7\tilde{\phi} \left[\frac{25}{4} \frac{\bar{\omega}^2}{a} A_4 - \frac{5}{2} \frac{\bar{\omega}^2}{a} A_4 - \frac{5}{2} \frac{\bar{\omega}}{a} A_2 e_{26} + \frac{1}{2a} A_4 e_{28} \right.$$

$$+ \frac{1}{2a} A_2 e_{29} \left] \right\} + a^4 \left\{ \ldots \right\} \tag{4.43}$$

It is possible to obtain Δ_2 and λ_2 from the first two of Eq's. (4.43) and to obtain u_{12} and u_{22} from the second two of Eq's. (4.43). By inspection it can be seen that Δ_2 and λ_2 include a factor of a^4 and u_{12} and u_{22} include a factor of a^5. Consequently when the amplitude a is small, the correction to the frequency, Δ_2, and the amplitude ratio, λ_2, as given by the second approximation are small compared with the corrections given by the first approximation. Similarly the amplitudes of the additional periodic terms, u_{12} and u_{22}, are small when a is

small. It is evident from the recursion process that has been used that the nth approximation will yield values of Δ_n and λ_n that include the factor a^{2n}; also u_{1n} and u_{2n} have the factor a^{2n+1}. The solution obtained in this fashion is an analytic function of the parameter σ and the equation can be satisfied to any desired accuracy for small amplitudes of motion, even when σ is relatively large, because the coefficient of increasing powers of σ can be made to converge at any desired rate by selection of a sufficiently small amplitude, a . When σ is set equal to 1 we obtain

$$\theta_1 = a \cos \Phi + \sum_{n=1}^{\infty} u_{1n} = a \cos \Phi + \sum_{m=1}^{\infty} B_{2m+1} \cos (2m + 1)\Phi$$

$$(4.44)$$

$$\theta_2 = a\lambda^{(i)} \sin \Phi + \sum_{n=1}^{\infty} u_{2n} = a\lambda^{(i)} \cos \Phi + \sum_{m=1}^{\infty} C_{2m+1} \sin (2m + 1)\Phi$$

in which

$$\Phi = (\overline{\omega}_i + \Delta_{i1} + \Delta_{i2} + \ldots)\tau + \delta$$

$$(i=1,2) \quad (4.45)$$

$$\lambda^{(i)} = \lambda_i + \lambda_{i1} + \lambda_{i2} + \ldots$$

In the above, δ is an arbitrary phase angle and the index i, denoting the natural frequency near which the principal oscillation takes place, has been reintroduced. We note that the frequencies present include only the odd multiples of Φ . Also it can be seen that no

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additional frequencies will be introduced if higher order terms (i.e. the 7th, 9th, etc. power) in the displacements and their derivatives had been included in the equations of motion, Eq's. (4.35).

4.7.3 Complete Nonlinear Periodic System

In view of the procedure shown in the preceding section it seems reasonable to consider the complete equations of motion, Eq's. (4.13), to be functions of the two parameters, σ and ϵ . Including only terms through the third power in the displacements and their derivatives, the equations of motion become

$$\theta_1'' + \beta_1 \theta_2' + \beta_2 \theta_1 = \sigma \left\{ \theta_2' \left[\theta_2^2 + \theta_1^2 (2 - \frac{1}{2} \ell r) - \theta_1 \theta_2' \right] \right.$$

$$+ \theta_1 \left[\theta_2^2 (\frac{1}{2} \ell r - 4 + 3r + \frac{9}{2} \epsilon^2 r - \frac{5}{2} \epsilon^2) + \theta_1^2 (\frac{1}{6} \ell r \right.$$

$$\left. - \frac{2}{3} - \frac{4}{3} \epsilon^2 \right] \right\} + \epsilon \left\{ c\tau \left[\theta_2' (- 4 + 2\theta_2^2 + 4\theta_1^2) \right. \right.$$

$$+ \theta_1 (- 4 - 2 \ell r + \theta_2^2 \left\langle \ell r - 13 + 9r \right\rangle$$

$$\left. \left. + \theta_1^2 \left\langle \frac{1}{3} \ell r - \frac{8}{3} \right\rangle) \right] + s\tau \left[\theta_2 (2 - \frac{1}{3} \theta_2^2) \right] \right\}$$

$$+ \epsilon^2 \left\{ c2\tau \left[\theta_2' (- 5 + \frac{5}{2} \theta_2^2 + 5\theta_1^2) + \theta_1 (7 - \frac{5}{2} \ell r \right. \right.$$

$$\left. \left. + \theta_2^2 \left\langle \frac{5}{4} \ell r - \frac{41}{2} + \frac{27}{2} r \right\rangle + \theta_1^2 \left\langle \frac{5}{12} \ell r - \frac{14}{3} \right\rangle) \right] \right.$$

$$+ \ s2\tau \left[5\theta_2 (1 \ - \ \tfrac{1}{6} \ \theta_2^2) \right] \Big] \Big\} \ + \ \epsilon^3 \left\{ \ldots \right\} \ + \ \ldots$$

$$\theta_2'' \ + \ \beta_3 \theta_1' \ + \ \beta_4 \theta_2 \ = \ \sigma \left\{ \ \theta_1^2 \theta_2'' \ + \ \theta_1' \left[2\theta_1 \theta_2' \ - \ \theta_2^2 \right. \right.$$

$$+ \ (\tfrac{\ell \ r}{2} \ - \ 2)\theta_1^2 \Big] \ + \ \theta_2 \left[\theta_2^2 (\tfrac{1}{6} \ \ell \ r \ - \ \tfrac{8}{3} \ + \ 2r \right.$$

$$- \ \tfrac{13}{3} \ \epsilon^2 \ + \ 3r\epsilon^2) \ + \ \theta_1^2 (\tfrac{1}{2} \ \ell \ r \ - \ 4 \ + \ 3r \ - \ \tfrac{13}{2} \ \epsilon^2$$

$$+ \ \tfrac{9}{2} \ \epsilon^2 r) \Big] \Big\} \ + \ \epsilon \left\{ \ s\tau \left[- \ 2\theta_1 (1 \ - \ \tfrac{2}{3} \ \theta_1^2 \ - \ \tfrac{1}{2} \ \theta_2^2) \right] \right.$$

$$+ \ c\tau \left[\ 4\theta_1' (1 \ - \ \tfrac{\theta_2^2}{2} \ - \ \theta_1^2) \ + \ \theta_2 (13 \ - \ 9r \ - \ 2 \ \ell \ r \right.$$

$$+ \ \theta_2^2 \tfrac{1}{3} \left\langle \ell \ r \ - \ 26 \ + \ 18r \right\rangle \ + \ \theta_1^2 \left\langle \ell \ r \ - \ 13 \right.$$

$$+ \ 9r \rangle) \Big] \Big\} \ + \ \epsilon^2 \left\{ \ s2\tau \left[- \ 5\theta_1 (1 \ - \ \tfrac{2}{3} \ \theta_1^2 \ - \ \tfrac{1}{2} \ \theta_2^2) \right] \right.$$

$$+ \ c2\tau \left[5\theta_1' (1 \ - \ \tfrac{1}{2} \ \theta_2^2 \ - \ \theta_1^2) \ + \ \theta_2 (\tfrac{1}{2} \left\langle 41 \ - \ 27r \right. \right.$$

$$- \ 5 \ \ell \ r \right\rangle \ + \ \theta_2^2 \tfrac{1}{3} \left\langle \tfrac{5}{4} \ \ell \ r \ - \ 41 \ + \ 27r \right\rangle \ + \ \theta_1^2 \tfrac{1}{2} \left\langle \tfrac{5}{2} \ \ell \ r \right.$$

$$- \ 41 \ + \ 27r \rangle \ ) \Big] \Big\} \ + \ \epsilon^3 \left\{ \ldots \right\} \ + \ \ldots \qquad\qquad (4.46)$$

We will again assume a solution including principle oscillations with frequencies near $\overline{w}_1$ and $\overline{w}_2$, but including frequency and amplitude

ratio terms that are functions of both ϵ and σ. In addition, the terms u_{ij} and v_{ij} are added and will include all terms of other frequencies.

$$\theta_1 = a\ c(\psi_1 \tau + \delta_1) + b\ c(\psi_2 \tau + \delta_2) + u_{10}\sigma + u_{20}\sigma^2$$

$$+ u_{01}\epsilon + u_{02}\epsilon^2 + u_{11}\epsilon\sigma + \ldots$$

$$\theta_2 = a(\lambda_1 + \lambda_{10}\sigma + \lambda_{20}\sigma^2 + \lambda_{01}\epsilon + \lambda_{02}\epsilon^2 + \lambda_{11}\epsilon\sigma$$

$$+ \ldots)\ s(\psi_1 \tau + \delta_1) + b(\lambda_2 + \gamma_{10}\sigma + \gamma_{20}\sigma^2$$

$$+ \gamma_{01}\epsilon + \gamma_{02}\epsilon + \gamma_{11}\epsilon\sigma + \ldots)\ s(\psi_2 \tau + \delta_2) + v_{10}\sigma$$

$$+ v_{20}\sigma^2 + v_{01}\epsilon + v_{02}\epsilon^2 + v_{11}\epsilon\sigma + \ldots \tag{4.47}$$

$$\psi_1 = \bar{\omega}_1 + \Delta_{10}\sigma + \Delta_{20}\sigma^2 + \Delta_{01}\epsilon + \Delta_{02}\epsilon^2 + \Delta_{11}\epsilon\sigma + \ldots$$

$$\psi_2 = \bar{\omega}_2 + \Gamma_{10}\sigma + \Gamma_{20}\sigma^2 + \Gamma_{01}\epsilon + \Gamma_{02}\epsilon^2 + \Gamma_{11}\epsilon\sigma + \ldots$$

The above periodic form of solution has been found to satisfy the equations of motion to any desired accuracy under the conditions of (a) non-resonance with arbitrary phase angles, δ_1 and δ_2, or (b) resonance with specific values of one of the phase angles. This latter case corresponds to constant amplitude resonant motion that is admitted on

the boundaries of the resonance instability regions as discussed in Section II. Analysis of the resonant oscillations would require the assumption of a time-dependent amplitude. However, in that case the frequencies of oscillation of the periodic terms would be the same as obtained in the constant amplitude expansion.

Substitution of the assumed solution, Eq's. (4.47), in the complete equations of motion can be shown to result in oscillatory motion including terms with the following frequencies

$$\pm \ell \pm m\psi_1 \pm n\psi_2 \, , \quad (\ell,m,n = 0,\pm 1,\pm 2,\ldots; \; m+n = \text{odd integer}) \qquad (4.48)$$

The implications of possible resonance with this multiple-infinite number of frequencies was discussed in Section 4.6.

The amplitudes of the terms on the right in the equations of motion that deal with the third and higher powers of θ_1, θ_2, and their derivatives can be made arbitrarily small by restricting our attention to sufficiently small amplitudes of oscillation. Thus many of the resonant frequencies predicted in expression (4.48) may be of small consequence. Eliminating all terms in the oscillation with coefficients raised to the fifth or higher power of the amplitude, a, it can be shown that the frequencies of oscillation are

$$n \pm \psi_1, \; n \pm \psi_2, \; n \pm 3\psi_1, \; n \pm 3\psi_2, \; n \pm 2\psi_1 \pm \psi_2, \; n \pm 2\psi_2 \pm \psi_1, \; (n=0,1,2,\ldots)$$

$$(4.49)$$

When ϵ is small, the corresponding resonance regions must occur where the natural frequencies take on the following approximate values

$$\bar{\omega}_1 \cong \frac{n}{4} \quad , \quad \bar{\omega}_2 \cong \frac{n}{4} \quad , \quad 3\bar{\omega}_1 \pm \bar{\omega}_2 = n \quad , \quad 3\bar{\omega}_2 \pm \bar{\omega}_1 = n \ ,$$

$$\bar{\omega}_1 \pm \bar{\omega}_2 \cong \frac{n}{2} \quad , \quad (n = 1,2,\dots) \tag{4.50}$$

If we further restrict our attention to the case in which all nonlinear terms are eliminated we see that the frequencies present in the periodic solution include only

$$\psi_1 \ , \ \psi_2 \ , \ n \pm \psi_1 \ , \ n \pm \psi_2 \ , \ (n = 1,2,\dots) \tag{4.51}$$

It was previously shown that, when ϵ is small, this results in resonance regions where the approximate value of one of the natural frequencies is

$$\bar{\omega}_1 \cong \frac{n}{2} \quad , \quad \bar{\omega}_2 \cong \frac{n}{2} \quad \text{or} \quad \bar{\omega}_1 \pm \bar{\omega}_2 \cong n \quad , \quad (n = 1,2,\dots) \tag{4.52}$$

It would be desirable to limit our attention to the linearized equations of motion due to the simpler mathematical forms that are involved. To determine the conditions under which this can be done, the nonlinear equations must be considered and the effect of nonlinearity must be understood both as to its effect on the linear resonance regions (resonances that appear in the linearized case) and the characteristics of the resonance regions that appear only in the nonlinear

case. For small oscillations about the equilibrium position, the nonlinear terms with the greatest magnitude will be of the third power of the displacements θ_1 and θ_2 and their derivatives. These are the nonlinear terms that were retained in Eq's. (4.46). The expansion of the complete assumed solution, Eq's. (4.47), in the equations of motion, Eq's. (4.46), is extremely lengthy and will not be shown here. Instead, the analysis is given here for a single resonance oscillation*, so that the assumed solution is

$$\theta_1 = a\, c(\psi_i \tau + \delta_i) + u_{10}\sigma + u_{20}\sigma^2 + u_{01}\epsilon + u_{02}\epsilon^2 + u_{11}\epsilon\sigma + \ldots$$

$$\theta_2 = a(\lambda_i + \lambda_{10}\sigma + \lambda_{20}\sigma^2 + \lambda_{01}\epsilon + \lambda_{02}\epsilon^2 + \lambda_{11})s(\psi_i \tau + \delta_i) + v_{10}\sigma + v_{20}\sigma^2$$

$$+ v_{01}\epsilon + v_{02}\epsilon^2 + v_{11}\epsilon\sigma + \ldots \tag{4.53}$$

$$\psi_i = \bar{\omega}_i + \triangle_{10}\sigma + \triangle_{20}\sigma^2 + \triangle_{01}\epsilon + \triangle_{02}\epsilon^2 + \triangle_{11}\epsilon\sigma + \ldots$$

where i may take on the values 1 or 2, corresponding to oscillation with the frequency of the principal oscillation near $\bar{\omega}_1$ or $\bar{\omega}_2$, respectively. Terms in the solution that involve coefficients with a

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* This selection greatly reduces the algebraic computations, but at the expense of making the analysis incapable of describing the effect of the nonlinear terms on resonances involving the sum or difference of natural frequencies. However, the influence of the nonlinear terms on these resonances is expected to be similar to the effects shown in the present analysis.

factor of the amplitude of oscillation, a, to the fifth or higher power
will be neglected, consistent with using as the equations of motion
Eq's. (4.46) which are truncated in a similar fashion. By comparison
with the preceding analysis of the unperturbed, nonlinear equations
we see that terms in σ to powers higher than the first can be
neglected and

$$\Delta_{10} = a^2 \frac{d_{11}(\beta_4 - \bar{\omega}_i^2) - e_{11}\beta_1\bar{\omega}_1}{(\beta_4 - \bar{\omega}_i^2)(\beta_1\lambda_i - 2\bar{\omega}_i) + \beta_1\bar{\omega}_i(\beta_3 + 2\lambda_i\bar{\omega}_i)}$$

$$\lambda_{10} = a^2 \frac{d_{11}(\beta_3 + 2\lambda_i\bar{\omega}_i) + e_{11}(\beta_1\lambda_i - 2\bar{\omega}_i)}{(\beta_4 - \bar{\omega}_i^2)(\beta_1\lambda_i - 2\bar{\omega}_i) + \beta_1\bar{\omega}_i(\beta_3 + 2\lambda_i\bar{\omega}_i)}$$

$$\tag{4.54}$$

$$u_{10} = A_1 c\ 3\psi_i$$

$$v_{10} = A_3 s\ 3\psi_i$$

where A_1 and A_3 are given by Eq's. (4.42).

Substitution of Eq's. (4.53) into the equations of motion,
Eq's. (4.46), and use of Eq's. (4.54) gives

$$\epsilon \left\{ a[\Delta_{01}(\beta_1\lambda_i - 2\bar{\omega}_i) + \lambda_{01}\beta_1\bar{\omega}_i]c(\psi_i\tau + \delta_i) + u''_{01} + \beta_1 v'_{01} + \beta_2 u_{01} \right\}$$

$$+ \epsilon^2 \left\{ a[\Delta_{02}(\beta_1\lambda_i - 2\bar{\omega}_i) + \lambda_{02}\beta_1\bar{\omega}_i]c(\psi_i\tau + \delta_i) + u''_{02} + \beta_1 v'_{02} + \beta_2 u_{02} \right\}$$

$$+ \epsilon^\sigma \left\{ a[\Delta_{11}(\beta_1\lambda_i - 2\bar{\omega}_i) + \lambda_{11}\beta_1\bar{\omega}_i]c(\psi_i\tau + \delta_i) + u''_{11} + \beta_1 v'_{11} + \beta_2 u_{11} \right\}$$

$$= \epsilon \left\{ a[p_{11} + p_{12}a^2]c(\tau + \psi_i\tau + \delta_i) + a[p_{13} + p_{14}a^2]c(\tau - \psi_i\tau - \delta_i) \right.$$

$$\left. + a^3 p_{15} c(\tau + 3\psi_i\tau + 3\delta_i) + a^3 p_{16} c(\tau - 3\psi_i\tau - 3\delta_i) \right\} + \epsilon^2 \{\ldots\} + \epsilon^\sigma \{\ldots\} + \ldots$$

$$(4.55)$$

$$\epsilon \left\{ a[\Delta_{01}(-\beta_3 - 2\bar{\omega}_i\lambda_i) + \lambda_{01}(\beta_4 - \bar{\omega}_i^2)]s(\psi_i\tau + \delta_i) + v''_{01} + \beta_3 u'_{01} + \beta_4 v_{01} \right\}$$

$$+ \epsilon^2 \left\{ a[\Delta_{02}(-\beta_3 - 2\bar{\omega}_i\lambda_i) + \lambda_{02}(\beta_4 - \bar{\omega}_i^2)]s(\psi_i\tau + \delta_i) + v''_{02} + \beta_3 u'_{02} + \beta_4 v_{02} \right\}$$

$$+ \epsilon\sigma \left\{ a[\Delta_{11}(-\beta_3 - 2\bar{\omega}_i\lambda_i) + \lambda_{11}(\beta_4 - \bar{\omega}_i^2)]s(\psi_i\tau + \delta_i) + v''_{11} + \beta_3 u'_{11} + \beta_4 v_{11} \right\}$$

$$= \epsilon \left\{ a[q_{11} + q_{12}a^2]s(\tau + \psi_i\tau + \delta_i) + a[q_{13} + q_{14}a^2]s(\tau - \psi_i\tau - \delta_i) \right.$$

$$\left. + a^3 q_{15} s(\tau + 3\psi_i\tau + 3\delta_i) + a^3 q_{16} s(\tau - 3\psi_i\tau - 3\delta_i) \right\} + \epsilon^2 \{\ldots\}$$

$$+ \epsilon^\sigma \{\ldots\} + \ldots$$

in which

$$p_{11} = 2 - 2\lambda_i\bar{\omega}_i - \ell r - \lambda_i$$

$$p_{12} = \frac{1}{8} \left[2\lambda_i^3\bar{\omega}_i + 12\lambda_i\bar{\omega}_i + \lambda_i^2(\ell r - 13 + 9r) + \ell r - 8 + \lambda_i^3 \right]$$

$$p_{13} = 2 - 2\lambda_i\bar{\omega}_i - \ell r + \lambda_i$$

$$p_{14} = \frac{1}{8} \left[2\lambda_i^3\bar{\omega}_i + 12\lambda_i\bar{\omega}_i + \lambda_i^2(\ell r - 13 + 9r) + \ell r - 8 - \lambda_i^3 \right]$$

$$p_{15} = \frac{1}{24} \left[-6\lambda_i^3\bar{\omega}_i + 12\lambda_i\bar{\omega}_i - 3\lambda_i^2(\ell r - 13 + 9r) + \ell r - 8 - \lambda_i^3 \right]$$

$$p_{16} = \frac{1}{24} \left[-6\lambda_i^3\bar{\omega}_i + 12\lambda_i\bar{\omega}_i - 3\lambda_i^2(\ell r - 13 + 9r) + \ell r - 8 + \lambda_i^3 \right]$$

$$(4.56)$$

$$q_{11} = \frac{1}{2} \lambda_i(13 - 9r - 2\ell r) - 1 - 2\bar{\omega}_i$$

$$q_{12} = \frac{1}{8} \left[4 + \lambda_i^2 + 6\lambda_i^2\bar{\omega}_i + 4\bar{\omega}_i + \lambda_i^3(\ell r - 26 + 18r) + \lambda_i(\ell r - 13 + 9r) \right]$$

$$q_{13} = -\frac{1}{2} \lambda_i(13 - 9r - 2\ell r) - 1 + 2\bar{\omega}_i$$

$$q_{14} = \frac{1}{8} \left[4 + \lambda_i^2 - 6\lambda_i^2\bar{\omega}_i - 4\bar{\omega}_i - \lambda_i^3(\ell r - 26 + 18r) - \lambda_i(\ell r - 13 + 9r) \right]$$

$$q_{15} = \frac{1}{24} \left[4 - 3\lambda_i^2 - 6\lambda_i^2\bar{\omega}_i + 12\bar{\omega}_i - \lambda_i^3(\ell r - 26 + 18r) + 3\lambda_i(\ell r - 13 + 9r) \right]$$

$$q_{16} = \frac{1}{24} \left[4 - 3\lambda_i^2 + 6\lambda_i^2\bar{\omega}_i - 12\bar{\omega}_i + \lambda_i^3(\ell r - 26 + 18r) - 3\lambda_i(\ell r - 13 + 9r) \right]$$

The asymptotic expansion can now proceed by equating the coefficients of ϵ in Eq's. (4.55) to zero. When this is done it is seen that resonance conditions, in which the frequency ψ_i appears on the right, occur when

$$\psi_i = 1 \pm \psi_i \quad , \quad \psi_i = 1 \pm 3\psi_i \qquad (4.57)$$

which correspond to

$$\text{Case A:} \quad \psi_i = 1/2 \quad (\text{or } \bar{\omega}_i \cong 1/2)$$

$$(4.58)$$

$$\text{Case B:} \quad \psi_i = 1/4 \quad (\text{or } \bar{\omega}_i \cong 1/4)$$

Analysis of Case A shows that

$$\Delta_{01}\bigg|_{\text{Case A}} = \pm \frac{(p_{13}+a^2 p_{14}+a^2 p_{16})(\beta_4-\bar{\omega}_i^2) - \beta_1\bar{\omega}_i(q_{13}+a^2 q_{14}-a^2 q_{16})}{(\beta_4-\bar{\omega}_i^2)(\beta_1\lambda_i-2\bar{\omega}_i) + \beta_1\bar{\omega}_i(\beta_3+2\bar{\omega}_i\lambda_i)}$$

$$(4.59)$$

where the plus sign corresponds to one boundary of the resonance region and the minus sign to the other. A resonance instability will occur, to first approximation accuracy, whenever the parameters of the problem are such that

$$0.5 - \left|\Delta_{01}\right|\bigg|_{\text{Case A}} \epsilon \quad < \bar{\omega}_i + \Delta_{10} < 0.5 + \left|\Delta_{01}\right|\bigg|_{\text{Case A}} \epsilon \qquad (4.60)$$

One can see that the nonlinear terms in the differential equations have introduced frequency correction terms that are proportional to the amplitude of oscillation to the second power, a^2, as given by $\triangle_{10}$ and the part of $\triangle_{01}$ that results from the coefficients p_{14}, p_{16}, q_{14}, and q_{16} . Figure 4.3 illustrates the effects of including the nonlinear terms for the case of $r = 1.5$ and $\epsilon = 0.1$ and 0.01 . Note that this is a resonance case that would occur even if nonlinear effects were omitted (see Eq's. (4.52)). The linear analysis shows that unbounded oscillation would occur for values of spin momentum, ℓ , identical with those predicted by the nonlinear analysis for $a \cong 0$. The nonlinear analysis, however, shows that the motion is not unbounded. With a constant spin momentum the amplitude of oscillation increases giving rise to a "stiffening" of the system; as a result an upper limit of the amplitude is reached. The maximum amplitude is a function of the width of the instability region. Inspection of Figure 4.3 shows that the amplitude would grow to a maximum value in excess of 0.5 radians for $\ell = 0.837$ when $\epsilon = 0.1$. On the other hand, the maximum amplitude for the narrower resonance region corresponding to $\epsilon = 0.01$ is approximately 0.15 radians and occurs when $\ell = 0.878$.

Case B of Eq's. (4.58) occurs when the natural frequency of the system falls within the range given by
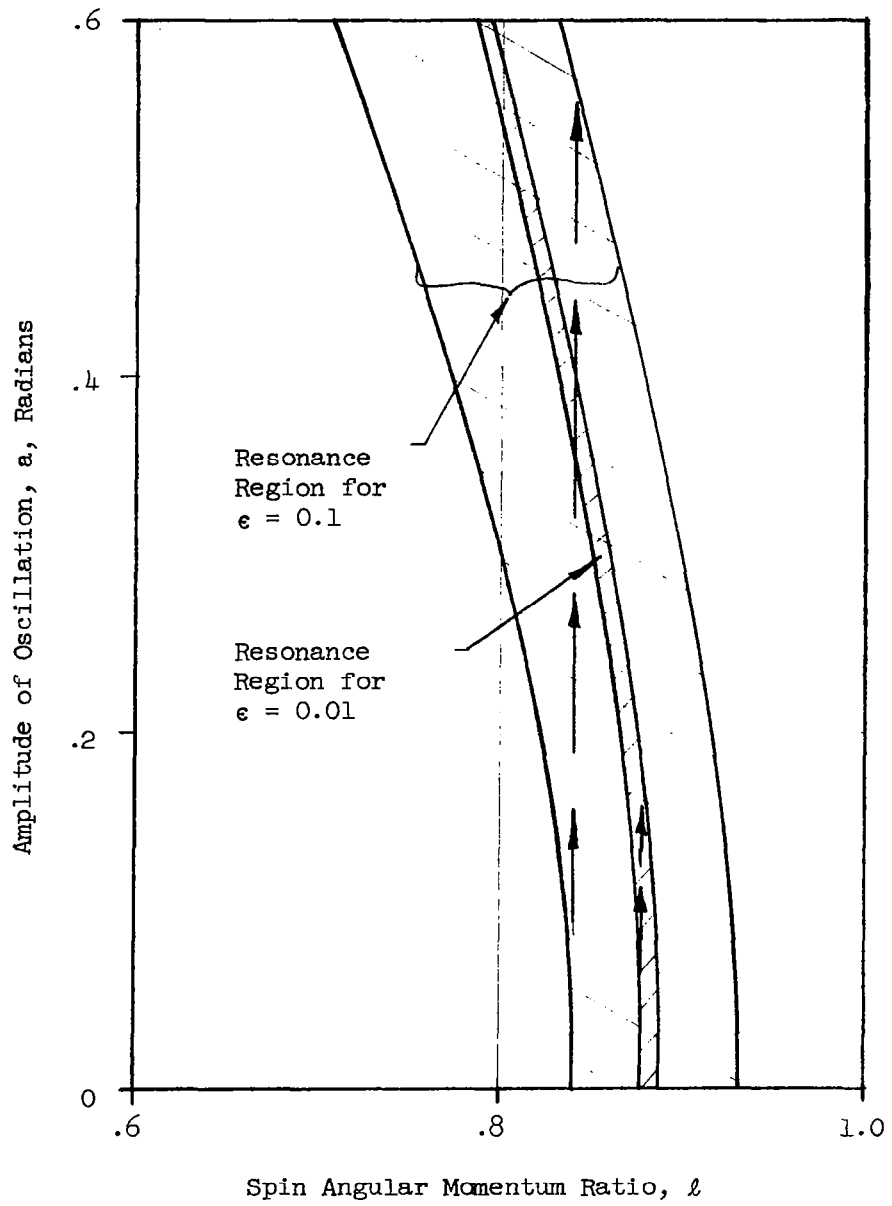
FIGURE 4.3

EFFECT OF NONLINEARITY ON THE

PRINCIPAL RESONANCE REGION FOR r = 1.5

$$0.25 - \left|\Delta_{01}\right|\bigg|_{\text{Case B}} \epsilon \quad < \bar{\omega}_i + \Delta_{10} < 0.25 + \left|\Delta_{01}\right|\bigg|_{\text{Case B}} \epsilon$$

$$(4.61)$$

in which

$$\Delta_{01}\bigg|_{\text{Case B}} = \pm\, a^2\, \frac{p_{16}(\beta_4 - \bar{\omega}_i^2) - q_{16}\beta_1\bar{\omega}_i}{(\beta_4 - \bar{\omega}_i^2)(\beta_1\lambda_i - 2\bar{\omega}_i) + \beta_1\bar{\omega}_i(\beta_3 + 2\bar{\omega}_i\lambda_i)} \qquad (4.62)$$

It can be seen that the frequency correction terms are proportional to a^2 , so that the width of the instability region is essentially zero for very small amplitudes. This type of resonance region is illustrated in Figure 4.4. Note that this is a resonance region that does not appear in the linearized case. Inspection of Figure 4.4 shows that the instability region has zero width at zero amplitude. Further, if a disturbance occurs so that the instability region is entered, a further limited increase in amplitude will occur (at constant ℓ) until the boundary of the resonance region is encountered.

When no resonance occurs in the ϵ-order terms, Δ_{10} is found to be zero. Resonance regions that occur at other frequencies given in Eq's. (4.49) can be found by considering higher order terms in the asymptotic expansion, Eq's. (4.53). In general these regions will be narrower than the two cases just described, since their width is measured in terms of Δ_{n0} multiplied by ϵ^n where n is equal to or larger than 2 and ϵ is a small parameter. Consequently, in these cases, the amplitude of resonance oscillation will be further restricted
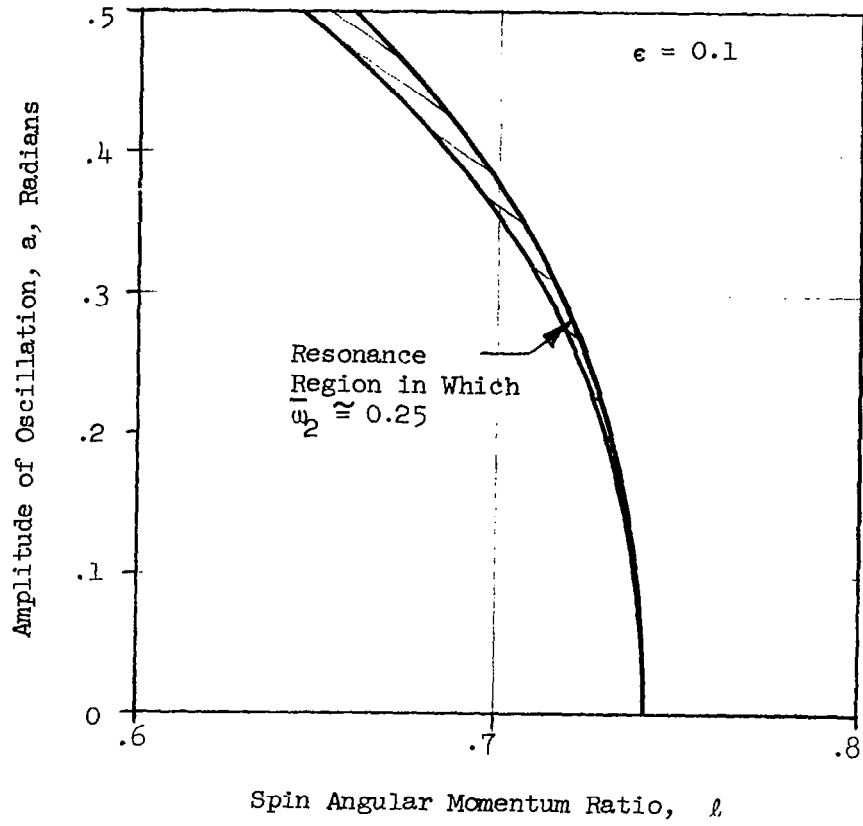
FIGURE 4.4

PRINCIPAL NONLINEAR RESONANCE REGION

FOR r = 1.5

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because of the phenomenon noted above, that is the tendency of the system to shift the frequency of resonant oscillation so as to limit the amplitude reached by the system for a given set of parameters.

We conclude that our principal interest is in the resonances given by the linearized system and, further, the most dangerous resonances will be the broadest resonance regions given by the lowest order approximation in ϵ. Hence, concentrating on the linearized system, we may write the linearized equations of motion

$$\theta_1'' + \beta_1 \theta_2' + \beta_2 \theta_1 = \epsilon \{c\tau[-4\theta_2' + \theta_1(4-2\ell r)] + s\tau(2\theta_2)\}$$

$$+ \epsilon^2\{c2\tau[-5\theta_2' + \theta_1(7 - 5/2\ \ell r)] + s2\tau(5\theta_2)\} + 0(\epsilon^3)$$

$$(4.63)$$

$$\theta_2'' + \beta_3 \theta_1' + \beta_4 \theta_2 = \epsilon\{c\tau[4\theta_1' + \theta_2(13-9r-2\ell r) - s\tau(2\theta_1)\}$$

$$+ \epsilon^2\{c2\tau[5\theta_1' + \theta_2\ 1/2\ (41-27r-5\ell r)] - s2\tau(5\theta_1)\} + 0(\epsilon^3)$$

and the assumed solution will be as given by Eq's. (4.47) but with σ set equal to zero.

$$\theta_1 = a\ c(\psi_1\tau + \delta_1) + b\ c(\psi_2\tau + \delta_2) + u_{01}\epsilon + u_{02}\epsilon^2 + \ldots$$

$$\theta_2 = a(\lambda_1 + \lambda_{01}\epsilon + \lambda_{02}\epsilon^2 + \ldots)\ s(\psi_1\tau + \delta_1) + b(\lambda_2 + \gamma_{01}\epsilon + \gamma_{02}\epsilon^2 + \ldots)$$

$$\cdot\ s(\psi_2\tau + \delta_2) + v_{01}\epsilon + v_{02}\epsilon^2 + \ldots$$

$$(4.64)$$

$$\psi_1 = \bar{\omega}_1 + \Delta_{01}\epsilon + \Delta_{02}\epsilon^2 + \cdots$$

$$\psi_2 = \bar{\omega}_2 + \Gamma_{01}\epsilon + \Gamma_{02}\epsilon^2 + \cdots$$

Substituting the assumed solution into the equations of motion and equating the coefficients of the first power of ϵ one obtains

$$a[\Delta_{01}(-2\bar{\omega}_1 + \beta_1\lambda_1) + \lambda_{01}\beta_1\bar{\omega}_1]\ c(\psi_1\tau + \delta_1) + b[\ \Gamma_{01}(-2\bar{\omega}_2 + \beta_1\Lambda_2)$$

$$+ \gamma_{01}\beta_1\bar{\omega}_2]\ c(\psi_2\tau + \delta_2) + u''_{01} + \beta_1 v'_{01} + \beta_2 u_{01}$$

$$= a[f_{11}\ c(\tau + \psi_1\tau + \delta_1) + f_{12}\ c(\tau - \psi_1\tau - \delta_1)]$$

$$+ b[f_{21}\ c(\tau + \psi_2\tau + \delta_2) + f_{22}\ c(\tau - \psi_2\tau - \delta_2)]$$

$$(4.65)$$

$$a[\Delta_{01}(-\beta_3 - 2\lambda_1\bar{\omega}_1) + \lambda_{01}(\beta_4 - \bar{\omega}_1^2)]\ s(\psi_1\tau + \delta_1) + b[\ \Gamma_{01}(-\beta_3 - 2\lambda_2\bar{\omega}_2)$$

$$+ \gamma_{01}(\beta_4 - \bar{\omega}_2^2)]\ s(\psi_2\tau + \delta_2) + v''_{01} + \beta_3 u'_{01} + \beta_4 v_{01}$$

$$= a[g_{11}\ s(\tau + \psi_1\tau + \delta_1) + g_{12}\ s(\tau - \psi_1\tau - \delta_1)]$$

$$+ b[g_{21}\ s(\tau + \psi_2\tau + \delta_2) + g_{22}\ s(\tau - \psi_2\tau - \delta_2)]$$

in which

$$f_{ij} = \frac{1}{2}\ [-4\bar{\omega}_i\lambda_i + 4 - 2\ell r + (-1)^j 2\lambda_i]$$

$$g_{ij} = \frac{1}{2}\ [(-1)^j 4\bar{\omega}_i + (-1)^j \lambda_i(2\ell r + 9r - 13) - 2]$$

$$(4.66)$$

Resonance may be seen to occur in Eq's. (4.65) when one of the frequencies on the right is equal to $\dot{\psi}_1$ or $\dot{\psi}_2$. Specifically this occurs when

$$\dot{\psi}_1 = 1 \pm \dot{\psi}_1 \; , \quad \dot{\psi}_2 = 1 \pm \dot{\psi}_1 \; , \quad \dot{\psi}_1 = 1 \pm \dot{\psi}_2 \; , \quad \dot{\psi}_2 = 1 \pm \dot{\psi}_2 \qquad (4.67)$$

Eliminating cases without physical meaning and repeated cases, the principal resonances are seen to occur for the following principal frequencies of oscillation

$$\text{Case 1:} \quad \dot{\psi}_1 = \bar{\omega}_1 + \Delta_{01}\epsilon + \ldots = 1/2$$

$$\text{Case 2:} \quad \dot{\psi}_2 = \bar{\omega}_2 + \Gamma_{01}\epsilon + \ldots = 1/2$$

$$(4.68)$$

$$\text{Case 3:} \quad \dot{\psi}_1 + \dot{\psi}_2 = \bar{\omega}_1 + \bar{\omega}_2 + \Delta_{01}\epsilon + \Gamma_{01}\epsilon + \ldots = 1$$

$$\text{Case 4:} \quad \dot{\psi}_1 - \dot{\psi}_2 = \bar{\omega}_1 - \bar{\omega}_2 + \Delta_{01}\epsilon - \Gamma_{01}\epsilon + \ldots = 1$$

The first case given by Eq's. (4.68) can be investigated by setting $\dot{\psi}_1 = 1/2$ in Eq's. (4.65). Our interest is principally in determining Δ_{01} which is done by equating the terms of equal frequency in Eq's. (4.65). This yields a solution when $\delta_1 = n\pi/2$, (n=0,1,2,...), corresponding to the condition of constant amplitude resonant oscillation on the boundaries of the resonance region. For this case, Δ_{01} has been evaluated as given below, where the positive sign corresponds to $\delta_1 = 0, \pi, 2\pi, \ldots$ and the negative sign corresponds to $\delta_1 = \frac{\pi}{2}, 3\frac{\pi}{2}, \ldots$

$$\Delta_{01}\Big|_{\text{Case 1}} = \pm \frac{f_{12}(\beta_4 - \bar{\omega}_1^2) - g_{12}\beta_1\bar{\omega}_1}{(\beta_4 - \bar{\omega}_1^2)(-2\bar{\omega}_1 + \beta_1\lambda_1) + \beta_1\bar{\omega}_1(\beta_3 + 2\lambda_1\bar{\omega}_1)} \tag{4.69}$$

Thus an unstable resonance condition will be encountered (within first approximation accuracy) when the parameters of the problem are such that the natural frequency falls within the range

$$\frac{1}{2} - |\Delta_{01}| \, \epsilon < \bar{\omega}_1 < \frac{1}{2} + |\Delta_{01}| \, \epsilon \tag{4.70}$$

Similarly for Case 2 of Eq's. (4.68) the unstable region is given by

$$\frac{1}{2} - |\Gamma_{01}| \, \epsilon < \bar{\omega}_2 < \frac{1}{2} + |\Gamma_{01}| \, \epsilon \tag{4.71}$$

in which

$$\Gamma_{01}\Big|_{\text{Case 2}} = \pm \frac{f_{22}(\beta_4 - \bar{\omega}_2^2) - g_{22}\beta_1\bar{\omega}_2}{(\beta_4 - \bar{\omega}_2^2)(-2\bar{\omega}_2 + \beta_1\lambda_2) + \beta_1\bar{\omega}_2(\beta_3 + 2\lambda_2\bar{\omega}_2)} \tag{4.72}$$

Case 3 allows two distinct resonance oscillations to take place, one with frequency near $\bar{\omega}_1$ and the other near $\bar{\omega}_2$. The frequencies of oscillations on the boundary depend on Δ_{01} and Γ_{01} which are found from Eq's. (4.69) to be

$$\Delta_{01}\Big|_{\text{Case } 3} = \pm \frac{b}{a} \frac{f_{22}(\beta_4 - \bar{\omega}_1^2) - g_{22}\beta_1\bar{\omega}_1}{(\beta_4 - \bar{\omega}_1^2)(-2\bar{\omega}_1 + \beta_1\lambda_1) + \beta_1\bar{\omega}_1(\beta_3 + 2\lambda_1\bar{\omega}_1)}$$

$$= \pm \frac{b}{a} \Delta_{01}^*\Big|_{\text{Case } 3}$$

$$(4.73)$$

$$\Gamma_{01}\Big|_{\text{Case } 3} = \pm \frac{a}{b} \frac{f_{12}(\beta_4 - \bar{\omega}_2^2) - g_{12}\beta_1\bar{\omega}_2}{(\beta_4 - \bar{\omega}_2^2)(-2\bar{\omega}_2 + \beta_1\lambda_2) + \beta_1\bar{\omega}_2(\beta_3 + 2\lambda_2\bar{\omega}_2)}$$

$$= \pm \frac{a}{b} \Gamma_{01}^*\Big|_{\text{Case } 3}$$

In each of the above equations the positive sign corresponds to oscillation on one boundary with $\delta_1 + \delta_2 = 0, 2\pi, \ldots$ whereas the negative sign corresponds to oscillation on the other boundary with $\delta_1 + \delta_2 = \pi, 3\pi, \ldots$

An unstable resonance condition will be encountered (within first approximation accuracy) when the parameters of the problem are such that the sum of the natural frequencies falls within the range

$$1 - \left|\Delta_{01} + \Gamma_{01}\right| \epsilon < \bar{\omega}_1 + \bar{\omega}_2 < 1 + \left|\Delta_{01} + \Gamma_{01}\right| \epsilon \qquad (4.74)$$

where the values of Δ_{01} and Γ_{01} used on each side of the inequality must be evaluated on the same boundary of the instability region.

Case 4 is similar to Case 3 in that two separate oscillations again take place. The unstable region is found to be given by

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$$1 - \left|\Delta_{01} - \Gamma_{01}\right| \epsilon < \bar{\omega}_1 - \bar{\omega}_2 < 1 + \left|\Delta_{01} - \Gamma_{01}\right| \epsilon \qquad (4.75)$$

where Δ_{01} and Γ_{01} on each side of the inequality must again be evaluated on the same instability boundary; their values are given by

$$\Delta_{01}\Big|_{\text{Case 4}} = \pm \frac{b}{a} \frac{f_{21}(\beta_4 - \bar{\omega}_1^2) - g_{21}\beta_1\bar{\omega}_1}{(\beta_4 - \bar{\omega}_1^2)(-2\bar{\omega}_1 + \beta_1\lambda_1) + \beta_1\bar{\omega}_1(\beta_3 + 2\lambda_1\bar{\omega}_1)}$$

$$= \pm \frac{b}{a} \Delta^*_{01}\Big|_{\text{Case 4}}$$

$$\qquad (4.76)$$

$$\Gamma_{01}\Big|_{\text{Case 4}} = \pm \frac{a}{b} \frac{f_{11}(\beta_4 - \bar{\omega}_2^2) - g_{11}\beta_1\bar{\omega}_2}{(\beta_4 - \bar{\omega}_2^2)(-2\bar{\omega}_2 + \beta_1\lambda_2) + \beta_1\bar{\omega}_2(\beta_3 + 2\lambda_2\bar{\omega}_2)}$$

$$= \pm \Gamma^*_{01}\Big|_{\text{Case 4}}$$

where the positive signs again correspond to the boundary for which $\delta_1 - \delta_2 = 0, 2\pi, \ldots$ and the negative signs to $\delta_1 - \delta_2 = \pi, 3\pi, \ldots$.

Cases 1 and 2 above furnish means of calculating the size of the resonance instability region for any orbit eccentricity. Cases 3 and 4, on the other hand, do not establish a unique resonance region, since the parameters Δ_{01} and Γ_{01} are functions of the relative amplitudes of the two principle oscillations. Specifically, in Cases 3 and 4, Δ_{01} is proportional to b/a and Γ_{01} is proportional to a/b. It can be shown for Case 3 that, when Δ^*_{01} and Γ^*_{01} have the same sign on a given instability boundary, the minimum width of the instability region is given by

$$b/a = \left[\Gamma^*_{01}\Big|_{\text{Case 3}} \Big/ \Delta^*_{01}\Big|_{\text{Case 3}} \right]^{1/2} \qquad (4.77)$$

The instability region may have zero width when Γ_{01}^{*} and Δ_{01}^{*} are of opposite sign. Similarly, in Case 4, when Δ_{01}^{*} and Γ_{01}^{*} have opposite signs on a given instability boundary the minimum width of the instability region occurs when

$$b/a = -[\Gamma_{01}^{*}\Big|_{\text{Case 3}} / \Delta_{01}^{*}\Big|_{\text{Case 3}}]^{1/2} \qquad (4.78)$$

In this case the region may be of zero width when Δ_{01}^{*} and Γ_{01}^{*} have the same sign.

Of course, the regions described above in the first approximation could be established in greater detail, if desired, by further expansion of the equations of motion and sequential solution for Δ_{02}, Γ_{02}, Δ_{03}, Γ_{03}, etc. To illustrate this procedure, the second approximation has been made for Case 2 ($\psi_2 = \frac{1}{2}$). To this end we must find γ_{01}, u_{01} and v_{01} from the first approximation. The expression for γ_{01} is

$$\gamma_{01} = \pm \frac{f_{22}(\beta_3 + 2\lambda_2\bar{\omega}_2) + g_{22}(\beta_1\lambda_2 - 2\bar{\omega}_2)}{(\beta_4 - \bar{\omega}_2^2)(\beta_1\lambda_2 - 2\bar{\omega}_2) + \beta_1\bar{\omega}_2(\beta_3 + 2\lambda_2\bar{\omega}_2)} \qquad (4.79)$$

where the positive sign corresponds to $\delta_2 = 0, \pi, \ldots$ and the negative sign corresponds to $\delta_2 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. Equating the coefficients of ϵ^2 in the asymptotic expansion of Eq's. (4.63), applying harmonic balance, and solving for Γ_{02} we obtain

$$\Gamma_{02} = \frac{f_3(\beta_4 - \bar{\omega}_2^2) - f_4\beta_1\bar{\omega}_2}{(\beta_4 - \bar{\omega}_2)^2(\beta_1\lambda_2 - 2\bar{\omega}_2) + \beta_1\bar{\omega}_2(\beta_3 + 2\lambda_2\bar{\omega}_2)} \qquad (4.80)$$

in which

$$f_3 = \Gamma_{01}^2 - \beta_1 \Gamma_{01}\gamma_{01} + \beta_{21}(-1 - 2\bar{\omega}_2) + A_{21}(2 - \ell r) \pm [\gamma_{01}\bar{\omega}_2 + 2\lambda_2\Gamma_{01}\gamma_{01}]$$

$$f_4 = 2\Gamma_{01}\gamma_{01}\bar{\omega}_2 - \lambda_2\Gamma_{01}^2 \mp 2\Gamma_{01} + (\frac{13}{2} - \frac{9}{2}r - \ell r)(B_{21} \pm \gamma_{01}) - (1 + \bar{\omega}_2)A_{21}$$

In the above definitions of f_3 and f_4 the positive sign corresponds to $\delta_2 = 0$, π, 2π, ... and the negative sign corresponds to $\delta_2 = \frac{\pi}{2}$, $\frac{\pi}{3}$, ... The coefficients A_{21} and B_{21} are identical with those derived below for the ϵ order terms.

When the parameters of the problem are such that no resonance regions in the ϵ order terms, as given in the first approximation by Eq's. (4.70), (4.71), (4.74), and (4.75), are encountered, then the equations of motion, Eq's. (4.65) yield the solution

$$\Delta_{01} = \Gamma_{01} = \lambda_{01} = \gamma_{01} = 0$$

$$u_{01} = a[A_{11}c(\tau + \psi_1\tau + \delta_1) + A_{12}c(\tau - \psi_1\tau - \delta_1)] +$$

$$b[A_{21}c(\tau + \psi_2\tau + \delta_2) + A_{22}c(\tau - \psi_2\tau - \delta_2)]$$

(4.82)

$$v_{01} = a[B_{11}s(\tau + \psi_1\tau + \delta_1) + B_{12}s(\tau - \psi_1\tau - \delta_1)] +$$

$$b[B_{21}s(\tau + \psi_2\tau + \delta_2) + B_{22}s(\tau - \psi_2\tau - \delta_2)]$$

in which

$$A_{ij} = \frac{f_{ij}\{\beta_4 - [1 - (-1)^j\psi_i]^2\} - g_{ij}\beta_1[1 - (-1)^j\psi_i]}{\{\beta_2 - [1 - (-1)^j\psi_i]^2\}\{\beta_4 - [1 - (-1)^j\psi_i]^2\} + \beta_1\beta_3[1 - (-1)^j\psi_i]^2}$$

(4.83)

$$B_{ij} = \frac{f_{ij}\beta_3[1 - (-1)^j\psi_i] + g_{ij}\{\beta_2 - [1 - (-1)^j\psi_i]^2\}}{\{\beta_2 - [1 - (-1)^j\psi_i]^2\}\{\beta_4 - [1 - (-1)^j\psi_i]^2\} + \beta_1\beta_3[1 - (-1)^j\psi_i]^2}$$

Equating the ϵ^2 terms in the asymptotic expansion of the equations of motion, using Eq's. (4.82), gives

$$a[\Delta_{02}(\beta_1\lambda_1 - 2\bar{\omega}_1) + \lambda_{02}\beta_1\bar{\omega}_1] \, c(\psi_1\tau + \delta_1) + b[\Gamma_{02}(\beta_1\lambda_2 - 2\bar{\omega}_2) +$$

$$\gamma_{02}\beta_1\bar{\omega}_2] \, c(\psi_2\tau + \delta_2) + u''_{02} + \beta_1 v'_{02} + \beta_2 u_{02} = a[h_1 c(\psi_1\tau +$$

$$\delta_1) + h_2 c(2\tau + \psi_1\tau + \delta_1) + h_3 c(2\tau - \psi_1\tau - \delta_1)] + b[h_4 c(\psi_2\tau +$$

$$\delta_2) + h_5 c(2\tau + \psi_2\tau + \delta_2) + h_6 c(2\tau - \psi_2\tau - \delta_2)]$$

$$(4.84)$$

$$a[\Delta_{02}(-\beta_3 - 2\lambda_1\bar{\omega}_1) + \lambda_{02}(\beta_4 - \bar{\omega}_1^2)] \, s(\psi_1\tau + \delta_1) + b[\Gamma_{02}(-\beta_3 -$$

$$2\lambda_2\bar{\omega}_2) + \gamma_{02}(\beta_4 - \bar{\omega}_2^2)] \, s(\psi_2\tau + \delta_2) + v''_{02} + \beta_3 u'_{02} + \beta_4 v_{02}$$

$$= a[k_1 s(\psi_1\tau + \delta_1) + k_2 s(2\tau + \psi_1\tau + \delta_1)] + k_3 s(2\tau - \psi_1\tau -$$

$$\delta_1)] + b[k_4 s(\psi_2\tau + \delta_2) + k_5 s(2\tau + \psi_2\tau + \delta_2) + k_6 s(2\tau -$$

$$\psi_2\tau - \delta_2)]$$

where the values of the constants h and k are

$$h_1 = (-1 - 2\bar{\omega}_1)B_{11} + (-1 + 2\bar{\omega}_1)B_{12} + (2 - \ell r)(A_{11} + A_{12})$$

$$h_2 = \frac{1}{2}(-5\lambda_1\bar{\omega}_1 + 7 - \frac{5}{2}\ell r - 5\lambda_1) + (-3 - 2\bar{\omega}_1)B_{11} + (2-\ell r)A_{11}$$

$$h_3 = \frac{1}{2}(-5\lambda_1\bar{\omega}_1 + 7 - \frac{5}{2}\ell r + 5\lambda_1) + (-3 + 2\bar{\omega}_1)B_{12} + (2-\ell r)A_{12}$$

$$h_4 = (-1 - 2\bar{\omega}_2)B_{21} + (-1 + 2\bar{\omega}_2)B_{22} + (2 - \ell r)(A_{21} + A_{22})$$

$$h_5 = \frac{1}{2}(-5\lambda_2\bar{\omega}_2 + 7 - \frac{5}{2}\ell r - 5\lambda_2) + (-3 - 2\bar{\omega}_2)B_{21} + (2 - \ell r)A_{21}$$

$$h_6 = \frac{1}{2}(-5\lambda_2\bar{\omega}_2 + 7 - \frac{5}{2}\ell r + 5\lambda_2) + (-3 + 2\bar{\omega}_2)B_{22} + (2 - \ell r)A_{22}$$

$$k_1 = (-1 - 2\bar{\omega}_1)A_{11} + (1 - 2\bar{\omega}_1)A_{12} + \frac{1}{2}(13 - 9r - 2\ell r)(B_{11} - B_{12})$$

$$k_2 = \frac{1}{2}[-5\bar{\omega}_1 + \frac{1}{2}\lambda_1(41 - 27r - 5\ell r) - 5] + (-3 - 2\bar{\omega}_1)A_{11} +$$

$$\frac{1}{2}(13 - 9r - 2\ell r)B_{11}$$

(4.85)

$$k_3 = \frac{1}{2}[5\bar{\omega}_1 - \frac{1}{2}\lambda_1(41 - 27r - 5\ell r) - 5] + (-3 + 2\bar{\omega}_1)A_{12} +$$

$$\frac{1}{2}(13 - 9r - 2\ell r)B_{12}$$

$$k_4 = (-1 - 2\bar{\omega}_2)A_{21} + (1 - 2\bar{\omega}_1)A_{22} + \frac{1}{2}(13 - 9r - 2\ell r)(B_{21} - B_{22})$$

$$k_5 = \frac{1}{2}[-5\bar{\omega}_2 + \frac{1}{2}\lambda_2(41 - 27r - 5\ell r) - 5] + (-3 - 2\bar{\omega}_2)A_{21} +$$

$$\frac{1}{2}(13 - 9r - 2\ell r)B_{21}$$

$$k_6 = \frac{1}{2}[5\bar{\omega}_2 - \frac{1}{2}\lambda_2(41 - 27r - 5\ell r) - 5] + (-3 + 2\bar{\omega}_2)A_{22} +$$

$$\frac{1}{2}(13 - 9r - 2\ell r)B_{22}$$

One can observe, based on inspection of Eq's. (4.84), that resonance regions exist for

$$\psi_1 = 2 \pm \psi_1 \, , \qquad \psi_1 = 2 \pm \psi_2 \, ,$$

$$\psi_2 = 2 \pm \psi_1 \, , \qquad \psi_2 = 2 \pm \psi_2$$

(4.86)

Eliminating cases without physical meaning and repeated cases, shows that the following four additional resonance regions appear

Case 5: $\quad \psi_1 = \bar{\omega}_1 + \Delta_{02}\epsilon^2 + \ldots = 1$

Case 6: $\quad \psi_2 = \bar{\omega}_2 + \Gamma_{02}\epsilon^2 + \ldots = 1$

Case 7: $\psi_1 + \psi_2 = \bar{\omega}_1 + \bar{\omega}_2 + \Delta_{02}\epsilon^2 + \Gamma_{02}\epsilon^2 + \dots = 2$ $\qquad$ (4.87)

Case 8: $\psi_1 - \psi_2 = \bar{\omega}_1 - \bar{\omega}_2 + \Delta_{02}\epsilon^2 + \Gamma_{02}\epsilon^2 + \dots = 2$ $\qquad$ (4.88)

Solving for the coefficient Δ_{02} corresponding to Case 5, one obtains

$$\Delta_{02}\Big|_{\text{Case 5}} = \frac{(h_1 \pm h_3)(\beta_4 - \bar{\omega}_1^2) - (k_1 \pm k_3)\beta_1\bar{\omega}_1}{(\beta_1\lambda_1 - 2\bar{\omega}_1)(\beta_4 - \bar{\omega}_1^2) + \beta_1\bar{\omega}_1(\beta_3 + 2\bar{\omega}_1\lambda_1)}$$

$$= m_1 \pm m_2 \qquad (4.88)$$

in which the positive signs yield the boundary of the instability re-
gion for which $\delta_1 = 0, \pi, \dots$ and the negative sign gives the boundary
corresponding to $\delta_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$. Instability can be seen to occur
within the accuracy of the second approximation, when the parameters of
the problem are such that

$$1 - (m_1 + |m_2|)\, \epsilon^2 < \bar{\omega}_1 < 1 - (m_1 - |m_2|)\, \epsilon^2 \qquad (4.89)$$

Similarly for Case 6

$$\Gamma_{02}\Big|_{\text{Case 6}} = \frac{(h_4 \pm h_6)(\beta_4 - \bar{\omega}_2^2) - (k_4 \pm k_6)\, \beta_1\bar{\omega}_2}{(\beta_1\lambda_2 - 2\bar{\omega}_2)(\beta_4 - \bar{\omega}_2^2) + \beta_1\bar{\omega}_2(\beta_3 + 2\bar{\omega}_2\lambda_2)}$$

$$= m_3 \pm m_4 \qquad (4.90)$$

and instability occurs when

$$1 - (m_3 + |m_4|)\, \epsilon^2 < \bar{\omega}_1 < 1 - (m_3 - |m_4|)\, \epsilon^2 \qquad (4.91)$$

Case 7 allows two distinct resonance oscillations to take place,
one with frequency near $\bar{\omega}_1$ and the other near $\bar{\omega}_2$. The frequencies of
oscillation on the boundary depend on Δ_{02} and Γ_{02} which can be found

from Eq's. (4.84) to be

$$\Delta_{02}\Big|_{\text{Case } 7} = \frac{(h_1 \pm \frac{b}{a} h_6)(\beta_4 - \bar{\omega}_1^2) - (k_1 \pm \frac{b}{a} k_6) \beta_1 \bar{\omega}_1}{(\beta_1 \lambda_1 - 2\bar{\omega}_1)(\beta_4 - \bar{\omega}_1^2) + \beta_1 \bar{\omega}_1 (\beta_3 + 2\bar{\omega}_1 \lambda_1)}$$

$$= m_5 \pm \frac{b}{a} m_6$$

(4.92)

$$\Gamma_{02}\Big|_{\text{Case } 7} = \frac{(h_4 \pm \frac{a}{b} h_3)(\beta_4 - \bar{\omega}_2^2) - (k_4 \pm \frac{a}{b} k_3) \beta_1 \bar{\omega}_2}{(\beta_1 \lambda_2 - 2\bar{\omega}_2)(\beta_4 - \bar{\omega}_2^2) + \beta_1 \bar{\omega}_2 (\beta_3 + 2\bar{\omega}_2 \lambda_2)}$$

$$= m_7 \pm \frac{a}{b} m_8$$

In each of the above equations the positive signs correspond to oscillation on one boundary for which $\delta_1 + \delta_2 = 0, 2\pi, \ldots$ while the negative signs correspond to oscillation on the other boundary of the instability region for which $\delta_1 + \delta_2 = \pi, 3\pi, \ldots$. An unstable resonance will be encountered, within second approximation accuracy, when the parameters of the problem are such that the sum of the natural frequencies falls between the values $2 - \Delta_{02}\Big|_{\text{Case } 7} - \Gamma_{02}\Big|_{\text{Case } 7}$ evaluated on one boundary of the instability region and $2 - \Delta_{02}\Big|_{\text{Case } 7} - \Gamma_{02}\Big|_{\text{Case } 7}$ evaluated on the other boundary.

It can again be noted that the width of the instability region is a function of the ratio of amplitudes, b/a. When m_6 and m_8 are of the same sign, it can be shown that the minimum width of the instability region occurs when $b/a = (m_8/m_6)^{1/2}$ so that the motion will always be unstable when

$$2 - [m_5 + m_7 + 2\sqrt{m_6 m_8}] \; \epsilon^2 < \bar{\omega}_1 + \bar{\omega}_2 < 2 - [m_5 + m_7 - 2\sqrt{m_6 m_8}] \; \epsilon^2$$

$$(4.93)$$

Although the above relation gives the minimum width of the instability region when m_6 and m_8 have the same sign, no similar relation is possible when m_6 and m_8 are of opposite sign, since the instability region may be of zero width in that case.

Similarly in Case 8, an unstable resonance will occur when $\bar{\omega}_1 - \bar{\omega}_2$ falls between the value $2 - \Delta_{02}\big|_{\text{Case } 8} - \Gamma_{02}\big|_{\text{Case } 8}$ evaluated on one boundary of the resonance region and $2 - \Delta_{02}\big|_{\text{Case } 8} - \Gamma_{02}\big|_{\text{Case } 8}$ on the other boundary, where Δ_{02} and Γ_{02} are obtained from the following expressions

$$\Delta_{02}\bigg|_{\text{Case } 8} = \frac{(h_1 \pm \frac{b}{a} h_5)(\beta_4 - \bar{\omega}_1^2) - (k_1 \pm \frac{b}{a} k_5)\, \beta_1 \bar{\omega}_1}{(\beta_1 \lambda_1 - 2\bar{\omega}_1)(\beta_4 - \bar{\omega}_1^2) + \beta_1 \bar{\omega}_1 (\beta_3 + 2\bar{\omega}_1 \lambda_1)}$$

$$= m_9 \pm \frac{b}{a} m_{10}$$

$$(4.94)$$

$$\Gamma_{02}\bigg|_{\text{Case } 8} = \frac{(h_4 \pm \frac{a}{b} h_2)(\beta_4 - \bar{\omega}_2^2) - (k_4 \pm \frac{a}{b} k_2)\, \beta_1 \bar{\omega}_2}{(\beta_1 \lambda_2 - 2\bar{\omega}_2)(\beta_4 - \bar{\omega}_2^2) + \beta_1 \bar{\omega}_2 (\beta_3 + 2\bar{\omega}_2 \lambda_2)}$$

$$= m_{10} \pm \frac{a}{b} m_{12}$$

In the above, the positive signs correspond to the boundary on which $\delta_1 - \delta_2 = 0,\ 2\pi,\ldots$ and the negative signs correspond to the boundary on which $\delta_1 - \delta_2 = \pi,\ 3\pi,\ \ldots$. The instability region may be of zero width when m_{10} and m_{12} are of the same sign. The minimum width of the

instability region, when m_{10} and m_{12} are of opposite sign, is given by

$$2 - [m_9 - m_{11} + 2\sqrt{-m_{10}m_{12}}] \epsilon^2 < \bar{\omega}_1 - \bar{\omega}_2 < 2 - [m_9 - m_{11} - 2\sqrt{-m_{10}m_{12}}] \epsilon^2$$

$$(4.95)$$

4.8 Results and Comparison with Other Research

The stability of attitude motion of a rigid, symmetric, spinning satellite in an elliptic orbit has been investigated. The problem has been formulated by regarding the ellipse eccentricity, ϵ, as a parameter. The orbital coordinates comprising the radial distance R and the angular position Θ are treated as known functions of time. A set of nonlinear equations of motion is derived containing periodic coefficients entering into the equations by virtue of R and Θ. A particular equilibrium position in which the spin axis is normal to the orbit plane has been identified and the stability of motion about this equilibrium position studied. The equations of motion have been studied in both the linear and nonlinear form using the methods of Section II. Instability regions have been located in the ℓ vs. r plane where ℓ is proportional to the spin angular momentum p_φ and r is the ratio of the moments of inertia about the spin axis and a transverse axis. The location and width of the instability regions have been determined for various values of ϵ in closed form. The important nonlinear phenomena are presented in Section 4.7. The appearance of new instability regions, not found in the linearized system, and the occurrence of nonlinear "stiffening" of

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the system tending to limit the amplitude of resonance oscillation are noted. Typical resonance regions corresponding to the nonlinear equations are shown in Figures 4.3 and 4.4 where these effects can be seen. The latter effect has been demonstrated both analytically and experimentally by Bolotin (see Reference 22, page 88). Thus consideration of the nonlinear effect shows mathematically bounded motion. However, the amplitude of motion can be rather substantial and the stability of motion must be defined in terms of the system performance requirements — bounded motion may be classified as unstable if the amplitude of oscillation is too large.

The orbit eccentricity ϵ has been treated throughout as a small parameter. A value of ϵ that may be considered as small from a mathematical standpoint is not necessarily small from a physical standpoint as it can be shown to cover the vast majority of earth orbiting dynamics problems. As an example an ellipse of apogee height 1000 miles and perigee height 100 miles has an orbit eccentricity, ϵ, slightly less than 0.1. This ratio of apogee to perigee heights, however, is regarded as high in most space applications.

No previous stability analysis of the complete nonlinear system of equations is known to have been performed. The analysis of the linearized system can be directly compared to the work of Kane and Barba [13] who also studied the stability of motion of a rigid, symmetric, spinning satellite in an elliptic orbit. Their analysis, based upon a method by Cesari [20], utilizes Floquet's theory together with numerical

integration of the linearized equations of motion to check the stability of motion at discrete points in the parameter space. The investigation of Reference [13] included checks of the stability for a number of combinations of parameters with an orbit eccentricity of 0.1, and can be compared with the results of the present more general analysis. Figure 4.5 shows the stability data of Reference [13], converted to the dimensionless groupings of the present research. Comparison with the stability boundaries dictated by the requirement for a positive definite Hamiltonian, Figure 4.1, shows that the analysis of Reference [13] predicts infinitesimally stable solutions for some cases of negative spin momentum (i.e. spin in the direction opposite to the orbit angular velocity vector). This region of infinitesimal stability is not predicted by the Liapounov type of stability analysis, when the Hamiltonian is used as the testing function, as has been reported previously [9], [10]. This stability region is known to be of little engineering significance since damping has the effect of causing divergent oscillations in this region [9].

Figure 4.6 shows the stability data of Reference [13] in the region of parameter space of principal interest in this investigation — specifically in the region where nonresonance oscillations are predicted to be stable for zero orbit eccentricity. Reference [13] shows that five of the points for which stability was predicted for zero eccentricity became unstable for an orbit eccentricity of 0.1. Three of these

Legend:

× Unstable for ε = 0 and ε = 0.1

⊗ Infinitesimally Stable for ε = 0 and Unstable for ε = 0.1
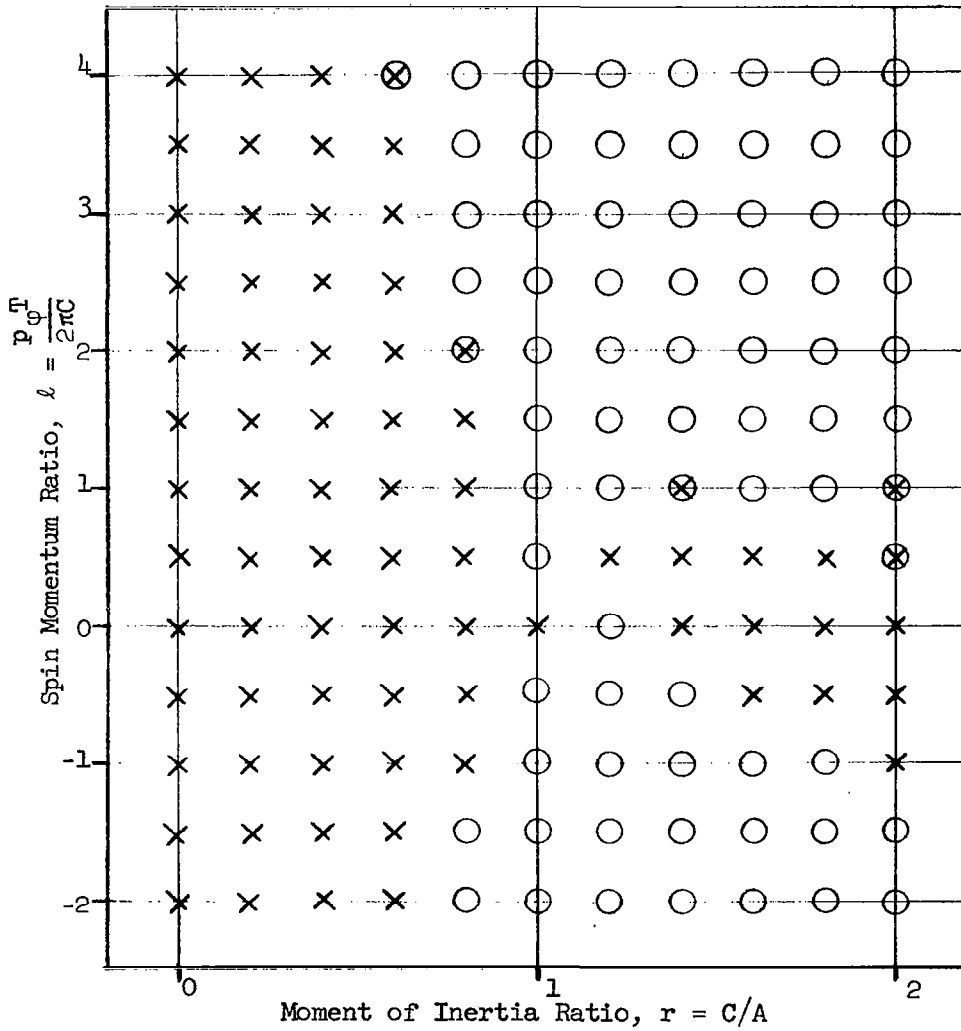
○ Infinitesimally Stable for ε = 0 and ε = 0.1

FIGURE 4.5

STABILITY DATA OF REFERENCE [13]
FOR ORBIT ECCENTRICITY OF 0.1

○ Infinitesimally Stable, per Reference [13], for ϵ = 0 and 0.1.

⊗ Infinitesimally Stable for ϵ = 0 and Unstable for ϵ = 0.1, per Reference [13].

The Hamiltonian is Positive Definite to the Right of the Curve for an Orbit Eccentricity of

0 ⟶

0.1 ⟶
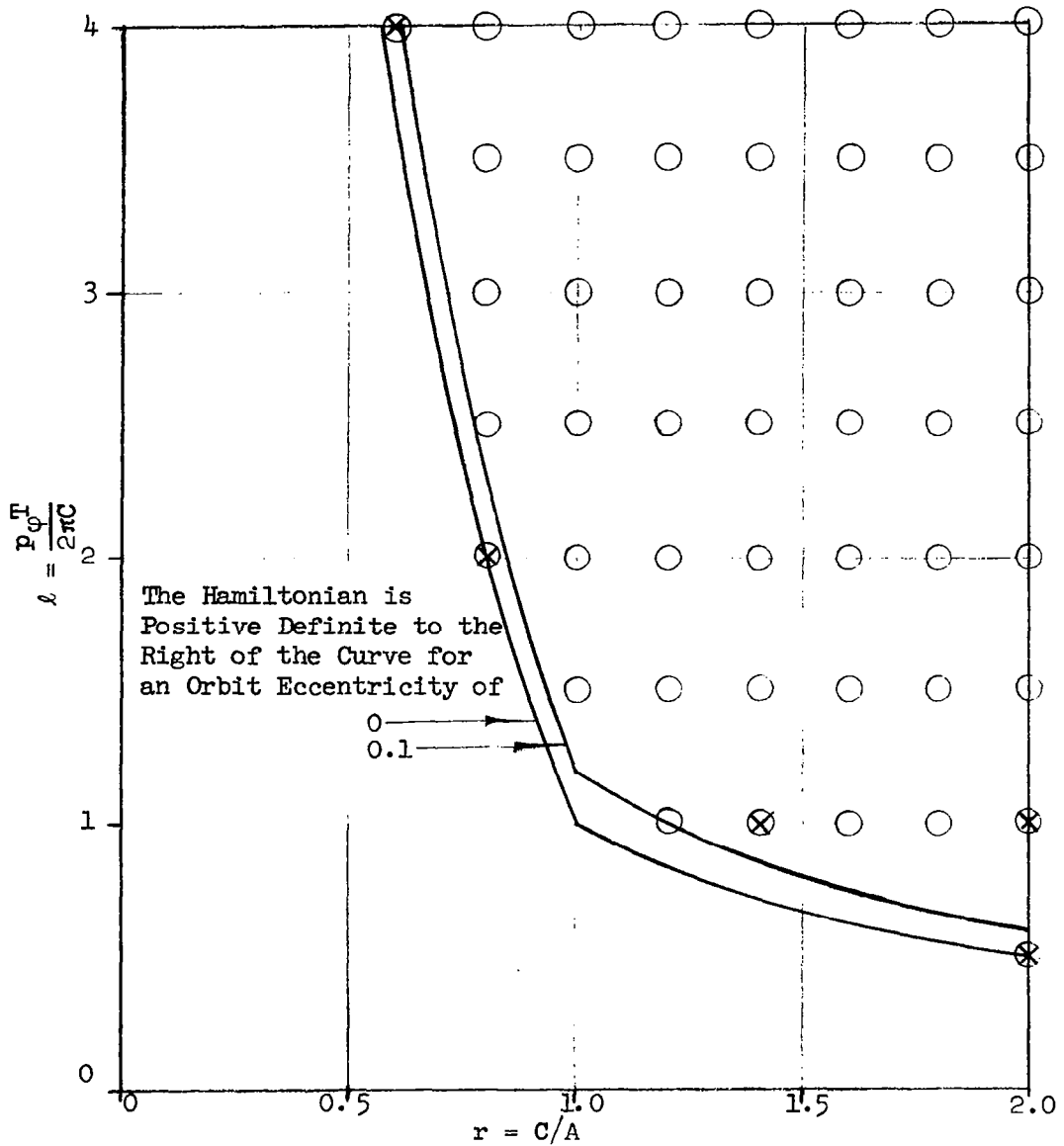
$\ell = \dfrac{p_\varphi T}{2\pi C}$

$r = C/A$

FIGURE 4.6

COMPARISON OF THE RESULTS OF THE PRESENT RESEARCH
WITH THE STABILITY DATA OF REFERENCE [13]
FOR AN ORBIT ECCENTRICITY OF 0.1.

unstable points are predicted in the present research because they do not satisfy the requirement that the Hamiltonian be positive definite for an orbit eccentricity of 0.1. To show complete agreement with Reference [13] the other two unstable points must fall within the resonance regions of the linearized system, whereas the stable points must lie outside of the resonance regions. Review of the approximate locations of the resonance regions for an extremely small orbit eccentricity, as given by Figure 4.2, indicates that a number of the points for which stability was predicted in Reference [13] for $\epsilon = 0.1$ could be in resonance regions. Notable the points at $r = 1$ could show instabilities. However these might not be detected by the numerical integration technique of Reference [13], since the rigid body in that case equivalent to a sphere and would not be subjected to disturbing torques. While an eccentricity of 0.1 cannot be considered vanishingly small, the comparison still appears valid because the general nature of Figure 4.2 is not expected to change substantially; the resonance regions shift locations slightly and increase their width with increasing ϵ. Some of the most significant instability regions have been defined for an orbit eccentricity of 0.1, as shown in Figure 4.7.

Concentrating on the two unstable points shown in Reference [13] at $\ell = 1.0$ and $r = 2.0$ and 1.4, we see that the first of these may be subjected to any one of several resonance conditions, the most important being a resonance oscillation for which $\bar{\omega}_2 \cong 1$ (see Figure 4.2). The existence of instability at the first of these two points was not confirmed in the present analysis. As the present analysis considered terms through the second power in ϵ, it must be concluded that the instability found by Kane and Barba arises from higher order terms.

On the other hand, instability at $\ell = 1.0$ and $r = 1.4$ was confirmed and it can be seen from Figure 4.7 that it belongs to the resonance region $\bar{\omega}_1 + \bar{\omega}_2 \cong 2$.
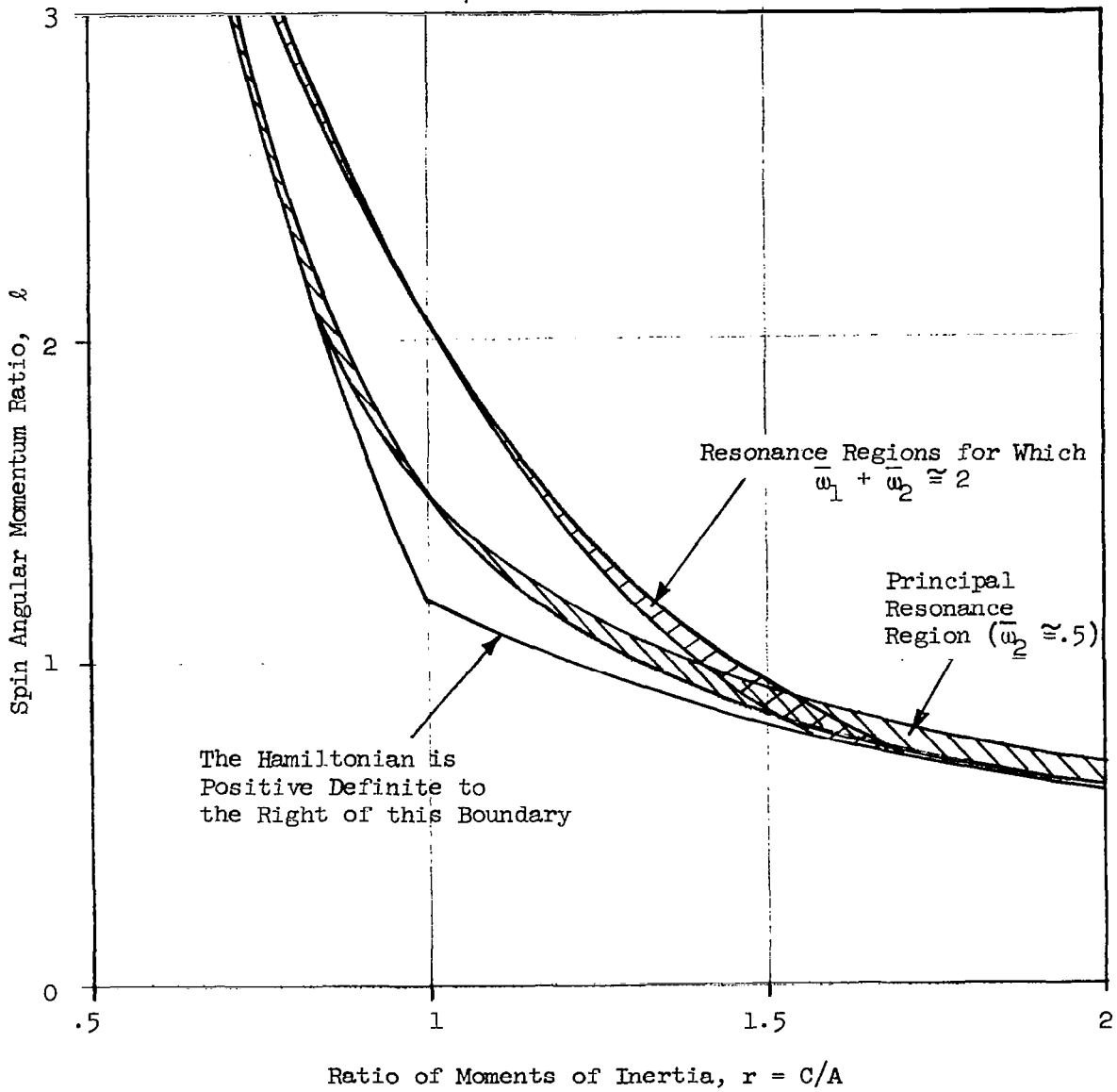
FIGURE 4.7

SELECTED INSTABILITY REGIONS

FOR ORBIT ECCENTRICITY $\epsilon = 0.1$

APPENDIX A

EXTENDED LIAPOUNOV STABILITY CRITERIA FOR

NONAUTONOMOUS SYSTEM

Theorem

Given a system described by the differential equations

$$\dot{x}_s = X_s(x_1, x_2, \ldots, x_{2n}, t), \quad (s=1, 2, \ldots, 2n) \qquad (A.1)$$

for which an equilibrium position, E, exists at $x_1 = x_2 = \ldots = x_{2n} = 0$,
the disturbed motion about this equilibrium position will be stable if a
continuous function V can be found such that

a. $V(x_1, \ldots, x_{2n}, t)$ is positive definite in the neighborhood of E
and zero at E , and

b. $\int_{t_0}^{t} \frac{dV}{dt} dt \le M(x_1^{(0)2} + x_2^{(0)2} + \ldots + x_{2n}^{(0)2})$ for motion subsequent

to $t = t_0$, in which M is a finite positive constant and

$x_1^{(0)}, x_2^{(0)}, \ldots, x_{2n}^{(0)}$ are initial, small displacements at
time $t = t_0$.

Definition

$x_{max} = \epsilon$ will be defined to mean a surface of cubic shape, with
geometric center at the origin, and with sides of length 2ϵ .

Proof

Let us assume that V is a continuous, positive definite function in the space and time domain given by

$$x_s \leq h \quad , \quad t \geq t_0 \qquad (A.2)$$

Contours of constant V in the $x_1, x_2, \ldots, x_{2n}$ space at any given instant are closed surfaces about the equilibrium position. Also, since V is a continuous function, the contours corresponding to different values of V do not intersect one another.

Let us choose an arbitrarily small positive number ϵ_1 with

$$\epsilon_1 < h \qquad (A.3)$$

We will designate as V_1 the smallest value of V that occurs at any time $t \geq t_0$ on the surface $x_{max} = \epsilon_1$.

The closed contour $V = V_1$ will change shape and size in the $x_1, x_2, \ldots, x_{2n}$ space since it is in general a function of time. However, it will remain a closed contour enclosing the origin since V is positive definite in this region for all time $t \geq t_0$. We will denote by $x_{max} = \epsilon_2$ the largest cubic surface centered about the origin that will be entirely enclosed in all contours $V = V_1$, for $t \geq t_0$. It can also be noted that V_1 is the largest value that V can take on this cubic surface. This is shown graphically for two dimensions in Figure A-1.

Note:
   Crosshatched Region is the
   Area Swept by the Curve
   $V = V_1$ for $t > t_0$

Contours of $V = V_1$
For Various Times

x_2 0

$x_{max} = \epsilon_2$

E

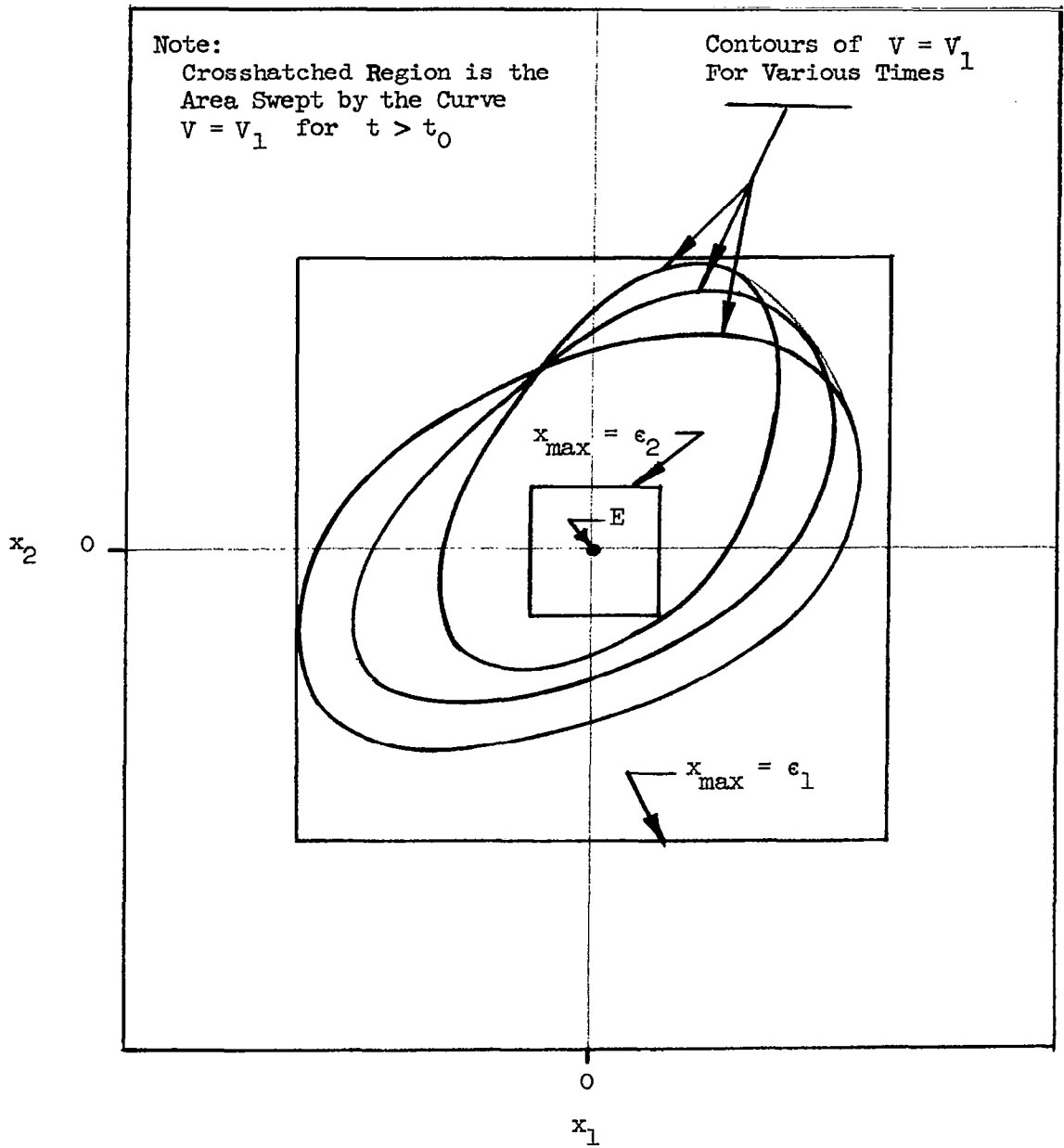$x_{max} = \epsilon_1$

0

x_1

FIGURE A.1

TWO DIMENSIONAL EXAMPLE OF THE TIME DEPENDENCE

OF $V = V_1$ FOR A NONAUTONOMOUS SYSTEM

Let us consider the integral

$$\int_{0}^{t} \frac{dV}{dt} \, dt$$

in which the integration is performed along the motion trajectory

subsequent to time $t = t_0$. The motion is assumed to be initiated by

means of a small disturbance from the equilibrium position

$x_1^{(0)}$, $x_2^{(0)}$,...,$x_{2n}^{(0)}$ occuring at time $t = t_0$. We see that

$$\int_{t_0}^{t} \frac{dV}{dt} \, dt = V(t) - V(t_0)$$

or

$$V(t) = V(t_0) + \int_{t_0}^{t} \frac{dV}{dt} \, dt \qquad (A.4)$$

The motion will certainly be stable if $V(t)$ at no time exceeds V_1,

since all motion will then take place within the arbitrarily small

region $x_s \leq \epsilon_1$. Therefore the stability requirement may be written

$$V(t_0) + \int_{t_0}^{t} \frac{dV}{dt} \, dt \leq V_1 \qquad (A.5)$$

We desire to show that if

$$\int_{t_0}^{t} \frac{dV}{dt} \, dt \leq M(x_1^{(0)2} + x_2^{(0)2} + \ldots + x_{2n}^{(0)2}) \qquad (A.6)$$

then the inequality given in Eq. (A.5) will be satisfied and the motion will be stable. To do this we must investigate the form of V in the neighborhood of E. Since V is continuous and is positive definite in the neighborhood of E and has zero value and zero first derivatives at E, the function V must behave like a quadratic function (or higher even power polynomial) of the coordinates x_s in a sufficiently small neighborhood of the origin. For sufficiently small ϵ_1 this will certainly be the case throughout the region $x_s \leq \epsilon_2$. In that case a value, c_0, can be selected so that the value of V within the region $x_s < \epsilon_2$ will be no greater than

$$V_{max} \leq c_0 V_1 (x_1^2 + x_2^2 + \ldots + x_{2n}^2) \qquad (A.7)$$

This can be shown by means of geometric arguments.

At the time $t = t_0$

$$V(t_0) \leq V_{max} (x_1^{(0)}, \ldots, x_n^{(0)}) \leq c_0 V_1 (x_1^{(0)2} + x_2^{(0)2} + \ldots + x_{2n}^{(0)2}) \qquad (A.8)$$

Consequently, adding Eqs. (A.6) and (A.8) we obtain

$$V(t_0) + \int_{t_0}^{t} \frac{dV}{dt} \, dt \leq (M + c_0 V_1)(x_1^{(0)2} + x_2^{(0)2} + \ldots + x_{2n}^{(0)2}) \qquad (A.9)$$

Hence if

$$\left(M + c_0 V_1 \right) \left(x_1^{(0)^2} + \ldots + x_{2n}^{(0)^2} \right) \leq V_1 \qquad (A.10)$$

then the stability requirement, Eq. (A.5), will certainly be satisfied. Furthermore the inequality (A.10) can be assured by suitable small choice of the initial disturbance $x_1^{(0)}, \ldots, x_{2n}^{(0)}$.

STABILITY ANALYSIS OF THE MATHIEU EQUATION

BY THE METHODS OF SECTION II

The methods developed in Section II are developed for application to multi-degree-of-freedom systems but can be applied to the Mathieu equation which has a single degree of freedom and includes a periodic coefficient. The Mathieu equation may be written

$$\ddot{y} + a(1-2q \cos 2\nu_0 t)y = 0 \tag{B.1}$$

and could be derived from a mechanical system with kinetic energy, potential energy, Lagrangian function, and Hamiltonian function of

$$KE = \frac{1}{2} \dot{y}^2 \tag{B.2}$$

$$PE = \frac{1}{2} a(1-2q \cos 2\nu_0 t)y^2 \tag{B.3}$$

$$L = \frac{1}{2} \dot{y}^2 - \frac{1}{2} a(1-2q \cos 2\nu_0 t)y^2 \tag{B.4}$$

$$H = \frac{1}{2} \dot{y}^2 + \frac{1}{2} a(1-2q \cos 2\nu_0 t)y^2 \tag{B.5}$$

by means of Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \qquad (B.6)$$

We note that an equilibrium position exists at the point $y = \dot{y} = 0$, since this point satisfies the equation of motion.

Liapounov Type of Analysis

Application of the first part of the stability theorem states that the motion can be stable if

$$\left.\frac{\partial^2 H}{\partial y^2}\right|_E = a(1-2q \cos 2\omega_0 t) > 0 \qquad (B.7)$$

for all time. Let us assume q to be a positive constant (which can be assured by a shift in the time reference) and Eq. (B.7) is satisfied if

$$\begin{aligned} a &> 0, \\ 2q &< 1 \end{aligned} \qquad (B.8)$$

The second part of the stability theorem states that the following inequality must be satisfied

$$\int_{t_0}^{t} \frac{dH}{dt}\, dt = \int_{t_0}^{t} \frac{\partial H}{\partial t}\, dt \le M(y^{(0)2} + \dot{y}^{(0)2}) \qquad (B.9)$$

for the equilibrium to be stable. But Eq. (B.1) is linear and Floquet's theorem tells us that the solution can be written in the form

$$y = \sum_{n=0}^{\infty} \sum_{j=1}^{2} e^{(\mu_j + i\nu_j)t} \left[a_{nj}\cos(2n\omega_0 t + \delta_{1n}) \right] \tag{B.10}$$

Differentiation with respect to t gives us $\dot{y}$. Also, at time $t = t_0$ we know that $y = y^{(0)}$ and $\dot{y} = \dot{y}^{(0)}$. We can redefine the coefficients in Eq. (B.10) so that

$$y = y^{(0)} \sum_{n=0}^{\infty} \sum_{j=1}^{2} e^{(\mu_j + i\nu_j)t} b_{nj}\cos(2n\omega_0 t + \delta_{1n})$$

$$\dot{y} = \dot{y}^{(0)} \sum_{n=0}^{\infty} \sum_{j=1}^{2} e^{(\mu_j + i\nu_j)t} c_{nj}\cos(2n\omega_0 t + \delta_{2n}) \tag{B.11}$$

and it will be assumed without proof that the Fourier coefficients b_n and c_n are bounded. Substitution of Eq.'s (B.11) and (B.5) into Eq. (B.9) can be expressed in the form

$$y^{(0)2} \sum_{n=0}^{\infty} \sum_{j=1}^{2} \sum_{k=1}^{2} \int_{t_0}^{t} e^{(\mu_j + \mu_k)t} e^{i(\nu_j + \nu_k)t} d_{nj}\cos(2n\omega_0 t + \delta_{3n})dt$$

$$+ \dot{y}^{(0)2} \sum_{n=0}^{\infty} \sum_{j=1}^{2} \int_{t_0}^{t} e^{(\mu_j + \mu_k)t} e^{i(\nu_j + \nu_k)t} e_{nj}\cos(2n\omega_0 t + \delta_{4n})dt \tag{B.12}$$

$$\leq M \, y(0)^2 + M \, \dot{y}(0)^2$$

where M is an arbitrary positive constant. This relation can be satisfied as long as the value of each of the integrals is bounded. We must investigate four cases

a. $\mu_j + \mu_k \leq 0$, $\nu_j + \nu_k \neq 2n\omega_0$

b. $\mu_j + \mu_k > 0$, $\nu_j + \nu_k \neq 2n\omega_0$

$$(B.13)$$

c. $\mu_j + \mu_k \leq 0$, $\nu_j + \nu_k = 2n\omega_0$

d. $\mu_j + \mu_k > 0$, $\nu_j + \nu_k = 2n\omega_0$

The first two cases correspond to nonresonant motion, and the value of the integral varies periodically with time. Case a. represents stable motion for which the inequality, Eq. (B.12), is satisfied whereas case b. is clearly impossible since, for large t , the Hamiltonian would oscillate with increasing amplitude. Cases c. and d. correspond to resonant motion. Case c. represents bounded motion and case d. represents divergent motion in which the Hamiltonian increases without bound with time.

We can conclude that unstable motion does not occur in the nonresonant case for which the Hamiltonian is positive definite. We

can use other techniques to define the resonance regions in detail, but for small q it is possible to find the approximate locations of all of the regions of resonance instability using the above information. We note that in the limit as $q \to 0$ the solution must approach

$$y = a_1 \, e^{i\sqrt{a}t} + a_2 \, e^{-i\sqrt{a}t} \tag{B.14}$$

Comparison with Eq. (B.10) shows us that as $q \to 0$ all the Fourier coefficients must approach zero except those corresponding to $n = 0$, the μ must approach zero, and

$$\nu_1 \to \sqrt{a}$$
$$\nu_2 \to -\sqrt{a} \tag{B.15}$$

The resonance regions according to the last of Eq.'s (B.13) must occur when the parametric resonance frequence is given by the approximation

$$2\sqrt{a} \cong 2n\omega_0 \, , \qquad (n=0,\pm1,\pm2,\dots) \tag{B.16}$$

or

$$a \cong n^2\omega_0^2 \, , \qquad (n=0,\pm1,\pm2,\dots) \tag{B.17}$$

This is valid only in the case of small q and is in agreement with the existing solutions of the Mathieu equation.

Method Based on Infinite Determinant [22]

At the boundary of each region of instability we will find that periodic motion exists and that this motion can be expressed in the form

$$y = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (B.18)$$

Substituting into Eq. (B.1) and equating coefficients of $\cos n\omega_0 t$ we obtain

$$a_1(\omega_0^2 - a + qa) + a_3 qa = 0$$

$$a_2[(2\omega_0)^2 - a] + a_4 qa = 0 \qquad (B.19)$$

$$a_{n-2} qa + a_n[(n\omega_0)^2 - a] + a_{n+2} qa = 0 \qquad ,(n = 3,4,\dots)$$

and equating coefficients of $\sin n\omega_0 t$ we get

$$b_1(\omega_0^2 - a - qa) + b_3 qa = 0$$

$$b_2[(2\omega_0)^2 - a] + b_4 qa = 0 \qquad (B.20)$$

$$b_{n-2} qa + b_n[(n\omega_0)^2 - a] + b_{n+2} qa = 0 \qquad ,(n = 3,4,\dots)$$

The determinant of the coefficients must be zero in order for a solution to exist. Actually four separate determinants can be written for the coefficients of even and odd a_n and b_n. The determinants for odd a_n and b_n have been found to describe the first region which has been called the region of "principal" instability.

$$
\begin{vmatrix}
\omega_0^2-a+qa & qa & 0 & 0 & \cdots \\
qa & (3\omega_0)^2-a & qa & 0 & \cdots \\
0 & qa & (5\omega_0)^2-a & qa & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{vmatrix} = 0 \qquad \text{(B.21)}
$$

$$
\begin{vmatrix}
\omega_0^2-a-qa & qa & 0 & 0 & \cdots \\
qa & (3\omega_0)^2-a & qa & 0 & \cdots \\
0 & qa & (5\omega_0)^2-a & qa & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{vmatrix} = 0 \qquad \text{(B.22)}
$$

Dividing through by $-n^2 a$ we obtain a form that is convergent [23].
The infinite determinant in this case can be evaluated in its entirety
and this is one method of determining the regions of instability of
the Mathieu equation. However, we are interested in developing techniques
for use in problems where this will not be possible. Evaluation of the
determinants, Eq. (B.21) and Eq. (B.22), by taking successively larger
principle minors starting from the upper left will result, for small
q , in successively better estimates of the regions of instability.
The first estimate gives us

$$\omega_0^2 - a = \pm \, qa \qquad \qquad (B.23)$$

which gives

$$a \cong \omega_0^2 \, (1 \pm q) \qquad \qquad (B.24)$$

or, for subsequent comparison, can be written in the form

$$\sqrt{a} \; \cong \omega_0 (1 \pm 1/2 \; q) \qquad \qquad (B.25)$$

The second approximation comes from

$$(\omega_0^2 - a \pm qa) \, [(3\omega_0)^2 - a] - q^2 a^2 = 0 \qquad \qquad (B.26)$$

and can be written as an improved approximation for the first region of instability.

$$\sqrt{a} \; \cong \omega_0 (1 \pm 1/2 \; q + 5/16 \; q^2) \qquad \qquad (B.27)$$

The second approximation also yields a first approximation of the third region of instability. In a similar fashion the boundaries of the odd regions of instability may be defined with increased accuracy as higher order minors of the infinite determinants are evaluated, and an additional region can be defined for each additional order is taken.

The even numbered regions of stability

$$(\text{near} \; \sqrt{a} \; = 2\omega_0 \, , \; 4\omega_0 \, , \; 6\omega_0 \, , \dots)$$

can be obtained from the determinants of the coefficients of even a_n and b_n in Eq's. (B.19) and (B.20).

Method Based on Asymptotic Expansion in Terms of a Small Parameter

We can also define the regions of instability by means of an asymptotic expansion of the equations of motion in terms of a small parameter. In this case no new information will be obtained and its application to the Mathieu equation is for the purpose of illustration of the technique.

Following the procedure outlined in Section II we will assume a resonance condition to exist with constant amplitude and phase angle

$$y = a_1 \cos(\omega_0 t + \delta) + q\, u_1(t) + q^2 u_2(t) + \ldots \qquad (B.28)$$

in which the terms u_1, u_2, ... include nonresonance terms. When q is a small parameter the first region of instability (for which $\omega_0 \cong \sqrt{a}$) will be defined in terms of the expansion

$$\sqrt{a} - \omega_0 = \Delta_1 q + \Delta_2 q^2 + \ldots \qquad (B.29)$$

Substituting into the differential equation, Eq. (B.1), one obtains for terms up to the second power of q

$$- a_1 \omega_0^2 \cos(\omega_0 t + \delta) + q\, \frac{d^2 u_1}{dt^2} + q^2 \frac{d^2 u_2}{dt^2} + a_1 a \cos(\omega_0 t + \delta)$$

$$\qquad (B.30)$$

$$+ a\, q u_1 + a\, q^2 u_2 - 2aq \cos 2\omega_0 t [a_1 \cos(\omega_0 t + \delta) + q\, u_1] = 0$$

From Eq. (B.29) we can write

$$a = \omega_0^2 + 2\Delta_1 \omega_0 q + (2\Delta_2 \omega_0 + \Delta_1^2)q^2 + \ldots \qquad (B.31)$$

which, when substituted into Eq. (B.31), gives

$$q \frac{d^2 u_1}{dt^2} + q^2 \frac{d^2 u_2}{dt^2} + 2a_1 \Delta_1 \omega_0 q \cos(\omega_0 t + \delta) + q^2(2\Delta_2 \omega_0 + \Delta_1^2)a_1 \cos(\omega_0 t + \delta)$$

$$+ \omega_0^2 q \, u_1 + 2\Delta_1 \omega_0 q^2 u_1 + \omega_0^2 q^2 u_2 - a(\omega_0^2 q + 2\Delta_1 \omega_0 q^2)[\cos(3\omega_0 t + \delta)$$

$$+ \cos(\omega_0 t - \delta)] - 2\omega_0^2 q^2 \cos 2\omega_0 t \, u_1 = 0 \qquad (B.32)$$

Equating the coefficients of q one obtains

$$\frac{d^2 u_1}{dt^2} + 2a_1 \Delta_1 \omega_0 \cos(\omega_0 t + \delta) + \omega_0^2 u_1$$

$$- a_1 \omega_0^2 [\cos(3\omega_0 t + \delta) + \cos(\omega_0 t - \delta)] = 0 \qquad (B.33)$$

Equating the coefficients of the terms in $\cos \omega_0 t$ so that the function u_1 does not include the resonance oscillation we see that this constant amplitude resonance oscillation can only take place when $\delta = \frac{n\pi}{2}$ such that

$$\Delta_1 = \frac{\omega_0}{2} \qquad \text{when} \qquad \delta = 0 \text{ or } \pi$$

$$\Delta_1 = -\frac{\omega_0}{2} \qquad \text{when} \qquad \delta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \tag{B.34}$$

Also, u_1 must satisfy the relation

$$\frac{d^2 u_1}{dt^2} + \omega_0^2 u_1 = a_1 \omega_0^2 \cos(3\omega_0 t + \delta) \tag{B.35}$$

from which

$$u_1 = -\frac{a_1}{8} \cos(3\omega_0 t + \delta) \tag{B.36}$$

Equating the coefficients of q^2 in Eq. (B.33) and including the appropriate substitutions for u_1, one obtains

$$\frac{d^2 u_2}{dt^2} + (2\Delta_2\omega_0 + \Delta_1^2)a_1 \cos(\omega_0 t + \delta) - \frac{1}{4}\Delta_1\omega_0 a_1 \cos(3\omega_0 t + \delta)$$

$$+ \omega_0^2 u_2 - 2a_1\Delta_1\omega_0\cos(3\omega_0 t + \delta) - 2a_1\Delta_1\omega_0\cos(\omega_0 t - \delta)$$

$$+ \frac{1}{8} a_1\omega_0^2 \cos(\omega_0 t - \delta) + \frac{1}{8} a_1\omega_0^2 \cos(5\omega_0 t + \delta) = 0 \tag{B.37}$$

Equating the coefficients of $\cos \omega_0 t$ so that u_2 will not include the resonance oscillation

$$a_1(2\triangle_2\omega_0 + \triangle_1^2)\,\cos(\omega_0 t + \delta) - 2a_1\triangle_1\omega_0\cos(\omega_0 t - \delta)$$

$$+ \frac{1}{8}\,a_1\omega_0^2\,\cos(\omega_0 t - \delta) = 0 \qquad\qquad (\text{B}.38)$$

This can be solved for $\triangle_2$ by appropriate substitution of $\triangle_1$ from the first approximation and the corresponding value of δ such that when

$$\triangle_1 = \frac{\omega_0}{2}\,, \qquad \delta = 0 \text{ or } \pi\,, \qquad \triangle_2 = \frac{5}{16}\,\omega_0$$

and when $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{B}.39)$

$$\triangle_1 = -\frac{\omega_0}{2}\,, \qquad \delta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\,, \qquad \triangle_2 = \frac{5}{16}\,\omega_0$$

Substituting the values derived above for $\triangle_1$ and $\triangle_2$ into Eq. (B.30) the results of the previous section are varified such that the boundaries of the first resonance region are given by

$$\sqrt{a}\ = \omega_0(1 \pm \frac{1}{2}\,q + \frac{5}{16}\,q^2 + \ldots) \qquad\qquad (\text{B}.40)$$

The above approximation could be carried further by initial assumption of more terms in the asymptotic expansion and repeating the above process.

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