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STUDY OF THE PRESSURE DISTRIBUTION ON OSCILLATING PANELS IN LOW SUPERSONIC FLOW WITH TURBULENT BOUNDARY LAYER

by E. F. E. Zeydel

Prepared by
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Ga.
for Ames Research Center

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PREFACE

This report covers research initiated by the National Aeronautics and Space Administration, Ames Research Center, Moffett Field, California, under Contract NAS2-2897. The work is administered by Mr. P. A. Gaspers.

The principal investigator of this program is Dr. E. F. E. Zeydel.

This report covers the work done under this contract for the period 11 June 1965 to 10 June 1966.

The author wishes to acknowledge the contribution of Mr. A. C. Bruce for part of the Appendix.

ABSTRACT

A theoretical analysis for the unsteady pressure distribution and generalized aerodynamic force on an oscillating wall of finite extent in the chordwise direction with a thick boundary layer is presented. The analysis is based on a two-dimensional model for the turbulent boundary layer of given velocity profile. Linearized potential flow theory is applied and the finite chord case is treated by using Fourier Integral techniques.

The theoretical equations for a thick, two-dimensional boundary layer with small perturbations are given in the Appendix.

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LIST OF SYMBOLS

A_n	= deflection amplitude defined by (7)
a	= panel chord
C_n, D_n	= defined by (11)
c_n	= speed of sound of n^{th} sublayer
E_n, F_n	= defined by (14)
$f(\lambda)$	= Fourier Transform of w_0, defined by (57)
$F(\lambda)$	= defined by (51)
G_n	= matrix defined by (36)
H	= matrix defined by (42)
$H_0^{(2)}$	= Hankel Function of the second kind and zero order
i^2	= - 1
J_0	= Bessel Function of the first kind and zero order
M_n	= Mach Number of n^{th} sublayer
P_n	= perturbation pressure coefficient defined by (9)
p	= static pressure
p_n	= perturbation pressure of n^{th} sublayer at $y_n = 0$
q_r	= r^{th} generalized coordinate, defined by (60)
Q_r	= r^{th} generalized aerodynamic force defined by (63)
R_{rs}	= defined by (64)
r,s	= integers
T	= absolute temperature
t	= time
U	= local potential flow velocity

U_n	=	uniform velocity of n^{th} sublayer
u,v	=	x and y velocity components, respectively
w_n	=	displacement of n^{th} layer at $y_n = 0$
x,y	=	reference coordinate system
Y_0	=	Bessel Function of the second kind and zero order
y_n	=	reference coordinate of n^{th} sublayer
z_0	=	deflection function defined by (55)
α_n	=	defined by (31)
$\bar{\alpha}_n$	=	defined by (34)
β_n	=	$(M_n^2 - 1)^{1/2}$
γ_n	=	$(1 - M_n^2)^{1/2}$
δ	=	boundary layer thickness
δ_n	=	thickness of n^{th} sublayer
$\bar{\delta}_r$	=	defined by (70)
ε_r	=	defined by (69)
λ	=	wave number defined by (1)
μ	=	coefficient of viscosity
ρ	=	fluid density
ρ_n	=	mass air density of n^{th} sublayer
σ_n	=	defined by (28)
τ_n	=	defined by (29)
τ_{ij}	=	stress in j^{th} direction on i^{th} plane
ϕ_n	=	velocity potential of n^{th} sublayer
ψ_r	=	r^{th} modeshape defined by (59)

Ψ_r = Fourier Transform of ψ_r, defined by (61)

ω = frequency of vibration

Superscripts and others:

t = total, sum of "mean" and fluctuations

' = fluctuation component or differentiation with respect to x

$-$ = mean component

I. INTRODUCTION

In recent years attention has been devoted to the description of the effects of a turbulent boundary layer on the flutter characteristics of aircraft and space vehicle skin panels [1,2]*. In these studies the unsteady pressure distribution on oscillating walls has been estimated either by using a viscous fluid model (McClure) or by using a shear flow model (Fung). Both authors treat only the case of a panel boundary of infinite extent in the chordwise direction.

In this report a theoretical analysis is presented for the unsteady pressure distribution on an oscillating wall of finite extent in the chordwise direction. The analysis is based on a two-dimensional model for the thick boundary layer of given velocity profile. Separating the boundary layer in N sublayers with uniform velocity, the pressure distribution in each sublayer is estimated by applying linearized potential flow theory. The finite chord case is treated by writing the panel boundary in Fourier Integral form and developing the appropriate transfer function for the unsteady pressure distribution. It is shown that the transfer function remains finite for all wavelenghts in the presence of a boundary layer so that well-known procedures can be utilized for numerical evaluation.

The theoretical equations for a thick, two-dimensional boundary layer with small perturbations are given in the Appendix. At present, these equations are not amenable to panel flutter analysis and further development seems warranted only after more detailed information on the shear model is collected.

A brief literature survey is also given in the Appendix.

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*Numbers in square brackets refer to the bibliography.

II. THE PRESSURE DISTRIBUTION ON A TRAVELING WAVY WALL WITH A THICK BOUNDARY LAYER

Consider a two-dimensional laminar flow in the x-direction with a given mean velocity profile $U(y)$. We assume that the mean flow is a boundary layer flow, which joins the external flow, U_N, at $y = \delta$ (see Figure 1).

Let the traveling wavy wall boundary at $y = 0$ be given by

$$w_0 = A_0 e^{i\lambda x} e^{i\omega t} \quad ; \quad -\infty < x < +\infty \qquad (1)$$

$$-\infty < \lambda < +\infty$$

The problem then becomes to determine the pressure perturbation p_0, at $y = 0$, and within the boundary layer in terms of w_0.

For simplicity, we divide the boundary layer δ into N sublayers of thicknesses, $\delta_0, \delta_1, \ldots, \delta_n, \ldots, \delta_{N-1}$ and assume that in each sublayer the mean velocity and density is uniform. We also assume that the amplitude A_0 is small compared to the thickness of each sublayer and that the perturbed flows in the sublayer and in the external flow are isentropic and irrotational. The linearized equations for the perturbation velocity potential ϕ can then be utilized to obtain the perturbation velocity and pressure fields within each sublayer. For generality, the external flow velocity U_N is assumed to be either sub- or supersonic so that the mean flow velocity in the sublayers are either sub- or supersonic.

The expressions between the displacements and pressures of a sub- or supersonic sublayer can be obtained as follows.

Let the uniform mean flow velocity, the fluid density, and Mach number of the nth sublayer of thickness, δ_n, be given by U_n, ρ_n, and M_n, respectively (see Figure 1).

The governing equation for the velocity potential ϕ_n in the sublayer is

$$\left(1 - M_n^2\right) \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial^2 \phi_n}{\partial y_n^2} - \frac{2M_n}{c_n} \frac{\partial^2 \phi_n}{\partial x \partial t} - \frac{1}{c_n^2} \frac{\partial^2 \phi_n}{\partial t^2} = 0 \qquad (2)$$

The boundary conditions for tangential flow are

$$\left.\frac{\partial \phi_n}{\partial y_n}\right|_{y_n = 0} = \frac{\partial w_n}{\partial t} + U_n \frac{\partial w_n}{\partial x} \tag{3}$$

$$\left.\frac{\partial \phi_n}{\partial y_n}\right|_{y_n = \delta_n} = \frac{\partial w_{n+1}}{\partial t} + U_n \frac{\partial w_{n+1}}{\partial x} \tag{4}$$

where w_n and w_{n+1} are the displacements at $y_n = 0$ and $y_n = \delta_n$, respectively.

The perturbation pressures at the edges of the sublayer are given by

$$p_n = - \rho_n \left.\left(\frac{\partial \phi_n}{\partial t} + U_n \frac{\partial \phi_n}{\partial x} \right)\right|_{y_n = 0} \tag{5}$$

$$p_{n+1} = - \rho_n \left.\left(\frac{\partial \phi_n}{\partial t} + U_n \frac{\partial \phi_n}{\partial x} \right)\right|_{y_n = \delta_n} \tag{6}$$

where p_n and p_{n+1} are the pressure perturbations at $y_n = 0$ and $y_n = \delta_n$, respectively.

Let for $-\infty < x < +\infty \quad , \quad -\infty < \lambda < +\infty$

$$w_n = A_n e^{i\lambda x} e^{i\omega t} \tag{7}$$

$$w_{n+1} = A_{n+1} e^{i\lambda x} e^{i\omega t} \tag{8}$$

$$p_n = - \rho_n U_n^2 P_n e^{i\lambda x} e^{i\omega t} \tag{9}$$

$$p_{n+1} = - \rho_{n+1} U_{n+1}^2 P_{n+1} e^{i\lambda x} e^{i\omega t} \tag{10}$$

where A_n, A_{n+1}, P_n and P_{n+1} are in general complex quantities.

If U_n is subsonic, $M_n < 1$ and equation (2) has the solution

$$\phi_n = U_n \left[C_n e^{-\gamma_n \sigma_n y} + D_n e^{+\gamma_n \sigma_n y} \right] e^{i\lambda x} e^{i\omega t} \qquad (11)$$

where

$$\sigma_n = \left\{ \left(\lambda - \frac{M_n \omega}{c_n \gamma_n^2} \right)^2 - \frac{\omega^2}{c_n^2 \gamma_n^4} \right\}^{1/2} \qquad (12)$$

and

$$\gamma_n = \sqrt{1 - M_n^2} \qquad (13)$$

If U_n is supersonic, so that $M_n > 1$, the solution of equation (2) becomes

$$\phi_n = U_n \left[E_n e^{-i\beta_n \tau_n y} + F_n e^{+i\beta_n \tau_n y} \right] e^{i\lambda x} e^{i\omega t} \qquad (14)$$

where

$$\tau_n = \left\{ \left(\lambda + \frac{M_n \omega}{c_n \beta_n^2} \right)^2 - \frac{\omega^2}{c_n^2 \beta_n^4} \right\}^{1/2} \qquad (15)$$

and

$$\beta_n = \sqrt{M_n^2 - 1} \qquad (16)$$

In the region $-\infty < \lambda < \infty$, σ_n and τ_n can be either real or pure imaginary. For the sublayers, where both the positive and negative root is applied, a proper selection of these roots for specific values of λ is not required. For the external flow region, $y > \delta$, however, a selection of the proper root must be based on the radiation and finiteness conditions at infinity.

Let the relative Mach number corresponding to the relative velocity between the traveling wave and the external flow velocity be

$$\bar{M}_N = M_N + \frac{\omega}{\lambda c_N} \tag{17}$$

Solutions of equation (2) with the proper behavior at ∞ then become

a) $\bar{M}_N > 1$; $\lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{-i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{18}$$

b) $\bar{M}_N > 1$; $\lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{+i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{19}$$

c) $-1 \leq \bar{M}_N \leq 1$; $\lambda \gtrless 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{- \left\{\lambda^2\left(1 - \bar{M}_N{}^2\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{20}$$

d) $\bar{M}_N < -1$; $\lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{+i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{21}$$

e) $\bar{M}_N < -1$; $\lambda < 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{-i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{22}$$

Interpreting (18) - (22) in the notation of (11) and (14), using only the first term in the square bracket, we find after a few algebraic manipulations, that the solution of equation (2) for the external flow region can be written as

a) For $M_N < 1$,

$$\phi_N = U_N C_N \, e^{-\gamma_N \sigma_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{23}$$

where

$$\sigma_N = \left\{ \left(\lambda - \frac{M_N \omega}{c_N \gamma_N^2} \right)^2 - \frac{\omega^2}{c_N^2 \gamma_N^4} \right\}^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N(1 - M_N)}$$

$$\text{or } \lambda < - \frac{\omega}{c_N(1 + M_N)}$$

$$= i \left\{ \frac{\omega^2}{c_N^2 \gamma_N^4} - \left(\lambda - \frac{\omega^2}{c_N \gamma_N^2} \right)^2 \right\}^{1/2} \quad ; \quad - \frac{\omega}{c_N(1 + M_N)} < \lambda < \frac{\omega}{c_N(1 - M_N)} \tag{24}$$

and

$$\gamma_N = \sqrt{1 - M_N^2} \tag{25}$$

b) For $M_N > 1$,

$$\phi_N = U_N E_N \, e^{-i\beta_N \tau_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{26}$$

where

$$\tau_N = \left\{\left(\lambda + \frac{M_N\omega}{c_N\beta_N{}^2}\right)^2 - \frac{\omega^2}{c_N{}^2\beta_N{}^4}\right\}^{1/2} \quad ; \quad \lambda > -\frac{\omega}{c_N(M_N+1)}$$

$$= -i\left\{\frac{\omega^2}{c_N{}^2\beta_N{}^4} - \left(\lambda + \frac{M_N\omega}{c_N\beta_N{}^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_N(M_N-1)} < \lambda < -\frac{\omega}{c_N(M_N+1)}$$

$$= -\left\{\left(\lambda + \frac{M_N\omega}{c_N\beta_N{}^2}\right)^2 - \frac{\omega^2}{c_N{}^2\beta_N{}^4}\right\}^{1/2} \quad ; \quad \lambda < -\frac{\omega}{c_N(M_N-1)}$$

$$\beta_N = \sqrt{M_N{}^2 - 1} \tag{27}$$

Although strictly speaking not required, it is convenient to define, in general

$$\sigma_n = \left\{\left(\lambda - \frac{M_n\omega}{c_n\gamma_n{}^2}\right)^2 - \frac{\omega^2}{c_n{}^2\gamma_n{}^4}\right\}^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_n(1-M_n)}$$

$$\text{or} \quad \lambda < -\frac{\omega}{c_n(1+M_n)}$$

$$= i\left\{\frac{\omega}{c_n{}^2\gamma_n{}^4} - \left(\lambda - \frac{M_n\omega}{c_n\gamma_n{}^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_n(1+M_n)} < \lambda < \frac{\omega}{c_n(1-M_n)} \tag{28}$$

and

$$\tau_n = \left\{\left(\lambda + \frac{M_n \omega}{c_n \beta_n^2}\right)^2 - \frac{\omega^2}{c_n^2 \beta_n^4}\right\}^{1/2} \quad ; \quad \lambda > -\frac{\omega}{c_n(M_n + 1)}$$

$$= -i\left\{\frac{\omega^2}{c_n^2 \beta_n^4} - \left(\lambda + \frac{M_n \omega}{c_n \beta_n^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_n(M_n - 1)} < \lambda < -\frac{\omega}{c_n(M_n + 1)}$$

$$= -\left\{\left(\lambda + \frac{M_n \omega}{c_n \beta_n^2}\right)^2 - \frac{\omega^2}{c_n^2 \beta_n^4}\right\}^{1/2} \quad ; \quad \lambda < -\frac{\omega}{c_n(M_n - 1)} \qquad (29)$$

for the sublayers as well as the external flow region.

The relation between the deflection and pressure coéfficients for the sublayers can be obtained as follows.

If U_n is subsonic, $M_n < 1$, we eliminate C_n and D_n, using (3) - (5), (7) - (9) and (11) and find

$$A_{n+1} = \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} A_n + \frac{\gamma_n \sigma_n}{\lambda^2} \frac{U_n^2}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2} P_n \qquad (30)$$

where

$$\alpha_n = \gamma_n \sigma_n \delta_n \qquad (31)$$

Similarly, using (3), (5) - (7), (9) - (11), yields

$$P_{n+1} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^2} \frac{\lambda^2}{\gamma_n \sigma_n} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2} A_n$$

$$+ \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} P_n \tag{32}$$

If U_n is supersonic, $M_n > 1$, we find using (3) - (5), (7) - (9) and (14),

$$A_{n+1} = \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} A_n + \frac{i\beta_n \tau_n}{\lambda^2} \frac{U_n^2}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2} P_n \tag{33}$$

where

$$\bar{\alpha}_n = i\beta_n \tau_n \delta_n \tag{34}$$

and using (3), (5) - (7), (9) - (11), and (14)

$$P_{n+1} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^2} \frac{\lambda^2}{i\beta_n \tau_n} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2} A_n$$

$$+ \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} P_n \tag{35}$$

For clarity, we write (30), (32), (33) and (35) in matrix form

$$\begin{vmatrix} A_{n+1} \\ \\ P_{n+1} \end{vmatrix} = \left[G_n\right] \begin{vmatrix} A_n \\ \\ P_n \end{vmatrix} \tag{36}$$

where the elements of G_n are given by , for $M_n \leq 1$,

$$g_{n_{11}} = \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2}$$

$$g_{n_{12}} = \frac{\gamma_n \sigma_n}{\lambda^2} \frac{U_n^{\ 2}}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2}$$

$$g_{n_{21}} = \frac{\rho_n U_n^{\ 2}}{\rho_{n+1} U_{n+1}^{\ 2}} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^{\ 2}} \frac{\lambda^2}{\gamma_n \sigma_n} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2}$$

$$g_{n_{22}} = \frac{\rho_n U_n^{\ 2}}{\rho_{n+1} U_{n+1}^{\ 2}} \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} \tag{37}$$

where

$$\alpha = \gamma_n \sigma_n \delta_n$$

$$\gamma_n = \sqrt{1 - M_n^{\ 2}}$$

σ_n is defined by (28) (38)

and, for $M_n > 1$,

$$g_{n_{11}} = \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{12}} = \frac{i\beta_n\tau_n}{\lambda^2} \frac{U_n^{\,2}}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{21}} = \frac{\rho_n U_n^{\,2}}{\rho_{n+1}\, U_{n+1}^{\,2}} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^{\,2}} \frac{\lambda^2}{i\beta_n\tau_n} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{22}} = \frac{\rho_n U_n^{\,2}}{\rho_{n+1}\, U_{n+1}^{\,2}} \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} \tag{39}$$

where

$$\bar{\alpha}_n = i\beta_n\tau_n\delta_n$$

$$\beta_n = \sqrt{M_n^{\,2} - 1}$$

τ_n is defined by (29) (40)

Consequently,

$$\begin{vmatrix} A_N \\ \\ P_N \end{vmatrix} = \left[H\right] \begin{vmatrix} A_0 \\ \\ P_0 \end{vmatrix} \tag{41}$$

where

$$\left[H\right] = \left[G_{N-1}\right] \left[G_{N-2}\right] \cdot \cdot \cdot \left[G_0\right] \tag{42}$$

If the flow external to the boundary layer is subsonic, so that $M_n < 1$, its velocity potential is given by (23) and we find, using (3) and (7),

$$C_N = - \frac{i\left(\frac{\omega}{\lambda} + U_N\right)\lambda}{\gamma_N \sigma_N U_N} A_N \tag{43}$$

so that from (5) and (9) there follows

$$P_N = \frac{\left(\frac{\omega}{\lambda} + U_N\right)^2 \lambda^2}{\gamma_N \sigma_N U_N{}^2} A_N \tag{44}$$

The pressure coefficient, P_0, is obtained from (41) and (44)

$$P_0 = - \frac{h_{21} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \ U_N{}^2} h_{11}}{h_{22} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \ U_N{}^2} h_{12}} A_0 \quad ; \quad M_N < 1 \tag{45}$$

where h_{11}, h_{12}, h_{21}, and h_{22} are the elements of the matrix H.

If the external flow is supersonic, $M_N > 1$, its velocity potential is given by (26) and there follows from (3) and (7),

$$E_N = - \frac{\left(\frac{\omega}{\lambda} + U_N\right)\lambda}{\beta_N \tau_N U_N} A_N \tag{46}$$

and from (5) and (9)

$$P_N = -\frac{i\left(\frac{\omega}{\lambda} + U_N\right)^2 \lambda^2}{\beta_N \tau_N U_N^{\ 2}} A_N \tag{47}$$

Using (41) and (44), the pressure coefficient, P_0, becomes

$$P_0 = -\frac{h_{21} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N\ U_N^{\ 2}}\ h_{11}}{h_{22} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N\ U_N^{\ 2}}\ h_{12}}\ A_0 \quad ; \quad M_N > 1 \tag{48}$$

The perturbation pressure at the surface of the traveling wavy wall

$$w_0 = A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad \begin{array}{l} -\infty < x < +\infty \\ -\infty < \lambda < +\infty \end{array} \tag{49}$$

is given by

$$p_0 = -\ \rho_0 U_0^{\ 2} P_0\ e^{i\lambda x}\ e^{i\omega t}$$

or

$$p_0 = -\ \rho_N U_N^{\ 2}\ F(\lambda) A_0\ e^{i\lambda x}\ e^{i\omega t} \tag{50}$$

where

$$F(\lambda) = -\frac{\rho_0 U_0^{\,2}}{\rho_N U_N^{\,2}} \; \frac{h_{21} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \; U_N^{\,2}} \; h_{11}}{h_{22} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \; U_N^{\,2}} \; h_{12}} \quad ; \quad M_N < 1$$

$$= -\frac{\rho_0 U_0^{\,2}}{\rho_N U_N^{\,2}} \; \frac{h_{21} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N \; U_N^{\,2}} \; h_{11}}{h_{22} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N \; U_N^{\,2}} \; h_{12}} \quad ; \quad M_N > 1 \qquad (51)$$

It is interesting to note the following characteristics of the function $F(\lambda)$,

(a) If $\delta \to 0$, $\delta_n \to 0$ and

$$g_{n_{11}} = 1 \quad ; \quad g_{n_{12}} = g_{n_{21}} = 0$$

$$g_{n_{22}} = \frac{\rho_n U_n^{\,2}}{\rho_{n+1} \; U_{n+1}^{\,2}}$$

Thus,

$$h_{11} = 1 \quad ; \quad h_{12} = h_{21} = 0$$

$$h_{22} = \frac{\rho_0 U_0^{\,2}}{\rho_N U_N^{\,2}}$$

and

$$F(\lambda) = + \frac{\left(\omega + U_N\lambda\right)^2}{\gamma_N\sigma_N\ U_N^{\ 2}} \quad ; \quad M_N < 1$$

$$= - \frac{i\left(\omega + U_N\lambda\right)^2}{\beta_N\tau_N\ U_N^{\ 2}} \quad ; \quad M_N > 1 \tag{52}$$

so that

$$P_0 = - \frac{\rho_N\left(\omega + U_N\lambda\right)^2}{\gamma_N\sigma_N} A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad M_N < 1$$

$$= + \frac{i\rho_N\left(\omega + U_N\lambda\right)^2}{\beta_N\tau_N} A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad M_N > 1 \tag{53}$$

which is the expected traveling wavy wall solution.

(b) If $\delta = 0$ and $\omega = 0$, we obtain the stationary wavy wall solution without a boundary layer

$$P_0 = - \frac{\rho_N U_N^{\ 2}|\lambda|}{\gamma_N} A_0\ e^{i\lambda x} \quad ; \quad M_N < 1$$

$$= \frac{i\rho_N U_N^{\ 2}\lambda}{\beta_N} A_0\ e^{i\lambda x} \quad ; \quad M_N > 1 \tag{54}$$

(c) If $\delta \neq 0$ and $\omega = 0$ the results correspond to those given in progress report No. 2 for a stationary wavy wall with a thick boundary layer.

Of importance for the numerical evaluation of $F(\lambda)$ is the fact that only if $\delta = 0$ and $\omega \neq 0$, $F(\lambda)$ contains singular points, since σ_N and τ_N become zero for specific values of λ. However, if $\delta \neq 0$, $F(\lambda)$ remains finite for all values of λ.

III. THE UNSTEADY PRESSURE DISTRIBUTION AND GENERALIZED FORCE ON AN OSCILLATING WALL OF FINITE CHORD LENGTH WITH A THICK BOUNDARY LAYER

The unsteady pressure distribution on an oscillating wall of finite chord length with a thick boundary layer can be obtained from the previous results by writing the wall boundary in Fourier integral form.

Let the oscillating wall boundary at y = 0 be given by

$$w_0(x,t) = z_0(x)\, e^{i\omega t} \quad ; \quad 0 \leq x \leq a$$

$$= 0 \quad ; \quad x < 0 \text{ or } x > a \tag{55}$$

In Fourier integral form, (51) becomes

$$w_0(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{56}$$

where

$$f(\lambda) = \int_{-\infty}^{+\infty} z_0(x)\, e^{-i\lambda x}\, dx = \int_0^a z_0(x)\, e^{-i\lambda x}\, dx \tag{57}$$

The unsteady pressure distributions at y = 0 follows directly from (49), (50), and (56)

$$p_0(x,t) = -\frac{\rho_N U_N^{\,2}}{2\pi} \int_{-\infty}^{+\infty} F(\lambda)\, f(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{58}$$

It is customary in panel flutter analysis to utilize a Ritz-Galerkin method to obtain the flutter boundaries and to satisfy the panel boundary conditions by choosing a suitable set of deflection functions. Therefore, let

$$z_0(x) = \sum_r q_r \, \psi_r(x) \qquad (59)$$

where $\Psi_r(x)$ satisfies the appropriate boundary conditions. From (57)

$$f(\lambda) = \sum_r q_r \; \Psi_r(x) \qquad (60)$$

where

$$\Psi_r(\lambda) = \int_0^a \psi_r(x) \, e^{-i\lambda x} \, dx \qquad (61)$$

The unsteady pressure distribution at y = 0 becomes

$$p_0(x,t) = -\frac{\rho_N U_N^2}{2\pi} \sum_r q_r \int_{-\infty}^{+\infty} F(\lambda) \; \Psi_r(\lambda) \; e^{i\lambda x} \, d\lambda \; e^{i\omega t} \qquad (62)$$

The generalized aerodynamic force is defined by

$$Q_r(t) = \int_0^a P_0(x,t) \, \psi_r(x) \, dx$$

$$= -\frac{\rho_N U_N^2}{2\pi} \sum_s q_s \, R_{rs} \, e^{i\omega t} \qquad (63)$$

where

$$R_{rs} = \int_{-\infty}^{+\infty} F(\lambda) \; \Psi_r(-\lambda) \; \Psi_s(\lambda) \, d\lambda \qquad (64)$$

For a panel with primed edges at the leading and trailing edges ($x = 0,a$),

$$\psi_r(x) = \sin \frac{r\pi}{a} x \tag{65}$$

and

$$\Psi_r(\lambda) = \int_0^a \sin \frac{r\pi}{a} x \, e^{-i\lambda x} \, dx = \frac{r\frac{\pi}{a}}{\left(r\frac{\pi}{a}\right)^2 - \lambda^2} \left[1 - (-1)^r e^{-i\lambda a}\right] \tag{66}$$

Thus,

$$R_{rs} = \frac{rs\pi^2}{a^2} \int_{-\infty}^{+\infty} F(\lambda) \frac{\left\{1 - (-1)^r e^{+i\lambda a}\right\}\left\{1 - (-1)^s e^{-i\lambda a}\right\}}{\left\{\left(r\frac{\pi}{a}\right)^2 - \lambda^2\right\}\left\{\left(s\frac{\pi}{a}\right)^2 - \lambda^2\right\}} d\lambda \tag{67}$$

where $F(\lambda)$ is defined by (51).

For a panel with clamped leading and trailing edges

$$\Psi_r(x) = \text{Cosh}\ \bar{\delta}_r x - \cos \bar{\delta}_r x - \varepsilon_r (\text{Sinh}\ \bar{\delta}_r x - \sin \bar{\delta}_r x) \tag{68}$$

where

$$\varepsilon_r = \frac{\text{Cosh}\ \bar{\delta}_r a - \cos \bar{\delta}_r a}{\text{Sinh}\ \bar{\delta}_r a - \sin \bar{\delta}_r a} \tag{69}$$

and δ_r is given by the characteristic equation

$$\text{Cosh}\ \bar{\delta}_r a \cos \bar{\delta}_r a = 1 \tag{70}$$

We write $\psi_r(x)$ in the form

$$\psi_r(x) = \tfrac{1}{2}(1 - \varepsilon_r)\, e^{\bar{\delta}_r x} + \tfrac{1}{2}(1 + \varepsilon_r)\, e^{-\bar{\delta}_r x}$$

$$- \tfrac{1}{2}(1 + \varepsilon_r)\, e^{+i\bar{\delta}_r x} - \tfrac{1}{2}(1 - \varepsilon_r)\, e^{-i\bar{\delta}_r x} \qquad (71)$$

Since

$$\int_0^a e^{\alpha x}\, e^{-i\lambda x}\, dx = \frac{1}{\alpha - i\lambda}\left[e^{(\alpha - i\lambda)a} - 1\right] \quad , \qquad (72)$$

$$\Psi_r(\lambda) = \int_0^a \psi_r(\lambda)\, e^{-i\lambda x}\, dx = \frac{(1 - \varepsilon_r)}{2(\bar{\delta}_r - i\lambda)}\left[e^{(\bar{\delta}_r - i\lambda)a} - 1\right]$$

$$- \frac{(1 + \varepsilon_r)}{2(\bar{\delta}_r + i\lambda)}\left[e^{-(\bar{\delta}_r + i\lambda)a} - 1\right] + \frac{i(1 + \varepsilon_r)}{2(\bar{\delta}_r - \lambda)}\left[e^{i(\bar{\delta}_r - \lambda)a} - 1\right]$$

$$- \frac{i(1 - \varepsilon_r)}{2(\bar{\delta}_r + \lambda)}\left[e^{-i(\bar{\delta}_r + \lambda)a} - 1\right] \qquad (73)$$

Using (73), R_{rs} for the clamped edge case follows again from (64).

The main difficulty in the foregoing analysis will be the evaluation of the infinite integral in the expression for R_{rs}. Since it is to be expected that $F(\lambda)$ as well as $\Psi_r(\lambda)$ are oscillatory with λ special care should be exercised to maintain accuracy if numerical techniques are applied. A cursory analysis, in which the effects of the boundary layer are ignored, shows that $F(\lambda)$ increases linearly with λ. If it is assumed that also with a thick boundary layer $F(\lambda)$ is proportional to λ, the kernel of the infinite integral in the expression for R_{rs} is at least proportional to $1/\lambda$, since $\Psi_r(\lambda)$ is at least proportional to $1/\lambda$.

In view of these circumstances it is proposed in the continuation of these research efforts to numerically evaluate the function R_{rs} for typical boundary layer profiles and panel design configurations to obtain a better understanding of the difficulties involved. Should the analysis prove promising, the effects of a thick boundary layer on the flutter characteristics of a two-dimensional panel configuration should be studied, followed by an extension of the present theories to the three-dimensional case.

IV. RECOMMENDATIONS AND FUTURE WORK

It is recommended to extend the theoretical developments presented in this report along the following lines:

1) Numerically evaluate the function R_{rs} for typical boundary layer profiles and panel configurations to obtain a better understanding of the general behavior of the function.

2) Numerically evaluate the pressure distribution on a finite stationary wavy wall and compare the results with available experimental data.

3) Conduct a flutter analysis using the two-dimensional aerodynamic theories developed and compare the results with available experimental data.

4) Extend the present theories to the three-dimensional case.

APPENDIX A

UNSTEADY POTENTIAL FLOW EQUATIONS

It is interesting to develop from the previous results the more familiar forms of the potential flow equations without boundary layer.

Unsteady Subsonic Flow ($\delta = 0$)

We have seen that for a traveling wavy wall boundary at y = 0 given by

$$w_0 = A_0 \, e^{i\lambda x} \, e^{i\omega t} \quad ; \quad -\infty < x < +\infty \tag{A-1}$$

the subsonic potential becomes [see (23) and (43)]

$$\phi_N = - \frac{i(\omega + \lambda U_N)}{\gamma_N \sigma_N} A_0 \, e^{-\gamma_N \sigma_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{A-2}$$

where σ_N and γ_N are given by (24) and (25), respectively.

On the other hand, when the oscillating wall boundary at y = 0 is given by [see (55)]

$$\begin{aligned} w_0(x,t) &= z_0(x) \, e^{i\omega t} \quad ; \quad 0 \leq x \leq a \\ &= 0 \quad ; \quad x < 0 \text{ or } x > a \end{aligned} \tag{A-3}$$

we find using (56) and (57) that

$$\phi_N = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i(\omega + \lambda U_N)}{\gamma_N \sigma_N} \, e^{-\gamma_N \sigma_N y} \, f(\lambda) \, e^{i\lambda x} \, d\lambda \, e^{i\omega t} \tag{A-4}$$

where $f(\lambda)$, given by (56), is the Fourier Transform of w_0.

Since

$$i\{\omega + \lambda U_N\}f(\lambda)\ e^{i\omega t} = \text{F.T.}\left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_0}{\partial x}\right) \qquad \text{(A-5)}$$

there follows from (A-4)

$$\text{F.T.}(\phi_N) = -\ \text{F.T.}\left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_0}{\partial x}\right)\ \text{F.T.}(Q) \qquad \text{(A-6)}$$

where

$$\text{F.T.}(Q) = \frac{e^{-\gamma_N \sigma_N y}}{\gamma_N \sigma_N} \qquad \text{(A-7)}$$

Using (24) we find that

$$\text{F.T.}\left\{e^{-i \frac{M_N \omega}{c_N \gamma_N^2} x}\ Q\right\} = \frac{e^{-\gamma_N \bar{\sigma}_N y}}{\gamma_N \bar{\sigma}_N} \qquad \text{(A-8)}$$

where

$$\bar{\sigma}_N = \left(\lambda^2 - \frac{\omega^2}{c_N^2 \gamma_N^4}\right)^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N \gamma_N^2} \quad \text{or} \quad \lambda < -\frac{\omega}{c_N \gamma_N^2}$$

$$= i\left(\frac{\omega^2}{c_N^2 \gamma_N^4} - \lambda^2\right)^{1/2} \quad ; \quad -\frac{\omega}{c_N \gamma_N^2} < \lambda < \frac{\omega}{c_N \gamma_N^2} \qquad \text{(A-9)}$$

Thus,

$$e^{-i\frac{M_N\omega}{c_N\gamma_N^2}x}\,Q = \frac{1}{\pi\gamma_N}\int_{\frac{\omega}{c_N\gamma_N^2}}^{\infty}\frac{e^{-\gamma_N\left(\lambda^2-\frac{\omega^2}{c_N^2\gamma_N^4}\right)^{1/2}y}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\gamma_N^4}\right)^{1/2}}\cos\lambda x\,d\lambda$$

$$-\frac{i}{\pi\gamma_N}\int_0^{\frac{\omega}{c_N\gamma_N^2}}\frac{\cos\left\{\gamma_N\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}\right\}}{\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}}\cos\lambda x\,d\lambda$$

$$-\frac{1}{\pi\gamma_N}\int_0^{\frac{\omega}{c_N\gamma_N^2}}\frac{\sin\left\{\gamma_N\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}\right\}}{\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}}\cos\lambda x\,d\lambda \tag{A-10}$$

Now [5 , p. 30]

$$\int_0^{\infty} g_1(x)\cos(xy)\,dx = \frac{\pi}{2}Y_0\left[a(b^2+y^2)^{1/2}\right]$$

where

$$g_1(x) = (a^2 - x^2)^{-1/2} \sin\left[b(a^2 - x^2)^{1/2}\right] \quad ; \quad 0 < x < a$$

$$= -(x^2 - a^2)^{-1/2} e^{-b(x^2 - a^2)^{1/2}} \quad ; \quad x > a$$

and

$$\int_0^\infty g_2(x) \cos(xy)\, dx = \frac{\pi}{2} J_0\left[a(b^2 + y^2)^{1/2}\right]$$

where

$$g_2(x) = (a^2 - x^2)^{-1/2} \cos\left[b(a^2 - x^2)^{1/2}\right] \quad ; \quad 0 < x < a$$

$$= 0 \quad ; \quad x > a$$

Thus,

$$Q = -\frac{1}{2\gamma_N} e^{i \frac{M\omega}{c_N \gamma_N^2} x} H_0^{(2)}\left[\frac{\omega}{c_N \gamma_N^2} (x^2 + \gamma_N^2 y^2)^{1/2}\right] \tag{A-11}$$

Taking the inverse transform of (A-5) and using (A-11), we find the well-known result [3]

$$\phi = \frac{i}{2\gamma_N} \int_0^a \left(\frac{\partial w_0(\xi,t)}{\partial t} + U_N \frac{\partial w_0(\xi,t)}{\partial \xi}\right) e^{+i \frac{M_N \omega}{c_N \gamma_N^2} (x - \xi)}$$
$$\times H_0^{(2)}\left\{\frac{\omega}{c_N \gamma_N^2} \left[(x - \xi)^2 + y^2\right]^{1/2}\right\} d\xi \tag{A-12}$$

If $\omega = 0$, (A-9) gives

$$\bar{\sigma}_N = |\lambda| \quad ; \quad -\infty < \lambda < +\infty$$

and

$$Q = \frac{1}{\pi\gamma_N} \int_0^\infty \frac{e^{-\gamma_N \lambda y}}{\gamma} \cos \lambda x \, d\lambda$$

$$= - \frac{1}{2\pi\gamma_N} \ell n \left[x^2 + \gamma_N^{\,2} y^2 \right] \qquad \text{(A-13)}$$

thus,

$$\phi_N = + \frac{1}{2\pi\gamma_N} \int_0^a U_N \frac{\partial w_0(\xi,t)}{\partial \xi} \ell n \left[(x - \xi)^2 + \gamma_N^{\,2} y^2 \right] d\xi \qquad \text{(A-14)}$$

{See [4 , p. 213] }.

Unsteady Supersonic Flow ($\delta = 0$)

The supersonic case can be obtained in a similar way using the supersonic potential corresponding to (A-1) { see (26) and (46) }

$$\phi_N = - \frac{(\omega + \lambda U_N)}{\beta_N \tau_N} A_0 \; e^{-i\beta_N \tau_N y} \; e^{i\lambda x} \; e^{i\omega t} \qquad \text{(A-15)}$$

The velocity potential corresponding to (A-3) becomes

$$\phi_N = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(\omega + \lambda U_N)}{\beta_N \tau_N} e^{-i\beta_N \tau_N y} f(\lambda) \, e^{i\lambda x} \, d\lambda \; e^{i\omega t} \qquad \text{(A-16)}$$

Using (A-5), there follows

$$\text{F.T.}(\phi_N) = \text{F.T.}\left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_N}{\partial t}\right) \text{F.T.}\{\bar{Q}\} \tag{A-17}$$

where

$$\text{F.T.}(\bar{Q}) = \frac{ie^{-i\beta_N \tau_N y}}{\beta_N \tau_N} \tag{A-18}$$

so that, using (27)

$$\text{F.T.}\left\{ e^{+i \frac{M_N \omega}{c_N \beta_N^2} x} \bar{Q} \right\} = \frac{ie^{-i\beta_N \tau_N y}}{\beta_N \bar{\tau}_N} \tag{A-19}$$

where

$$\bar{\tau}_N = \left(\lambda^2 - \frac{\omega^2}{c_N^2 \beta_N^4}\right)^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N \beta_N^2}$$

$$= - i\left(\frac{\omega^2}{c_N^2 \beta_N^4} - \lambda^2\right)^{1/2} \quad ; \quad - \frac{\omega}{c_N \beta_N^2} < \lambda < \frac{\omega}{c_N{}_N^{\,2}}$$

$$= - \left(\lambda^2 - \frac{\omega^2}{c_N^2 \beta_N^4}\right)^{1/2} \quad ; \quad \lambda < - \frac{\omega}{c_N \beta_N^2} \tag{A-20}$$

Consequently,

$$e^{+i\frac{M_N\omega}{c_N\beta_N^2}x}\,\bar{Q} = -\frac{1}{\pi\beta_N}\int_0^{\frac{\omega}{c_N\beta_N^2}} \frac{e^{-\beta_N\left(\frac{\omega^2}{c_N^2\beta_N^4}-\lambda^2\right)^{1/2}y}}{\left(\frac{\omega^2}{c_N^2\beta_N^4}-\lambda^2\right)^{1/2}}\cos\lambda x\, d\lambda$$

$$-\frac{1}{\pi\beta_N}\int_{\frac{\omega}{c_N\beta_N^2}}^{\infty} \frac{\cos\left\{\beta_N\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}\right\}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}}\sin\lambda x\, d\lambda$$

$$+\frac{1}{\pi\beta_N}\int_{\frac{\omega}{c_N\beta_N^2}}^{\infty} \frac{\sin\left\{\beta_N\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}\right\}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}}\cos\lambda x\, d\lambda \qquad \text{(A-21)}$$

Now, [5, p. 30]

$$\int_0^\infty g_3(x)\cos(xy)\,dx = \frac{\pi}{2}J_0\left[a(y^2-b^2)^{1/2}\right] \quad ; \quad y > b$$

$$= 0 \quad ; \quad y < b$$

where

$$g_3(x) = (a^2 - x^2)^{-1/2} \, e^{-b(a^2 - x^2)^{1/2}} \quad ; \quad 0 < x < a$$

$$= - (x^2 - a^2)^{-1/2} \sin \left[b(a^2 - x^2)^{1/2} \right] \quad ; \quad x > a$$

and [5, p. 142]

$$\int_0^\infty g_4(x) \sin (xy) \, dx = \frac{\pi}{2} J_0 \left[a(y^2 - b^2)^{1/2} \right] \quad ; \quad y > b$$

$$= 0 \quad ; \quad y < b$$

where

$$g_4(x) = 0 \quad ; \quad 0 < x < a$$

$$= (x^2 - a^2)^{-1/2} \cos \left[b(x^2 - a^2)^{1/2} \right] \quad ; \quad x > a$$

Thus,

$$Q = -\frac{1}{\beta_N} \, e^{i \frac{M_N \omega}{c_N \beta_N^2} x} \, J_0 \left\{ \frac{\omega}{c_N \beta_N^2} \left(x^2 - \beta_N^2 y^2 \right)^{1/2} \right\} ; \quad x > \beta_N y$$

$$= 0 \quad ; \quad x < \beta_N y$$

(A-22)

Taking the inverse transform of (A-17) yields {see [3] }

$$\phi_N = -\frac{1}{\beta_N} \int_0^{x-\beta_N y} \left\{ \frac{\partial w_0(\xi,t)}{\partial t} + U_N \frac{\partial w_0(\xi,t)}{\partial \xi} \right\} e^{-i \frac{M\omega}{c_N \beta_N^2}(x-\xi)}$$

$$\times \; J_0 \left\{ \frac{\omega}{c_N \beta_N^2} \left[(x-\xi)^2 - \beta_N^2 y^2 \right]^{1/2} \right\} d\xi \qquad \text{(A-23)}$$

If $\omega = 0$, we find from (A-23)

$$\phi_N = -\frac{1}{\beta_N} \int_0^{x-\beta_N y} U_N \frac{\partial w_0(\xi,t)}{\partial \xi} \, d\xi$$

$$= -\frac{U_N}{\beta_N} \left[w_0(x - \;{}_N y, t) - w_0(0,t) \right] \qquad \text{(A-24)}$$

APPENDIX B

LITERATURE SURVEY

The major purposes of this survey are to examine current literature pertaining to the unsteady flow associated with oscillating panels exposed to thick turbulent boundary layers with low supersonic Mach numbers and literature dealing with the obvious complicating mechanisms of unsteadiness, turbulence, compressibility, and transonic perturbations. Accordingly, the references are tabulated and briefly discussed in this order.

General Problem

The influence of the presence of the boundary layer on the flow past wavy walls is considered from the viewpoints of various authors in references [1 - 6]*. These papers are of prime importance in that they represent the various current approaches to approximation of the problem at hand. Miles [1] utilizes an inviscid, parallel shear flow model to investigate traveling waves; Benjamin [2] also considers traveling waves but with viscous effects included; and Mercer [3] considers standing waves with high frequency such as to omit consideration of a critical layer within the turbulent boundary layer. Fung [5] considers a somewhat crude model of the boundary layer represented as an inviscid subsonic uniform flow region beneath a uniform supersonic free stream, his primary objective being to include some effect of the boundary layer Mach number distribution. McClure [4,6] attempts to consider the general problem involving viscosity, compressibility, and turbulent boundary layer in a remarkable analysis. The complication of the theory, however, renders it difficult to utilize in a general investigation with a large number of parameters involved.

References [7] - [22] include work dealing with a number of related problems in unsteady boundary layer flows whereas references [23] - [32] pertain to turbulent boundary layers. References [33] - [39] include the effects of compressibility on the viscous flow and finally references [40] and [41] deal with the transonic flow past wave-shaped walls.

*Numbers in square brackets refer to the bibliography at the end of this Appendix.

Publications Examined

1. Miles, J. W., "On Panel Flutter in the Presence of a Boundary Layer," Journ. of Aerospace Sciences, vol. 26, p. 81, February 1959.

2. Benjamin, T. B., "Shearing Flow Over a Wavy Boundary," Journ. of Fluid Mech., vol. 6, p. 161, 1959.

3. Mercer, A. G., "Turbulent Boundary Layer Flow Over a Flat Plate Vibrating with Transverse Standing Waves," University of Minnesota, St. Anthony Falls Hydraulic Laboratory, Tech. Paper No. 41, Series B, September 1962.

4. McClure, J. D., "Stability of Finite Chord Panels Exposed to Low Supersonic Flows with a Turbulent Boundary Layer," IAS Paper No. 63-21, 1963.

5. Fung, Y. C., "Some Recent Contributions to Panel Flutter Research," AIAA Journal, vol. 1, no. 4, pp. 898-909, April 1963.

6. McClure, J. D., "Regular Disturbances in the Turbulent Boundary Layer," Boeing Scientific Research Laboratories, Flight Sciences Laboratory Report No. 92. January 1965.

7. Stewartson, K., "On the Impulsive Motion of a Flat Plate in a Viscous Fluid," Quart. Journ. of Mech. and Appl. Math., vol. 4, p. 182, 1951.

8. Moore, F. K., "Unsteady Laminar Boundary Layer Flow," NACA TN 2471, September 1951.

9. Ostrach, S., "Note on the Aerodynamic Heating of an Oscillating Surface," NACA TN 3146, April 1954.

10. Lighthill, M. J., "The Response of Laminar Skin Friction and Heat Transfer to Fluctuations in the Stream Velocity," Proc. of the Roy. Soc., Series A., vol. 224, p. 1, July 1954.

11. Cheng, Sim. I. and Elliott, D., "The Unsteady Laminar Boundary Layer on a Flat Plate," Princeton Report No. 318 (AFOSR TN 56-115), September 1955.

12. Ostrach, S., "Compressible Laminar Boundary Layer and Heat Transfer for Unsteady Motions of a Flat Plate," NACA TN 3569, November 1955.

13. Glauert, M. B., "The Laminar Boundary Layer on Oscillating Plates and Cylinders," J. Fluid Mech., vol. 1, pp. 97-110, 1956.

14. Wu, T. Yao-Tsu, "Small Perturbations in the Unsteady Flow of a Compressible Viscous and Heat Conducting Fluid," Journ. of Math. and Physics, vol. 35, 1956.

15. Carrier, G. F. and DiPrima, R. C., "On the Unsteady Motion of a Viscous Fluid Past a Semi-Infinite Flat Plate," Journ. of Math. and Physics, vol. 35, 1956.

16. Ostrach, S. and Moore, F. K., "Average Properties of Compressible Laminar Boundary Layer on Flat Plate with Unsteady Flight Velocity," NACA TN 3886, December 1956.

17. Moore, F. K. and Ostrach, S., "Displacement Thickness of the Unsteady Boundary Layer," Readers Forum, J.A.S., vol. 24, p. 77, January 1957.

18. Cheng, Sim. I., "Some Aspects of Unsteady Laminar Boundary Layer Flows," Quarterly of Applied Mathematics, vol. XIV, no. 4, January 1957.

19. Moore, F. K., "Aerodynamic Effects of Boundary Layer Unsteadiness," Sixth Anglo-American Aeronautical Conference, Folkestone, September 1957, Royal Aero Society 1959.

20. Segel, L. A., "A Uniformly Valid Asymptotic Expansion of the Solution to an Unsteady Boundary Layer Problem," Journal of Math. and Physics, vol. 39, pp. 189-197, 1960.

21. Jones, W. P., "Trends in Unsteady Aerodynamics," Sixth Lanchester Memorial Lecture, Journal. Roy. Aero. Soc., vol. 67, no. 627, p. 137, March 1963.

22. Rosenhead, L., (Edt.), "Fluid Motion Memoirs, Laminar Boundary Layers," Oxford, 1963. Chapters VI, VII.

23. Van Driest, E. R., "Turbulent Boundary Layer in Compressible Fluids," J.A.S., vol. 18, no. 3, p. 145, March 1951.

24. Goldschmied, F. R., "Skin Friction of Incompressible Turbulent Boundary Layers Under Adverse Pressure Gradients," NACA TN 2431, August 1951.

25. Klebanoff, P. S. and Diehl, Z. W., "Some Features of Artificially Thickened Fully Developed Turbulent Boundary Layers with Zero Pressure Gradient," NACA TN 2475, October 1951.

26. Turckenbrodt, E., "A Method of Quadrature for Calculation of the Laminar and Turbulent Boundary Layer in Case of Plane and Rotationally Symmetrical Flow," NACA TM 1379, May 1955.

27. Townsend, A. A., "The Properties of Equilibrium Boundary Layers," Journ. Fluid Mech., vol. 1, pp. 564-573, 1956.

28. Steiger, M. H. and Bloom, M. H., "Thick Boundary Layers Over Slender Bodies with Some Effects of Heat Transfer, Mass Transfer, and Pressure Gradient, ARL Tech. Report #6, December 1960.

29. Shore, P. S., Dixon, S. C., and Griffith, G. E., "Experimental Pressures and Turbulent Heat-Transfer Coefficients Associated with Sinusoidal Proturberances on a Flat Plate at a Mach Number of 3," NASA TN D-1626, March 1963.

30. Moore, D. R. and Harkness, J., "Experimental Investigations of the Compressible Turbulent Boundary Layer at Very High Reynolds Numbers," AIAA Journal, vol. 3, no. 4, April 1965.

31. Burton, R. A., "A Simple Universal Velocity Profile Equation," AIAA Journal, vol. 3, no. 4, p. 784, April 1965.

32. Thompson, B. G. J., "Calculations of the Turbulent Boundary Layer," AIAA Journal, vol. 3, no. 4, p. 746, April 1965.

33. Jarre, Gianni, "Inductive Theory of Shear Flows," International Council--Aeronautical Sciences, p. 787.

34. Cope, W. F. and Hartree, D. R., "The Laminar Boundary Layer in Compressible Flow," Phil. Trans. of Roy. Soc. of London, Math. and Phys. Sciences, Series A, vol. 241, 1948-49.

35. Chapman, D. R. and Rubesin, M. W., "Temperature and Velocity Profiles in the Compressible Laminar Boundary Layer with Arbitrary Distribution of Surface Temperature," J.A.S., vol. 16, no. 9, p. 547, September 1949.

36. Lagerstrom, P. A., Cole, J. D., and Trilling, L., "Problems in the Theory of Viscous Compressible Fluids," ONR-GALCIT Report, Cal. Tech., October 1950.

37. Tani, I., "On the Approximate Solution of the Laminar Boundary Layer Equations," J.A.S., vol. 21, no. 7, p. 487, July 1954.

38. Kaplun, S., "The Role of Coordinate Systems in Boundary Layer Theory," Zeitschrift für augewandte Mathematik und Physik, vol. V, p. 111, 1954.

39. Howarth, L., "Laminar Boundary Layers," Encyclopedia of Physics, vol. VIII/1, Fluid Dynamics I; Edt. S. Flugge, Springer-Verlag, Berlin, 1959.

40. Kaplan, Carl, "On a Solution of the Nonlinear Differential Equation for Transonic Flow Past a Wave-Shaped Wall," NACA Report 1069, 1952.

41 Hosakawa, I., "Transonic Flow Past a Wavy Wall," Journal of the Physical Society of Japan, vol. 15, no. 11, pp. 2080-2086, November 1960.

APPENDIX C

MEAN FLOW EQUATIONS WITH SMALL PERTURBATIONS

The pressure distribution on a sinusoidally oscillating plate exposed to a thick turbulent boundary layer must be considered from the standpoint of the governing differential equations of continuity, momentum, and energy, together with an equation of state. This complex problem will first be considered on the form of a two-dimensional flow with emphasis on the resulting form of the continuity and momentum equations for small perturbations. The applicable governing equations are then:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{C-1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{C-2}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{C-3}$$

where the definitions employed are:

$$\left.\begin{aligned}
p &= -\frac{1}{3}(p_x + p_y + p_z) \\
p + p_x &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} \\
p + p_y &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} \\
p + p_z &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{yx} = \tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \tag{C-4}$$

By using the continuity equation (C-1), the momentum equations (C-2) and (C-3) may be conveniently rewritten as

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{C-5}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{C-6}$$

The turbulent flow may then be represented as composed of "mean" and fluctuating components in all dependent variables where it is understood that the fluctuating components are random in nature and not to be confused with the regular fluctuations due to wall oscillations which are by definition periodic and part of the "mean" flow. The fluctuations due to turbulence are then defined by primed quantities as follows:

$$\left.\begin{aligned}
&u = \bar{u} + u' \;,\ldots \\
&\rho u = \overline{\rho u} + (\rho u)' \;,\ldots \\
&\rho = \bar{\rho} + \rho' \\
&p_i = \overline{p_i} + p_i' \quad ; \quad i = x,y,z \\
&\tau_{ij} = \overline{\tau_{ij}} + \tau_{ij}' \quad ; \quad i = x,y,z \quad ; \quad i \neq j
\end{aligned}\right\} \tag{C-7}$$

These definitions are then utilized in equations (C-1), (C-5), and (C-6), and the resulting equations time averaged over a period large in comparison to the time of random fluctuations but small in comparison to the time of "mean" variations such as the period of regular disturbances due to wall oscillations. The resulting "mean" flow equations are as given by Van Driest 23, Appendix B .

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\,(\overline{\rho u}) + \frac{\partial}{\partial y}\,(\overline{\rho v}) = 0 \tag{C-8}$$

$$\bar{\rho}\,\frac{\partial \bar{u}}{\partial t} + \overline{\rho u}\,\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\,\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial t}\,(-\,\overline{\rho' u'}) + \frac{\partial}{\partial x}\left[\bar{p}_x - \overline{(\rho u)' u'}\right]$$

$$+ \frac{\partial}{\partial y}\left[\bar{\tau}_{yx} - \overline{(\rho v)' u'}\right] \tag{C-9}$$

$$\bar{\rho}\,\frac{\partial \bar{v}}{\partial t} + \overline{\rho u}\,\frac{\partial \bar{v}}{\partial x} + \overline{\rho v}\,\frac{\partial \bar{v}}{\partial y} = \frac{\partial}{\partial t}\,(-\,\overline{\rho' v'}) + \frac{\partial}{\partial y}\left[\bar{p}_y - \overline{(\rho v)' v'}\right]$$

$$+ \frac{\partial}{\partial x}\left[\bar{\tau}_{xy} - \overline{(\rho u)' v'}\right] \tag{C-10}$$

where the barred quantities represent the "mean" values as given by the time average. If in addition to the shear stress, equation (C-4), the normal stresses are conventionally defined, one obtains:

$$\left.\begin{aligned}
\tau_{xx} &+ 2\mu\,\frac{\partial u}{\partial x} - \frac{2}{3}\,\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{yy} &= 2\mu\,\frac{\partial v}{\partial y} - \frac{2}{3}\,\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \tag{C-11}$$

These definitions allow equations (C-9) and (C-10) to be written in terms of the mean pressure, $\bar{p}$, by use of

$$\left.\begin{aligned}\bar{p}_x &= -\bar{p} + \overline{\tau_{xx}} \\ \bar{p}_y &= -\bar{p} + \overline{\tau_{yy}}\end{aligned}\right\} \qquad \text{(C-12)}$$

It is then noted that the "mean" values of the products of fluctuating quantities enter equations (C-9) and (C-10) as two alternations to the analogous laminar flow equations. The first of these is the appearance of the apparent stress terms $\overline{(\rho u)'u'}$, $\overline{(\rho v)'u'}$, $\overline{(\rho v)'v'}$, and $\overline{(\rho u)'v'}$; the second appears as the unsteadiness of the "mean" products $\rho'u'$ and $\rho'v'$.

This first alternation is handled in a more compact form by defining the "total" stresses as:

$$\left.\begin{aligned}\overline{\tau_{xx}} - \overline{(\rho u)'u'} &\equiv \overline{\tau_{xx}}^{t} \\ \overline{\tau_{yy}} - \overline{(\rho v)'v'} &\equiv \overline{\tau_{yy}}^{t} \\ \overline{\tau_{yx}} - \overline{(\rho v)'u'} &\equiv \overline{\tau_{yx}}^{t} \\ \overline{\tau_{xy}} - \overline{(\rho u)'v'} &\equiv \overline{\tau_{xy}}^{t} \neq \overline{\tau_{yx}}^{t}\end{aligned}\right\} \qquad \text{(C-13)}$$

The second alternation is, in effect, hidden by use of the identity

$$\frac{\partial}{\partial t}(\overline{\rho u_i}) = \frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i) + \frac{\partial}{\partial t}(\overline{\rho' u_i'}) \qquad \text{(C-14)}$$

and the continuity equation (C-8) multiplied by the appropriate velocity component of the mean flow $\bar{u}$ or $\bar{v}$. With these improvements the system of equations governing the flow is:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\,(\overline{\rho u}) + \frac{\partial}{\partial y}\,(\overline{\rho v}) = 0 \qquad \text{(C-15)}$$

$$\frac{\partial}{\partial t}\,(\overline{\rho u}) + \frac{\partial}{\partial x}\,(\overline{\rho u}\;\bar{u}) + \frac{\partial}{\partial y}\,(\overline{\rho v}\;\bar{u}) + \frac{\partial \bar{p}}{\partial x} = \frac{\partial}{\partial x}\,\overline{\tau_{xx}}^{\,t} + \frac{\partial}{\partial y}\,\overline{\tau_{yx}}^{\,t}$$

(C-16)

$$\frac{\partial}{\partial t}\,(\overline{\rho v}) + \frac{\partial}{\partial x}\,(\overline{\rho u}\;\bar{v}) + \frac{\partial}{\partial y}\,(\overline{\rho v}\;\bar{v}) + \frac{\partial \bar{p}}{\partial y} = \frac{\partial}{\partial y}\,\overline{\tau_{yy}}^{\,t} + \frac{\partial}{\partial x}\,\overline{\tau_{xy}}^{\,t}$$

(C-17)

These equations, as expressed, are in their least objectional form from the standpoint of the turbulent nature of the flow, in that the "mean" of products of the fluctuating quantities do not explicitly appear. Equations (C-16) and (C-17) are simply the Navier-Stokes equations for the "mean" flow written in the form of equations (C-5) and (C-6); as such they are not amenable to exact solution and recourse is made to approximate methods.

Small Perturbations

An obvious simplification to the preceeding general formulation may be accompolished by incorporating two reasonable assumptions:

1) The disturbances due to harmonic wall oscillations are assumed to be small. In effect, the flow is described as the sum of a "mean" steady flow plus a "mean" regular fluctuation which permits all flow variables to be defined as:

$$\underset{\text{"mean" flow}}{\overline{(\;)}} = \underset{\text{steady "mean" flow}}{\overline{(\;)}} + \underset{\text{unsteady regular fluctuation}}{\tilde{(\;)}}$$

2) The variations of the steady "mean" flow components (the flow present in absence of wall oscillations) in the panel chordwise direction are neglected.

These assumptions are consistent with the approach used in inviscid flow over wavy walls of infinite extent wherein mathematical simplification is desirable and permissible. Further simplifications in the inviscid theories, by order of magnitude analysis, lead to the linearized theories for subsonic and supersonic flow and the nonlinear theory for transonic flow.

A detailed discussion of these assumptions and their consequences is given in the following development.

Steady Mean Turbulent Flow

The neglect of chordwise variations of flow variables in the "mean" steady flow component requires that

$$\frac{\partial}{\partial x}(\ ) = 0 \qquad \text{(C-18)}$$

In addition,

$$\frac{\partial}{\partial t}(\ ) = 0$$

by definition and the governing equations (C-15) - (C-17) then reduce to the simple equations that follow:

$$\text{continuity} \qquad \frac{\partial}{\partial y}\left(\overline{\rho v}\right) = 0 \qquad \text{(C-19)}$$

$$\text{x-momentum} \qquad \frac{\partial}{\partial y}\left(\overline{\rho v}\ \bar{u}\right) = \frac{\partial}{\partial y}\ \bar{\tau}^{t}_{yx} \qquad \text{(C-20)}$$

$$\text{y-momentum} \qquad \frac{\partial}{\partial y}\left(\overline{\rho v}\ \bar{v}\right) + \frac{\partial \bar{p}}{\partial y} = \frac{\partial}{\partial y}\ \bar{\tau}^{t}_{yy} \qquad \text{(C-21)}$$

A detailed examination of these equations yields some insight into the structure of the steady "mean" flow component, which is desirable, inasmuch as deviations from this state must be due to the wall oscillation. Consideration of each of these equations follows:

Continuity, Steady Mean Flow. As a result of equation (C-19) and the boundary condition

$$\overline{\rho v}\Big|_{wall} = 0$$

one obtains

$$\overline{\rho v} = 0 \qquad \text{(C-22)}$$

Further, by definition

$$\overline{\rho v} = \bar{\rho}\bar{v} + \overline{\rho' v'}$$

so that

$$\bar{\rho}\bar{v} = - \overline{\rho' v'} \qquad \text{(C-23)}$$

x-Momentum, Steady Mean Flow. By definition

$$\bar{\tau}_{yx}^{t} \equiv \bar{\tau}_{yx} - \overline{(\rho v)' u'}$$

$$\equiv \mu \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) - \overline{(\rho v)' u'}$$

so by virtue of equation (C-19) or (C-22) and the boundary condition

$$\overline{(\rho v)' u'} \Big|_{wall} = 0$$

equation (C-20) yields for the assumed steady "mean" flow:

$$\mu \frac{\partial \bar{u}}{\partial y} = \mu \frac{\partial \bar{u}}{\partial y} \Bigg|_{wall} + \overline{(\rho v)' u'} \qquad \text{(C-24)}$$

y-Momentum, Steady Mean Flow. By definition

$$\bar{\tau}_{yy}^{t} = \bar{\tau}_{yy} - \overline{(\rho v)' v'}$$

$$= 2\mu \frac{\partial \bar{v}}{\partial y} - \frac{2}{3} \mu \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) - \overline{(\rho v)' v'}$$

so by virtue of equation (C-19) or (C-22) and the boundary condition

$$\overline{(\rho v)'v'}\Big|_{wall} = 0$$

equation (C-21) yields for the assumed steady "mean" flow:

$$\bar{p} = \bar{p}_{wall} + \frac{4}{3}\mu \left. \frac{\partial\bar{v}}{\partial y} - \frac{\partial\bar{v}}{\partial y}\right|_{wall} - \overline{(\rho v)'v'} \tag{C-25}$$

The resulting steady "mean" flow is then the compressible counterpart of a steady, parallel, incompressible shear flow. While a solution for the "mean" steady flow is not sought, it is both interesting and necessary to formulate the consequences of the neglect of chordwise variations of flow as given by equations (C-22), (C-23), (C-24), and (C-25). In particular, it is evident that a pressure variation through the thick turbulent boundary layer is present and contains, according to equation (C-25), contributions due the turbulent character of the flow and a combination viscous compressibility term. It is intuitively assumed that this pressure variation is small; in any case, it is the deviation from this state, due to the wall oscillations, in the presence of the boundary layer that is sought.

Small Perturbation Equations

The results of assuming small perturbations to the steady "mean" turbulent flow due to regular wall oscillations are found by rewriting the governing equations (C-15) - (C-17) replacing the total "mean" flow variables with their corresponding sum of steady and regular fluctuations. The deviations from the steady "mean" flow may then be examined in the form of governing equations for the perturbations inasmuch as the steady "mean" flow satisfies equations (C-19) - (C-21).

Continuity, Perturbed Mean Flow. Inasmuch as no products of variables appear in the continuity equation (C-15), the corresponding perturbation equation is given by

$$\frac{\partial\tilde{\rho}}{\partial t} + \frac{\partial}{\partial x}(\widetilde{\rho u}) + \frac{\partial}{\partial y}(\widetilde{\rho v}) = 0 \tag{C-26}$$

x-Momentum, Perturbed Mean Flow. Equation (C-16) becomes upon substitution

$$\frac{\partial}{\partial t}\left(\overline{\rho u}\right) + \frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho u}}\right) + \frac{\partial}{\partial x}\left[\left(\overline{\rho u} + \widetilde{\widetilde{\rho u}}\right)\left(\bar{u} + \tilde{u}\right)\right] + \frac{\partial}{\partial y}\left[\left(\overline{\rho v} + \widetilde{\widetilde{\rho v}}\right)\left(\bar{u} + \tilde{u}\right)\right]$$

$$+ \frac{\partial \bar{p}}{\partial x} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\left(\bar{\tau}_{xx}^{\,t} + \tilde{\tau}_{xx}^{\,t}\right) + \frac{\partial}{\partial y}\left(\bar{\tau}_{yx}^{\,t} + \tilde{\tau}_{yx}^{\,t}\right)$$

where the symbol $(\bar{\ })$ now refers only to the steady component of the "mean" flow. Neglecting products of small terms and utilizing equations (C-19) and (C-20) this equation reduces to the corresponding x-momentum perturbation equation:

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho u}}\right) + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \bar{u}\,\frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{u}}{\partial y} + \bar{u}\,\frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\,\tilde{\tau}_{xx}^{\,t} + \frac{\partial}{\partial y}\,\tilde{\tau}_{yx}^{\,t}$$

(C-27)

y-Momentum, Perturbed Mean Flow. Similarly equation (C-17) is expanded by substitution to obtain:

$$\frac{\partial}{\partial t}\left(\overline{\rho v}\right) + \frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) + \frac{\partial}{\partial x}\left[\left(\overline{\rho u} + \widetilde{\widetilde{\rho u}}\right)\left(\bar{v} + \tilde{v}\right)\right] + \frac{\partial}{\partial y}\left[\left(\overline{\rho v} + \widetilde{\widetilde{\rho v}}\right)\left(\bar{v} + \tilde{v}\right)\right]$$

$$+ \frac{\partial \bar{p}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\left(\bar{\tau}_{yy}^{\,t} + \tilde{\tau}_{yy}^{\,t}\right) + \frac{\partial}{\partial x}\left(\bar{\tau}_{xy}^{\,t} + \tilde{\tau}_{xy}^{\,t}\right)$$

and reduced in like manner to the y-momentum equation

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tau}_{yy}^{\,t} + \frac{\partial}{\partial x}\,\tilde{\tau}_{xy}^{\,t}$$

(C-28)

<u>Perturbation Equations, Alternate Forms</u>. Collecting the perturbation equations, one has

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x} (\widetilde{\rho u}) + \frac{\partial}{\partial y} (\widetilde{\rho v}) = 0 \tag{C-26}$$

$$\frac{\partial}{\partial t} (\widetilde{\rho u}) + \overline{\rho u} \frac{\partial \tilde{u}}{\partial x} + \bar{u} \frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho v} \frac{\partial \bar{u}}{\partial y} + \bar{u} \frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x} \tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y} \tilde{\tau}_{yx}{}^{t} \tag{C-27}$$

$$\frac{\partial}{\partial t} (\widetilde{\rho v}) + \overline{\rho u} \frac{\partial \tilde{v}}{\partial x} + \bar{v} \frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho u} \frac{\partial \bar{v}}{\partial y} + \bar{v} \frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y} \tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x} \tilde{\tau}_{xy}{}^{t} \tag{C-28}$$

Alternately, using equation (C-26), equations (C-27) and (C-28) may be rewritten as

$$\frac{\partial}{\partial t} (\widetilde{\rho u}) - \bar{u} \frac{\partial \tilde{\rho}}{\partial t} + \overline{\rho u} \frac{\partial \tilde{u}}{\partial x} + \widetilde{\rho v} \frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x} \tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y} \tilde{\tau}_{yx}{}^{t} \tag{C-27a}$$

$$\frac{\partial}{\partial t} (\widetilde{\rho v}) - \bar{v} \frac{\partial \tilde{\rho}}{\partial t} + \overline{\rho u} \frac{\partial \tilde{v}}{\partial x} + \widetilde{\rho v} \frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y} \tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x} \tilde{\tau}_{xy}{}^{t} \tag{C-28a}$$

Expanding the unsteady terms in equations (C-27a) and (C-28a) shows that

$$\frac{\partial}{\partial t} (\widetilde{\rho u}) - \bar{u} \frac{\partial \tilde{\rho}}{\partial t} = \bar{\rho} \frac{\partial \tilde{u}}{\partial t}$$

and

$$\frac{\partial}{\partial t}(\widetilde{\rho v}) - \bar{v}\frac{\partial \tilde{\rho}}{\partial t} = \bar{\rho}\frac{\partial \tilde{v}}{\partial t}$$

and the momentum equations (C-27) and (C-28) are most conveniently written in the form

$$\bar{\rho}\frac{\partial \tilde{u}}{\partial t} + \overline{\rho u}\frac{\partial \tilde{u}}{\partial x} + \widetilde{\rho v}\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y}\tilde{\tau}_{yx}{}^{t} \tag{C-27b}$$

$$\bar{\rho}\frac{\partial \tilde{v}}{\partial t} + \overline{\rho u}\frac{\partial \tilde{v}}{\partial x} + \widetilde{\rho v}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x}\tilde{\tau}_{xy}{}^{t} \tag{C-28b}$$

where, neglecting products of small terms

$$\widetilde{\rho v} = \bar{\rho}\tilde{v} + \tilde{\rho}\bar{v} \tag{C-29}$$

Utilizing equation (C-29) and a similar result for $\widetilde{\rho u}$, the convective derivatives in equation (C-26) may be expanded. The expansion of the latter yields

$$\frac{\partial}{\partial x}(\widetilde{\rho u}) = \bar{\rho}\frac{\partial \tilde{u}}{\partial x} + \tilde{u}\frac{\partial \bar{\rho}}{\partial x} + \tilde{\rho}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \tilde{\rho}}{\partial x}$$

but the second and third terms on the right are zero by definition of the mean flow so that

$$\frac{\partial}{\partial x}(\widetilde{\rho u}) = \bar{\rho}\frac{\partial \tilde{u}}{\partial x} + \bar{u}\frac{\partial \tilde{\rho}}{\partial x} \tag{C-30}$$

and likewise

$$\frac{\partial}{\partial y}(\widetilde{\rho v}) = \bar{\rho}\frac{\partial \tilde{v}}{\partial y} + \tilde{v}\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\frac{\partial \bar{v}}{\partial y} + \bar{v}\frac{\partial \tilde{\rho}}{\partial y} \tag{C-31}$$

The continuity equation (C-26) may then be written after substitution of equations (C-30) and (C-31) as

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\rho}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\rho}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + \tilde{v}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\,\frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(C-32)}$$

Perturbation Equations, Final Form. Finally, collecting results, equations (C-32), (C-27b), and (C-28b) provide the governing equations of momentum and continuity in their most tractable form:

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\rho}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\rho}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + \tilde{v}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\,\frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(C-33a)}$$

$$\bar{\rho}\,\frac{\partial \tilde{u}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \tilde{\rho}\tilde{v}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\,\tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tau}_{yx}{}^{t} \qquad \text{(C-33b)}$$

$$\bar{\rho}\,\frac{\partial \tilde{v}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \tilde{\rho}\tilde{v}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tau}_{xy}{}^{t} \qquad \text{(C-33c)}$$

with

$$\tilde{\rho}\tilde{v} = \bar{\rho}\tilde{v} + \tilde{\rho}\bar{v} \qquad \text{(C-33d)}$$

and

$$\left.\begin{aligned}
\tilde{\tau}_{xx}{}^{t} &= 2\mu\,\frac{\partial \tilde{u}}{\partial x} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) \\
\tilde{\tau}_{yx}{}^{t} &= \tilde{\tau}_{xy}{}^{t} = \mu\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right) \\
\tilde{\tau}_{yy}{}^{t} &= 2\mu\,\frac{\partial \tilde{v}}{\partial y} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)
\end{aligned}\right\} \qquad \text{(C-33e)}$$

In formulating these perturbation equations, it has been assumed, as evidenced by equations (C-33e), that the perturbations do not influence the character of the turbulence, that is:

$$\widetilde{(\rho u)'u'} = 0$$

$$\widetilde{(\rho v)'u'} = 0$$

$$\widetilde{(\rho v)'v'} = 0$$

$$\widetilde{(\rho u)'v'} = 0$$

This assumption effectively eliminates the indeterminate situation due to the presence of turbulence and its corresponding "apparent" stress terms and allows one to consider the present problem as a pseudo-laminar flow. Examination of the governing equations, on the other hand, reveals that the mathematical situation consists of three equations in which the "mean" steady flow properties

$$\bar{\rho},\ \bar{u},\ \bar{v},\ \overline{\rho u},\ \mu$$

are regarded as known or prescribed coefficients and the perturbation quantities

$$\tilde{\rho},\ \tilde{u},\ \tilde{v},\ \tilde{p}$$

are regarded as the dependent variables. Obviously, except in the case of incompressible perturbations, this situation necessitates consideration of the energy equation and an equation of state.

REFERENCES

(1) McClure, J. D., "Stability of Finite Chord Panels Exposed to Low Supersonic Flows with a Turbulent Boundary Layer," I.A.S. Paper No. 63-21, New York Meeting, January 1963.

(2) Fung, Y. C., "Some Recent Contributions to Panel Flutter Research," AIAA Journal, vol. 1, no. 4, pp. 898-909, April 1963.

(3) Garrick, I. E., "High Speed Aerodynamics and Jet Propulsion," vol. VII, Section F, Princeton University Press, Princeton, New Jersey, 1957.

(4) Bisplinghoff, R. L., Ashley, H., Halfman, R. L., Aeroelasticity, Chapter 5, Addison-Wesley Publishing Company, Cambridge, 1955.

(5) Von Oberhettinger, F., Tabellen zur Fourier Transformation , Springer-Verlag, 1957.

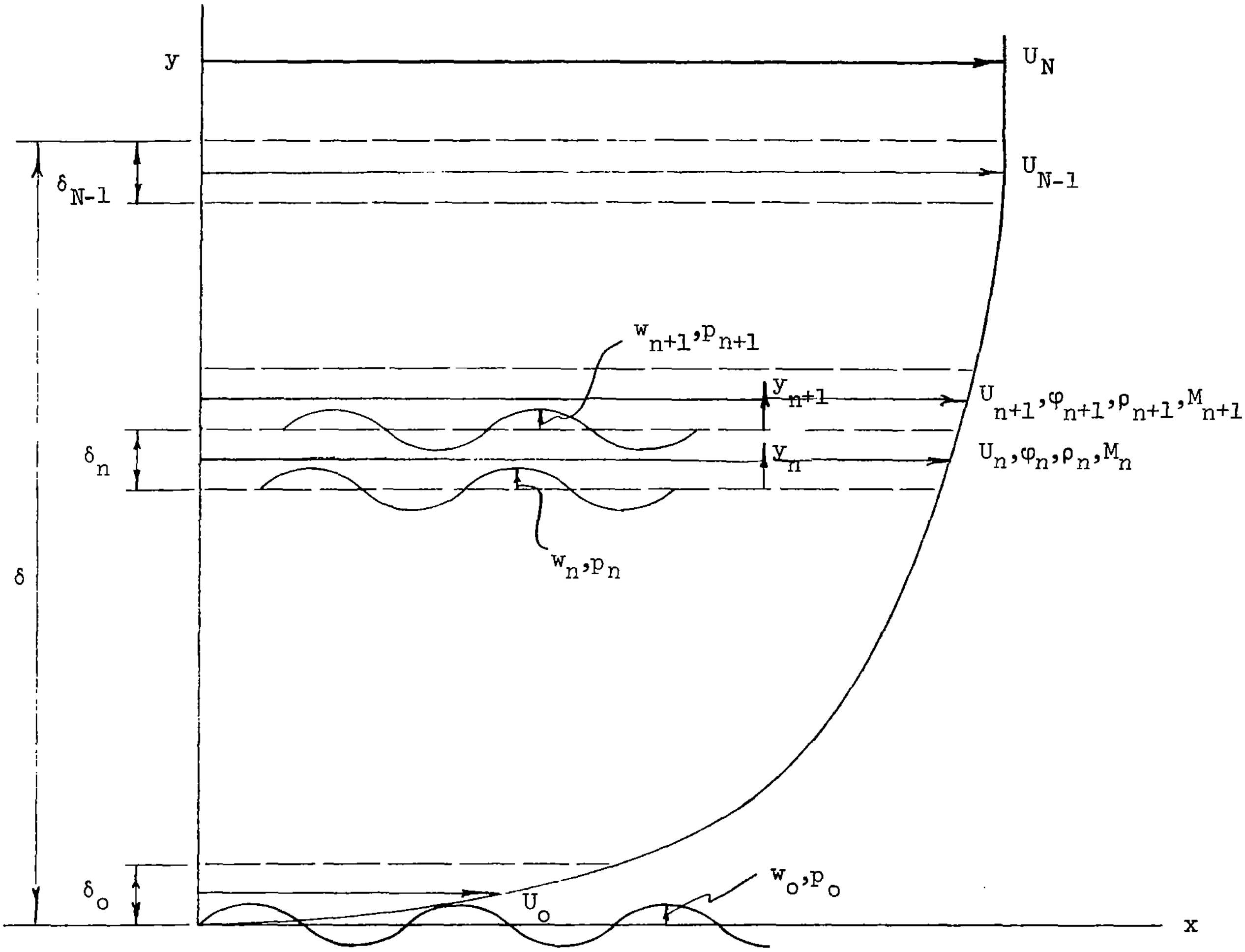

Fig. 1. Idealized Boundary Layer with Notations.

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