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THE GENERATION OF A GAUSSIAN RANDOM
PROCESS IN A POSITION PARAMETER

By David L. Finn

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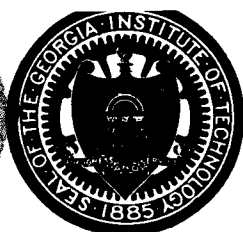
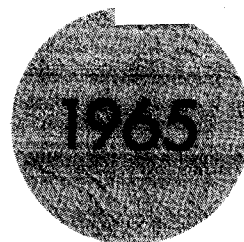
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ABSTRACT

A method is presented for approximating, by use of analog computer components, any prescribed stationary Gaussian random process depending only on a position parameter x . The output of the analog mechanization system simulates the effect of the prescribed random process on some physical device or sensing element whose variable location is specified by the position parameter x . The two inputs to the mechanization system are Gaussian white noise and the first derivative of the function of time describing the position of the sensing element. Best approximation of the prescribed random process is obtained when the second derivative of the function of time describing the position of the sensing element is restricted to small values.

PREFACE

The research work reported in this technical note was done by Dr. D. L. Finn and Mr. W. A. Yates in association with Project No. A-588 of the Engineering Experiment Station at the Georgia Institute of Technology. This is in support of the program of the Simulation Branch of the Computation Laboratory at the Marshall Space Flight Center. The work done by Mr. Yates, a doctoral fellow in the School of Electrical Engineering, has been carried out as a portion of his doctoral thesis research.

CONTENTS

	<u>Page</u>
Title Page	
Abstract	i
Preface	ii
List of Illustrations	iv
I. INTRODUCTION AND SUMMARY	1
II. RANDOM PROCESSES DEPENDENT ON A POSITION PARAMETER	3
2.1 <u>General Description</u>	3
2.2 <u>Transformation into Random Processes Dependent on a Time Parameter</u>	3
2.3 <u>Mathematical Characterization</u>	5
III. THE MECHANIZATION PROCEDURE	8
3.1 <u>Derivation of an Equation Characterizing the Mechanization System</u>	8
3.2 <u>Adjustment of Level of White Noise Generator</u>	13
3.3 <u>A Mechanization System</u>	13
IV. EXAMPLE, EXPONENTIAL AUTOCORRELATION FUNCTION	17
V. CONCLUSIONS	24
Bibliography	25

LIST OF ILLUSTRATIONS

Figure	Page
2.1 Sample Functions of a Random Process Depending on a Position Parameter	4
2.2 Transformation of a Position Parameter Process $g(x)$ into a Time Parameter Process $g(x(t))$ by Specification of Position as a Function of Time $x(t)$	6
3.1 Mechanization of the Composite Process $g(x(t))$ with $x(t) = t$	10
3.2 Mechanization of the Composite Process $g(x(t))$ with $x(t) = vt + K$	10
3.3 Approximation of the Composite Process $g(x(t))$	10
3.4 A Mechanization System for the Composite Process $g(x(t))$	15
4.1 Mechanization System for Exponential Autocorrelation Function	18
4.2 Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 1	20
4.3 Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 2	21
4.4 Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 3	22
4.5 Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 4	23

I. INTRODUCTION AND SUMMARY

In Technical Note No. 5 for this project a special class of stochastic processes was defined. These processes were designated as partially stationary stochastic processes. This class has potential usefulness in the development of methods of generating nonstationary stochastic processes for simulating the random wind disturbances that affect a rocket in flight.

A stochastic process representing random wind disturbances necessarily depends on one or more position parameters as well as time. For example, the wind disturbance affecting a rocket in flight might be represented by a random process $g(x, t)$. Here, x is taken to represent the altitude of the rocket at the time instant t . For a particular rocket flight the altitude is described by a function of time $x(t)$. In this case, the wind disturbance affecting the rocket is represented by the composite process $g(x(t))$. The composite process $g(x(t))$ depends only on a time parameter.

In Technical Note No. 3 for this project a synthesis method was given for the generation of nonstationary time-parameter random processes having a prescribed first and second moment. This synthesis method can be used in the simulation of the random disturbance affecting a rocket in flight following a prescribed trajectory. The present research effort has been directed toward the development of simulation methods that do not require a prior specification of the flight trajectory.

It appears that the most difficult aspect of the mechanization of a process of the type $g(x, t)$ mentioned above is obtaining the required dependence on the position parameter x . For this reason the preliminary work that is reported in this technical note has been directed toward the generation of a random process that depends only on a position parameter. A study of possible applications listed in Chapter II indicates that the mechanization procedure presented in this technical note has substantial potential usefulness in its own right. More important, the present study has provided techniques that appear to be applicable in the more general problem of generating a random process depending on both a position and time parameter.

Chapter II of this technical note is devoted to a general discussion of random processes that depend on a position parameter. Chapter III presents a derivation of the transfer function that is used as a basis for the approximation of a stationary Gaussian random process depending on a single position parameter x . In general for good approximation the position parameter, which

is taken to describe the position of some physical element, is constrained to have a small second derivative with respect to time. This means that the physical element must have a small acceleration.

Section 3.3 of this technical note presents one mechanization system that generates a random process of the type described in Section 3.1. Chapter IV contains the results of a theoretical and experimental study of a mechanization system that generates a stationary Gaussian process having an exponential autocorrelation function. It is shown that this system theoretically provides an exact realization of the desired random process.

II. RANDOM PROCESSES DEPENDENT ON A POSITION PARAMETER

A general description of random processes depending on a position parameter is given in this chapter. Several applications of possible interest in analog simulation studies are mentioned for random processes depending on a position parameter. The transformation of a given position-parameter process into a composite process depending on time is discussed. The composite process evaluates the effect of the position-parameter process on some physical element whose position varies with time.

2.1 General Description

A random process (or stochastic process) $g(x)$, $-\infty < x < \infty$ is an indexed collection of random variables. The process may be characterized as an ensemble of sample functions associated with a probability measure. In this technical note the sample functions of the process $g(x)$ are considered to depend on a position parameter x .

Sample functions of a random process of the type under consideration might typically have an appearance as shown in Figure 2.1. Many different physical processes conceptually might be represented as random processes depending on a position parameter. The sample functions depicted in Figure 2.1 might represent the random surface variations of an airport runway or a highway. They might represent surface irregularities in a channel or tube that is guiding the movement of a liquid or gaseous substance. They possibly could depict small random variations in the density or hardness of a solid material that is being processed by a saw, lathe, or drill press. The sample functions might simulate variations in the width or thickness of a long ribbon of material subjected to a rolling operation in an industrial plant. In all cases, the sample functions are taken to represent some type of random irregularity whose value is dependent only upon its position with respect to a spatial coordinate system.

2.2 Transformation into Random Processes Dependent on a Time Parameter

For the physical processes mentioned in the previous section a physical device or sensing element of some kind whose location in space may be described by a position coordinate x is affected by a random process $g(x)$ depending only on the position coordinate. As the sensing element changes position a function of time $x(t)$ is generated that describes the instantaneous location of the element. At the position x the effect of the random process on the sensing element is described by the value of $g(x)$. When the position is a function of

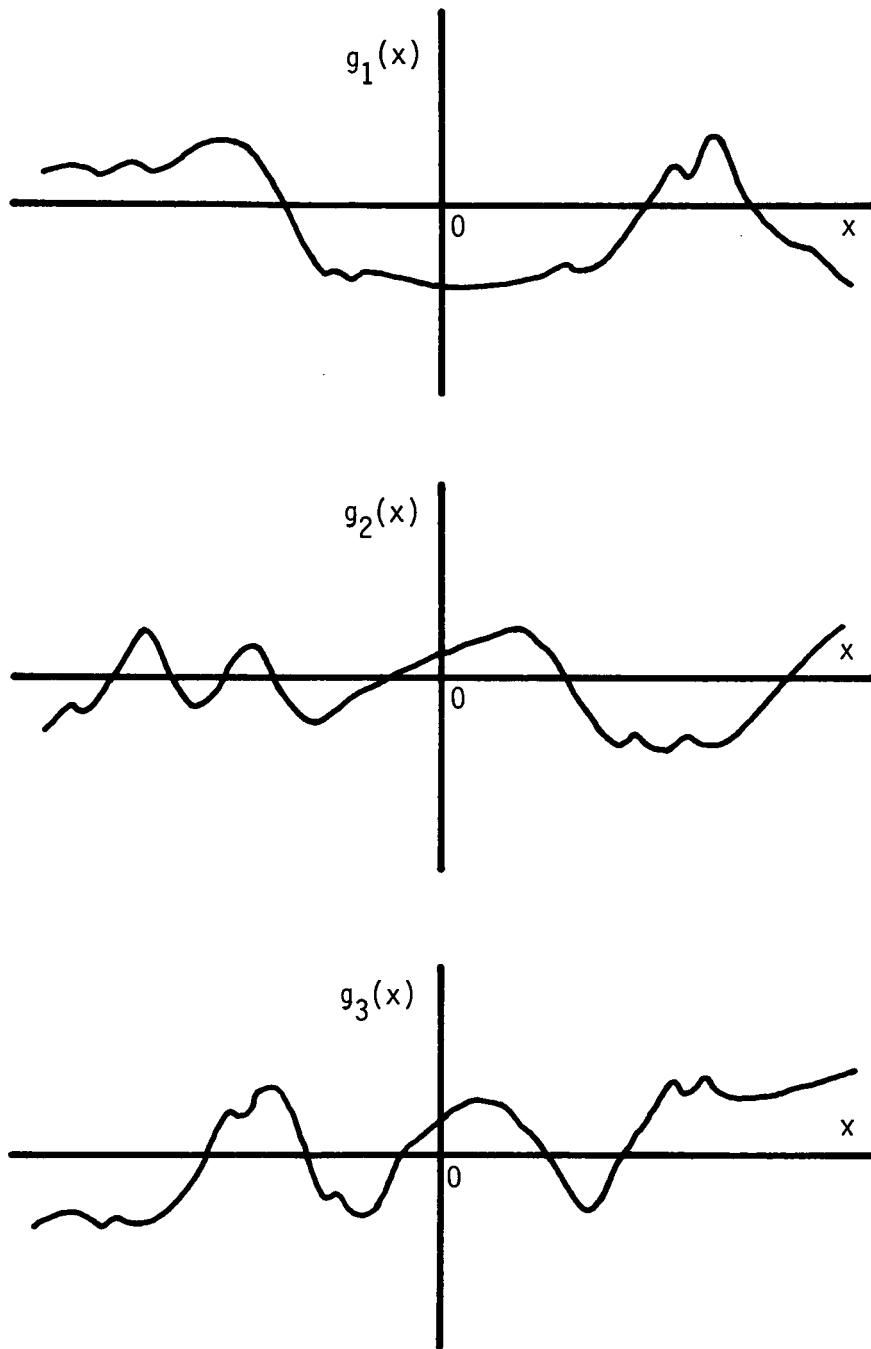


Figure 2.1. Sample Functions of a Random Process Depending on a Position Parameter.

time $x(t)$ the effect of the random process on the sensing element at the instant of time t is described by the value of the composite process $g(x(t))$. A typical sample function for a process $g(x)$ is shown in Figure 2.2. Also, the corresponding sample function for the composite time dependent process $g(x(t))$ is shown for an assumed time variation of position $x(t)$.

The fact that the composite process $g(x(t))$ is dependent upon a time parameter provides the potentiality that the effect of the process $g(x)$ on a sensing element having position $x(t)$ may be simulated by an analog computer system operating in the time domain.

2.3 Mathematical Characterization

Gaussian random processes are utilized in the mechanization system whose synthesis is presented in this technical note.* A Gaussian process is uniquely characterized by specification of its mean and its autocorrelation function.

The mean m_g of a random process $g(x)$ is defined as the expected value of the random variable generated by the process for a fixed position x .

$$m_g = E(g(x)). \quad (2.1)$$

The autocorrelation function $R_g(x_1, x_2)$ is defined as

$$R_g(x_1, x_2) = E(g(x_1)g(x_2)). \quad (2.2)$$

A random process is said to be stationary in the wide-sense if its mean and autocorrelation function are independent of the origin for position x . For a wide-sense stationary random process there results:

$$m_g = C. \quad (2.3)$$

$$R_g(x_1, x_2) = E(g(x_2+\tau)g(x_2)) = R_g(\tau), \quad \tau = x_1 - x_2.$$

Here, the mean is a constant and the autocorrelation function depends only on the position difference $\tau = x_1 - x_2$.

*

Reference 2, page 145.

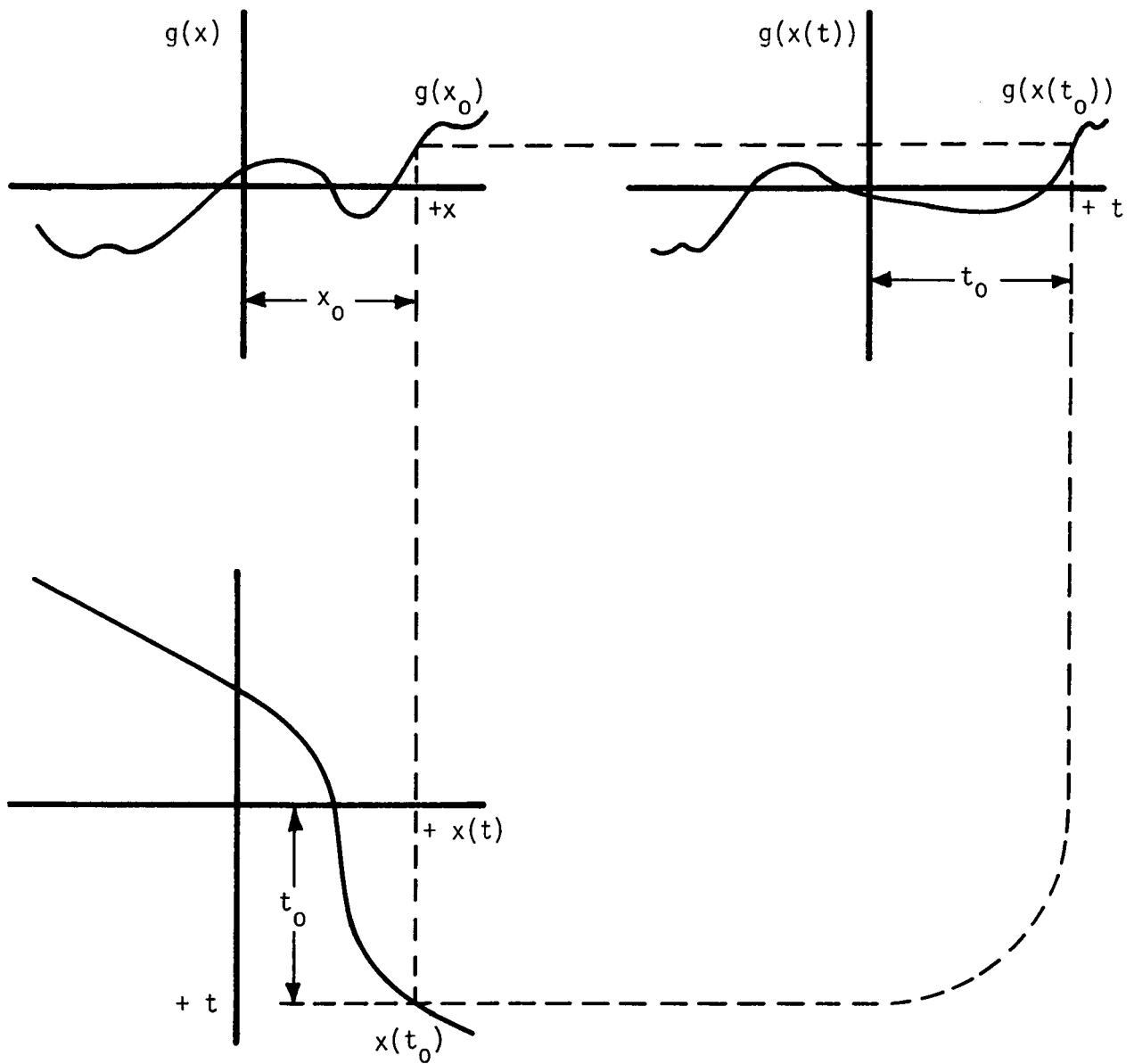


Figure 2.2. Transformation of a Position Parameter Process $g(x)$ into a Time Parameter Process $g(x(t))$ by Specification of Position as a Function of Time $x(t)$.

The power spectral density $S_g(\omega)$ for a wide-sense stationary random process is defined as*

$$S_g(\omega) = \int_{-\infty}^{\infty} R_g(\tau) e^{-j\omega\tau} d\tau. \quad (2.4)$$

The autocorrelation function may be expressed in terms of the power spectral density as

$$R_g(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega) e^{j\omega\tau} d\omega. \quad (2.5)$$

*Reference 2, page 104; Reference 7, page 338.

III. THE MECHANIZATION PROCEDURE

In this chapter a synthesis procedure is presented for a mechanization system to approximate composite random processes of the type described in Chapter II. Such a composite process represents a random effect, depending only on a space coordinate x , that is experienced by a physical device or sensing element having a variable position specified by the coordinate x . This variable position is described by a function of time $x(t)$.

The procedure is restricted to the generation of a Gaussian random process $g(x)$ that is stationary in a position parameter x . The associated composite process $g(x(t))$ is in general nonstationary in time. In general the function of time $x(t)$ should be constrained to have a small second derivative with respect to time in order for the output of the mechanization system to provide a close approximation to the composite process $g(x(t))$. This means that the physical device or sensing element should have a small acceleration.

It is known that for any random process having given mean and autocorrelation function, there exists a Gaussian process with identical mean and autocorrelation function.* Thus, if only the first two statistical moments of an arbitrary wide-sense stationary random process are of interest in a simulation study, the generation of an appropriate Gaussian process will suffice.

3.1 Derivation of an Equation Characterizing the Mechanization System

Without loss of generality the synthesis technique presented in this technical note is concerned with the realization of processes $g(x)$ having mean equal to zero. Processes having nonzero mean can be realized as the sum of the random process generated by this technique and the output of a source having constant output numerically equal to m_g .

It is assumed that the stationary Gaussian process $g(x)$ is such that if x is replaced by t then the time dependent process $g(t)$ can be realized by the application of Gaussian white noise to a linear filter. Further, the power spectral density is assumed to be expressed as a ratio of two polynomials in ω . Thus restricted, the power spectral density may be expressed as**

$$S_g(\omega) = \left| H_g(p) \right|_{p=j\omega}^2 \quad (3.1)$$

* Reference 8, page 3.

** Reference 2, page 227.

Here, $H_g(p)$ is the transfer function of a time invariant linear filter. This transfer function may be expressed as a ratio of two polynomials in complex frequency p .

$$H_g(p) = \frac{a_0 + a_1 p + a_2 p^2 + \dots + a_m p^m}{b_0 + b_1 p + b_2 p^2 + \dots + b_n p^n} \quad (3.2)$$

A technique for determining the a_k and b_k in the transfer function of equation (3.2) is discussed in Reference 2, page 233. These are evaluated for a simple example in Chapter 4 of this technical note.

The transfer function $H_g(p)$ of (3.2) may be realized in a variety of ways by the use of analog computer elements.* Specifically, this transfer function $H_g(p)$ may be realized by an appropriate mechanization of the differential equation shown in equation (3.3).

$$\begin{aligned} b_0 e + b_1 \frac{de}{dt} + b_2 \frac{d^2 e}{dt^2} + \dots + b_n \frac{d^n e}{dt^n} \\ = a_0 w + a_1 \frac{dw}{dt} + a_2 \frac{d^2 w}{dt^2} + \dots + a_m \frac{d^m w}{dt^m} \end{aligned} \quad (3.3)$$

If a Gaussian white noise random process $w(t)$ having a power spectral density of unity is applied to a linear filter having the transfer function $H_g(p)$ of (3.2) the output of the filter is a Gaussian random process $e(t)$ having power spectral density equal to $S_g(\omega)$ of (3.1) and corresponding auto-correlation function $R_g(\tau)$. Thus, the filter output $e(t)$ is a realization of the random process $g(x)$ where x has been replaced by t . It is clear that the output $e(t)$ thus represents a realization of the composite random process $g(x(t))$ where the position of the sensing element is described by the function $x(t) = t$. This mechanization is depicted in Figure 3.1.

The mechanization of the composite process $g(x(t))$ when the position of the sensing element is described by the equation $x(t) = vt + K$ may be accomplished in a straightforward manner. Here, v and K are assumed to be arbitrary constants. It is noted that v is the derivative of $x(t)$ with respect to time.

The mechanization just described is again accomplished by applying Gaussian white noise $w(t)$ to a linear filter having output $e(t)$. The output of the filter

* Reference 4, Reference 6.

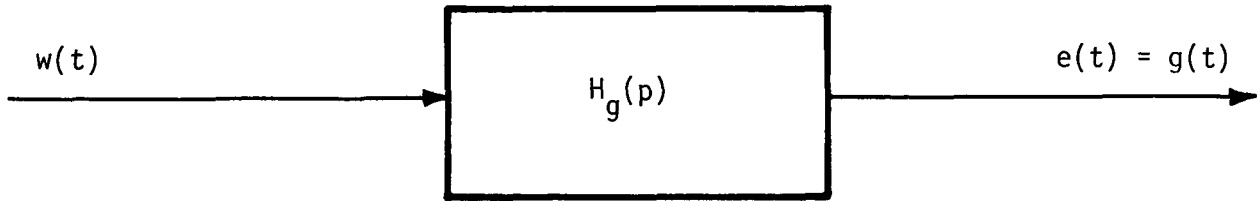


Figure 3.1. Mechanization of the Composite Process $g(x(t))$ with $x(t) = t$.

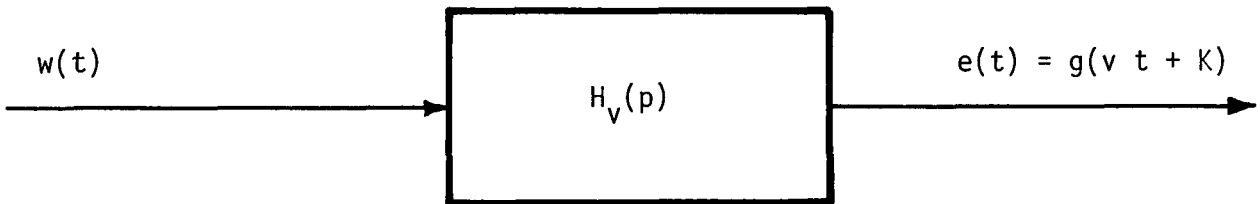


Figure 3.2. Mechanization of the Composite Process $g(x(t))$ with $x(t) = v t + K$.

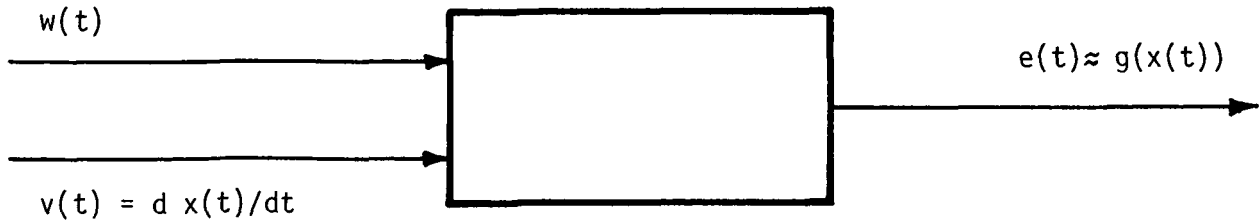


Figure 3.3. Approximation of the Composite Process $g(x(t))$.

is to be

$$e(t) = g(x(t)) = g(vt + K). \quad (3.4)$$

The autocorrelation function of the output is

$$R_v(t_2 + \tau, t_2) = E(g(vt_2 + v\tau + K)g(vt_2 + K)) = R_g(v\tau). \quad (3.5)$$

Thus, mechanization may be accomplished by providing a filter whose output $e(t)$ has autocorrelation function:

$$R_v(\tau) = R_g(v\tau). \quad (3.6)$$

The power spectral density for $e(t)$ is

$$\begin{aligned} S_v(\omega) &= \int_{-\infty}^{\infty} R_v(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} R_g(v\tau) e^{-j\omega\tau} d\tau \\ &= \frac{1}{|v|} \int_{-\infty}^{\infty} R_g(\alpha) e^{-j\frac{\omega}{v}\alpha} d\alpha \\ &= \frac{1}{|v|} S_g\left(\frac{\omega}{v}\right). \end{aligned} \quad (3.7)$$

The mechanization may be accomplished by applying a Gaussian white noise random process $w(t)$ having $S_w(\omega) = 1$ to a linear filter having transfer function $H_v(p)$ such that

$$\left| H_v(p) \right|_{p=j\omega}^2 = S_v(\omega) = \frac{1}{|v|} S_g\left(\frac{\omega}{v}\right) = \frac{1}{|v|} \left| H_g\left(\frac{p}{v}\right) \right|_{p=j\omega}^2. \quad (3.8)$$

Thus, a satisfactory transfer function to accomplish the mechanization is

$$H_v(p) = \frac{1}{\sqrt{|v|}} H_g\left(\frac{p}{v}\right) = \frac{1}{\sqrt{|v|}} \frac{a_0 + a_1 p/v + \dots + a_m p^m/v^m}{b_0 + b_1 p/v + \dots + b_n p^n/v^n}. \quad (3.9)$$

This transfer function may, again, be realized by a variety of methods. In particular, $H_v(p)$ may be realized by a mechanization of the following differential equation.

$$\sqrt{|v|} (b_0 v^n e + b_1 v^{n-1} \frac{de}{dt} + \dots + \frac{d^n e}{dt^n}) = a_0 v^n w + a_1 v^{n-1} \frac{dw}{dt} + \dots + a_{n-1} v \frac{d^{n-1} w}{dt^{n-1}}. \quad (3.10)$$

Here, it has been assumed, without loss of generality, that $m = n-1$ and that $b_n = 1$. This mechanization is depicted in Figure 3.2. It is noted that the transfer function $H_v(p)$ may be realized by the use of a fixed parameter linear filter. However, a different filter must be utilized for each selection of v .

Consideration of the differential equation of (3.10) shows a possible means for synthesizing a system that generates a random process that approximates the composite process $g(x(t))$ when $x(t)$ is an arbitrary function of time having a second derivative restricted to small magnitudes. If the v in (3.10) is re-defined as

$$v(t) = \frac{dx(t)}{dt} \quad (3.11)$$

$$x(t) = \int_0^t x(\alpha) d\alpha + x(0),$$

the differential equation of (3.10) becomes

$$\sqrt{|v(t)|} (b_0 v(t)^n e + b_1 v(t)^{n-1} \frac{de}{dt} + \dots + \frac{d^n e}{dt^n}) \quad (3.12)$$

$$= a_0 v(t)^n w + a_1 v(t)^{n-1} \frac{dw}{dt} + \dots + a_{n-1} v(t) \frac{d^{n-1} w}{dt^{n-1}}.$$

The differential equation of (3.12) may be mechanized by the use of standard analog computer elements such as multipliers and operational amplifiers.* Such a mechanization is depicted in Figure 3.3.

*

Reference 4, Reference 6.

The previous discussion has shown that the mechanization system depicted in Figure 3.3 generates exactly the composite process $g(x(t))$ for the case that $x(t) = vt + K$ regardless of the fixed value assigned to v . A change in the value of v is achieved merely by changing the value of the input $v(t)$.

Because the system of Figure 3.3 provides an exact realization of $g(x(t))$ for any constant value of $v(t)$ it is expected to provide a close approximation to a correct realization for slowly varying functions of time $v(t)$ defined by (3.11). Thus, in general, the second derivative of $x(t)$ is to be kept to a small value for best approximation.

It is to be emphasized that in general any analog computer realization of the transfer function of (3.9) provides a possibility of adaptation for approximating the composite process $g(x(t))$. To accomplish this adaptation, multipliers and other components are added to the system to allow $v(t)$ to be applied as a system input. One method of accomplishing this is presented in the next section.

In general the errors obtained in approximating $g(x(t))$ are expected to depend on the procedure used to implement the transfer function $H_v(p)$. A general analysis has not been made to evaluate errors in approximation. However, for the example presented in Chapter 4 it is shown that an exact realization of the composite process is obtained for arbitrary non-negative functions $v(t)$.

3.2 Adjustment of Level of White Noise Generator

The random process $g(x)$ is assumed to be stationary. Thus, the statistical mean squared value $E(g(x(t))^2)$ of the composite process $g(x(t))$ is independent of the waveform $x(t)$. This fact provides a convenient method of adjusting the white noise input of the system of Figure 3.3 to provide the desired mean squared value of $g(x(t))$. To accomplish this, the input $v(t)$ is adjusted to any convenient fixed value. The output of the system is then a stationary Gaussian process in the time parameter t . The timewise average of the squared output is then equal to the statistical average of the squared output. Thus an rms meter may be used to determine $E(g(x(t))^2)$. The levels of the white noise input may be varied until the rms meter at the output indicates the correct mean squared value for the composite process.

3.3 A Mechanization System

It was noted in the previous section that any procedure that realizes the transfer function $H_v(p)$ of (3.9) provides a possibility of adaptation for the approximation of the composite process $g(x(t))$. To accomplish this adaptation

it is necessary to modify the mechanization system so that v , the time derivative of the position x , can be applied as a time-function input to the system.

One method of realizing the transfer function $H_v(p)$ is to define a new set of equations equivalent to (3.10) for fixed positive values of v .*

$$\begin{aligned}
 e_1 &= e \\
 \frac{de_1}{dt} &= \sqrt{|v|} (e_2 + a_{n-1} w - b_{n-1} \sqrt{|v|} e_1) \\
 \frac{de_2}{dt} &= v(e_3 + a_{n-2} w - b_{n-2} \sqrt{|v|} e_1) \\
 \frac{de_3}{dt} &= v(e_4 + a_{n-3} w - b_{n-3} \sqrt{|v|} e_1) \\
 \frac{de_4}{dt} &= v(e_5 + a_{n-4} w - b_{n-4} \sqrt{|v|} e_1) \\
 &\quad \circ \quad \circ \quad \circ \\
 \frac{de_{n-1}}{dt} &= v(e_n + a_1 w - b_1 \sqrt{|v|} e_1) \\
 \frac{de_n}{dt} &= v(a_0 w - b_0 \sqrt{|v|} e_1).
 \end{aligned} \tag{3.13}$$

A mechanization for this set of equations is shown in Figure 3.4. The output of the system of Figure 3.4 provides an exact realization of the composite process $g(x(t))$ when the position parameter varies according to the equation $x(t) = vt + K$. This realization is exact for any fixed value of v . In accordance with the discussion of the preceding section the output of the system of Figure 3.4 is expected to provide a close approximation to the composite process $g(x(t))$ whenever the second derivative of $x(t)$ is small.

To accomplish the mechanization the input v to the system is set equal to the derivative of $x(t)$.

$$v = \frac{dx(t)}{dt} \tag{3.14}$$

*Reference 8, page 8.

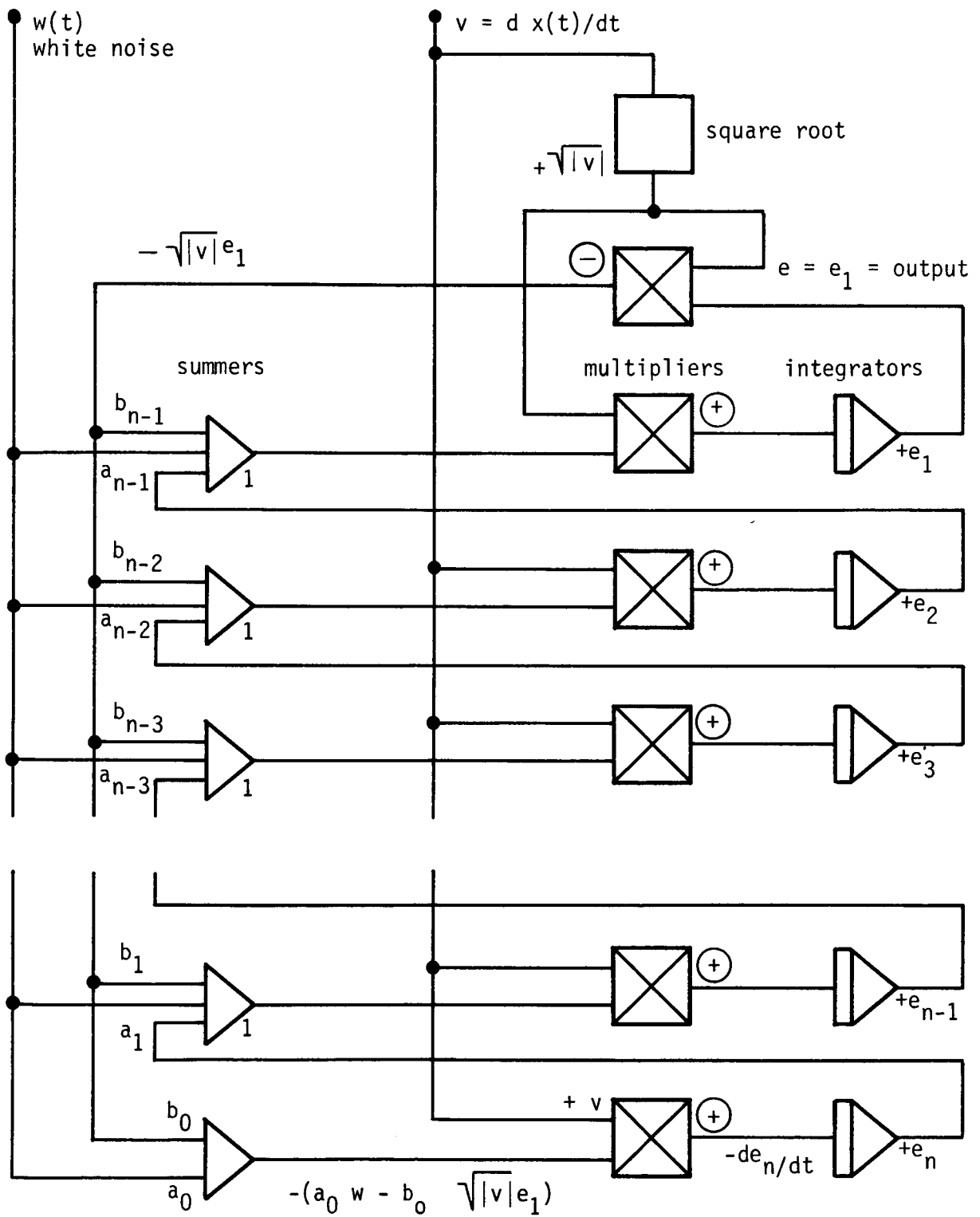


Figure 3.4. A Mechanization System for the Composite Process $g(x(t))$.

The a_K and b_K in the equations of (3.13) are found by use of the power spectral density $S_g(\omega)$ of the random process described in the discussion of equation (3.1).

$$S_g(\omega) = \left| H_g(p) \right|_{p=j\omega}^2$$

(3.15)

$$H_g(p) = \frac{a_0 + a_1 p + a_2 p^2 + \dots + a_{n-1} p^{n-1}}{b_0 + b_1 p + b_2 p^2 + \dots + b_{n-1} p^{n-1} + p^n}.$$

Here, the b_n and m in equation (3.2) have been assumed to be $b_n = 1$ and $m = n-1$.

IV. EXAMPLE, EXPONENTIAL AUTOCORRELATION FUNCTION

In this chapter a mechanization system is synthesized for the generation of the composite process $g(x(t))$ associated with a stationary Gaussian process $g(x)$ having an exponential autocorrelation function. Curves are presented showing the measured autocorrelation functions for the output of the mechanization system compared with the autocorrelation function for $g(x(t))$. The measurements were processed by the procedure discussed in Technical Note No. 3. Ensembles of 200 sample functions each were used in computation of the measured value of the autocorrelation function.

The autocorrelation function for the stationary Gaussian process $g(x)$ is assumed to be

$$R_g(\tau) = \frac{\alpha^2}{2\beta} e^{-\beta|\tau|} \quad (4.1)$$

The mean of $g(x)$ is assumed to be zero. The power spectral density for $g(x)$ is found by use of (2.4) to be

$$S_g(\omega) = \frac{\alpha^2}{\omega^2 + \beta^2} \quad (4.2)$$

By use of (3.1) the transfer function $H_g(p)$ is

$$H_g(p) = \frac{\alpha}{\beta + p} \quad (4.3)$$

The transfer function $H_v(p)$ defined by (3.9) is for the present example

$$H_v(p) = \frac{1}{\sqrt{|v|}} \frac{\alpha}{\beta + \frac{p}{v}} \quad (4.4)$$

The transfer function $H_v(p)$ may be realized by mechanization of the following differential equation.

$$\beta v e + \frac{de}{dt} = \alpha \sqrt{|v|} w \quad (4.5)$$

A simplified diagram of an analog computer network that mechanizes (4.5) is shown in Figure 4.1. The system of Figure 4.1 provides an exact realization

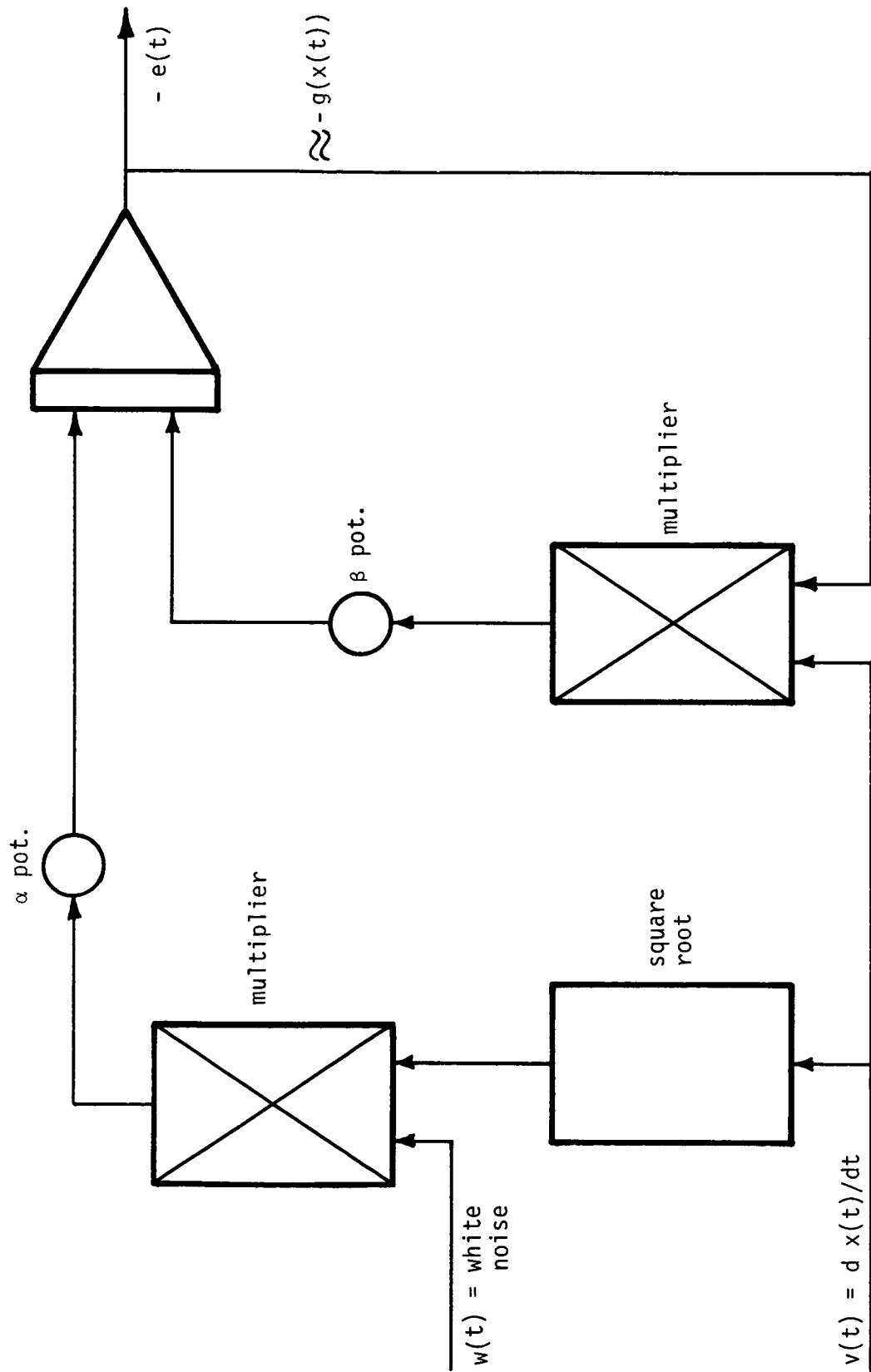


Figure 4.1. Mechanization System for Exponential Autocorrelation Function.

of $g(x)$ for a position variable $x(t) = vt + K$. In accordance with the discussion in Chapter 3 the system is expected to approximate the composite process $g(x(t))$ when the position parameter is a function of time $x(t)$ having a small second derivative.

The quality of the approximation of $g(x(t))$ may be found by obtaining an analytical solution of the differential equation of (4.5).^{*} If $v(t)$ is restricted to be a non-negative function of time, the general solution $e(t)$ for the differential equation of (4.5) is

$$e(t) = e^{-\int_0^t \beta v(\alpha) d\alpha} \int_0^t e^{\int_0^{\gamma} \beta v(\alpha) d\alpha} \sqrt{\alpha^2 v(\gamma) w(\gamma)} d\gamma + C e^{-\int_0^t \beta v(\alpha) d\alpha} \quad (4.6)$$

Assuming $w(t)$ to be a stationary Gaussian white noise random process having power spectral density of unity, the autocorrelation function for $e(t)$ is found to be

$$\begin{aligned} E(e(t_1) e(t_2)) &= \frac{\alpha^2}{2\beta} e^{-\beta \left| \int_0^{t_1} v(\alpha) d\alpha - \int_0^{t_2} v(\alpha) d\alpha \right|} \\ &= \frac{\alpha^2}{2\beta} e^{-\beta |x(t_1) - x(t_2)|}, \quad t_1 > t_2. \end{aligned}$$

This is exactly the autocorrelation function for $g(x(t))$. Thus, the mechanization of Figure 4.1 provides an exact realization of the composite process $g(x(t))$ when the first derivative of $x(t)$ is restricted to be non-negative. Thus for the mechanization considered in this example, the second derivative of the position function $x(t)$ need not be restricted to small values.

A comparison of theoretical values of the autocorrelation function of $g(x(t))$ with values determined experimentally from an implementation of the system of Figure 4.1 is shown in Figures 4.2 through 4.5. For this example the parameters of the power spectral density of (4.2) used were $\alpha = 0.016$ and $\beta = 0.005$.

^{*}Reference 2, page 43.

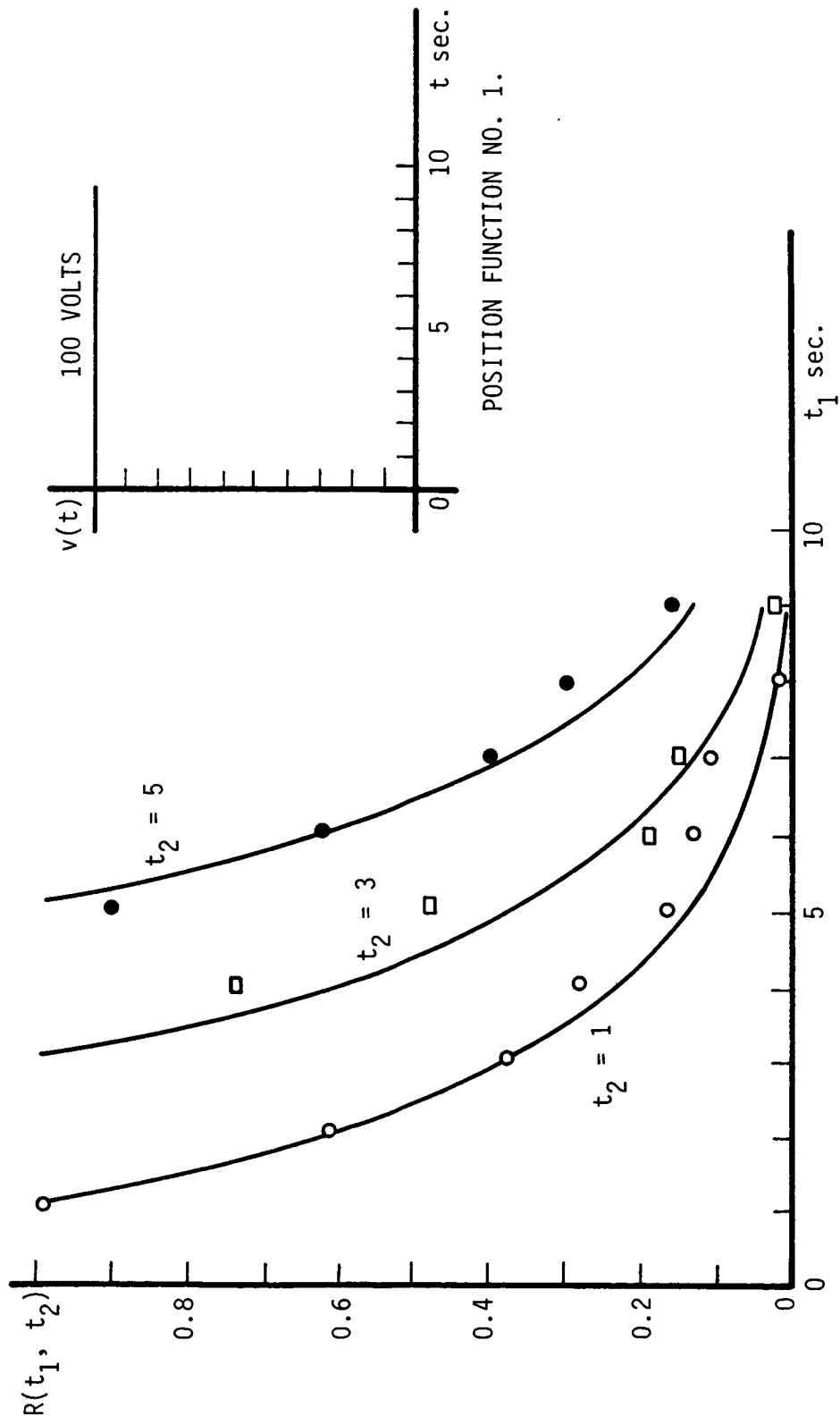


Figure 4.2. Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 1.

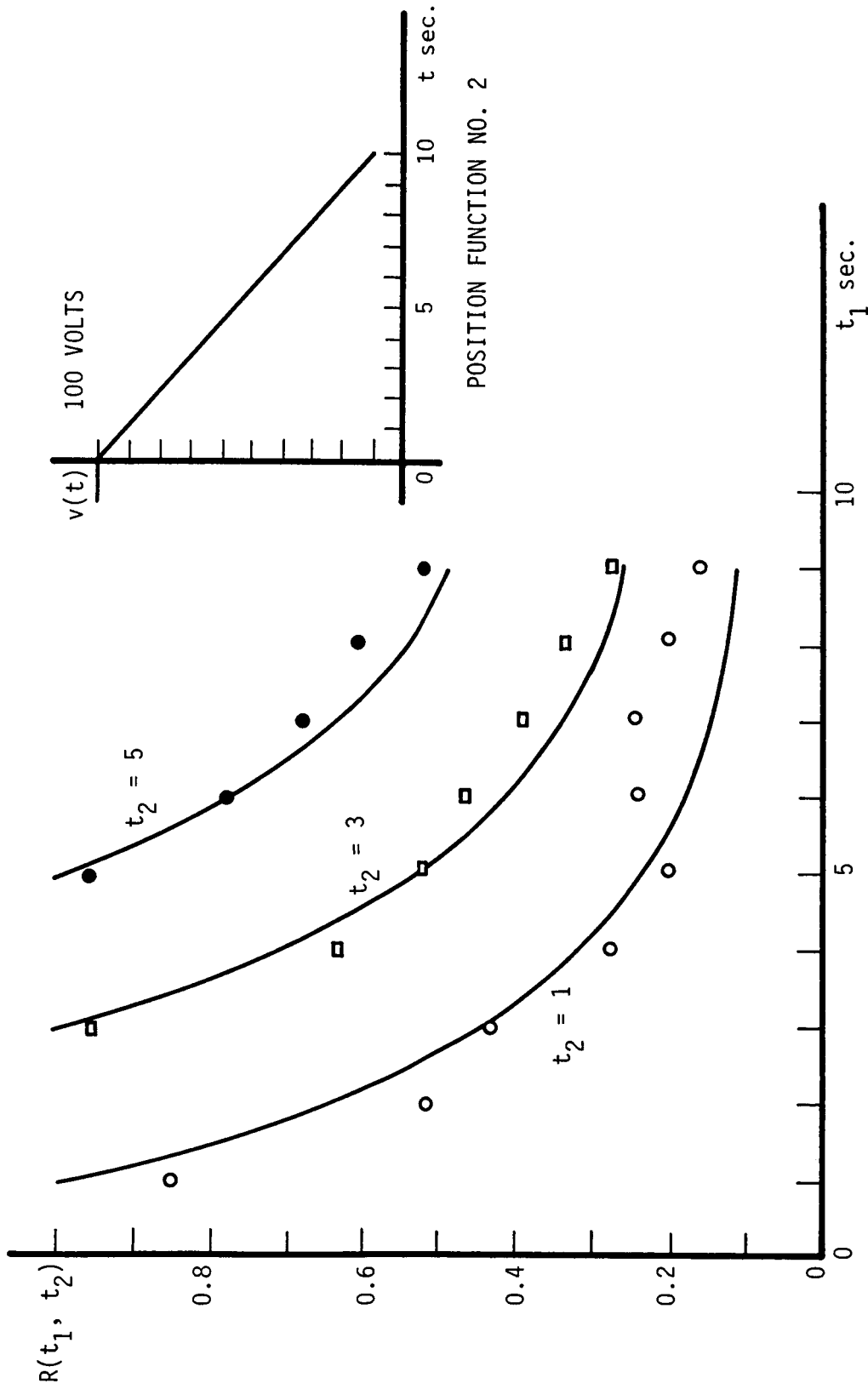


Figure 4.3. Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 2.

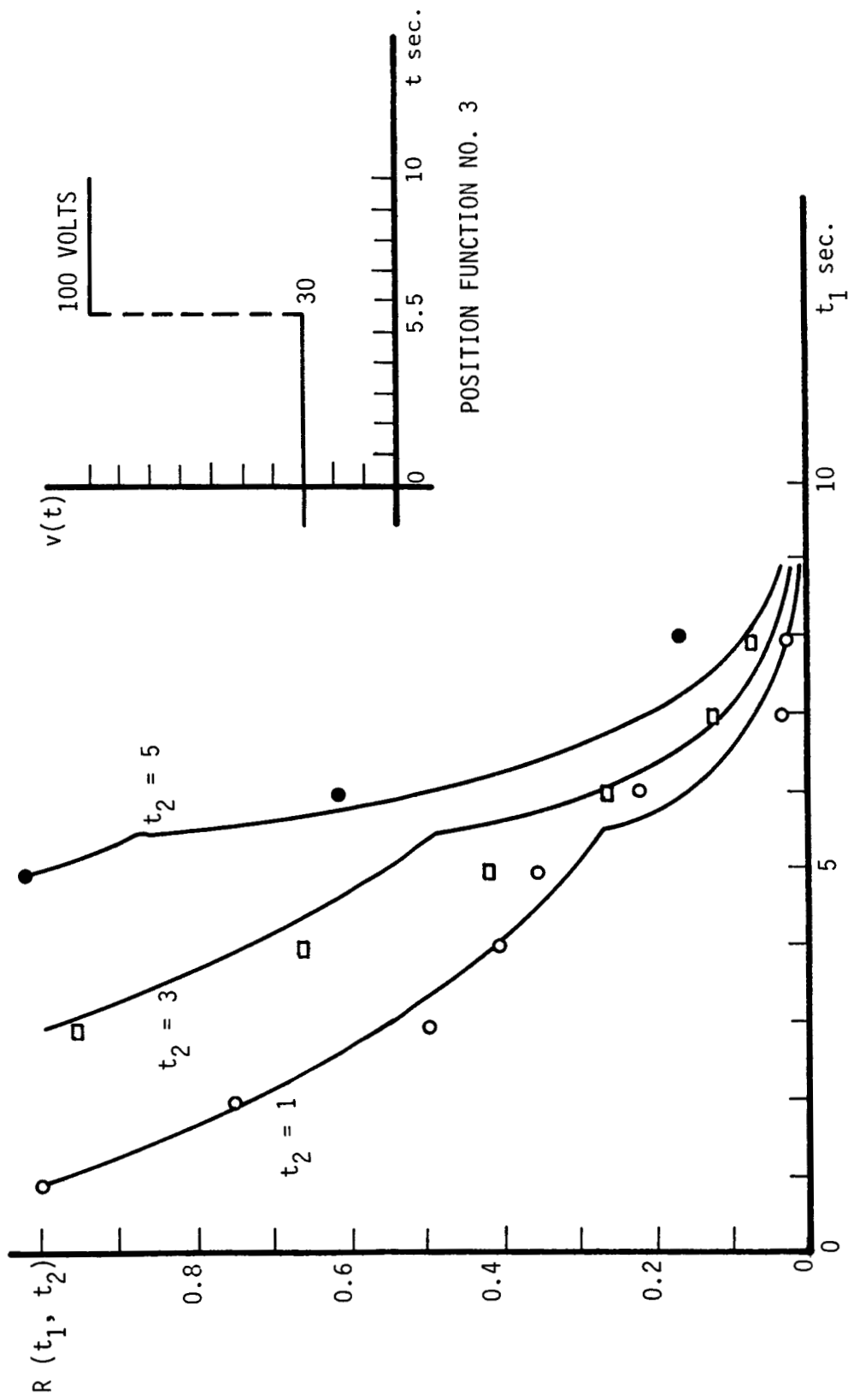


Figure 4.4. Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 3.

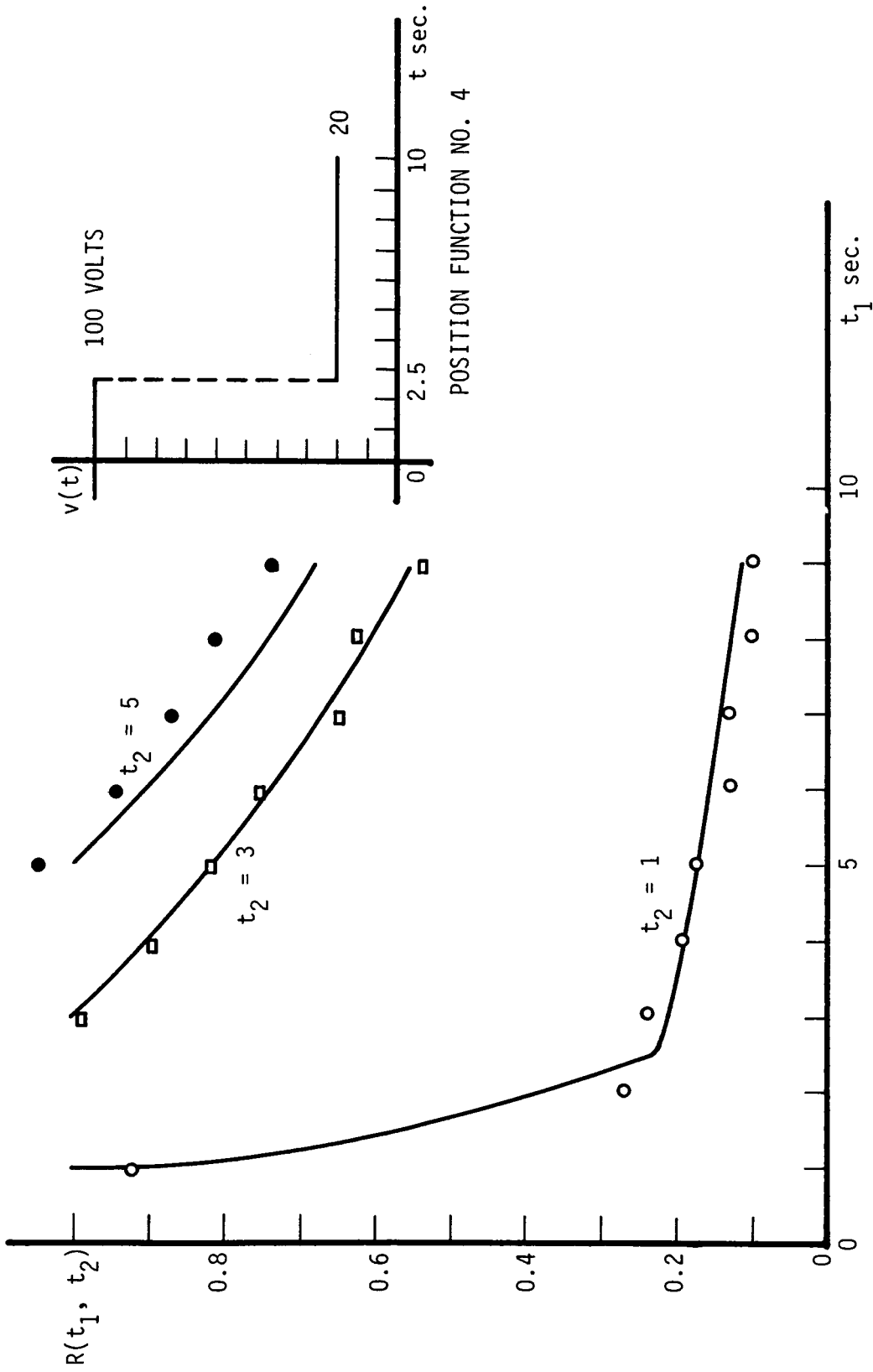


Figure 4.5. Normalized Autocorrelation Function $R(t_1, t_2)$ for Position Function No. 4.

V. CONCLUSIONS

In this report a mechanization procedure has been presented for the approximation of a Gaussian random process in a position parameter x . The mechanization system simulates the effect of the random process on a sensing element whose variable position is specified by the parameter x . The variation in position described by the function $x(t)$ is constrained to have a small second derivative.

A synthesis procedure for obtaining a particular mechanization system for approximating the desired random process is presented. Experimental results obtained from measurements on a physical mechanization system are given to indicate the quality of approximation to the desired random process.

The work reported in this technical note is part of a research effort directed toward the generation of a random process depending on both a time and position parameter. Such a random process has utility for the simulation of the random wind disturbances affecting a rocket in flight. It is felt that progress has been made during this investigation on the general problem of approximating a Gaussian random process depending on both a time and position parameter. Also, progress has been made toward the synthesis of a mechanization system to generate a random process depending on a single position parameter without applying the constraint that the second derivative of the position parameter must be small.

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