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Formulas Through the Eighth Order

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H. A. Luther and J. A. Smitherman†

The Runge-Kutta expressions considered are to be both the explicit and the implicit. The notation chosen is as follows:

$$y_{n+1} = y_n + \sum_{i=1}^{\nu} R_i k_i ,$$

$$(1) \quad k_i = h f(x_n + a_i h, y_n + \sum_{j=1}^{\nu} b_{ij} k_j)$$

$$a_i = \sum_{j=1}^{\nu} b_{ij} .$$

The differential system is of course

$$(2) \quad \frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 .$$

Here y is a column matrix of length m , as is $f(x,y)$, so that we are considering m simultaneous first order ordinary equations in the variable x . The constant ν may be any integer as large, or larger, than m .

The requirement $\nu \geq m$ should be considered as pertaining only to formal presentation. If (1) is to properly approximate an explicit solution of (2) in the Runge-Kutta sense, for $m > 4$ we must have $\nu > m$. A more precise description has been given by Butcher [2].

The following tables are based on the exposition of Butcher [1]; the elementary weights such as $[\phi]$, $[[\phi]^2]$, etc., found in the tables, are those defined by him.

Table One can be used to write down the equations which the k_i , R_i , and b_{ij} must satisfy. Each elementary weight is to be equated to the proper constant as described in [1]. For a specific choice of R_i , k_i , and b_{ij} in (1), and for a remainder of order $p+1$ (a Runge-Kutta formula of order p) all relations listed through the first p orders must be

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satisfied. Thus to get a third order R-K formula we must satisfy the restrictions implied by the first-order relation, the second-order relation, and the two third-order relations, namely

$$1 = \sum_{i_1=1}^{\nu} R_{i_1}; \quad 1/2 = \sum_{i_1=1}^{\nu} R_{i_1} a_{i_1}; \quad 1/6 = \sum_{i_1=1}^{\nu} \sum_{i_2=1}^{\nu} R_{i_1} b_{i_1 i_2} a_{i_2};$$

$$1/3 = \sum_{i_1=1}^{\nu} R_{i_1} a_{i_1}^2.$$

The summation convention is of course employed in the table, and all indices extend from 1 to ν . It is observed that these equations serve to define an implicit formulation, in general. That is, in applying (1), the k_i are defined implicitly.

If we require $b_{ij} = 0$ for $j \geq i$, then (1) defines each k_i explicitly. Table Two is based on this assumption. For this table, ν has been arbitrarily taken as 9. As explained in [2], this choice gives more parameters than needed for orders one through six. It gives a minimum number for order seven, and is not sufficient for order eight. However, the pattern seems clear, and there is no difficulty in increasing (or decreasing) the value of ν for the orders shown. Certain abbreviations for Table Two are found at its beginning. The equations listed are for orders one through seven only.

REFERENCES

- [1]. Butcher, J. C., Coefficients for the Study of Runge-Kutta Integration Processes, *J. Austral. Math. Soc.* 3(1963), 185-201.
- [2]. _____, On the Attainable Order of Runge-Kutta Methods, *Math. Comp.* 19(1965), 408-417.

TABLE ONE

The Elementary Weights for the Runge-Kutta Formulas
of the First Eight Orders

FIRST ORDER

1.
$$\phi = \sum_{i_1=1}^v R_{i_1}$$

SECOND ORDER

1.
$$[\phi] = R_{i_1} a_{i_1}$$

THIRD ORDER

1.
$$[_2\phi]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}$$

2.
$$[\phi^2] = R_{i_1} a_{i_1}^2$$

FOURTH ORDER

1.
$$[_3\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}$$

2.
$$[_2\phi^2]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^2$$

3.
$$[[\phi]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}$$

4.
$$[\phi^3] = R_{i_1} a_{i_1}^3$$

FIFTH ORDER

1. $[_4\phi]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_4}$

2. $[_3\phi^2]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}^2$

3. $[_2[\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2} a_{i_3}$

4. $[_2\phi^3]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^3$

5. $[[_2\phi]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_3}$

6. $[[\phi^2]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}^2$

7. $[[\phi]^2] = R_{i_1} (b_{i_1 i_2} a_{i_2})^2$

8. $[[\phi]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}^2 a_{i_2}$

9. $[\phi^4] = R_{i_1} a_{i_1}^4$

SIXTH ORDER

1. $[\phi]_5 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_5}$
2. $[\phi^2]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_4}^2$
3. $[\phi\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3} a_{i_4}$
4. $[\phi^3]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}^3$
5. $[\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_4}$
6. $[\phi^2\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2} a_{i_3}^2$
7. $[\phi^2]_2 = R_{i_1} b_{i_1 i_2} (b_{i_2 i_3} a_{i_3})^2$
8. $[\phi\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^2 a_{i_3}$
9. $[\phi^4]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^4$
10. $[\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_4}$

11. $[[\phi^2]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_3}^2$
12. $[[[\phi]\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2} a_{i_3}$
13. $[[\phi^3]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}^3$
14. $[[\phi^2]_2[\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3} a_{i_4}$
15. $[[\phi^2][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_2}^3 a_{i_3}$
16. $[[\phi^2]_2\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_3}$
17. $[[\phi^2]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}^2 a_{i_2}^2$
18. $[[\phi]^2\phi] = R_{i_1} a_{i_1} (b_{i_1 i_2} a_{i_2})^2$
19. $[[\phi]\phi^3] = R_{i_1} b_{i_1 i_2} a_{i_1}^3 a_{i_2}$
20. $[\phi^5] = R_{i_1} a_{i_1}^5$

SEVENTH ORDER

1. $[_6\phi]_6 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_6}$
2. $[_5\phi^2]_5 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_5}^2$
3. $[_4[\phi]\phi]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_4} a_{i_5}$
4. $[_4\phi^3]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_4}^3$
5. $[_3[2\phi]_2\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_3} a_{i_5}$
6. $[_3[\phi^2]\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3} a_{i_4}^2$
7. $[_3[\phi]^2]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} (b_{i_3 i_4} a_{i_4})^2$
8. $[_3[\phi]\phi^2]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3}^2 a_{i_4}$
9. $[_3\phi^4]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}^4$
10. $[_2[_3\phi]_3\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_2} a_{i_5}$
11. $[_2[_2\phi^2]_2\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_4}^2$
12. $[_2[[\phi]\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_3} a_{i_4}$

13. $[_2[\phi^3]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2} a_{i_3}^3$
14. $[_2[_2\phi]_2[\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4} a_{i_5}$
15. $[_2[\phi^2][\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_3}^2 a_{i_4}$
16. $[_2[_2\phi]_2\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2}^2 a_{i_4}$
17. $[_2[\phi^2]\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^2 a_{i_3}^2$
18. $[_2[\phi]^2\phi]_2 = R_{i_1} b_{i_1 i_2} a_{i_2} (b_{i_2 i_3} a_{i_3})^2$
19. $[_2[\phi]\phi^3]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^3 a_{i_3}$
20. $[_2\phi^5]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^5$
21. $[[_4\phi]_4\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1} a_{i_5}$
22. $[[_3\phi^2]_3\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_4}^2$
23. $[[_2[\phi]\phi]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_3} a_{i_4}$
24. $[[_2\phi^3]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_3}^3$

25. $[[[{}_2\phi]_2\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_2} a_{i_4}$
26. $[[[\phi^2]\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2} a_{i_3}^2$
27. $[[[\phi]^2]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} (b_{i_2 i_3} a_{i_3})^2$
28. $[[[\phi]\phi^2]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2}^2 a_{i_3}$
29. $[[\phi^4]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}^4$
30. $[[{}_3\phi]_3[\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_3} a_{i_5}$
31. $[[{}_2\phi^2]_2[\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3} a_{i_4}^2$
32. $[[[[\phi]\phi][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_2} a_{i_3} a_{i_4}$
33. $[[\phi^3][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_2}^3 a_{i_3}$
34. $[[{}_3\phi]_3\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1}^2 a_{i_4}$
35. $[[{}_2\phi^2]_2\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_3}^2$
36. $[[[[\phi]\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_2} a_{i_3}$

37. $[[\phi^3]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}^2 a_{i_2}^3$
38. $[[{}_2\phi]_2^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4} a_{i_5}$
39. $[[{}_2\phi]_2[\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3}^2 a_{i_4}$
40. $[[\phi^2]_2^2] = R_{i_1} (b_{i_1 i_2} a_{i_2}^2)^2$
41. $[[{}_2\phi]_2[\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1} a_{i_3} a_{i_4}$
42. $[[\phi^2][\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_1} a_{i_2}^2 a_{i_3}$
43. $[[{}_2\phi]_2\phi^3] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^3 a_{i_3}$
44. $[[\phi^2]\phi^3] = R_{i_1} b_{i_1 i_2} a_{i_1}^3 a_{i_2}^2$
45. $[[\phi]^3] = R_{i_1} (b_{i_1 i_2} a_{i_2})^3$
46. $[[\phi]^2\phi^2] = R_{i_1} a_{i_1}^2 (b_{i_1 i_2} a_{i_2})^2$
47. $[[\phi]\phi^4] = R_{i_1} b_{i_1 i_2} a_{i_1}^4 a_{i_2}$
48. $[\phi^6] = R_{i_1} a_{i_1}^6$

EIGHTH ORDER

1. $[_7\phi]_7 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} b_{i_6 i_7} a_{i_7}$
2. $[_6\phi^2]_6 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_6}^2$
3. $[_5[\phi]\phi]_5 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_5} a_{i_6}$
4. $[_5\phi^3]_5 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_5}^3$
5. $[_4[_2\phi]_2\phi]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_4} a_{i_6}$
6. $[_4[\phi^2]\phi]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_4} a_{i_5}^2$
7. $[_4[\phi]^2]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} (b_{i_4 i_5} a_{i_5})^2$
8. $[_4[\phi]\phi^2]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_4}^2 a_{i_5}$
9. $[_4\phi^4]_4 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_4}^4$
10. $[_3[_3\phi]_3\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_3} a_{i_6}$
11. $[_3[_2\phi^2]_2\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_3} a_{i_5}^2$
12. $[_3[[\phi]\phi]\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_3} a_{i_4} a_{i_5}$

13. $[_3[\phi^3]\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3} a_{i_4}^3$
14. $[_3[_2\phi]_2[\phi]]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_3 i_5} b_{i_4 i_6} a_{i_5} a_{i_6}$
15. $[_3[\phi^2][\phi]]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_3 i_5} a_{i_4}^2 a_{i_5}$
16. $[_3[_2\phi]_2\phi^2]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_3}^2 a_{i_5}$
17. $[_3[\phi]^2\phi^2]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3}^2 a_{i_4}^2$
18. $[_3[\phi]^2\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3} (b_{i_3 i_4} a_{i_4})^2$
19. $[_3[\phi]\phi^3]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_3}^3 a_{i_4}$
20. $[_3\phi^5]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}^5$
21. $[_2[_4\phi]_4\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_2} a_{i_6}$
22. $[_2[_3\phi^2]_3\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_2} a_{i_5}^2$
23. $[_2[_2[\phi]\phi]_2\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_2} a_{i_4} a_{i_5}$
24. $[_2[_2\phi^3]_2\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_4}^3$

25. $[_2[[_2\phi]_2\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_2} a_{i_3} a_{i_5}$
26. $[_2[[\phi^2]\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_3} a_{i_4}^2$
27. $[_2[[\phi]^2]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2} (b_{i_3 i_4} a_{i_4})^2$
28. $[_2[[\phi]\phi^2]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2} a_{i_3}^2 a_{i_4}$
29. $[_2[\phi^4]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2} a_{i_3}^4$
30. $[_2[_3\phi]_3[\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} b_{i_5 i_6} a_{i_4} a_{i_6}$
31. $[_2[_2\phi^2]_2[\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4} a_{i_5}^2$
32. $[_2[[\phi]\phi][\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_3} a_{i_4} a_{i_5}$
33. $[_2[\phi^3][\phi]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_3}^3 a_{i_4}$
34. $[_2[_3\phi]_3\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_2}^2 a_{i_5}$
35. $[_2[_2\phi^2]_2\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2}^2 a_{i_4}^2$
36. $[_2[[\phi]\phi]\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_2}^2 a_{i_3} a_{i_4}$

37. $[_2[\phi^3]\phi^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^2 a_{i_3}^3$
38. $[_2[_2\phi]_2^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} b_{i_4 i_6} a_{i_5} a_{i_6}$
39. $[_2[_2\phi]_2[\phi^2]]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4}^2 a_{i_5}$
40. $[_2[\phi^2]^2]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_3}^2 a_{i_4}^2$
41. $[_2[_2\phi]_2[\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_2} a_{i_4} a_{i_5}$
42. $[_2[\phi^2][\phi]\phi]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_2} a_{i_3}^2 a_{i_4}$
43. $[_2[_2\phi]_2\phi^3]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_2}^3 a_{i_4}$
44. $[_2[\phi^2]\phi^3]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^3 a_{i_3}^2$
45. $[_2[\phi]^3]_2 = R_{i_1} b_{i_1 i_2} (b_{i_2 i_3} a_{i_3})^3$
46. $[_2[\phi]^2\phi^2]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^2 (b_{i_2 i_3} a_{i_3})^2$
47. $[_2[\phi]\phi^4]_2 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_2}^4 a_{i_3}$
48. $[_2\phi^6]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}^6$

49. $[[5\phi]_5\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_1} a_{i_6}$
50. $[[4\phi^2]_4\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1} a_{i_5}^2$
51. $[[3[\phi]\phi]_3\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1} a_{i_4} a_{i_5}$
52. $[[3\phi^3]_3\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_4}^3$
53. $[[2[2\phi]_2\phi]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1} a_{i_3} a_{i_5}$
54. $[[2[\phi^2]\phi]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_3} a_{i_4}^2$
55. $[[2[\phi]^2]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} (b_{i_3 i_4} a_{i_4})^2$
56. $[[2[\phi]\phi^2]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_3}^2 a_{i_4}$
57. $[[2\phi^4]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_3}^4$
58. $[[[3\phi]_3\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1} a_{i_2} a_{i_5}$
59. $[[[2\phi^2]_2\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_2} a_{i_4}^2$
60. $[[[[\phi]\phi]\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_2} a_{i_3} a_{i_4}$

61. $[[[\phi^3]\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2} a_{i_3}^3$
62. $[[[_2\phi]_2[\phi]]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_1} a_{i_4} a_{i_5}$
63. $[[[\phi^2][\phi]]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_2 i_4} a_{i_1} a_{i_3}^2 a_{i_4}$
64. $[[[_2\phi]_2\phi^2]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1} a_{i_2}^2 a_{i_4}$
65. $[[[\phi^2]\phi^2]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2}^2 a_{i_3}^2$
66. $[[[\phi]^2\phi]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2} (b_{i_2 i_3} a_{i_3})^2$
67. $[[[\phi]\phi^3]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2}^3 a_{i_3}$
68. $[[\phi^5]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}^5$
69. $[[[_4\phi]_4[\phi]]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} b_{i_5 i_6} a_{i_3} a_{i_6}$
70. $[[[_3\phi^2]_3[\phi]]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_3} a_{i_5}^2$
71. $[[[_2[\phi]\phi]_2[\phi]]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_3} a_{i_4} a_{i_5}$
72. $[[[_2\phi^3]_2[\phi]]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3} a_{i_4}^3$

73. $[[[{}_2\phi]_2\phi][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_2} a_{i_3} a_{i_5}$
74. $[[[\phi^2]\phi][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_2} a_{i_3} a_{i_4}^2$
75. $[[[\phi]^2][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_3} (b_{i_2 i_4} a_{i_4})^2$
76. $[[[\phi]\phi^2][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_2}^2 a_{i_3} a_{i_4}$
77. $[[\phi^4][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_2}^4 a_{i_3}$
78. $[[{}_4\phi]_4\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} b_{i_4 i_5} a_{i_1}^2 a_{i_5}$
79. $[[{}_3\phi^2]_3\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1}^2 a_{i_4}^2$
80. $[[{}_2[\phi]\phi]_2\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1}^2 a_{i_3} a_{i_4}$
81. $[[{}_2\phi^3]_2\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_3}^3$
82. $[[[{}_2\phi]_2\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1}^2 a_{i_2} a_{i_4}$
83. $[[[\phi^2]\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_2} a_{i_3}^2$
84. $[[[\phi]^2]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}^2 (b_{i_2 i_3} a_{i_3})^2$

85. $[[[\phi]\phi^2]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_2}^2 a_{i_3}$
86. $[[\phi^4]\phi^2] = R_{i_1} a_{i_1}^3 a_{i_2}^4$
87. $[[{}_3\phi]_3[{}_2\phi]_2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} b_{i_3 i_6} a_{i_5} a_{i_6}$
88. $[[{}_3\phi]_3[\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_3}^2 a_{i_5}$
89. $[[{}_2\phi^2]_2[{}_2\phi]_2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4}^2 a_{i_5}$
90. $[[{}_2\phi^2]_2[\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3}^2 a_{i_4}^2$
91. $[[[\phi]\phi][{}_2\phi]_2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_2} a_{i_4} a_{i_5}$
92. $[[[\phi]\phi][\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_2} a_{i_3}^2 a_{i_4}$
93. $[[\phi^3][{}_2\phi]_2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_3 i_4} a_{i_2}^3 a_{i_4}$
94. $[[\phi^3][\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_2}^3 a_{i_3}^2$
95. $[[{}_3\phi]_3[\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_4 i_5} a_{i_1} a_{i_3} a_{i_5}$
96. $[[{}_2\phi^2]_2[\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1} a_{i_3} a_{i_4}^2$

97. $[[[\phi]\phi][\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1} a_{i_2} a_{i_3} a_{i_4}$
98. $[[\phi^3][\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_1} a_{i_2}^3 a_{i_3}$
99. $[[{}_3\phi]_3\phi^3] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} b_{i_3 i_4} a_{i_1}^3 a_{i_4}$
100. $[[{}_2\phi^2]_2\phi^3] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^3 a_{i_3}^2$
101. $[[[\phi]\phi]\phi^3] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^3 a_{i_2} a_{i_3}$
102. $[[\phi^3]\phi^3] = R_{i_1} b_{i_1 i_2} a_{i_1}^3 a_{i_2}^3$
103. $[[{}_2\phi]_2^2\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_1} a_{i_4} a_{i_5}$
104. $[[{}_2\phi]_2[\phi^2]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1} a_{i_3}^2 a_{i_4}$
105. $[[\phi^2]_2^2\phi] = R_{i_1} a_{i_1} (b_{i_1 i_2} a_{i_2}^2)^2$
106. $[[{}_2\phi]_2[\phi]^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_1 i_4} b_{i_2 i_5} a_{i_3} a_{i_4} a_{i_5}$
107. $[[\phi^2][\phi]^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_1 i_4} a_{i_2}^2 a_{i_3} a_{i_4}$
108. $[[{}_2\phi]_2[\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1}^2 a_{i_3} a_{i_4}$

$$109. \quad [[\phi^2][\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_1}^2 a_{i_2}^2 a_{i_3}$$

$$110. \quad [[{}_2\phi]{}_2\phi^4] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^4 a_{i_3}$$

$$111. \quad [[\phi^2]\phi^4] = R_{i_1} b_{i_1 i_2} a_{i_1}^4 a_{i_2}^2$$

$$112. \quad [[\phi]^3\phi] = R_{i_1} a_{i_1} (b_{i_1 i_2} a_{i_2})^3$$

$$113. \quad [[\phi]^2\phi^3] = R_{i_1} a_{i_1}^3 (b_{i_1 i_2} a_{i_2})^2$$

$$114. \quad [[\phi]\phi^5] = R_{i_1} b_{i_1 i_2} a_{i_1}^5 a_{i_2}$$

$$115. \quad [\phi^7] = R_{i_1} a_{i_1}^7$$

TABLE TWO

Runge-Kutta Relations
for Explicit Formulas Through Order Seven

The following notation is employed:

$$c_i = \sum_{j=2}^{i+1} a_j b_{i+2,j}$$

$$d_i = \sum_{j=2}^{i+1} a_j^2 b_{i+2,j}$$

$$e_i = \sum_{j=2}^{i+1} a_j^3 b_{i+2,j}$$

$$f_i = \sum_{j=2}^{i+1} a_j^4 b_{i+2,j}$$

$$g_i = \sum_{j=2}^{i+1} a_j^5 b_{i+2,j}$$

FIRST ORDER

$$\phi = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 = 1$$

SECOND ORDER

$$[\phi] = a_2 R_2 + a_3 R_3 + a_4 R_4 + a_5 R_5 + a_6 R_6 + a_7 R_7 + a_8 R_8 + a_9 R_9 = \frac{1}{2}$$

THIRD ORDER

$$[{}_2\phi]_2 = c_1 R_3 + c_2 R_4 + c_3 R_5 + c_4 R_6 + c_5 R_7 + c_6 R_8 + c_7 R_9 = \frac{1}{6}$$

$$[\phi^2] = a_2^2 R_2 + a_3^2 R_3 + a_4^2 R_4 + a_5^2 R_5 + a_6^2 R_6 + a_7^2 R_7 + a_8^2 R_8 + a_9^2 R_9 = \frac{1}{3}$$

FOURTH ORDER

$$\begin{aligned} [{}_3\phi]_3 &= c_1 b_{43} R_4 + (c_1 b_{53} + c_2 b_{54}) R_5 + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) R_6 \\ &+ (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) R_7 + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} \\ &+ c_4 b_{86} + c_5 b_{87}) R_8 + (c_1 b_{93} + c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} \\ &+ c_6 b_{98}) R_9 = \frac{1}{24} \end{aligned}$$

$$[{}_2\phi^2]_2 = d_1 R_3 + d_2 R_4 + d_3 R_5 + d_4 R_6 + d_5 R_7 + d_6 R_8 + d_7 R_9 = \frac{1}{12}$$

$$[[\phi]\phi] = a_3 c_1 R_3 + a_4 c_2 R_4 + a_5 c_3 R_5 + a_6 c_4 R_6 + a_7 c_5 R_7 + a_8 c_6 R_8 + a_9 c_7 R_9 = \frac{1}{8}$$

$$[\phi^3] = a_2 {}^3R_2 + a_3 {}^3R_3 + a_4 {}^3R_4 + a_5 {}^3R_5 + a_6 {}^3R_6 + a_7 {}^3R_7 + a_8 {}^3R_8 + a_9 {}^3R_9 = \frac{1}{4}$$

FIFTH ORDER

$$\begin{aligned} [4\phi]_4 = & c_1 b_{43} b_{54} R_5 + [c_1 b_{43} b_{64} + (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6 + [c_1 b_{43} b_{74} \\ & + (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{76}] R_7 + [c_1 b_{43} b_{84} \\ & + (c_1 b_{53} + c_2 b_{54}) b_{85} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + (c_1 b_{73} + c_2 b_{74} \\ & + c_3 b_{75} + c_4 b_{76}) b_{87}] R_8 + [c_1 b_{43} b_{94} + (c_1 b_{53} + c_2 b_{54}) b_{95} + (c_1 b_{63} \\ & + c_2 b_{64} + c_3 b_{65}) b_{96} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) b_{97} \\ & + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87}) b_{98}] R_9 = \frac{1}{120} \end{aligned}$$

$$\begin{aligned} [3\phi^2]_3 = & b_{43} d_1 R_4 + (b_{53} d_1 + b_{54} d_2) R_5 + (b_{63} d_1 + b_{64} d_2 + b_{65} d_3) R_6 \\ & + (b_{73} d_1 + b_{74} d_2 + b_{75} d_3 + b_{76} d_4) R_7 + (b_{83} d_1 + b_{84} d_2 + b_{85} d_3 \\ & + b_{86} d_4 + b_{87} d_5) R_8 + (b_{93} d_1 + b_{94} d_2 + b_{95} d_3 + b_{96} d_4 + b_{97} d_5 \\ & + b_{98} d_6) R_9 = \frac{1}{60} \end{aligned}$$

$$\begin{aligned} [2[\phi]\phi]_2 = & a_3 b_{43} c_1 R_4 + (a_3 b_{53} c_1 + a_4 b_{54} c_2) R_5 + (a_3 b_{63} c_1 + a_4 b_{64} c_2 \\ & + a_5 b_{65} c_3) R_6 + (a_3 b_{73} c_1 + a_4 b_{74} c_2 + a_5 b_{75} c_3 + a_6 b_{76} c_4) R_7 \\ & + (a_3 b_{83} c_1 + a_4 b_{84} c_2 + a_5 b_{85} c_3 + a_6 b_{86} c_4 + a_7 b_{87} c_5) R_8 \\ & + (a_3 b_{93} c_1 + a_4 b_{94} c_2 + a_5 b_{95} c_3 + a_6 b_{96} c_4 + a_7 b_{97} c_5 \\ & + a_8 b_{98} c_6) R_9 = \frac{1}{40} \end{aligned}$$

$$[{}_2\phi^3]_2 = e_1R_3 + e_2R_4 + e_3R_5 + e_4R_6 + e_5R_7 + e_6R_8 + e_7R_9 = \frac{1}{20}$$

$$\begin{aligned} [[{}_2\phi]_2\phi] &= a_4b_{43}c_1R_4 + a_5(b_{53}c_1 + b_{54}c_2)R_5 + a_6(b_{63}c_1 + b_{64}c_2 + b_{65}c_3)R_6 \\ &+ a_7(b_{73}c_1 + b_{74}c_2 + b_{75}c_3 + b_{76}c_4)R_7 + a_8(b_{83}c_1 + b_{84}c_2 + b_{85}c_3 \\ &+ b_{86}c_4 + b_{87}c_5)R_8 + a_9(b_{93}c_1 + b_{94}c_2 + b_{95}c_3 + b_{96}c_4 \\ &+ b_{97}c_5 + b_{98}c_6)R_9 = \frac{1}{30} \end{aligned}$$

$$\begin{aligned} [[\phi^2]\phi] &= a_3d_1R_3 + a_4d_2R_4 + a_5d_3R_5 + a_6d_4R_6 + a_7d_5R_7 + a_8d_6R_8 \\ &+ a_9d_7R_9 = \frac{1}{15} \end{aligned}$$

$$[[\phi]^2] = c_1^2R_3 + c_2^2R_4 + c_3^2R_5 + c_4^2R_6 + c_5^2R_7 + c_6^2R_8 + c_7^2R_9 = \frac{1}{20}$$

$$\begin{aligned} [[\phi]\phi^2] &= a_3^2c_1R_3 + a_4^2c_2R_4 + a_5^2c_3R_5 + a_6^2c_4R_6 + a_7^2c_5R_7 + a_8^2c_6R_8 \\ &+ a_9^2c_7R_9 = \frac{1}{10} \end{aligned}$$

$$[\phi^4] = a_2^4R_2 + a_3^4R_3 + a_4^4R_4 + a_5^4R_5 + a_6^4R_6 + a_7^4R_7 + a_8^4R_8 + a_9^4R_9 = \frac{1}{5}$$

SIXTH ORDER

$$\begin{aligned} [5\phi]_5 &= c_1 b_{43} b_{54} b_{65} R_6 + \{c_1 b_{43} b_{54} b_{75} + [c_1 b_{43} b_{64} + (c_1 b_{53} \\ &+ c_2 b_{54}) b_{65}] b_{76}\} R_7 + \{c_1 b_{43} b_{54} b_{85} + [c_1 b_{43} b_{64} + (c_1 b_{53} \\ &+ c_2 b_{54}) b_{65}] b_{86} + [c_1 b_{43} b_{74} + (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} \\ &+ c_2 b_{64} + c_3 b_{65}) b_{76}] b_{87}\} R_8 + \{c_1 b_{43} b_{54} b_{95} + [c_1 b_{43} b_{64} \\ &+ (c_1 b_{53} + c_2 b_{54}) b_{65}] b_{96} + [c_1 b_{43} b_{74} + (c_1 b_{53} + c_2 b_{54}) b_{75} \\ &+ (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{76}] b_{97} + [c_1 b_{43} b_{84} + (c_1 b_{53} \\ &+ c_2 b_{54}) b_{85} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + (c_1 b_{73} + c_2 b_{74} \\ &+ c_3 b_{75} + c_4 b_{76}) b_{87}] b_{98}\} R_9 = \frac{1}{720} \end{aligned}$$

$$\begin{aligned} [4\phi^2]_4 &= d_1 b_{43} b_{54} R_5 + [d_1 b_{43} b_{64} + (d_1 b_{53} + d_2 b_{54}) b_{65}] R_6 \\ &+ [d_1 b_{43} b_{74} + (d_1 b_{53} + d_2 b_{54}) b_{75} + (d_1 b_{63} + d_2 b_{64} \\ &+ d_3 b_{65}) b_{76}] R_7 + [d_1 b_{43} b_{84} + (d_1 b_{53} + d_2 b_{54}) b_{85} \\ &+ (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{86} + (d_1 b_{73} + d_2 b_{74} \\ &+ d_3 b_{75} + d_4 b_{76}) b_{87}] R_8 + [d_1 b_{43} b_{94} + (d_1 b_{53} + d_2 b_{54}) b_{95} \\ &+ (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{96} + (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} \\ &+ d_4 b_{76}) b_{97} + (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) b_{98}] R_9 = \frac{1}{360} \end{aligned}$$

$$\begin{aligned}
{}_3[\phi]\phi]_3 &= a_3c_1b_{43}b_{54}R_5 + [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]R_6 \\
&+ [a_3c_1b_{43}b_{74} + (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64} \\
&+ a_5c_3b_{65})b_{76}]R_7 + [a_3c_1b_{43}b_{84} + (a_3c_1b_{53} + a_4c_2b_{54})b_{85} \\
&+ (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{86} + (a_3c_1b_{73} + a_4c_2b_{74} \\
&+ a_5c_3b_{75} + a_6c_4b_{76})b_{87}]R_8 + [a_3c_1b_{43}b_{94} + (a_3c_1b_{53} \\
&+ a_4c_2b_{54})b_{95} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{96} \\
&+ (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{97} + (a_3c_1b_{83} \\
&+ a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} + a_7c_5b_{87})b_{98}]R_9 = \frac{1}{240}
\end{aligned}$$

$$\begin{aligned}
{}_3\phi^3]_3 &= b_{43}e_1R_4 + (b_{53}e_1 + b_{54}e_2)R_5 + (b_{63}e_1 + b_{64}e_2 + b_{65}e_3)R_6 \\
&+ (b_{73}e_1 + b_{74}e_2 + b_{75}e_3 + b_{76}e_4)R_7 + (b_{83}e_1 + b_{84}e_2 + b_{85}e_3 \\
&+ b_{86}e_4 + b_{87}e_5)R_8 + (b_{93}e_1 + b_{94}e_2 + b_{95}e_3 + b_{96}e_4 + b_{97}e_5 \\
&+ b_{98}e_6)R_9 = \frac{1}{120}
\end{aligned}$$

$$\begin{aligned}
[{}_2[{}_2\phi]_2\phi]_2 &= a_4c_1b_{43}b_{54}R_5 + [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]R_6 \\
&+ [a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64} \\
&+ c_3b_{65})b_{76}]R_7 + [a_4c_1b_{43}b_{84} + a_5(c_1b_{53} + c_2b_{54})b_{85} \\
&+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})b_{87}]R_8 + [a_4c_1b_{43}b_{94} + a_5(c_1b_{53} + c_2b_{54})b_{95} \\
&+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + a_7(c_1b_{73} + c_2b_{74} \\
&+ c_3b_{75} + c_4b_{76})b_{97} + a_8(c_1b_{83} + c_2b_{84} + c_3b_{85} \\
&+ c_4b_{86} + c_5b_{87})b_{98}]R_9 = \frac{1}{180}
\end{aligned}$$

$$\begin{aligned}
[{}_2[\phi^2]_2\phi]_2 &= a_3b_{43}d_1R_4 + (a_3b_{53}d_1 + a_4b_{54}d_2)R_5 + (a_3b_{63}d_1 + a_4b_{64}d_2 \\
&+ a_5b_{65}d_3)R_6 + (a_3b_{73}d_1 + a_4b_{74}d_2 + a_5b_{75}d_3 + a_6b_{76}d_4)R_7 \\
&+ (a_3b_{83}d_1 + a_4b_{84}d_2 + a_5b_{85}d_3 + a_6b_{86}d_4 + a_7b_{87}d_5)R_8 \\
&+ (a_3b_{93}d_1 + a_4b_{94}d_2 + a_5b_{95}d_3 + a_6b_{96}d_4 + a_7b_{97}d_5 \\
&+ a_8b_{98}d_6)R_9 = \frac{1}{90}
\end{aligned}$$

$$\begin{aligned}
[{}_2[\phi]_2^2]_2 &= c_1^2b_{43}R_4 + (c_1^2b_{53} + c_2^2b_{54})R_5 + (c_1^2b_{63} + c_2^2b_{64} + c_3^2b_{65})R_6 \\
&+ (c_1^2b_{73} + c_2^2b_{74} + c_3^2b_{75} + c_4^2b_{76})R_7 + (c_1^2b_{83} + c_2^2b_{84} \\
&+ c_3^2b_{85} + c_4^2b_{86} + c_5^2b_{87})R_8 + (c_1^2b_{93} + c_2^2b_{94} + c_3^2b_{95} \\
&+ c_4^2b_{96} + c_5^2b_{97} + c_6^2b_{98})R_9 = \frac{1}{120}
\end{aligned}$$

$$\begin{aligned}
[{}_2[\phi]\phi^2]_2 &= a_3^2 c_1 b_{43} R_4 + (a_3^2 b_{53} c_1 + a_4^2 b_{54} c_2) R_5 + (a_3^2 b_{63} c_1 + a_4^2 b_{64} c_2 \\
&\quad + a_5^2 b_{65} c_3) R_6 + (a_3^2 b_{73} c_1 + a_4^2 b_{74} c_2 + a_5^2 b_{75} c_3 + a_6^2 b_{76} c_4) R_7 \\
&\quad + (a_3^2 b_{83} c_1 + a_4^2 b_{84} c_2 + a_5^2 b_{85} c_3 + a_6^2 b_{86} c_4 + a_7^2 b_{87} c_5) R_8 \\
&\quad + (a_3^2 b_{93} c_1 + a_4^2 b_{94} c_2 + a_5^2 b_{95} c_3 + a_6^2 b_{96} c_4 + a_7^2 b_{97} c_5 \\
&\quad + a_8^2 b_{98} c_6) R_9 = \frac{1}{60}
\end{aligned}$$

$$[{}_2\phi^4]_2 = f_1 R_3 + f_2 R_4 + f_3 R_5 + f_4 R_6 + f_5 R_7 + f_6 R_8 + f_7 R_9 = \frac{1}{30}$$

$$\begin{aligned}
[{}_3\phi]_3\phi &= a_5 c_1 b_{43} b_{54} R_5 + a_6 [c_1 b_{43} b_{64} + (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6 \\
&\quad + a_7 [c_1 b_{43} b_{74} + (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64} \\
&\quad + c_3 b_{65}) b_{76}] R_7 + a_8 [c_1 b_{43} b_{84} + (c_1 b_{53} + c_2 b_{54}) b_{85} \\
&\quad + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&\quad + c_4 b_{76}) b_{87}] R_8 + a_9 [c_1 b_{43} b_{94} + (c_1 b_{53} + c_2 b_{54}) b_{95} \\
&\quad + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{96} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&\quad + c_4 b_{76}) b_{97} + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} \\
&\quad + c_5 b_{87}) b_{98}] R_9 = \frac{1}{144}
\end{aligned}$$

$$\begin{aligned}
[[2\phi^2]_2\phi] &= a_4 d_1 b_{43} R_4 + a_5 (d_1 b_{53} + d_2 b_{54}) R_5 + a_6 (d_1 b_{63} + d_2 b_{64} \\
&\quad + d_3 b_{65}) R_6 + a_7 (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) R_7 + a_8 (d_1 b_{83} \\
&\quad + d_2 b_{84} + d_3 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) R_8 + a_9 (d_1 b_{93} \\
&\quad + d_2 b_{94} + d_3 b_{95} + d_4 b_{96} + d_5 b_{97} + d_6 b_{98}) R_9 = \frac{1}{72}
\end{aligned}$$

$$\begin{aligned}
[[[\phi]\phi]\phi] &= a_3 a_4 c_1 b_{43} R_4 + a_5 (a_3 c_1 b_{53} + a_4 c_2 b_{54}) R_5 + a_6 (a_3 c_1 b_{63} \\
&\quad + a_4 c_2 b_{64} + a_5 c_3 b_{65}) R_6 + a_7 (a_3 c_1 b_{73} + a_4 c_2 b_{74} + a_5 c_3 b_{75} \\
&\quad + a_6 c_4 b_{76}) R_7 + a_8 (a_3 c_1 b_{83} + a_4 c_2 b_{84} + a_5 c_3 b_{85} + a_6 c_4 b_{86} \\
&\quad + a_7 c_5 b_{87}) R_8 + a_9 (a_3 c_1 b_{93} + a_4 c_2 b_{94} + a_5 c_3 b_{95} + a_6 c_4 b_{96} \\
&\quad + a_7 c_5 b_{97} + a_8 c_6 b_{98}) R_9 = \frac{1}{48}
\end{aligned}$$

$$\begin{aligned}
[[\phi^3]\phi] &= a_3 e_1 R_3 + a_4 e_2 R_4 + a_5 e_3 R_5 + a_6 e_4 R_6 + a_7 e_5 R_7 + a_8 e_6 R_8 + a_9 e_7 R_9 \\
&= \frac{1}{24}
\end{aligned}$$

$$\begin{aligned}
[[2\phi]_2[\phi]] &= c_1 c_2 b_{43} R_4 + (c_1 b_{53} + c_2 b_{54}) c_3 R_5 + (c_1 b_{63} + c_2 b_{64} \\
&\quad + c_3 b_{65}) c_4 R_6 + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) c_5 R_7 \\
&\quad + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87}) c_6 R_8 + (c_1 b_{93} \\
&\quad + c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} + c_6 b_{98}) c_7 R_9 = \frac{1}{72}
\end{aligned}$$

$$[[\phi^2][\phi]] = c_1 d_1 R_3 + c_2 d_2 R_4 + c_3 d_3 R_5 + c_4 d_4 R_6 + c_5 d_5 R_7 + c_6 d_6 R_8 \\ + c_7 d_7 R_9 = \frac{1}{36}$$

$$[[2\phi]_2\phi^2] = a_4^2 c_1 b_{43} R_4 + a_5^2 (b_{53} c_1 + b_{54} c_2) R_5 + a_6^2 (b_{63} c_1 + b_{64} c_2 \\ + b_{65} c_3) R_6 + a_7^2 (b_{73} c_1 + b_{74} c_2 + b_{75} c_3 + b_{76} c_4) R_7 \\ + a_8^2 (b_{83} c_1 + b_{84} c_2 + b_{85} c_3 + b_{86} c_4 + b_{87} c_5) R_8 \\ + a_9^2 (b_{93} c_1 + b_{94} c_2 + b_{95} c_3 + b_{96} c_4 + b_{97} c_5 + b_{98} c_6) R_9 = \frac{1}{36}$$

$$[[\phi^2]\phi^2] = a_3^2 d_1 R_3 + a_4^2 d_2 R_4 + a_5^2 d_3 R_5 + a_6^2 d_4 R_6 + a_7^2 d_5 R_7 + a_8^2 d_6 R_8 \\ + a_9^2 d_7 R_9 = \frac{1}{18}$$

$$[[\phi]^2\phi] = a_3 c_1^2 R_3 + a_4 c_2^2 R_4 + a_5 c_3^2 R_5 + a_6 c_4^2 R_6 + a_7 c_5^2 R_7 + a_8 c_6^2 R_8 \\ + a_9 c_7^2 R_9 = \frac{1}{24}$$

$$[[\phi]\phi^3] = a_3^3 c_1 R_3 + a_4^3 c_2 R_4 + a_5^3 c_3 R_5 + a_6^3 c_4 R_6 + a_7^3 c_5 R_7 + a_8^3 c_6 R_8 \\ + a_9^3 c_7 R_9 = \frac{1}{12}$$

$$[\phi^5] = a_2^5 R_2 + a_3^5 R_3 + a_4^5 R_4 + a_5^5 R_5 + a_6^5 R_6 + a_7^5 R_7 + a_8^5 R_8 \\ + a_9^5 R_9 = \frac{1}{6}$$

SEVENTH ORDER

$$\begin{aligned}
 [6\phi]_6 = & c_1 b_{43} b_{54} b_{65} b_{76} R_7 + \{c_1 b_{43} b_{54} b_{65} b_{86} + [c_1 b_{43} b_{54} b_{75} \\
 & + (c_1 b_{43} b_{64} + \overline{c_1 b_{53} + c_2 b_{54} b_{65}}) b_{76}] b_{87}\} R_8 + \{c_1 b_{43} b_{54} b_{65} b_{96} \\
 & + [c_1 b_{43} b_{54} b_{75} + (c_1 b_{43} b_{64} + \overline{c_1 b_{53} + c_2 b_{54} b_{65}}) b_{76}] b_{97} \\
 & + [c_1 b_{43} b_{54} b_{85} + (c_1 b_{43} b_{64} + \overline{c_1 b_{53} + c_2 b_{54} b_{65}}) b_{86} \\
 & + (c_1 b_{43} b_{74} + \overline{c_1 b_{53} + c_2 b_{54} b_{75}} \\
 & + \overline{c_1 b_{63} + c_2 b_{64} + c_3 b_{65} b_{76}}) b_{87}] b_{98}\} R_9 = \frac{1}{5040}
 \end{aligned}$$

$$\begin{aligned}
 [5\phi^2]_5 = & d_1 b_{43} b_{54} b_{65} R_6 + \{d_1 b_{43} b_{54} b_{75} + [d_1 b_{43} b_{64} + (d_1 b_{53} \\
 & + d_2 b_{54}) b_{65}] b_{76}\} R_7 + \{d_1 b_{43} b_{54} b_{85} + [d_1 b_{43} b_{64} + (d_1 b_{53} \\
 & + d_2 b_{54}) b_{65}] b_{86} + [d_1 b_{43} b_{74} + (d_1 b_{53} + d_2 b_{54}) b_{75} + (d_1 b_{63} \\
 & + d_2 b_{64} + d_3 b_{65}) b_{76}] b_{87}\} R_8 + \{d_1 b_{43} b_{54} b_{95} + [d_1 b_{43} b_{64} \\
 & + (d_1 b_{53} + d_2 b_{54}) b_{65}] b_{96} + [d_1 b_{43} b_{74} + (d_1 b_{53} + d_2 b_{54}) b_{75} \\
 & + (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{76}] b_{97} + [d_1 b_{43} b_{84} + (d_1 b_{53} \\
 & + d_2 b_{54}) b_{85} + (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{86} + (d_1 b_{73} + d_2 b_{74} \\
 & + d_3 b_{75} + d_4 b_{76}) b_{87}] b_{98}\} R_9 = \frac{1}{2520}
 \end{aligned}$$

$$\begin{aligned}
[{}_4[\phi]\phi]_4 &= a_3c_1b_{43}b_{54}b_{65}R_6 + \{a_3c_1b_{43}b_{54}b_{75} + [a_3c_1b_{43}b_{64} \\
&+ (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_3c_1b_{43}b_{54}b_{85} \\
&+ [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{86} + [a_3c_1b_{43}b_{74} \\
&+ (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64} \\
&+ a_5c_3b_{65})b_{76}]b_{87}\}R_8 + a_3c_1b_{43}b_{54}b_{95} + [a_3c_1b_{43}b_{64} \\
&+ (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{96} + [a_3c_1b_{43}b_{74} + (a_3c_1b_{53} \\
&+ a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{76}]b_{97} \\
&+ [a_3c_1b_{43}b_{84} + (a_3c_1b_{53} + a_4c_2b_{54})b_{85} + (a_3c_1b_{63} + a_4c_2b_{64} \\
&+ a_5c_3b_{65})b_{86} + (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} \\
&+ a_6c_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{1680}
\end{aligned}$$

$$\begin{aligned}
[{}_4\phi^3]_4 &= e_1b_{43}b_{54}R_5 + [e_1b_{43}b_{64} + (e_1b_{53} + e_2b_{54})b_{65}]R_6 + [e_1b_{43}b_{74} \\
&+ (e_1b_{53} + e_2b_{54})b_{75} + (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{76}]R_7 + [e_1b_{43}b_{84} \\
&+ (e_1b_{53} + e_2b_{54})b_{85} + (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{86} + (e_1b_{73} \\
&+ e_2b_{74} + e_3b_{75} + e_4b_{76})b_{87}]R_8 + [e_1b_{43}b_{94} + (e_1b_{53} + e_2b_{54})b_{95} \\
&+ (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{96} + (e_1b_{73} + e_2b_{74} + e_3b_{75} \\
&+ e_4b_{76})b_{97} + (e_1b_{83} + e_2b_{84} + e_3b_{85} + e_4b_{86} \\
&+ e_5b_{87})b_{98}]R_9 = \frac{1}{840}
\end{aligned}$$

$$\begin{aligned}
[{}_3[{}_2\phi]_2\phi]_3 &= a_4c_1b_{43}b_{54}b_{65}R_6 + \{a_4c_1b_{43}b_{54}b_{75} + [a_1c_4b_{43}b_{64} + a_5(c_1b_{53} \\
&+ c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_4c_1b_{43}b_{54}b_{85} + [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} \\
&+ c_2b_{54})b_{65}]b_{86} + [a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} \\
&+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}R_8 + \{a_4c_1b_{43}b_{54}b_{95} \\
&+ [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]b_{96} + [a_4c_1b_{43}b_{74} \\
&+ a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97} \\
&+ [a_4c_1b_{43}b_{84} + a_5(c_1b_{53} + c_2b_{54})b_{85} + a_6(c_1b_{63} + c_2b_{64} \\
&+ c_3b_{65})b_{86} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{1260}
\end{aligned}$$

$$\begin{aligned}
[{}_3[\phi^2]\phi]_3 &= a_3d_1b_{43}b_{54}R_5 + [a_3d_1b_{43}b_{64} + (a_3d_1b_{53} + a_4d_2b_{54})b_{65}]R_6 \\
&+ [a_3d_1b_{43}b_{74} + (a_3d_1b_{53} + a_4d_2b_{54})b_{75} + (a_3d_1b_{63} + a_4d_2b_{64} \\
&+ a_5d_3b_{65})b_{76}]R_7 + [a_3d_1b_{43}b_{84} + (a_3d_1b_{53} + a_4d_2b_{54})b_{85} \\
&+ (a_3d_1b_{63} + a_4d_2b_{64} + a_5d_3b_{65})b_{86} + (a_3d_1b_{73} + a_4d_2b_{74} \\
&+ a_5d_3b_{75} + a_6d_4b_{76})b_{87}]R_8 + [a_3d_1b_{43}b_{94} + (a_3d_1b_{53} \\
&+ a_4d_2b_{54})b_{95} + (a_3d_1b_{63} + a_4d_2b_{64} + a_5d_3b_{65})b_{96} \\
&+ (a_3d_1b_{73} + a_4d_2b_{74} + a_5d_3b_{75} + a_6d_4b_{76})b_{97} + (a_3d_1b_{83} \\
&+ a_4d_2b_{84} + a_5d_3b_{85} + a_6d_4b_{86} + a_7d_5b_{87})b_{98}]R_9 = \frac{1}{630}
\end{aligned}$$

$$\begin{aligned}
[{}_3[\phi]^2]_3 &= c_1^2 b_{43} b_{54} R_5 + [c_1^2 b_{43} b_{64} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{65}] R_6 \\
&+ [c_1^2 b_{43} b_{74} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{75} + (c_1^2 b_{63} + c_2^2 b_{64} \\
&+ c_3^2 b_{65}) b_{76}] R_7 + [c_1^2 b_{43} b_{84} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{85} \\
&+ (c_1^2 b_{63} + c_2^2 b_{64} + c_3^2 b_{65}) b_{86} + (c_1^2 b_{73} + c_2^2 b_{74} \\
&+ c_3^2 b_{75} + c_4^2 b_{76}) b_{87}] R_8 + [c_1^2 b_{43} b_{94} + (c_1^2 b_{53} \\
&+ c_2^2 b_{54}) b_{95} + (c_1^2 b_{63} + c_2^2 b_{64} + c_3^2 b_{65}) b_{96} \\
&+ (c_1^2 b_{73} + c_2^2 b_{74} + c_3^2 b_{75} + c_4^2 b_{76}) b_{97} + (c_1^2 b_{83} \\
&+ c_2^2 b_{84} + c_3^2 b_{85} + c_4^2 b_{86} + c_5^2 b_{87}) b_{98}] R_9 = \frac{1}{840}
\end{aligned}$$

$$\begin{aligned}
[{}_3[\phi]\phi^2]_3 &= a_3^2 c_1 b_{43} b_{54} R_5 + [a_3^2 c_1 b_{43} b_{64} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{65}] R_6 \\
&+ [a_3^2 c_1 b_{43} b_{74} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{75} + (a_3^2 c_1 b_{63} \\
&+ a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{76}] R_7 + [a_3^2 c_1 b_{43} b_{84} + (a_3^2 c_1 b_{53} \\
&+ a_4^2 c_2 b_{54}) b_{85} + (a_3^2 c_1 b_{63} + a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{86} \\
&+ (a_3^2 c_1 b_{73} + a_4^2 c_2 b_{74} + a_5^2 c_3 b_{75} + a_6^2 c_4 b_{76}) b_{87}] R_8 \\
&+ [a_3^2 c_1 b_{43} b_{94} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{95} + (a_3^2 c_1 b_{63} \\
&+ a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{96} + (a_3^2 c_1 b_{73} + a_4^2 c_2 b_{74} \\
&+ a_5^2 c_3 b_{75} + a_6^2 c_4 b_{76}) b_{97} + (a_3^2 c_1 b_{83} + a_4^2 c_2 b_{84} \\
&+ a_5^2 c_3 b_{85} + a_6^2 c_4 b_{86} + a_7^2 c_5 b_{87}) b_{98}] R_9 = \frac{1}{420}
\end{aligned}$$

$$\begin{aligned}
[3\phi^4]_3 &= b_{43}f_1R_4 + (b_{53}f_1 + b_{54}f_2)R_5 + (b_{63}f_1 + b_{64}f_2 + b_{65}f_3)R_6 \\
&+ (b_{73}f_1 + b_{74}f_2 + b_{75}f_3 + b_{76}f_4)R_7 + (b_{83}f_1 + b_{84}f_2 + b_{85}f_3 \\
&+ b_{86}f_4 + b_{87}f_5)R_8 + (b_{93}f_1 + b_{94}f_2 + b_{95}f_3 + b_{96}f_4 + b_{97}f_5 \\
&+ b_{98}f_6)R_9 = \frac{1}{210}
\end{aligned}$$

$$\begin{aligned}
[2[3\phi]_3\phi]_2 &= a_5c_1b_{43}b_{54}b_{65}R_6 + \{a_5c_1b_{43}b_{65}b_{75} + a_6[c_1b_{43}b_{64} + (c_1b_{53} \\
&+ c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_5c_1b_{43}b_{54}b_{85} + a_6[c_1b_{43}b_{64} + (c_1b_{53} \\
&+ c_2b_{54})b_{65}]b_{86} + a_7[c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} \\
&+ (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}R_8 + \{a_5c_1b_{43}b_{54}b_{95} \\
&+ a_6[c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{96} + a_7[c_1b_{43}b_{74} \\
&+ (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97} \\
&+ a_8[c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} \\
&+ c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]b_{98}\}R_9 \\
&= \frac{1}{1008}
\end{aligned}$$

$$\begin{aligned}
[{}_2[{}_2\phi^2]_2\phi]_2 &= a_4 d_1 b_{43} b_{54} R_5 + [a_4 d_1 b_{43} b_{64} + a_5 (d_1 b_{53} + d_2 b_{54}) b_{65}] R_6 \\
&+ [a_4 d_1 b_{43} b_{74} + a_5 (d_1 b_{53} + d_2 b_{54}) b_{75} + a_6 (d_1 b_{63} + d_2 b_{64} \\
&+ d_3 b_{65}) b_{76}] R_7 + [a_4 d_1 b_{43} b_{84} + a_5 (d_1 b_{53} + d_2 b_{54}) b_{85} \\
&+ a_6 (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{86} + a_7 (d_1 b_{73} + d_2 b_{74} \\
&+ d_3 b_{75} + d_4 b_{76}) b_{87}] R_8 + [a_4 d_1 b_{43} b_{94} + a_5 (d_1 b_{53} \\
&+ d_2 b_{54}) b_{95} + a_6 (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{96} + a_7 (d_1 b_{73} \\
&+ d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) b_{97} + a_8 (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} \\
&+ d_4 b_{86} + d_5 b_{87}) b_{98}] R_9 = \frac{1}{504}
\end{aligned}$$

$$\begin{aligned}
[{}_2[[\phi]\phi]\phi]_2 &= a_3 a_4 c_1 b_{43} b_{54} R_5 + [a_3 a_4 c_1 b_{43} b_{64} + a_5 (a_3 c_1 b_{53} \\
&+ a_4 c_2 b_{54}) b_{65}] R_6 + [a_3 a_4 c_1 b_{43} b_{74} + a_5 (a_3 c_1 b_{53} + a_4 c_2 b_{54}) b_{75} \\
&+ a_6 (a_3 c_1 b_{63} + a_4 c_2 b_{64} + a_5 c_3 b_{65}) b_{76}] R_7 + [a_3 a_4 c_1 b_{43} b_{84} \\
&+ a_5 (a_3 c_1 b_{53} + a_4 c_2 b_{54}) b_{85} + a_6 (a_3 c_1 b_{63} + a_4 c_2 b_{64} \\
&+ a_5 c_3 b_{65}) b_{86} + a_7 (a_3 c_1 b_{73} + a_4 c_2 b_{74} + a_5 c_3 b_{75} \\
&+ a_6 c_4 b_{76}) b_{87}] R_8 + [a_3 a_4 c_1 b_{43} b_{94} + a_5 (a_3 c_1 b_{53} \\
&+ a_4 c_2 b_{54}) b_{95} + a_6 (a_3 c_1 b_{63} + a_4 c_2 b_{64} + a_5 c_3 b_{65}) b_{96} \\
&+ a_7 (a_3 c_1 b_{73} + a_4 c_2 b_{74} + a_5 c_3 b_{75} + a_6 c_4 b_{76}) b_{97} \\
&+ a_8 (a_3 c_1 b_{83} + a_4 c_2 b_{84} + a_5 c_3 b_{85} + a_6 c_4 b_{86} \\
&+ a_7 c_5 b_{87}) b_{98}] R_9 = \frac{1}{336}
\end{aligned}$$

$$\begin{aligned}
[{}_2[\phi^3]\phi]_2 &= a_3b_{43}e_1R_4 + (a_3b_{53}e_1 + a_4b_{54}e_2)R_5 + (a_3b_{63}e_1 + a_4b_{64}e_2 \\
&+ a_5b_{65}e_3)R_6 + (a_3b_{73}e_1 + a_4b_{74}e_2 + a_5b_{75}e_3 + a_6b_{76}e_4)R_7 \\
&+ (a_3b_{83}e_1 + a_4b_{84}e_2 + a_5b_{85}e_3 + a_6b_{86}e_4 + a_7b_{87}e_5)R_8 \\
&+ (a_3b_{93}e_1 + a_4b_{94}e_2 + a_5b_{95}e_3 + a_6b_{96}e_4 + a_7b_{97}e_5 \\
&+ a_8b_{98}e_6)R_9 = \frac{1}{168}
\end{aligned}$$

$$\begin{aligned}
[{}_2[{}_2\phi]_2[\phi]]_2 &= c_1c_2b_{43}b_{54}R_5 + [c_1c_2b_{43}b_{64} + (c_1b_{53} + c_2b_{54})c_3b_{65}]R_6 \\
&+ [c_1c_2b_{43}b_{74} + (c_1b_{53} + c_2b_{54})c_3b_{75} + (c_1b_{63} + c_2b_{64} \\
&+ c_3b_{65})c_4b_{76}]R_7 + [c_1c_2b_{43}b_{84} + (c_1b_{53} + c_2b_{54})c_3b_{85} \\
&+ (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})c_5b_{87}]R_8 + [c_1c_2b_{43}b_{94} + (c_1b_{53} + c_2b_{54})c_3b_{95} \\
&+ (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})c_5b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} \\
&+ c_5b_{87})c_6b_{98}]R_9 = \frac{1}{504}
\end{aligned}$$

$$\begin{aligned}
[{}_2[\phi^2][\phi]]_2 &= c_1d_1b_{43}R_4 + (c_1d_1b_{53} + c_2d_2b_{54})R_5 + (c_1d_1b_{63} + c_2d_2b_{64} \\
&+ c_3d_3b_{65})R_6 + (c_1d_1b_{73} + c_2d_2b_{74} + c_3d_3b_{75} + c_4d_4b_{76})R_7 \\
&+ (c_1d_1b_{83} + c_2d_2b_{84} + c_3d_3b_{85} + c_4d_4b_{86} + c_5d_5b_{87})R_8 \\
&+ (c_1d_1b_{93} + c_2d_2b_{94} + c_3d_3b_{95} + c_4d_4b_{96} + c_5d_5b_{97} \\
&+ c_6d_6b_{98})R_9 = \frac{1}{252}
\end{aligned}$$

$$\begin{aligned}
{}_2[{}_2\phi]_2\phi^2]_2 &= a_4^2 c_1 b_{43} b_{54} R_5 + [a_4^2 c_1 b_{43} b_{64} + a_5^2 (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6 \\
&+ [a_4^2 c_1 b_{43} b_{74} + a_5^2 (c_1 b_{53} + c_2 b_{54}) b_{75} + a_6^2 (c_1 b_{63} + c_2 b_{64} \\
&+ c_3 b_{65}) b_{76}] R_7 + [a_4^2 c_1 b_{43} b_{84} + a_5^2 (c_1 b_{53} + c_2 b_{54}) b_{85} \\
&+ a_6^2 (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + a_7^2 (c_1 b_{73} + c_2 b_{74} \\
&+ c_3 b_{75} + c_4 b_{76}) b_{87}] R_8 + [a_4^2 c_1 b_{43} b_{94} + a_5^2 (c_1 b_{53} + c_2 b_{54}) b_{95} \\
&+ a_6^2 (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{96} + a_7^2 (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&+ c_4 b_{76}) b_{97} + a_8^2 (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} \\
&+ c_5 b_{87}) b_{98}] R_9 = \frac{1}{252}
\end{aligned}$$

$$\begin{aligned}
{}_2[\phi^2]_2\phi^2]_2 &= a_3^2 b_{43} d_1 R_4 + (a_3^2 b_{53} d_1 + a_4^2 b_{54} d_2) R_5 + (a_3^2 b_{63} d_1 + a_4^2 b_{64} d_2 \\
&+ a_5^2 b_{65} d_3) R_6 + (a_3^2 b_{73} d_1 + a_4^2 b_{74} d_2 + a_5^2 b_{75} d_3 + a_6^2 b_{76} d_4) R_7 \\
&+ (a_3^2 b_{83} d_1 + a_4^2 b_{84} d_2 + a_5^2 b_{85} d_3 + a_6^2 b_{86} d_4 + a_7^2 b_{87} d_5) R_8 \\
&+ (a_3^2 b_{93} d_1 + a_4^2 b_{94} d_2 + a_5^2 b_{95} d_3 + a_6^2 b_{96} d_4 + a_7^2 b_{97} d_5 \\
&+ a_8^2 b_{98} d_6) R_9 = \frac{1}{126}
\end{aligned}$$

$$\begin{aligned}
{}_2[\phi]_2\phi]_2 &= a_3 c_1^2 b_{43} R_4 + (a_3 c_1^2 b_{53} + a_4 c_2^2 b_{54}) R_5 + (a_3 c_1^2 b_{63} + a_4 c_2^2 b_{64} \\
&+ a_5 c_3^2 b_{65}) R_6 + (a_3 c_1^2 b_{73} + a_4 c_2^2 b_{74} + a_5 c_3^2 b_{75} + a_6 c_4^2 b_{76}) R_7 \\
&+ (a_3 c_1^2 b_{83} + a_4 c_2^2 b_{84} + a_5 c_3^2 b_{85} + a_6 c_4^2 b_{86} + a_7 c_5^2 b_{87}) R_8 \\
&+ (a_3 c_1^2 b_{93} + a_4 c_2^2 b_{94} + a_5 c_3^2 b_{95} + a_6 c_4^2 b_{96} + a_7 c_5^2 b_{97} \\
&+ a_8 c_6^2 b_{98}) R_9 = \frac{1}{168}
\end{aligned}$$

$$\begin{aligned}
{}_2[\phi]\phi^3]_2 &= a_3^2 b_{43} c_1 R_4 + (a_3^3 b_{53} c_1 + a_4^3 b_{54} c_2) R_5 + (a_3^3 b_{63} c_1 + a_4^3 b_{64} c_2 \\
&+ a_5^3 b_{65} c_3) R_6 + (a_3^3 b_{73} c_1 + a_4^3 b_{74} c_2 + a_5^3 b_{75} c_3 + a_6^3 b_{76} c_4) R_7 \\
&+ (a_3^3 b_{83} c_1 + a_4^3 b_{84} c_2 + a_5^3 b_{85} c_3 + a_6^3 b_{86} c_4 + a_7^3 b_{87} c_5) R_8 \\
&+ (a_3^3 b_{93} c_1 + a_4^3 b_{94} c_2 + a_5^3 b_{95} c_3 + a_6^3 b_{96} c_4 + a_7^3 b_{97} c_5 \\
&+ a_8^3 b_{98} c_6) R_9 = \frac{1}{84}
\end{aligned}$$

$${}_2\phi^5]_2 = g_1 R_3 + g_2 R_4 + g_3 R_5 + g_4 R_6 + g_5 R_7 + g_6 R_8 + g_7 R_9 = \frac{1}{42}$$

$$\begin{aligned}
[[_4\phi]_4\phi] &= a_6 c_1 b_{43} b_{54} b_{65} R_6 + \{c_1 b_{43} b_{54} b_{75} + [c_1 b_{43} b_{64} + (c_1 b_{53} \\
&+ c_2 b_{54}) b_{65}] b_{76}\} a_7 R_7 + \{c_1 b_{43} b_{54} b_{85} + [c_1 b_{43} b_{64} + (c_1 b_{53} \\
&+ c_2 b_{54}) b_{65} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65})] b_{86} + [c_1 b_{43} b_{74} \\
&+ (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{76}] b_{87}\} a_8 R_8 \\
&+ \{c_1 b_{43} b_{54} b_{95} + [c_1 b_{43} b_{64} + (c_1 b_{53} + c_2 b_{54}) b_{65}] b_{96} + [c_1 b_{43} b_{74} \\
&+ (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{76}] b_{97} \\
&+ [c_1 b_{43} b_{84} + (c_1 b_{53} + c_2 b_{54}) b_{85} + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} \\
&+ (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) b_{87}\} a_9 R_9 = \frac{1}{840}
\end{aligned}$$

$$\begin{aligned}
[[_3\phi^2]_3\phi] &= a_5 d_1 b_{43} b_{54} R_5 + a_6 [d_1 b_{43} b_{64} + (d_1 b_{53} + d_2 b_{54}) b_{65}] R_6 \\
&+ a_7 [d_1 b_{43} b_{74} + (d_1 b_{53} + d_2 b_{54}) b_{75} + (d_1 b_{63} + d_2 b_{64} \\
&+ d_3 b_{65}) b_{76}] R_7 + a_8 [d_1 b_{43} b_{84} + (d_1 b_{53} + d_2 b_{54}) b_{85} \\
&+ (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) b_{86} + (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} \\
&+ d_4 b_{76}) b_{87}] R_8 + a_9 [d_1 b_{43} b_{94} + (d_1 b_{53} + d_2 b_{54}) b_{95} + (d_1 b_{63} \\
&+ d_2 b_{64} + d_3 b_{65}) b_{96} + (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) b_{97} \\
&+ (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) b_{98}] R_9 = \frac{1}{420}
\end{aligned}$$

$$\begin{aligned}
[[_2[\phi]\phi]_2\phi] &= a_3 a_5 c_1 b_{43} b_{54} R_5 + a_6 [a_3 c_1 b_{43} b_{64} + (a_3 c_1 b_{53} + a_4 c_2 b_{54}) b_{65}] R_6 \\
&+ a_7 [a_3 c_1 b_{43} b_{74} + (a_3 c_1 b_{53} + a_4 c_2 b_{54}) b_{75} + (a_3 c_1 b_{63} \\
&+ a_4 c_2 b_{64} + a_5 c_3 b_{65}) b_{76}] R_7 + a_8 [a_3 c_1 b_{43} b_{84} + (a_3 c_1 b_{53} \\
&+ a_4 c_2 b_{54}) b_{85} + (a_3 c_1 b_{63} + a_4 c_2 b_{64} + a_5 c_3 b_{65}) b_{86} + (a_3 c_1 b_{73} \\
&+ a_4 c_2 b_{74} + a_5 c_3 b_{75} + a_6 c_4 b_{76}) b_{87}] R_8 + a_9 [a_3 c_1 b_{43} b_{94} \\
&+ (a_3 c_1 b_{53} + a_4 c_2 b_{54}) b_{95} + (a_3 c_1 b_{63} + a_4 c_2 b_{64} + a_5 c_3 b_{65}) b_{96} \\
&+ (a_3 c_1 b_{73} + a_4 c_2 b_{74} + a_5 c_3 b_{75} + a_6 c_4 b_{76}) b_{97} + (a_3 c_1 b_{83} \\
&+ a_4 c_2 b_{84} + a_5 c_3 b_{85} + a_6 c_4 b_{86} + a_7 c_5 b_{87}) b_{98}] R_9 = \frac{1}{280}
\end{aligned}$$

$$\begin{aligned}
[[_2\phi^3]_2\phi] &= a_4 e_1 b_{43} R_4 + a_5 (b_{53} e_1 + b_{54} e_2) R_5 + a_6 (b_{63} e_1 + b_{64} e_2 \\
&+ b_{65} e_3) R_6 + a_7 (b_{73} e_1 + b_{74} e_2 + b_{75} e_3 + b_{76} e_4) R_7 + a_8 (b_{83} e_1 \\
&+ b_{84} e_2 + b_{85} e_3 + b_{86} e_4 + b_{87} e_5) R_8 + a_9 (b_{93} e_1 + b_{94} e_2 + b_{95} e_3 \\
&+ b_{96} e_4 + b_{97} e_5 + b_{98} e_6) R_9 = \frac{1}{140}
\end{aligned}$$

$$\begin{aligned}
[[[{}_2\phi]_2\phi]\phi] &= a_4 a_5 c_1 b_{43} b_{54} R_5 + a_6 [a_4 c_1 b_{43} b_{64} + a_5 (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6 \\
&+ a_7 [a_4 c_1 b_{43} b_{74} + a_5 (c_1 b_{53} + c_2 b_{54}) b_{75} + a_6 (c_1 b_{63} + c_2 b_{64} \\
&+ c_3 b_{65}) b_{76}] R_7 + a_8 [a_4 c_1 b_{43} b_{84} + a_5 (c_1 b_{53} + c_2 b_{54}) b_{85} \\
&+ a_6 (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + a_7 (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&+ c_4 b_{76}) b_{87}] R_8 + a_9 [a_4 c_1 b_{43} b_{94} + a_5 (c_1 b_{53} + c_2 b_{54}) b_{95} \\
&+ a_6 (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{96} + a_7 (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&+ c_4 b_{76}) b_{97} + a_8 (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} \\
&+ c_5 b_{87}) b_{98}] R_9 = \frac{1}{210}
\end{aligned}$$

$$\begin{aligned}
[[[\phi^2]\phi]\phi] &= a_3 a_4 d_1 b_{43} R_4 + (a_3 d_1 b_{53} + a_4 d_2 b_{54}) a_5 R_5 + (a_3 d_1 b_{63} + a_4 d_2 b_{64} \\
&+ a_5 d_3 b_{65}) a_6 R_6 + (a_3 d_1 b_{73} + a_4 d_2 b_{74} + a_5 d_3 b_{75} + a_6 d_4 b_{76}) a_7 R_7 \\
&+ (a_3 d_1 b_{83} + a_4 d_2 b_{84} + a_5 d_3 b_{85} + a_6 d_4 b_{86} + a_7 d_5 b_{87}) a_8 R_8 \\
&+ (a_3 d_1 b_{93} + a_4 d_2 b_{94} + a_5 d_3 b_{95} + a_6 d_4 b_{96} + a_7 d_5 b_{97} \\
&+ a_8 d_6 b_{98}) a_9 R_9 = \frac{1}{105}
\end{aligned}$$

$$\begin{aligned}
[[[\phi]^2]\phi] &= a_4 c_1^2 b_{43} R_4 + (c_1^2 b_{53} + c_2^2 b_{54}) a_5 R_5 + (c_1^2 b_{63} + c_2^2 b_{64} \\
&+ c_3^2 b_{65}) a_6 R_6 + (c_1^2 b_{73} + c_2^2 b_{74} + c_3^2 b_{75} + c_4^2 b_{76}) a_7 R_7 \\
&+ (c_1^2 b_{83} + c_2^2 b_{84} + c_3^2 b_{85} + c_4^2 b_{86} + c_5^2 b_{87}) a_8 R_8 \\
&+ (c_1^2 b_{93} + c_2^2 b_{94} + c_3^2 b_{95} + c_4^2 b_{96} + c_5^2 b_{97} \\
&+ c_6^2 b_{98}) a_9 R_9 = \frac{1}{140}
\end{aligned}$$

$$\begin{aligned}
[[[\phi]\phi^2]\phi] &= a_4 a_3^2 b_{43} c_1 R_4 + a_5 (a_3^2 b_{53} c_1 + a_4^2 b_{54} c_2) R_5 + a_6 (a_3^2 b_{63} c_1 \\
&+ a_4^2 b_{64} c_2 + a_5^2 b_{65} c_3) R_6 + a_7 (a_3^2 b_{73} c_1 + a_4^2 b_{74} c_2 + a_5^2 b_{75} c_3 \\
&+ a_6^2 b_{76} c_4) R_7 + a_8 (a_3^2 b_{83} c_1 + a_4^2 b_{84} c_2 + a_5^2 b_{85} c_3 + a_6^2 b_{86} c_4 \\
&+ a_7^2 b_{87} c_5) R_8 + a_9 (a_3^2 b_{93} c_1 + a_4^2 b_{94} c_2 + a_5^2 b_{95} c_3 + a_6^2 b_{96} c_4 \\
&+ a_7^2 b_{97} c_5 + a_8^2 b_{98} c_6) R_9 = \frac{1}{70}
\end{aligned}$$

$$\begin{aligned}
[[\phi^4]\phi] &= a_3 f_1 R_3 + a_4 f_2 R_4 + a_5 f_3 R_5 + a_6 f_4 R_6 + a_7 f_5 R_7 + a_8 f_6 R_8 + a_9 f_7 R_9 \\
&= \frac{1}{35}
\end{aligned}$$

$$\begin{aligned}
[[{}_3\phi]_3[\phi]] &= c_1 c_3 b_{43} b_{54} R_5 + c_4 [c_1 b_{43} b_{64} + (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6 \\
&+ c_5 [c_1 b_{43} b_{74} + (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64} \\
&+ c_3 b_{65}) b_{76}] R_7 + c_6 [c_1 b_{43} b_{84} + (c_1 b_{53} + c_2 b_{54}) b_{85} \\
&+ (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} \\
&+ c_4 b_{76}) b_{87}] R_8 + c_7 [c_1 b_{43} b_{94} + (c_1 b_{53} + c_2 b_{54}) b_{95} + (c_1 b_{63} \\
&+ c_2 b_{64} + c_3 b_{65}) b_{96} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) b_{97} \\
&+ (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87}) b_{98}] R_9 = \frac{1}{336}
\end{aligned}$$

$$\begin{aligned}
[[{}_2\phi^2]_2[\phi]] &= d_1 b_{43} c_2 R_4 + (d_1 b_{53} + d_2 b_{54}) c_3 R_5 + (d_1 b_{63} + d_2 b_{64} \\
&+ d_3 b_{65}) c_4 R_6 + (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) c_5 R_7 \\
&+ (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) c_6 R_8 + (d_1 b_{93} \\
&+ d_2 b_{94} + d_3 b_{95} + d_4 b_{96} + d_5 b_{97} + d_6 b_{98}) c_7 R_9 = \frac{1}{168}
\end{aligned}$$

$$\begin{aligned}
[[[\phi]\phi][\phi]] &= a_3c_1b_{43}c_2R_4 + (a_3c_1b_{53} + a_4c_2b_{54})c_3R_5 + (a_3c_1b_{63} \\
&+ a_4c_2b_{64} + a_5c_3b_{65})c_4R_6 + (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} \\
&+ a_6c_4b_{76})c_5R_7 + (a_3c_1b_{83} + a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} \\
&+ a_7c_5b_{87})c_6R_8 + (a_3c_1b_{93} + a_4c_2b_{94} + a_5c_3b_{95} + a_6c_4b_{96} \\
&+ a_7c_5b_{97} + a_8c_6b_{98})c_7R_9 = \frac{1}{112}
\end{aligned}$$

$$\begin{aligned}
[[\phi^3][\phi]] &= c_1e_1R_3 + c_2e_2R_4 + c_3e_3R_5 + c_4e_4R_6 + c_5e_5R_7 + c_6e_6R_8 \\
&+ c_7e_7R_9 = \frac{1}{56}
\end{aligned}$$

$$\begin{aligned}
[[{}_3\phi]_3\phi^2] &= a_5^2c_1b_{43}b_{54}R_5 + a_6^2[c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]R_6 \\
&+ a_7^2[c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} \\
&+ c_3b_{65})b_{76}]R_7 + a_8^2[c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} \\
&+ (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})b_{87}]R_8 + a_9^2[c_1b_{43}b_{94} + (c_1b_{53} + c_2b_{54})b_{95} \\
&+ (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} \\
&+ c_4b_{76})b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} \\
&+ c_5b_{87})b_{98}]R_9 = \frac{1}{168}
\end{aligned}$$

$$\begin{aligned}
[[2\phi^2]_2\phi^2] &= a_4^2 d_1 b_{43} R_4 + a_5^2 (d_1 b_{53} + d_2 b_{54}) R_5 + a_6^2 (d_1 b_{63} + d_2 b_{64} \\
&\quad + d_3 b_{65}) R_6 + a_7^2 (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) R_7 \\
&\quad + a_8^2 (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) R_8 \\
&\quad + a_9^2 (d_1 b_{93} + d_2 b_{94} + d_3 b_{95} + d_4 b_{96} + d_5 b_{97} \\
&\quad + d_6 b_{98}) R_9 = \frac{1}{84}
\end{aligned}$$

$$\begin{aligned}
[[[\phi]\phi] \phi^2] &= a_3 a_4^2 b_{43} c_1 R_4 + a_5^2 (a_3 b_{53} c_1 + a_4 b_{54} c_2) R_5 + a_6^2 (a_3 b_{63} c_1 \\
&\quad + a_4 b_{64} c_2 + a_5 b_{65} c_3) R_6 + a_7^2 (a_3 b_{73} c_1 + a_4 b_{74} c_2 + a_5 b_{75} c_3 \\
&\quad + a_6 b_{76} c_4) R_7 + a_8^2 (a_3 b_{83} c_1 + a_4 b_{84} c_2 + a_5 b_{85} c_3 + a_6 b_{86} c_4 \\
&\quad + a_7 b_{87} c_5) R_8 + a_9^2 (a_3 b_{93} c_1 + a_4 b_{94} c_2 + a_5 b_{95} c_3 + a_6 b_{96} c_4 \\
&\quad + a_7 b_{97} c_5 + a_8 b_{98} c_6) R_9 = \frac{1}{56}
\end{aligned}$$

$$\begin{aligned}
[[\phi^3]\phi^2] &= a_3^2 e_1 R_3 + a_4^2 e_2 R_4 + a_5^2 e_3 R_5 + a_6^2 e_4 R_6 + a_7^2 e_5 R_7 + a_8^2 e_6 R_8 \\
&\quad + a_9^2 e_7 R_9 = \frac{1}{28}
\end{aligned}$$

$$\begin{aligned}
[[2\phi]_2^2] &= (c_1 b_{43})^2 R_4 + (c_1 b_{53} + c_2 b_{54})^2 R_5 + (c_1 b_{63} + c_2 b_{64} \\
&\quad + c_3 b_{65})^2 R_6 + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76})^2 R_7 \\
&\quad + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87})^2 R_8 \\
&\quad + (c_1 b_{93} + c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} + c_6 b_{98})^2 R_9 = \frac{1}{252}
\end{aligned}$$

$$\begin{aligned}
[[2\phi]_2[\phi^2]] &= c_1 d_2 b_{43} R_4 + (c_1 b_{53} + c_2 b_{54}) d_3 R_5 + (c_1 b_{63} + c_2 b_{64} \\
&+ c_3 b_{65}) d_4 R_6 + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) d_5 R_7 \\
&+ (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87}) d_6 R_8 \\
&+ (c_1 b_{93} + c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} \\
&+ c_6 b_{98}) d_7 R_9 = \frac{1}{126}
\end{aligned}$$

$$[[\phi^2]^2] = d_1^2 R_3 + d_2^2 R_4 + d_3^2 R_5 + d_4^2 R_6 + d_5^2 R_7 + d_6^2 R_8 + d_7^2 R_9 = \frac{1}{63}$$

$$\begin{aligned}
[[2\phi]_2[\phi]\phi] &= a_4 c_1 c_2 b_{43} R_4 + a_5 (c_1 b_{53} + c_2 b_{54}) c_3 R_5 + a_6 (c_1 b_{63} + c_2 b_{64} \\
&+ c_3 b_{65}) c_4 R_6 + a_7 (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76}) c_5 R_7 \\
&+ a_8 (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87}) c_6 R_8 + a_9 (c_1 b_{93} \\
&+ c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} + c_6 b_{98}) c_7 R_9 = \frac{1}{84}
\end{aligned}$$

$$\begin{aligned}
[[\phi^2][\phi]\phi] &= a_3 c_1 d_1 R_3 + a_4 c_2 d_2 R_4 + a_5 c_3 d_3 R_5 + a_6 c_4 d_4 R_6 + a_7 c_5 d_5 R_7 \\
&+ a_8 c_6 d_6 R_8 + a_9 c_7 d_7 R_9 = \frac{1}{42}
\end{aligned}$$

$$\begin{aligned}
[[{}_2\phi]_2\phi^3] &= a_4^3 b_{43} c_1 R_4 + a_5^3 (b_{53} c_1 + b_{54} c_2) R_5 + a_6^3 (b_{63} c_1 + b_{64} c_2 \\
&\quad + b_{65} c_3) R_6 + a_7^3 (b_{73} c_1 + b_{74} c_2 + b_{75} c_3 + b_{76} c_4) R_7 + a_8^3 (b_{83} c_1 \\
&\quad + b_{84} c_2 + b_{85} c_3 + b_{86} c_4 + b_{87} c_5) R_8 + a_9^3 (b_{93} c_1 + b_{94} c_2 + b_{95} c_3 \\
&\quad + b_{96} c_4 + b_{97} c_5 + b_{98} c_6) R_9 = \frac{1}{42}
\end{aligned}$$

$$\begin{aligned}
[[\phi^2]\phi^3] &= a_3^3 d_1 R_3 + a_4^3 d_2 R_4 + a_5^3 d_3 R_5 + a_6^3 d_4 R_6 + a_7^3 d_5 R_7 + a_8^3 d_6 R_8 \\
&\quad + a_9^3 d_7 R_9 = \frac{1}{21}
\end{aligned}$$

$$[[\phi]^3] = c_1^3 R_3 + c_2^3 R_4 + c_3^3 R_5 + c_4^3 R_6 + c_5^3 R_7 + c_6^3 R_8 + c_7^3 R_9 = \frac{1}{56}$$

$$\begin{aligned}
[[\phi]^2\phi^2] &= a_3^2 c_1^2 R_3 + a_4^2 c_2^2 R_4 + a_5^2 c_3^2 R_5 + a_6^2 c_4^2 R_6 + a_7^2 c_5^2 R_7 \\
&\quad + a_8^2 c_6^2 R_8 + a_9^2 c_7^2 R_9 = \frac{1}{28}
\end{aligned}$$

$$\begin{aligned}
[[\phi]\phi^4] &= a_3^4 c_1 R_3 + a_4^4 c_2 R_4 + a_5^4 c_3 R_5 + a_6^4 c_4 R_6 + a_7^4 c_5 R_7 + a_8^4 c_6 R_8 \\
&\quad + a_9^4 c_7 R_9 = \frac{1}{14}
\end{aligned}$$

$$\begin{aligned}
[\phi^6] &= a_2^6 R_2 + a_3^6 R_3 + a_4^6 R_4 + a_5^6 R_5 + a_6^6 R_6 + a_7^6 R_7 + a_8^6 R_8 \\
&\quad + a_9^6 R_9 = \frac{1}{7}
\end{aligned}$$