CONVERGENT-DIVERGENT NOZZLE FLOWS

by

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02874-6002-R000

30 December 1966

Prepared for

National Aeronautics and Space Administratic..
Manned Spacecraft Center

Under Contract NAS9-4358

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ACKNOWLEDGMENT

The numerical results given in this report were obtained with the assistance of T. J. McCarron, J. E. Melde, G. R. Nickerson and M. L. Sprankle of the Computation and Data Reduction Center, TRW Systems.

CONTENTS

ILLUSTRATIONS

ILLUSTRATIONS (Continued)

TABLES

Table | Page

NOMENCLATURE

a^*, $\overline{a^*}$	=	sonic velocity
A	=	nozzle cross-sectional area
m, $\overline{m}$	=	mass flow
P, $\overline{P}$	=	pressure
r	=	transformed coordinate, y/y^*
R	=	nozzle throat wall radius of curvature normalized with respect to nozzle throat half-height; gas constant
T	=	temperature
u, $\overline{u}$	=	velocity in x-direction normalized with respect to sonic velocity
v, $\overline{v}$	=	velocity in y-direction normalized with respect to sonic velocity
x	=	axial distance from nozzle throat plane
$\overline{X}$	=	fraction of mass flow in outer zone of a two-zone expansion
y	=	nozzle half-height
y^*	=	nozzle throat half-height
z	=	transformed coordinate, $x/R^{1/2}\ y^*$
γ, $\overline{\gamma}$	=	ratio of specific heats
ε	=	nozzle contraction ratio
ρ, $\overline{\rho}$	=	density
σ	=	1 for circular arc throats, 0 for parabolic throats, -R for hyperbolic throats
ω	=	1 for axisymmetric nozzle, 0 for planar nozzle

Superscript

*	=	sonic condition; throat condition

Subscripts

c	=	chamber condition
o	=	stagnation condition
s	=	at the streamline dividing the two flow zones
t	=	throat condition
w	=	at the wall of a nozzle

1. INTRODUCTION

This report contains the results of a study of uniform two-zone perfect gas expansions in convergent-divergent nozzles. This study was performed by TRW Systems Group for NASA (MSC) under contract NAS 9-4358, Improvement of Analytical Predictions of Delivered Specific Impulse.

The objective of this contract was to develop a family of four computer programs to calculate inviscid, one-dimensional and axisymmetric nonequilibrium nozzle flow fields accounting for the nonequilibrium effects of finite rate chemical reactions between gaseous combustion products and velocity and thermal lags between gaseous and condensed combustion products.

The four programs developed under this contract are:

- A one-dimensional program which calculates the equilibrium, frozen and kinetic performance of propellant systems having gaseous exhaust products containing the elements carbon, hydrogen, oxygen, nitrogen, fluorine and chlorine.
- A one-dimensional program which calculates the equilibrium, frozen and kinetic performance of systems having gaseous and condensed exhaust products containing the elements carbon, hydrogen, oxygen, nitrogen, fluorine, chlorine and one metal element, either aluminum, beryllium, boron or lithium.
- An axisymmetric program which calculates the kinetic performance of propellant systems having gaseous exhaust products containing the elements carbon, hydrogen, oxygen, nitrogen, fluorine and chlorine. On option, this program considers either the expansion of a uniform mixture (the ideal engine case) or of a two-zoned mixture (the film cooled engine case).
- An axisymmetric program which calculates the kinetic performance of propellant systems having gaseous and condensed exhaust products containing the elements carbon, hydrogen, oxygen, nitrogen, fluorine, chlorine and one metal element, either aluminum, beryllium, boron or lithium. This program considers only the expansion of a uniform mixture (the ideal engine case).

These programs differ in a number of ways from previous programs developed to calculate nonequilibrium nozzle expansions.

In particular:

- The programs are completely self-contained, requiring specification of only the propellant system (elemental composition and heat of formation), relaxation rates and nozzle geometry to run a case.

- The chemical species considered by the programs have been selected to allow accurate equilibrium, frozen and kinetic performance analyses of cryogenic, space storable, prepackaged, hybrid and solid propellant systems of current and projected operational use.

- All dissociation-recombination and binary exchange reactions between the gaseous species present in the exhaust are considered by the programs allowing complete kinetic expansion calculations.

- The programs utilize TRW Systems' implicit integration method which allows rapid integration of the chemical and gas-particle relaxation equations from equilibrium chamber conditions. Typical run times are three minutes for the one-dimensional programs and ten minutes for the axisymmetric programs on an IBM 7094 Mod II computer.

- The programs allow analysis of the performance loss associated with film cooling in propellant systems having all gaseous exhaust products.

- The programs allow simultaneous consideration of both chemical and gas-particle relaxation losses in propellant systems having condensed exhaust products.

- The one-dimensional programs allow equilibrium, frozen and kinetic performance calculations to be performed during a single machine run.

- The programs are written in machine independent language (FORTRAN IV), allowing their use on all standard computers

The study described in this report was performed to determine the appropriate transonic initial conditions for the two axisymmetric characteristic programs developed under NASA (MSC) contract NAS 9-4358. Since the study resulted in a new method of analyzing both uniform and two-zone convergent-divergent nozzle flows and revealed the nature and interrelationship of previous nozzle analyses, the results of this study are believed to be of sufficient general interest to merit publication as a separate contract report. The results of this study are presented in the following sections without reference to their use in the axisymmetric programs.

2. UNIFORM EXPANSIONS

The equations governing the inviscid isentropic expansion of a perfect gas through a convergent-divergent nozzle are

$$\left(1 - u^2 - \frac{\gamma-1}{\gamma+1} v^2\right) \frac{\partial u}{\partial x} + \left(1 - v^2 - \frac{\gamma-1}{\gamma+1} u^2\right) \frac{\partial v}{\partial y} + \left[1 - \frac{\gamma-1}{\gamma+1}(u^2 + v^2)\right] \frac{\omega v}{y}$$

$$- \frac{4}{\gamma+1} uv \frac{\partial u}{\partial y} = 0 \tag{2-1}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{2-2}$$

where the velocities have been normalized with respect to the throat sonic velocity and ω equals 0 or 1 depending on whether the nozzle is planar or axisymmetric. In seeking solutions of the above equations, it is desirable to choose a set of non-dimensional coordinates such that the various velocity derivatives are independent of the nozzle scale. For large values of the normalized throat wall radius of curvature, the flow velocities asymptotically approach those obtained from the one-dimensional channel flow equations. It can be shown from the channel flow equations (see Appendix A) that for choked flows

$$u = 1 + \sqrt{\frac{\omega+1}{\gamma+1} \frac{1}{R}} \frac{x}{y^*} + \ldots \tag{2-3}$$

at the nozzle throat, where x is the distance from the throat plane, y^* is the throat half height, and R is the normalized throat wall radius of curvature. Examination of this equation reveals that the axial nozzle coordin x must be normalized with respect to the distance $\sqrt{R}\, y^*$ in order for the dimensionless axial velocity gradient to remain of order one at the nozzle throat independent of the nozzle scale. Since the nozzle scale perpendicular to the nozzle axis is set by the throat half height y^*, it is apparent that the perpendicular coordinate y should be normalized with respect to the distance y^*. Thus, solutions of the above equations should be sought in terms of the normalized coordinates

$$z = \sqrt{\frac{1}{R}} \frac{x}{y^*} \tag{2-4}$$

$$r = \frac{y}{y^*} \tag{2-5}$$

rather than in the x,y coordinate system for large values of the normalized throat wall radius of curvature.

The above axial coordinate choice differs from Hall's[1] by a factor $\sqrt{\frac{\omega+1}{\gamma+1}R}$, since the axial coordinate used by Hall is

$$z_H = \sqrt{\frac{\omega+1}{\gamma+1} R} \, \frac{x}{y^*} \tag{2-6}$$

As will be shown later, the above choice results in the present solution being uniformly valid for all (subsonic, transonic and supersonic) nozzle flow regimes, while Hall's choice limits the validity of his solution to the transonic throat region.

In the r,z coordinate system, the above equations become

$$\sqrt{\frac{1}{R}}\left(1 - u^2 - \frac{\gamma-1}{\gamma+1} v^2\right)\frac{\partial u}{\partial z} + \left(1 - v^2 - \frac{\gamma-1}{\gamma+1} u^2\right)\frac{\partial v}{\partial r} + \left[1 - \frac{\gamma-1}{\gamma+1}(u^2 + v^2)\right]\frac{\omega v}{r}$$

$$- \frac{4}{\gamma+1} uv \frac{\partial u}{\partial y} = 0 \tag{2-7}$$

$$\sqrt{\frac{1}{R}} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} = 0 \tag{2-8}$$

The boundary conditions are

$$v(o,z) = 0 \tag{2-9}$$

and

$$\frac{v(r_w,z)}{u(r_w,z)} = \sqrt{\frac{1}{R}} \frac{dr_w}{dz} \tag{2-10}$$

At the nozzle throat,

$$r_w = 1 + \frac{x^2}{2R} + \ldots$$

$$= 1 + \frac{z^2}{2} + \ldots \tag{2-11}$$

for all throat sections. Thus, both u and $\frac{dr_w}{dz}$ are O(1) at the throat and v must be $O(R^{-1/2})$. This suggests that the velocity components can be expressed as expansions in inverse power of R, i.e.,

$$u = u_o(r,z) + \frac{u_1(r,z)}{R} + \frac{u_2(r,z)}{R^2} + \ldots \tag{2-12}$$

$$v = \sqrt{\frac{1}{R}}\,[v_o(r,z) + \frac{v_1(r,z)}{R} + \frac{v_2(r,z)}{R^2} + \ldots] \tag{2-13}$$

Substituting into equations (2-7) and (2-8) and equating powers of R^{-1} separately yields two sets of equations:

$$\left(1 - u_o^2\right)\frac{\partial u_o}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}u_o^2\right)\left(\frac{\partial v_o}{\partial r} + \frac{\omega v_o}{r}\right) = 0 \tag{2-14}$$

$$\frac{\partial u_o}{\partial r} = 0 \tag{2-15}$$

$$\left(1 - u_o^2\right)\frac{\partial u_n}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}u_o^2\right)\left(\frac{\partial v_n}{\partial r} + \frac{\omega v_n}{r}\right) - \frac{4}{\gamma+1}u_o v_o \frac{\partial u_n}{\partial r} = \phi_n, \quad n \geq 1 \tag{2-16}$$

$$\frac{\partial v_{n-1}}{\partial z} - \frac{\partial u_n}{\partial r} = 0 \qquad n \geq 1 \tag{2-17}$$

where

$$\phi_1 = \left(2u_ou_1 + \frac{\gamma-1}{\gamma+1}v_o^2\right)\frac{\partial u_o}{\partial z} + \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1}u_ou_1\right)\frac{\partial v_o}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_ou_1 + v_o^2\right)\frac{\omega v_o}{r} \tag{2-18}$$

$$\begin{aligned}\phi_2 = {} & \left(2u_ou_2 + u_1^2 + 2\frac{\gamma-1}{\gamma+1}v_ov_1\right)\frac{\partial u_o}{\partial z} + \left[2v_ov_1 + \frac{\gamma-1}{\gamma+1}(2u_ou_2 + u_1^2)\right]\frac{\partial v_o}{\partial r} \\ & + \frac{\gamma-1}{\gamma+1}\left(2u_ou_2 + u_1^2 + 2v_ov_1\right)\frac{\omega v_o}{r} + \left(2u_ou_1 + \frac{\gamma-1}{\gamma+1}v_o^2\right)\frac{\partial u_1}{\partial z} + \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1}u_ou_1\right)\frac{\partial v_1}{\partial r} \\ & + \frac{\gamma-1}{\gamma+1}\left(2u_ou_1 + v_o^2\right)\frac{\omega v_1}{r} + \frac{4}{\gamma+1}\left(u_ov_1 + u_1v_o\right)\frac{\partial u_1}{\partial r}\end{aligned} \tag{2-19}$$

....

From equations (2-9), (2-10), (2-12) and (2-13), it is found that the boundary conditions are

$$v_n(o,z) = 0 \qquad n \geq 0 \tag{2-20}$$

and

$$v_n(r_w,z) = u_n(r_w,z)\,\frac{dr_w}{dz} \qquad n \geq 0 \tag{2-21}$$

Equation (2-15) shows that $u_o(r,z)$ is a function of z alone. Thus,

$$u_o(r,z) = a_o(z) \tag{2-22}$$

Equation (2-14) is satisfied if $v_o(r,z)$ is of the form,

$$v_o(r,z) = a_1(z)r + \omega a_3(z)r^{-1} + (1-\omega)\, a_5(z) \tag{2-23}$$

From the axis and wall boundary conditions [equations (2-20) and (2-21)], it is easily shown that

$$a_1 = \frac{a_o}{r_w}\frac{dr_w}{dz} \tag{2-24}$$

$$a_3 = 0 \tag{2-25}$$

$$a_5 = 0 \tag{2-26}$$

Substituting the above results into equation (2-14) yields

$$\left(1 - a_o^2\right)\frac{da_o}{dz} + (\omega + 1)\left(1 - \frac{\gamma-1}{\gamma+1}a_o^2\right)\frac{a_o}{r_w}\frac{dr_w}{dz} = 0 \tag{2-27}$$

which is the one-dimensional channel flow equation. The solution of the above equations defines the one-dimensional velocity distribution (u_o and v_o) through the nozzle. Since the one-dimensional solution is valid for all (subsonic, transonic and supersonic) nozzle flow regimes, the present solution will also be valid for all nozzle flow regimes. The one-dimensional throat boundary conditions are that

$$a_o(0) = 1 \tag{2-28}$$

$$a_1(0) = 0 \tag{2-29}$$

for both planar and axisymmetric nozzle flows, since

$$\left.\frac{dr_w}{dz}\right|_o = 0 \tag{2-30}$$

at the nozzle throat.

The first order equations are

$$\left(1 - u_o^2\right)\frac{\partial u_1}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1} u_o^2\right)\left(\frac{\partial v_1}{\partial r} + \frac{\omega v_1}{r}\right) - \frac{4}{\gamma+1} u_o v_o \frac{\partial u_1}{\partial r}$$

$$= \left(2u_o u_1 + \frac{\gamma-1}{\gamma+1} v_o^2\right)\frac{\partial u_o}{\partial z} + \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1} u_o u_1\right)\frac{\partial v_o}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_o u_1 + v_o^2\right)\frac{\omega v_o}{r} \qquad (2\text{-}31)$$

$$\frac{\partial v_o}{\partial z} - \frac{\partial u_1}{\partial r} = 0 \qquad (2\text{-}32)$$

From equations (2-23) and (2-32), it is easily shown that

$$u_1 = b_o(z) + b_2(z) r^2 \qquad (2\text{-}33)$$

where

$$b_2 = \frac{1}{2}\frac{da_1}{dz} \qquad (2\text{-}34)$$

From equations (2-20), (2-31), and (2-33), it can be shown that

$$v_1 = b_1(z) r + b_3(z) r^3 \qquad (2\text{-}35)$$

where

$$\left(1 - a_o^2\right)\frac{db_o}{dz} + (\omega + 1)\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right) b_1 = 2a_o b_o\left[\frac{da_o}{dz} + \frac{\gamma-1}{\gamma+1}(\omega + 1) a_1\right] \qquad (2\text{-}36)$$

$$\left(1 - a_o^2\right)\frac{db_2}{dz} + (\omega + 3)\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right) b_3 - \frac{8}{\gamma-1} a_o a_1 b_2$$

$$= 2a_o b_2\left[\frac{da_o}{dz} + \frac{\gamma-1}{\gamma+1}(\omega + 1) a_1\right] + a_1^2\left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \left(1 + \frac{\gamma-1}{\gamma+1}\omega\right) a_1\right] \qquad (2\text{-}37)$$

From the wall boundary condition [equation (2-21)], it can be shown that

$$b_1 = \left(b_o + b_2 r_w^2\right)\frac{1}{r_w}\frac{dr_w}{dz} - b_3 r_w^2 \qquad (2\text{-}38)$$

The solution of the above equations defines the first order velocity components (u_1 and v_1) through the nozzle.

Examination of equations (2-36) and (2-37) reveals that they are singular at the nozzle throat (where $a_o = 1$). Thus, the above equations are algebraic at the throat and can be solved directly for $b_o(0)$, $b_1(0)$ and $b_3(0)$, yielding

$$b_o(0) = -\frac{1}{4} \tag{2-39}$$

$$b_1(0) = -\frac{1}{4}\sqrt{\frac{\gamma+1}{2}} \tag{2-40}$$

$$b_3(0) = \frac{1}{4}\sqrt{\frac{\gamma+1}{2}} \tag{2-41}$$

for axisymmetric flows and

$$b_o(0) = -\frac{1}{6} \tag{2-42}$$

$$b_1(0) = -\frac{1}{6}\sqrt{\gamma+1} \tag{2-43}$$

$$b_3(0) = \frac{1}{6}\sqrt{\gamma+1} \tag{2-44}$$

for planar flows. From equations (2-24) and (2-34) it can be shown that

$$b_2(0) = \frac{1}{2} \tag{2-45}$$

for both axisymmetric and planar flows.

The above first order throat conditions are identical to those obtained by Sauer[2] and Hall[1]. The two results differ away from the throat plane, however, due to the different functional dependence of the coefficients on the axial coordinate.

Examination of equation (2-16) reveals that it is also singular at the nozzle throat (where $u_o = 1$). Thus, the boundary conditions for all orders are set at the nozzle throat, and the various order throat conditions can be determined directly. The fact that the boundary conditions are set at the throat for all orders is mathematical proof that the nozzle throat plane sets the choked flow through the nozzle.

Examination of equations (2-24), (2-34) and (2-37) shows that b_2 depends on $\frac{d^2 r_w}{dz^2}$ and b_3 depends on $\frac{d^3 r_w}{dz^3}$. Thus, if $\frac{d^3 r_w}{dz^3}$ is discontinuous, the first order solution will be discontinuous. Thus, in general, if the wall derivative $\frac{d^{2n+1} r_w}{dz^{2n+1}}$ is nonanalytic, the nth order solution of the above equations will be discontinuous. The complete solution of the above equations will be analytic only if the wall is analytic.

The second order equations are

$$\left(1 - u_o^2\right)\frac{\partial u_2}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1} u_o^2\right)\left(\frac{\partial v_2}{\partial r} + \frac{\omega v_2}{r}\right) - \frac{4}{\gamma+1} u_o v_o \frac{\partial u_2}{\partial r}$$

$$= \left(2u_o u_2 + u_1^2 + 2\frac{\gamma-1}{\gamma+1} v_o v_1\right)\frac{\partial u_o}{\partial z} + \left[2v_o v_1 + \frac{\gamma-1}{\gamma+1}(2u_o u_2 + u_1^2)\right]\frac{\partial v_o}{\partial r}$$

$$+ \frac{\gamma-1}{\gamma+1}\left(2u_o u_2 + u_1^2 + 2v_o v_1\right)\frac{\omega v_o}{r} + \left(2u_o u_1 + \frac{\gamma-1}{\gamma+1} v_o^2\right)\frac{\partial u_1}{\partial r}$$

$$+ \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1} u_o u_1\right)\frac{\partial v_1}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_o u_1 + v_o^2\right)\frac{\omega v_1}{r} + \frac{4}{\gamma+1}\left(u_o v_1 + u_1 v_o\right)\frac{\partial u_1}{\partial r} \quad (2\text{-}46)$$

$$\frac{\partial v_1}{\partial z} - \frac{\partial u_2}{\partial r} = 0 \quad (2\text{-}47)$$

From equations (2-35) and (2-47), it is easily shown that

$$u_2 = c_o(z) + c_2(z)r^2 + c_4(z)r^4 \quad (2\text{-}48)$$

where

$$c_2 = \frac{1}{2}\frac{db_1}{dz} \quad (2\text{-}49)$$

$$c_4 = \frac{1}{4}\frac{db_3}{dz} \quad (2\text{-}50)$$

From equations (2-20), (2-46) and (2-48), it can be shown that

$$v_2 = c_1(z)r + c_3(z)r^3 + c_5(z)r^5 \tag{2-51}$$

where

$$\left(1 - a_o^2\right)\frac{dc_o}{dz} + (\omega + 1)\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)c_1 = \left(2a_o c_o + b_o^2\right)\left[\frac{da_o}{dz} + \frac{\gamma-1}{\gamma+1}(1+\omega)a_1\right]$$

$$+ 2a_o b_o \left[\frac{db_o}{dz} + \frac{\gamma-1}{\gamma+1}(1 + \omega)b_1\right] \tag{2-52}$$

$$\left(1 - a_o^2\right)\frac{dc_2}{dz} + (\omega + 3)\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)c_3 - \frac{8}{\gamma+1} a_o a_1 c_2$$

$$= 2(a_o c_2 + b_o b_2)\left[\frac{da_o}{dz} + \frac{\gamma-1}{\gamma+1}(1 + \omega)a_1\right] + 2a_1 b_1 \left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \left(1 + \frac{\gamma-1}{\gamma+1}\omega\right)a_1\right]$$

$$+ \frac{4}{\gamma+1}\left(a_o b_1 + a_1 b_o\right)\frac{da_1}{dz} + 2a_o b_o \left[\frac{db_2}{dz} + \frac{\gamma-1}{\gamma+1}(3 + \omega)b_3\right]$$

$$+ 2a_o b_2 \left[\frac{db_o}{dz} + \frac{\gamma-1}{\gamma+1}(1 + \omega)b_1\right] + a_1^2\left[\frac{\gamma-1}{\gamma+1}\frac{db_o}{dz} + \left(1 + \frac{\gamma-1}{\gamma+1}\omega\right)b_1\right] \tag{2-53}$$

$$\left(1 - a_o^2\right)\frac{dc_4}{dz} + (\omega + 5)\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)c_5 - \frac{16}{\gamma+1} a_o a_1 c_4$$

$$= \left(2a_o c_4 + b_2^2\right)\left[\frac{da_o}{dz} + \frac{\gamma-1}{\gamma+1}(1 + \omega)a_1\right] + 2a_1 b_3 \left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \left(1 + \frac{\gamma-1}{\gamma+1}\omega\right)a_1\right]$$

$$+ \frac{4}{\gamma+1}\left(a_o b_3 + a_1 b_2\right)\frac{da_1}{dz} + 2a_o b_2 \left[\frac{db_2}{dz} + \frac{\gamma-1}{\gamma+1}(3 + \omega)b_3\right]$$

$$+ a_1^2 \left[\frac{\gamma-1}{\gamma+1}\frac{db_2}{dz} + \left(3 + \frac{\gamma-1}{\gamma+1}\omega\right)b_3\right] \tag{2-54}$$

From the wall boundary condition [equation (2-21)], it can be shown that

$$c_1 = \left(c_o + c_2 r_w^2 + c_4 r_w^4\right)\frac{1}{r_w}\frac{dr_w}{dz} - c_3 r_w^2 - c_5 r_w^4 \tag{2-55}$$

The solution of the above equations defines the second order velocity components (u_2 and v_2) through the nozzle.

As previously discussed, the above equations are singular at the throat and can be solved directly to determine the second order throat conditions. Thus,

$$c_o(0) = \frac{10\gamma + 57}{288} \tag{2-56}$$

$$c_1(0) = \frac{28\gamma + 93}{288}\sqrt{\frac{\gamma+1}{2}} \tag{2-57}$$

$$c_2(0) = -\frac{4\gamma + 15}{24} \tag{2-58}$$

$$c_3(0) = -\frac{20\gamma + 63}{96}\sqrt{\frac{\gamma+1}{2}} \tag{2-59}$$

$$c_4(0) = \frac{2\gamma + 9}{24} \tag{2-60}$$

$$c_5(0) = \frac{\gamma + 3}{9}\sqrt{\frac{\gamma+1}{2}} \tag{2-61}$$

for axisymmetric flows and

$$c_o(0) = \frac{\gamma + 30}{270} \tag{2-62}$$

$$c_1(0) = \frac{34\gamma + 195}{1080}\sqrt{\frac{\gamma+1}{2}} \tag{2-63}$$

$$c_2(0) = -\frac{2\gamma + 9}{18} \tag{2-64}$$

$$c_3(0) = -\frac{5\gamma + 21}{54}\sqrt{\frac{\gamma+1}{2}} \tag{2-65}$$

$$c_4(0) = \frac{\gamma+6}{18} \tag{2-66}$$

$$c_5(0) = \frac{22\gamma + 75}{360}\sqrt{\frac{\gamma+1}{2}} \tag{2-67}$$

for planar flows.

The above second order throat conditions are identical to those obtained by Hall[1]. Both the first and second order throat conditions are independent of the nozzle shape and are thus universally valid for all nozzle flows. The solution away from the throat depends on the nozzle shape for all orders, however.

The third and higher order equations can be similarly obtained. The throat boundary conditions for these equations depend on the nozzle shape, and are thus not universally valid for all nozzle flows. The solutions of these equations are polynomial in r of order 2n and 2n+1, respectively. for u_n and v_n. Thus, studies of nozzle flows using numerical or integral techniques which assume that the velocity components can be represented by polynomials in r are mathematically equivalent to the present analysis and will have errors of the same order $\left(\frac{1}{R^{n+1}}\right)$ when terms containing higher powers of r in u and v are neglected. The second order solution given by Oswatitsch[3] does not contain terms of r^4 and r^5 in his u_2 and v_2, respectively. This solution is thus not truly second order, but contains errors of order $\frac{1}{R^2}$ due to neglecting these terms. This explains the discrepancy between Oswatitsch's and Hall's second order results noted by Hall[1]. It is noted that a number of previous analyses[4,5,6] have utilized terminated polynomials in r of order 2n and 2n-1 for u and v, respectively, and that these analyses are of inconsistent order, being of order n in u and n-1 in v.

Figures 2-1 and 2-2 show the results of the present analysis for the flow of air (γ = 1.4) through axisymmetric and planar hyperbolic nozzles having a normalized throat wall radius of curvature of 5. Tables 2-1 through 2-4 tabulate the velocity distribution along the axis and wall in these nozzles. In general, the convergence of the solution is fastest in the subsonic region and slowest in the supersonic region. This is to be expected, since the deviation from one-dimensional flow increases through the nozzle and is greatest in the supersonic section.

Figures 2-3 through 2-11 show the first and second order throat wall velocities, the throat axis velocities and the sonic point displacements as a function of the normalized throat wall radius of curvature in axisymmetric hyperbolic nozzles for flows with specific heat ratios of 1.2, 1.4, and 1.6. The throat wall and axis velocity variations are identical to Hall's results since the two analyses have the same throat boundary conditions. The sonic point displacement differs, however, and Hall's results are included for comparison. Also included on the velocity plots are second order rational fraction approximations (see Appendix B), which represent the probable true solution. Examination of the figures reveals only a weak dependence of the transonic results on gamma. Comparison of the second order solution with the second order rational fraction approximation shows that this solution probably represents the true solution quite accurately up to normalized throat wall radii of curvatures of three, and gives reasonable estimates of the transonic flow conditions up to normalized throat wall radii of curvatures of two. For a normalized throat wall radius of curvature less than one, the second order solution predicts that the throat axis velocity is supersonic, which is physically impossible. It is concluded that use of the second order solution should probably be limited to normalized throat wall radii of curvatures greater than two.

Table 2-1. Axis Velocity in an Axisymmetric Hyperbolic Nozzle (γ = 1.4, R = 5)

$\frac{x}{y^*}$	u_o	$u_o + \frac{u_1}{R}$	$u_o + \frac{u_1}{R} + \frac{u_2}{R^2}$
-1.0	0.6252	0.6231	0.6235
-0.5	0.8011	0.7772	0.7821
-0.4	0.8396	0.8106	0.8166
-0.3	0.8789	0.8446	0.8517
-0.2	0.9188	0.8793	0.8874
-0.1	0.9593	0.9144	0.9235
0.0	1.0000	0.9500	0.9599
0.1	1.0408	0.9858	0.9964
0.2	1.0816	1.0219	1.0330
0.3	1.1221	1.0579	1.0695
0.4	1.1622	1.0940	1.1057
0.5	1.2017	1.1298	1.1416
1.0	1.3858	1.3024	1.3123

Table 2-2. Wall Velocity in an Axisymmetric Hyperbolic Nozzle (γ = 1.4, R = 5)

$\frac{x}{y^*}$	u_o	$u_o + \frac{u_1}{R}$	$u_o + \frac{u_1}{R} + \frac{u_2}{R^2}$
-1.0	0.6252	0.6328	0.6295
-0.5	0.8011	0.8306	0.8262
-0.4	0.8396	0.8737	0.8692
-0.3	0.8789	0.9175	0.9128
-0.2	0.9188	0.9617	0.9569
-0.1	0.9593	1.0059	1.0011
0.0	1.0000	1.0500	1.0452
0.1	1.0408	1.0936	1.0888
0.2	1.0816	1.1364	1.1316
0.3	1.1221	1.1783	1.1734
0.4	1.1622	1.2189	1.2140
0.5	1.2017	1.2581	1.2532
1.0	1.3858	1.4290	1.4242

Table 2-3. Axis Velocity in a Planar Hyperbolic Nozzle (γ = 1.4, R = 5,

$\frac{x}{y^*}$	u_o	$u_o + \frac{u_1}{R}$	$u_o + \frac{u_1}{R} + \frac{u_2}{R^2}$
-1.0	0.7303	0.7260	0.726[illegible]
-0.5	0.8586	0.8404	0.8431
-0.4	0.8861	0.8649	0.8681
-0.3	0.9141	0.8898	0.3934
-0.2	0.9425	0.9152	0.9192
-0.1	0.9712	0.9408	0.9451
0.0	1.0000	0.9667	0.9713
0.1	1.0287	0.9927	0.9976
0.2	1.0577	1.0187	1.0239
0.3	1.0864	1.0451	1.0502
0.4	1.1148	1.0712	1.0763
0.5	1.1428	1.0972	1.1023
1.0	1.2749	1.2230	1.2270

Table 2-4. Wall Velocity in a Planar Hyperbolic Nozzle (γ = 1.4, R = 5)

$\frac{x}{y^*}$	u_o	$u_o + \frac{u_1}{R}$	$u_o + \frac{u_1}{R} + \frac{u_2}{R^2}$
-1.0	0.7303	0.7509	0.7454
-0.5	0.8586	0.9045	0.8985
-0.4	0.8861	0.9369	0.9311
-0.3	0.9141	0.9696	0.9639
-0.2	0.9425	1.0022	0.9967
-0.1	0.9712	1.0346	1.0293
0.0	1.0000	1.0667	1.0615
0.1	1.0287	1.0981	1.0932
0.2	1.0577	1.1287	1.1240
0.3	1.0864	1.1584	1.1538
0.4	1.1148	1.1871	1.1826
0.5	1.1428	1.2145	1.2102
1.0	1.2749	1.3327	1.3286

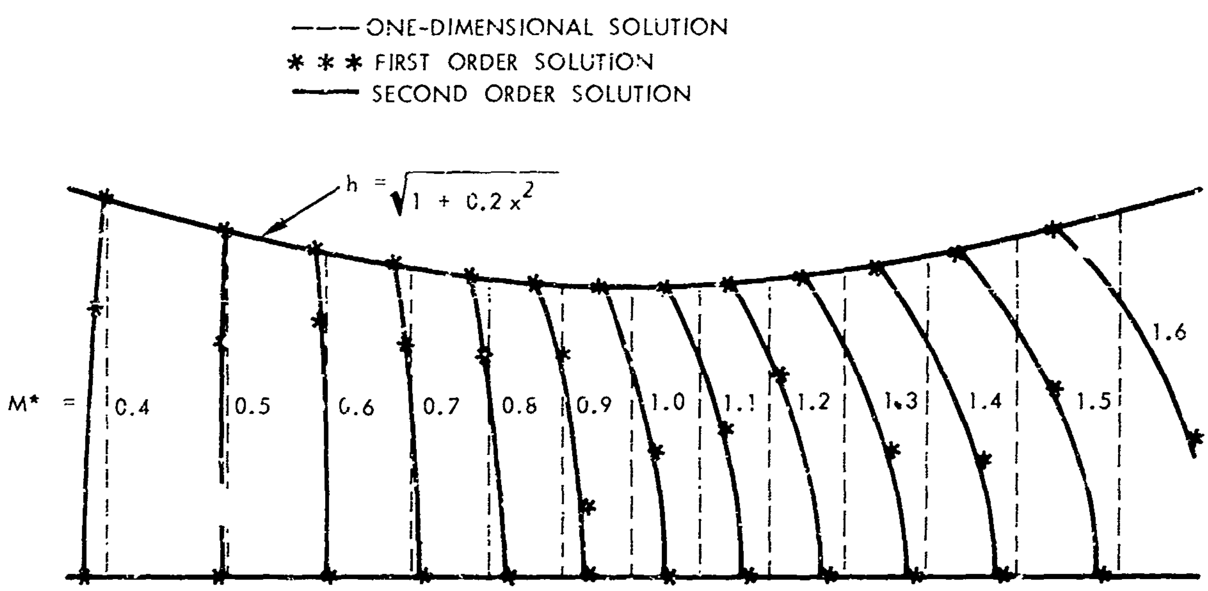

Figure 2-1. Contours of Constant Speed in Axisymmetric Hyperbolic Nozzle with $\gamma = 1.4$ and R = 5.

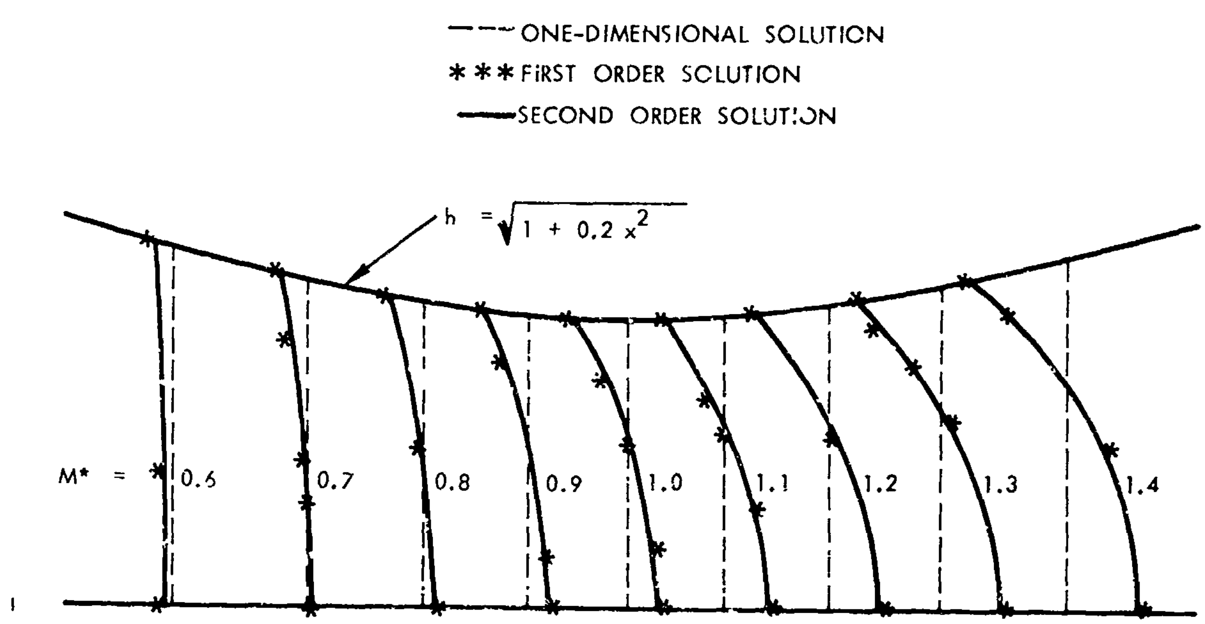

Figure 2-2. Contours of Constant Speed in Planar Hyperbolic Nozzle with $\gamma = 1.4$ and R = 5.

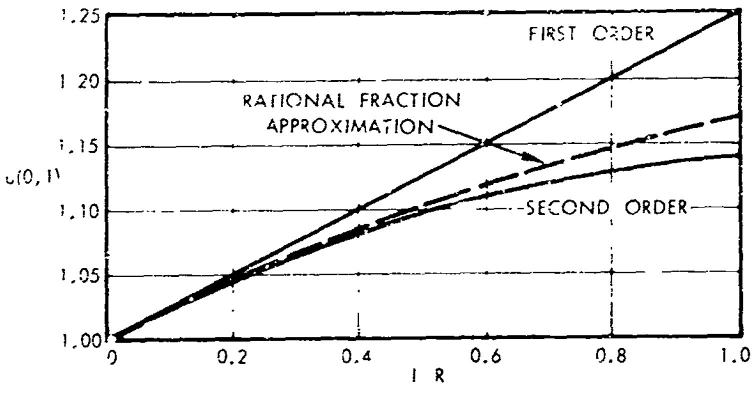

Figure 2-3. Throat Wall Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.2$.

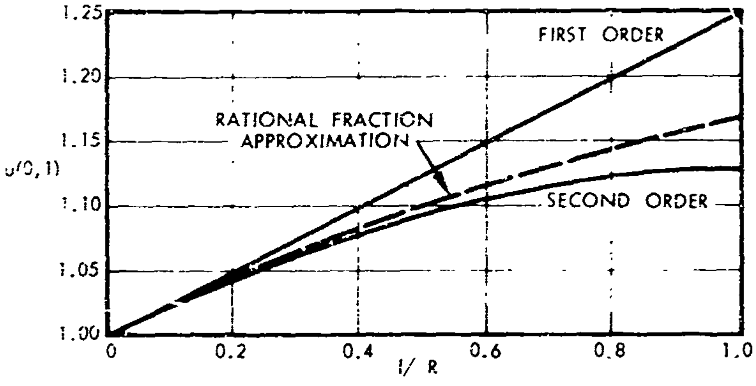

Figure 2-4. Throat Wall Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.4$.

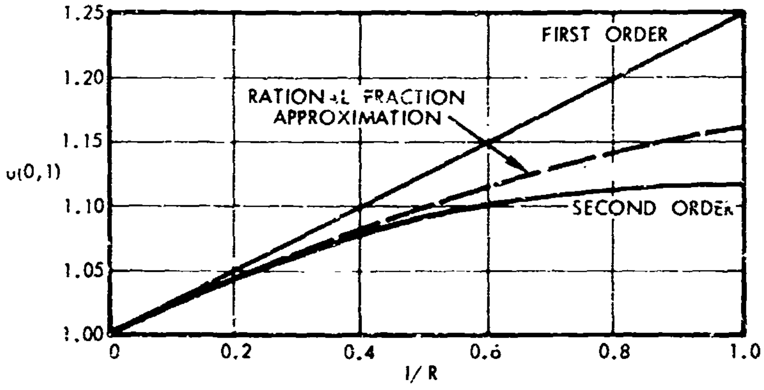

Figure 2-5. Throat Wall Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.67$.

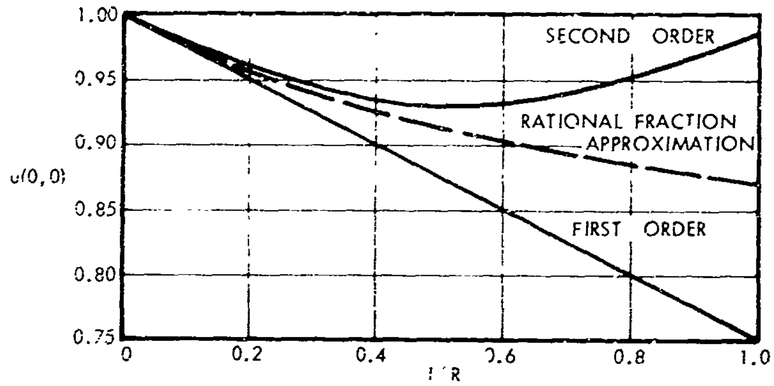

Figure 2-6. Throat Axis Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.2$.

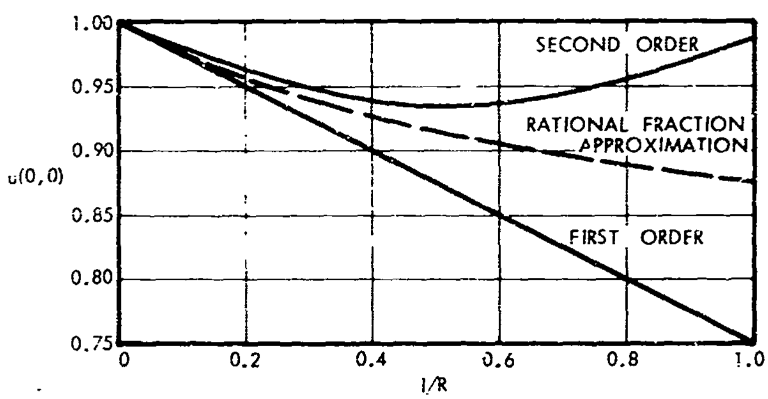

Figure 2-7. Throat Axis Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.4$.

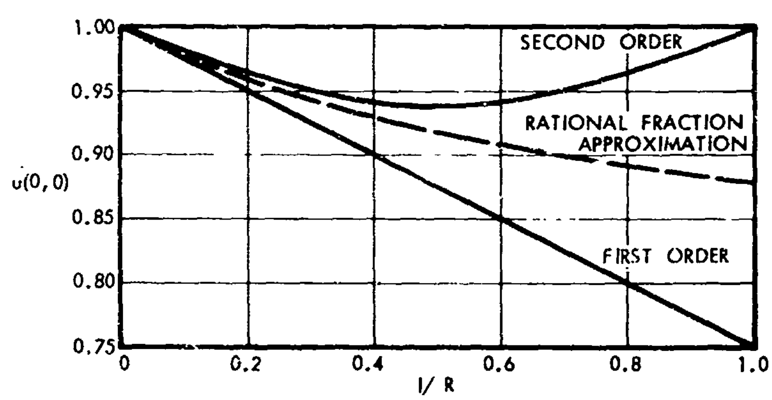

Figure 2-8. Throat Axis Velocity in Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.67$.

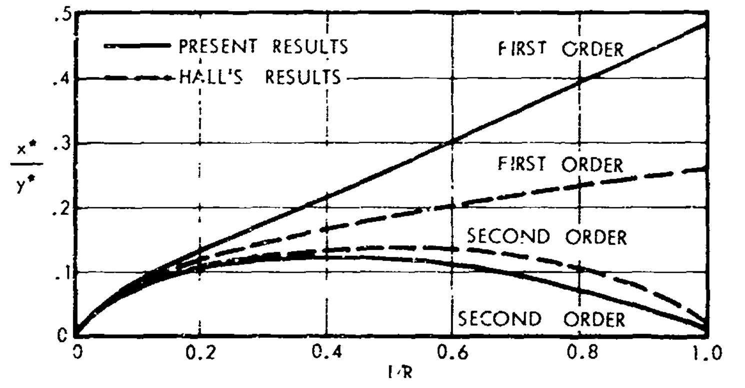

Figure 2-9. Sonic Point Displacement on Axis of Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.2$.

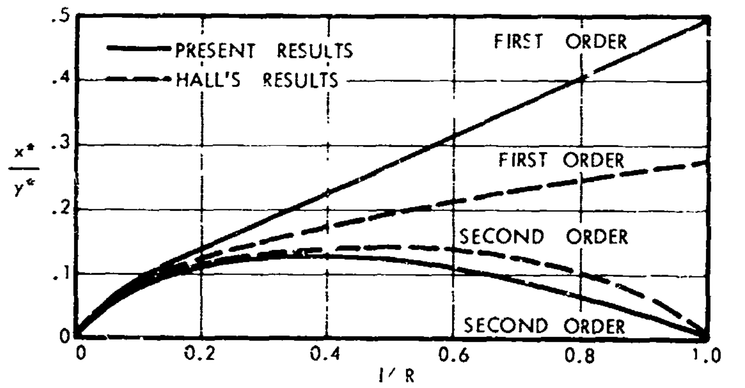

Figure 2-10. Sonic Point Displacement on Axis of Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.4$.

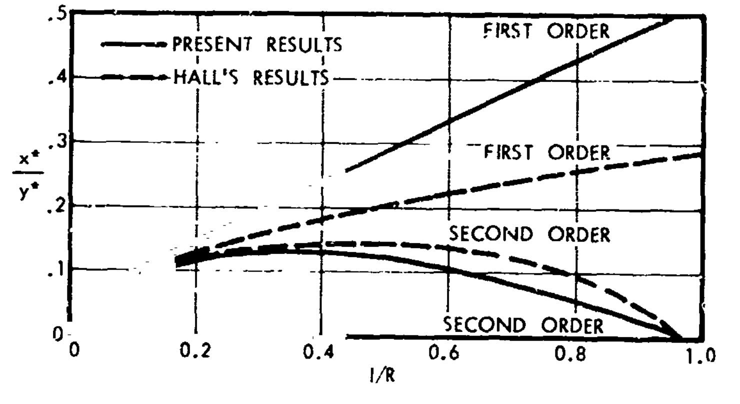

Figure 2-11. Sonic Point Displacement on Axis of Axisymmetric Hyperbolic Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.67$.

3. UNCHOKED NOZZLE FLOWS

Since unchoked shock free nozzle flows were of secondary interest during this study, only the applicability of the previous analysis to nozzle flows of this type will be shown and the throat boundary conditions given. Since the channel flow equations (see Appendix A) show that for unchoked (symmetric) flows

$$u = u^* - \frac{\omega + 1}{2}\left(1 - \frac{\gamma - 1}{\gamma + 1} u^{*2}\right)\frac{u^*}{1 - u^{*2}} \frac{x^2}{Ry^{*2}} + \ldots \tag{3-1}$$

at the nozzle throat, it is apparent that the axial nozzle coordinate x must be normalized with respect to the distance $\sqrt{R}\ y^*$ when analyzing unchoked nozzle flows in order for the dimensionless axial velocity gradients to remain of order one at the nozzle throat independent of the nozzle scale. Since the nozzle perpendicular to the axis is set by the throat half height y^* for all nozzle flows, it is apparent that the perpendicular coordinate y must be normalized with respect to the distance y^* when analyzing unchoked nozzle flows. Thus solutions of the equations governing the inviscid isentropic expansion of a perfect gas through a convergent-divergent nozzle [equations (2-1) and (2-2)] should be sought in terms of the normalized coordinates

$$z = \sqrt{\frac{1}{R}}\ \frac{x}{y^*} \tag{3-2}$$

$$r = \frac{y}{y^*} \tag{3-3}$$

rather than the x,y coordinate system for large values of the normalized throat wall radius of curvature when analyzing either choked or unchoked nozzle flows. Thus the preceeding analysis is also valid for unchoked nozzle flows.

Since the flow is symmetric, the throat boundary conditions for unchoked nozzle flows are that the axial derivatives of the various order axial velocity coefficients (a_o, b_o, b_2, etc.) are zero at the throat. It can be simply shown from the equations in the preceeding section that the one-dimensional, first order and second order unchoked throat conditions are

$$a_o(0) = a_o^* \tag{3-4}$$

$$a_1(0) = 0 \tag{3-5}$$

$$b_1(0) = 0 \tag{3-6}$$

$$b_2(0) = \frac{1}{2} a_o^* \tag{3-7}$$

$$b_3(0) = 0 \tag{3-8}$$

$$c_1(0) = 0 \tag{3-9}$$

$$c_3(0) = 0 \tag{3-10}$$

$$c_5(0) = 0 \tag{3-11}$$

It is noted that unlike the choked flow case, the throat boundary conditions are incompletely specified for unchoked nozzle flows. Physically this occurs because unchoked nozzle flows are not unique, there being an infinite family of such flows, the flow of interest being specified by an external constraint, the nozzle pressure ratio. This lack of uniqueness appears in the one-dimensional equations as the unspecified throat velocity a^*_o and in the equations governing the various order coefficients by the fact that the axis coefficients (b_o, c_o, etc.) drop from the equations at the throat due to the symmetrical nature of the flow. The uniqueness of the choked flow solution appears in the equations as a singularity which is missing in the equations for unchoked flows.

Since an external constraint must be specified in order to obtain a unique unchoked flow solution, it is desirable to specify a constraint such as to uniquely determine the throat conditions. The most natural such constraint is that the mass flow through the nozzle equals the one-dimensional mass flow through the nozzle. Thus specifying that

$$\int_o^1 \frac{(2\pi r)\, \rho(r,o)u(r,o)}{\rho^* a^*} \, dr = \pi^{\omega} a_o^* \left[\frac{\gamma + 1}{2} \left(1 - \frac{\gamma - 1}{\gamma + 1} a_o^{*2} \right) \right]^{\frac{1}{\gamma - 1}} \tag{3-12}$$

and expanding the integral as a power series in R^{-1} and equating the coefficients of the various powers of R^{-1} to zero yields the unique set of throat conditions

$$b_o(0) = -\frac{\omega + 1}{2(\omega + 3)} a_o^* \tag{3-13}$$

$$c_o(0) = \frac{(\omega + 1)^2 a_o^*}{16(\omega + 3)^2 (\omega + 5)} \left\{ \frac{16(3 - a_o^{*2}) a_o^{*2}}{(\omega + 1)(\gamma + 1)\left(1 - \frac{\gamma - 1}{\gamma + 1} a_o^{*2}\right)(1 - a^{*2})} - \frac{8(\omega + 5)}{\omega + 1} + \frac{(\omega + 1)(3 - 5a_o^{*2})}{1 - a_o^{*2}} - \frac{3(1 - \frac{\sigma}{R})(1 - a_o^{*2}) + 2\left[\frac{\gamma - 1}{\gamma + 1}(\omega + 1) + \frac{4}{\gamma + 1}\right] a_o^{*2}}{1 - \frac{\gamma - 1}{\gamma + 1} a_o^{*2}} \right\} \tag{3-14}$$

$$c_2(0) = \frac{a_o^*}{16(\omega + 3)} \left\{ 8 + \frac{(\omega + 1)(3 - 5a_o^{*2})}{1 - a_o^{*2}} - \frac{3(1 - \frac{\sigma}{R})(1 - a_o^{*2}) + 2\left[\frac{\gamma - 1}{\gamma + 1}(\omega + 1) + \frac{4}{\gamma + 1}\right] a_o^{*2}}{1 - \frac{\gamma - 1}{\gamma + 1} a_o^{*2}} \right\} \tag{3-15}$$

$$c_4(0) = \frac{a_o^*}{8(\omega + 3)} \left\{ - \frac{(\omega + 1)(3 - 5a_o^{*2})}{1 - a_o^{*2}} + \frac{3(1 - \frac{\sigma}{R})(1 - a_o^{*2}) + 2\left[\frac{\gamma - 1}{\gamma + 1}(\omega + 1) + \frac{4}{\gamma + 1}\right] a_o^{*2}}{1 - \frac{\gamma-1}{\gamma+1} a_o^{*2}} \right\} \tag{3-16}$$

where

$$\sigma = \begin{cases} 0 & \text{for parabolic throats} \\ 1 & \text{for circular arc throats} \\ -R & \text{for hyperolic throats} \end{cases} \tag{3-17}$$

It is interesting to note that the second order throat conditions depend on the wall shape. Thus only the first order throat conditions are universally valid for all nozzles. The solution away from the throat depends on the wall shape for all orders, however.

4. RELATIONSHIP TO HALL'S TRANSONIC SOLUTION

It is evident from the previous discussion that the nozzle flow solution presented in this report and Hall's transonic solution are closely related. The exact relationship between the two solutions can be easily seen by transforming Hall's solution to the coordinate system utilized in the present analysis. Thus, in the r,z coordinate system, Hall's first and second order axisymmetric solutions are:

$$u = 1 + \sqrt{\frac{2}{\gamma + 1}}\, z + \frac{1}{R}\left[-\frac{1}{4} + \frac{1}{2} r^2\right] \tag{4-1}$$

$$v = \sqrt{\frac{1}{R}}\left\{zr + \frac{1}{R}\left[-\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}}\, r + \frac{1}{4}\sqrt{\frac{\gamma + 1}{2}}\, r^3\right]\right\} \tag{4-2}$$

and

$$\begin{aligned} u = {} & 1 + \sqrt{\frac{2}{\gamma + 1}}\, z + \frac{3 - 2\gamma}{3(\gamma + 1)} z^2 + \frac{1}{R}\left[-\frac{1}{4} - \frac{5}{8}\sqrt{\frac{2}{\gamma + 1}}\, z \right. \\ & \left. + \left(\frac{1}{2} + \sqrt{\frac{2}{\gamma + 1}}\, z\right) r^2\right] + \frac{1}{R^2}\left[\frac{10\gamma + 57}{288} - \frac{4\gamma + 15}{24} r^2 + \frac{2\gamma + 9}{24} r^4\right] \end{aligned} \tag{4-3}$$

$$\begin{aligned} v = {} & \sqrt{\frac{1}{R}}\left\{zr + \sqrt{\frac{2}{\gamma + 1}}\, z^2 r + \frac{1}{R}\left[\left(-\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}} - \frac{4\gamma + 15}{12} z\right) r \right.\right. \\ & \left. + \left(\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}} + \frac{2\gamma + 9}{6} z\right) r^3\right] + \frac{1}{R^2}\left[\frac{28\gamma + 93}{288}\sqrt{\frac{\gamma + 1}{2}}\, r \right. \\ & \left.\left. - \frac{20\gamma + 63}{96}\sqrt{\frac{\gamma + 1}{2}}\, r^3 + \frac{\gamma + 3}{9}\sqrt{\frac{\gamma + 1}{2}}\, r^5\right]\right\} \end{aligned} \tag{4-4}$$

In the present analysis, the corresponding solutions are

$$u = a_o(z) + \frac{1}{R}\left[b_o(z) + b_2(z) r^2\right] \tag{4-5}$$

$$v = \sqrt{\frac{1}{R}}\left\{a_1(z) r + \frac{1}{R}\left[b_1(z) r + b_3(z) r^3\right]\right\} \tag{4-6}$$

and

$$u = a_o(z) + \frac{1}{R}[b_o(z) + b_2(z)r^2] + \frac{1}{R^2}[c_o(z) + c_2(z)r^2 + c_4(z)r^4] \quad (4\text{-}7)$$

$$v = \sqrt{\frac{1}{R}}\{a_1(z)r + \frac{1}{R}[b_1(z)r + b_3(z)r^3] + \frac{1}{R^2}[c_1(z)r + c_3(z)r^3 + c_5(z)r^5]\} \quad (4\text{-}8)$$

Expanding the various functions of z in the above equations as power series about the throat, the above solutions become

$$u = 1 + \sqrt{\frac{2}{\gamma + 1}}\, z + \frac{1}{R}[-\frac{1}{4} + \frac{1}{2} r^2] \quad (4\text{-}9)$$

$$v = \sqrt{\frac{1}{R}}\{ zr + \frac{1}{R}[-\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}}\, r + \frac{1}{4}\sqrt{\frac{\gamma+1}{2}}\, r^3]\} \quad (4\text{-}10)$$

and

$$u = 1 + \sqrt{\frac{2}{\gamma + 1}}\, z + \frac{3 - 2\gamma}{3(\gamma + 1)} z^2 + \frac{1}{R}[-\frac{1}{4} - \frac{5}{8}\sqrt{\frac{2}{\gamma + 1}}\, z + (\frac{1}{2} + \sqrt{\frac{2}{\gamma + 1}}\, z)\, r^2] + \frac{1}{R^2}[\frac{10\gamma + 57}{288} - \frac{4\gamma + 15}{24} r^2 + \frac{2\gamma + 9}{24} r^4] \quad (4\text{-}11)$$

$$v = \sqrt{\frac{1}{R}}\{zr + \sqrt{\frac{2}{\gamma + 1}}\, z^2 r + \frac{1}{R}[(-\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}} - \frac{4\gamma + 15}{12} z)r + (\frac{1}{4}\sqrt{\frac{\gamma + 1}{2}} + \frac{2\gamma + 9}{6} z)\, r^3] + \frac{1}{R^2}[\frac{28\gamma + 93}{288}\sqrt{\frac{\gamma + 1}{2}}\, r - \frac{20\gamma + 63}{96}\sqrt{\frac{\gamma + 1}{2}}\, r^3 + \frac{\gamma + 3}{9}\sqrt{\frac{\gamma + 1}{2}}\, r^5]\} \quad (4\text{-}12)$$

for axisymmetric flows. Comparison of Hall's solutions [equations (4-1) through (4-4)] with the above equations reveals that Hall's solutions are contained in the present solution and consist of expanding the various functions of z as power series about the throat and terminating the expansions at the $n - m^{th}$ term where n is the order of the solution desired and m is the order of the term in which the function appears. Thus, Hall's solutions are actually double expansion solutions, being expansions in both $\frac{1}{R}$ and z. The z expansion limits the validity of Hall's solution to the transonic region near the throat ($z \ll 1$). Figures 4-1 through 4-12 compare Hall's results with the present solution for the flow of air through hyperbolic nozzles.

In these figures, curve A refers to the present solution, curve B refers to Hall's first approximation and curve C refers to Hall's second approximation. Examination of the figures reveals that although both solutions are identical at the throat, there are considerable differences (especially in the higher order coefficients) away from the throat. This explains the difference in sonic point location between the present analysis and Hall's noted in Figures 2-9 through 2-11. In particular, the sonic point displacement in the present analysis depends on the throat shape for all orders while Hall's first and second order results are independent of the throat shape. Tables 4-1 through 4-3 compare Hall's sonic point displacement results with those of the present analysis for hyperbolic, parabolic and circular arc throat shapes as a function of $\frac{1}{R}$ for flows with specific heat ratios of 1.2, 1.4 and 1.67. Examination of the tables reveals that there is a noticeable effect of wall shape on the sonic point displacement. Comparison of the first order and second order results between themselves reveals that Hall's results do not fall between those for the three nozzle shapes. It would appear that Hall's analysis is applicable only to regions very near the throat and that his results away from the throat in the neighborhood of the sonic point are valid only for values of the normalized wall radius of curvature of five or greater.

Comparison of Hall's results with the present solution for planar flows reveals the same relationship between the two first order and second order analyses as was shown for axisymmetric flows. Although the present third order solution has not been completely worked out, comparison of Hall's third order results with the third order throat boundary conditions obtained for the present analysis reveals that the third order throat condition obtained from the two analyses differ. In particular, the third order throat conditions obtained for the present analysis depend on the wall shape (whether parabolic, hyperbolic or circular arc) while Hall's results do not. Since the present solution is uniformly valid for all nozzle flow regimes while Hall's solution is limited to the throat region, it appears that Hall's third and higher order solutions may be of mixed order in relationship to the present analysis. Resolution of this point is beyond the scope of the current study but it would appear that Hall's third order results may be severely limited in their applicability.

Table 4-1. Sonic Point Displacement on Axis of Axisymmetric Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature ($\gamma = 1.2$)

$\frac{1}{R}$	Circular Throat	Parabolic Throat	Hyperbolic Throat	Hall's Results
	$\frac{x^*}{y^*}$ from First Order Solution			
0.1	0.0883	0.0883	0.0882	0.0829
0.2	0.1338	0.1337	0.1330	0.1173
0.3	0.1775	0.1767	0.1741	0.1436
0.5	0.2863	0.2746	0.2576	0.1854
0.8	___*	0.6059	0.3928	0.2345
1.0	___*	___*	0.4804	0.2622
	$\frac{x^*}{y^*}$ from Second Order Solution			
0.1	0.0790	0.0791	0.0794	0.0798
0.2	0.1033	0.1037	0.1057	0.1078
0.3	0.1121	0.1137	0.1195	0.1249
0.5	0.0970	0.1039	0.1215	0.1386
0.8	0.0411	0.0502	0.0717	0.1075
1.0	0.0065	0.0088	0.0135	0.0289

* No solution obtained.

Table 4-2. Sonic Point Displacement on Axis of Axisymmetric Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature ($\gamma = 1.4$)

$\frac{1}{R}$	Circular Throat	Parabolic Throat	Hyperbolic Throat	Hall's Results
		$\frac{x^*}{y^*}$ from First Order Solution		
0.1	0.0924	0.0924	0.0922	0.0856
0.2	0.1403	0.1402	0.1393	0.1225
0.3	0.1866	0.1857	0.1827	0.1500
0.5	0.3049	0.2907	0.2706	0.1936
0.8	___*	0.6808	0.4094	0.2449
1.0	___*	___*	0.4953	0.2739
		$\frac{x^*}{y^*}$ from Second Order Solution		
0.1	0.0824	0.0824	0.0828	0.0832
0.2	0.1070	0.1075	0.1098	0.1122
0.3	0.1150	0.1168	0.1234	0.1297
0.5	0.0962	0.1036	0.1229	0.1422
0.8	0.0370	0.0455	0.0659	0.1028
1.0	0.0021	0.0029	0.0045	0.0101

* No solution obtained.

Table 4-3. Sonic Point Displacement on Axis of Axisymmetric Nozzle as a Function of Inverse Normalized Throat Wall Radius of Curvature ($\gamma = 1.67$)

$\frac{1}{R}$	Circular Throat	Parabolic Throat	Hyperbolic Throat	Hall's Results
	$\frac{x^*}{y^*}$ from First Order Solution			
0.1	0.0977	0.0977	0.0975	0.0913
0.2	0.1488	0.1486	0.1476	0.1292
0.3	0.1987	0.1975	0.1939	0.1582
0.5	0.3308	0.3124	0.2876	0.2043
0.8	___*	___*	0.4309	0.2584
1.0	___*	___*	0.5148	0.2889
	$\frac{x^*}{y^*}$ from Second Order Solution			
0.1	0.0866	0.0866	0.0870	0.0876
0.2	0.1116	0.1121	0.1148	0.1178
0.3	0.1181	0.1202	0.1279	0.1355
0.5	0.0943	0.1021	0.1233	0.1461
0.8	0.0314	0.0389	0.0569	0.0944
1.0	-0.0036	-0.0046	-0.0075	-0.0181

* No solution obtained.

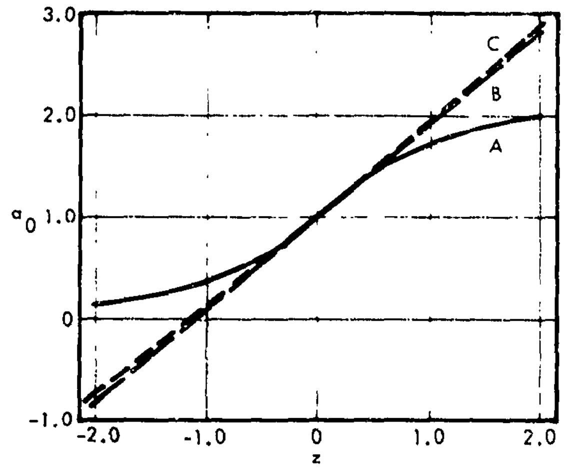

Figure 4-1. a_o vs. z

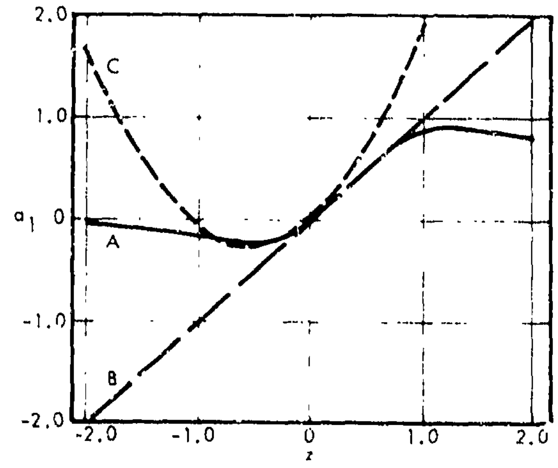

Figure 4-2. a_1 vs. z

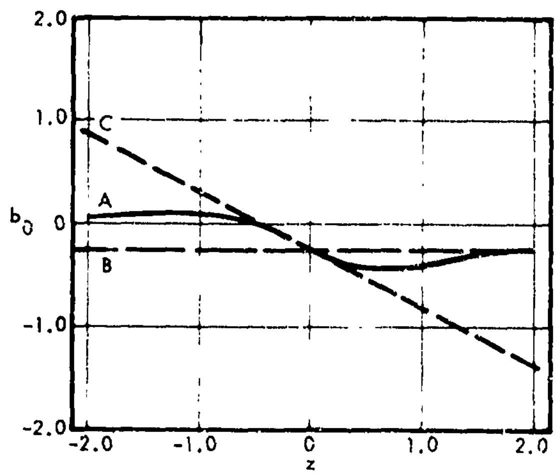

Figure 4-3. b_o vs. z

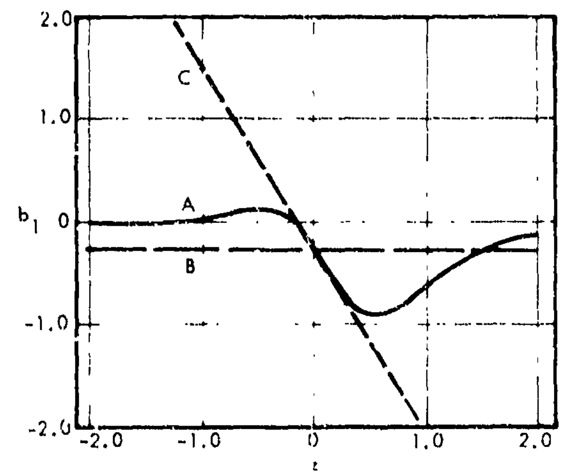

Figure 4-4. b_1 vs. z

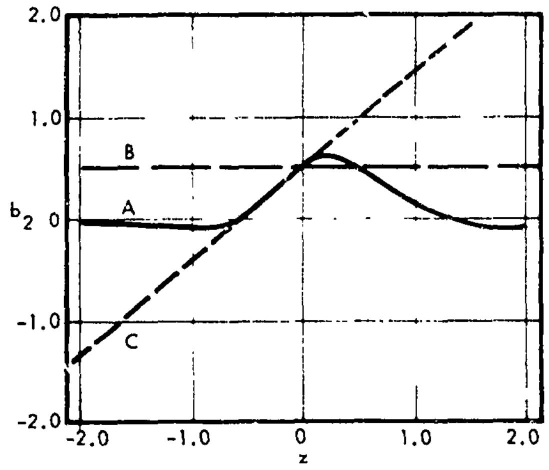

Figure 4-5. b_2 vs. z

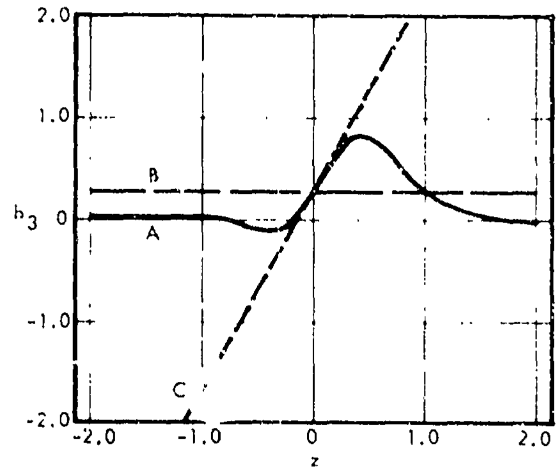

Figure 4-6. b_3 vs. z

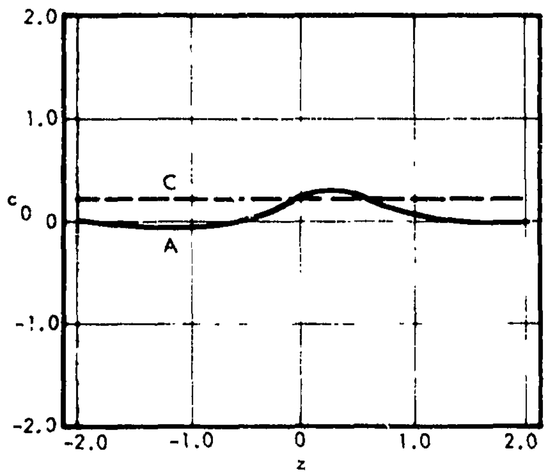

Figure 4-7. c_o vs. z

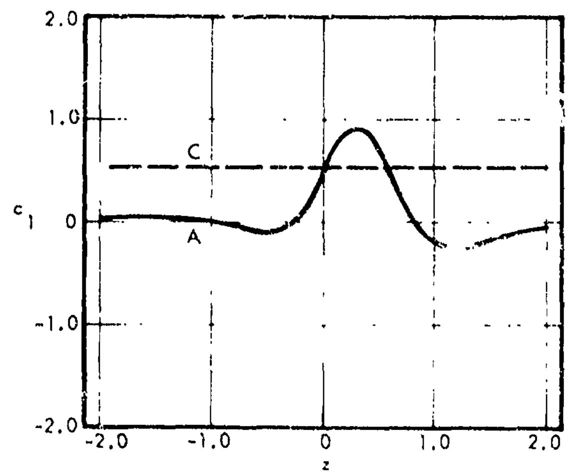

Figure 4-8. c_1 vs. z

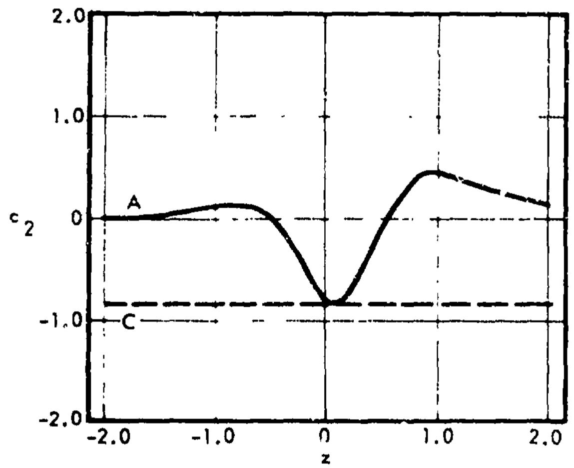

Figure 4-9. c_2 vs. z

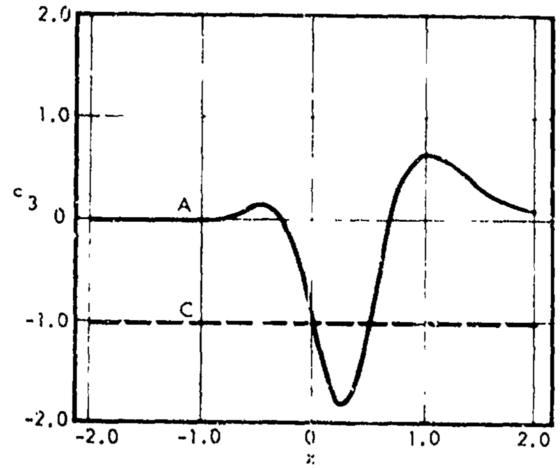

Figure 4-10. c_3 vs. z

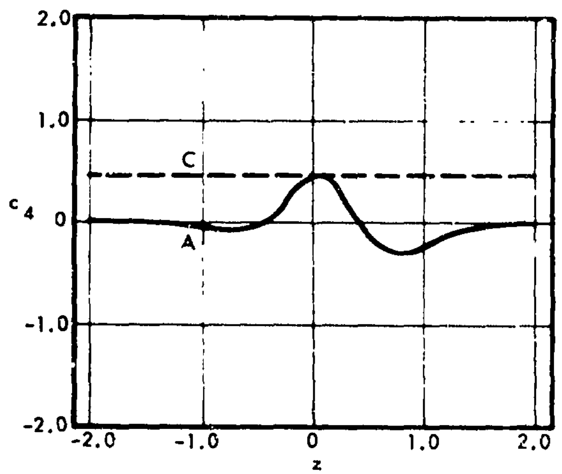

Figure 4-11. c_4 vs. z

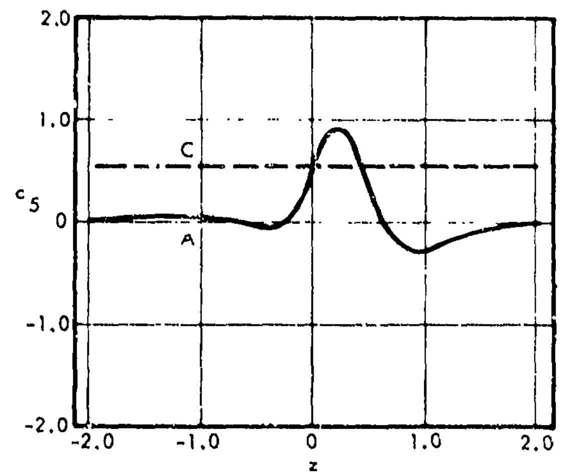

Figure 4-12. c_5 vs. z

5. TWO-ZONE NOZZLE EXPANSIONS

Since most rocket engines operate with a cool "barrier" zone near the wall to protect the thrust chamber from the hot "core" gases, the exhaust gas expansion through rocket engines can generally be represented as a two-zone expansion as shown in Figure 5-1. In order to simplify the analysis, the barrier zone is assumed to be confined to an annular ring. Thus the flow is axisymmetric in both zones. (The analysis is also applicable to two-dimensional nozzle flows in which the outer zone is planar.) Although it will be shown that the equations governing the two-zone expansion reduce to those for a uniform expansion, the two-zone solution will be derived separately.

The equations governing the inviscid isentropic expansion of two perfect gases through a nozzle are

$$\left(1 - u^2 - \frac{\gamma-1}{\gamma+1} v^2\right)\frac{\partial u}{\partial x} + \left(1 - v^2 - \frac{\gamma-1}{\gamma+1} u^2\right)\frac{\partial v}{\partial y} + \left[1 - \frac{\gamma-1}{\gamma+1} (u^2 + v^2)\right] \frac{\omega v}{y}$$

$$- \frac{4}{\gamma+1} uv \frac{\partial u}{\partial y} = 0 \tag{5-1}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{5-2}$$

in the inner zone and

$$\left(1 - \bar{u}^2 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{v}^2\right)\frac{\partial \bar{u}}{\partial x} + \left(1 - \bar{v}^2 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{u}^2\right)\frac{\partial \bar{v}}{\partial y} + \left[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1} (\bar{u}^2 + \bar{v}^2)\right]\frac{\omega \bar{v}}{y}$$

$$- \frac{4}{\bar{\gamma}+1} \bar{u}\bar{v} \frac{\partial \bar{u}}{\partial y} = 0 \tag{5-3}$$

$$\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} = 0 \tag{5-4}$$

in the outer zone where the velocities have been normalized with respect to the appropriate throat sonic velocity and ω equals 0 or 1, depending on whether the nozzle is planar or axisymmetric. As in the previous analysis, we shall seek solutions of the above equations in nondimensional coordinates chosen from the channel flow solutions such that the various velocity derivatives are independent of the nozzle scale for large values of the normalized throat wall radius of curvature. It can be shown from the two-zone channel flow solutions (see Appendix A) that for choked flows

$$u = 1 + \sqrt{\frac{\omega+1}{\gamma+1}\frac{k}{R}}\;\frac{x}{y^*} + \ldots \tag{5-5}$$

$$\bar{u} = 1 + \frac{\gamma}{\bar{\gamma}}\sqrt{\frac{\omega+1}{\gamma+1}\frac{k}{R}}\;\frac{x}{y^*} + \ldots \tag{5-6}$$

at the nozzle throat where x is the distance from the throat plane, y^* is the throat half height, R is the normalized throat wall radius of curvature and k is a dimensionless constant of order one. Examination of these equations reveals that as in the previous analysis, the axial nozzle coordinate x must be normalized with respect to the distance $\sqrt{R}\,y^*$ in order for the dimensionless axial velocity gradients to remain of order one at the nozzle throat independent of the nozzle scale. Similarly, since the nozzle scale perpendicular to the nozzle axis is set by the throat half height y^*, the perpendicular coordinate y should be normalized with respect to the distance y^*. Thus, solutions to the above equations for large values of the normalized throat wall radius of curvature should again be sought in terms of the normalized coordinates

$$z = \sqrt{\frac{1}{R}}\;\frac{x}{y^*} \tag{5-7}$$

$$r = \frac{y}{y^*} \tag{5-8}$$

rather than in the x,y coordinate system.

In the r,z coordinate system, the above equations become

$$\sqrt{\frac{1}{R}}\left(1 - u^2 - \frac{\gamma-1}{\gamma+1}v^2\right)\frac{\partial u}{\partial z} + \left(1 - v^2 - \frac{\gamma-1}{\gamma+1}u^2\right)\frac{\partial v}{\partial r} + \left[1 - \frac{\gamma-1}{\gamma+1}(u^2 + v^2)\right]\frac{\omega v}{r}$$

$$- \frac{4}{\gamma+1}uv\frac{\partial u}{\partial r} = 0 \tag{5-9}$$

$$\sqrt{\frac{1}{R}}\;\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} = 0 \tag{5-10}$$

in the inner zone and

$$\sqrt{\frac{1}{R}}\left(1 - \bar{u}^2 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{v}^2\right)\frac{\partial \bar{u}}{\partial z} + \left(1 - \bar{v}^2 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}^2\right)\frac{\partial \bar{v}}{\partial r} + \left[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}(\bar{u}^2 + \bar{v}^2)\right]\frac{\omega \bar{v}}{r}$$

$$- \frac{4}{\bar{\gamma}+1}\bar{u}\bar{v}\frac{\partial \bar{u}}{\partial r} = 0 \tag{5-11}$$

$$\sqrt{\frac{1}{R}}\ \frac{\partial \bar{v}}{\partial z} - \frac{\partial \bar{u}}{\partial r} = 0 \qquad (5\text{-}12)$$

in the outer zone.

The boundary conditions on the axis and at the wall are

$$v(o,z) = 0 \qquad (5\text{-}13)$$

$$\frac{\bar{v}(r_w,z)}{\bar{u}(r_w,z)} = \sqrt{\frac{1}{R}}\ \frac{dr_w}{dz} \qquad (5\text{-}14)$$

Since the flow angle and pressure match at the streamline dividing the two zones, the boundary conditions at the dividing streamline are

$$\frac{v(r_s,z)}{u(r_s,z)} = \frac{\bar{v}(r_s,z)}{\bar{u}(r_s,z)} = \sqrt{\frac{1}{R}}\ \frac{dr_s}{dz} \qquad (5\text{-}15)$$

$$P^* \left\{\frac{\gamma+1}{2}\left[1 - \frac{\gamma-1}{\gamma+1}(u(r_s,z)^2 + v(r_s,z)^2)\right]\right\}^{\frac{\gamma}{\gamma-1}}$$

$$= \bar{P}^* \left\{\frac{\bar{\gamma}+1}{2}\left[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}(\bar{u}(r_s,z)^2 + \bar{v}(r_s,z)^2)\right]\right\}^{\frac{\bar{\gamma}}{\bar{\gamma}-1}} \qquad (5\text{-}16)$$

where r_s is the radial position of the dividing streamline.

The sonic pressure is equal in both zones (see Appendix A) since this condition maximizes the mass flow through the nozzle and the throat plane then sets the flow through the nozzles. There are other families of solutions to the above equations for different pressure conditions (such as the total pressure in both zones being equal). In these solutions, the flow is not set at the throat plane but is set elsewhere in the flow system. These solutions (which correspond to nozzle flows with controlled external bleed such as occur in jet engines or ducted rockets) will not be further considered in this report.

Since u, $\bar{u}$, $\frac{dr_s}{dz}$ and $\frac{dr_w}{dz}$ are $O(1)$ at the throat, v and $\bar{v}$ must both be $O(R^{-1/2})$. This suggests that the velocity components in both zones can be expressed as expansions in inverse power of R for large values of the normalized throat wall radius of curvature, i.e.,

$$u = u_o(r,z) + \frac{u_1(r,z)}{R} + \frac{u_2(r,z)}{R^2} + \ldots \tag{5-17}$$

$$v = \sqrt{\frac{1}{R}}\,[v_o(r,z) + \frac{v_1(r,z)}{R} + \frac{v_2(r,z)}{R^2} + \ldots] \tag{5-18}$$

$$\bar{u} = \bar{u}_o(r,z) + \frac{\bar{u}_1(r,z)}{R} + \frac{\bar{u}_2(r,z)}{R^2} + \ldots \tag{5-19}$$

$$\bar{v} = \sqrt{\frac{1}{R}}\,[\bar{v}_o(r,z) + \frac{\bar{v}_1(r,z)}{R} + \frac{v_2(r,z)}{R^2} + \ldots] \tag{5-20}$$

Substituting into equations (5-9) through (5-12) and equating powers of R^{-1} separately gives the following sets of equations:

$$\left(1 - u_o^2\right)\frac{\partial u_o}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}u_o^2\right)\left(\frac{\partial v_o}{\partial r} + \frac{\omega v_o}{r}\right) = 0 \tag{5-21}$$

$$\frac{\partial u_o}{\partial r} = 0 \tag{5-22}$$

$$\left(1 - u_o^2\right)\frac{\partial u_n}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}u_o^2\right)\left(\frac{\partial v_n}{\partial r} + \frac{\omega v_n}{r}\right) - \frac{4}{\gamma+1}u_o v_o\frac{\partial u_n}{\partial r} = \phi_n, \quad n \geq 1 \tag{5-23}$$

$$\frac{\partial v_{n-1}}{\partial z} - \frac{\partial u_n}{\partial r} = 0, \quad n \geq 1 \tag{5-24}$$

in the inner zone where

$$\phi_1 = \left(2u_ou_1 + \frac{\gamma-1}{\gamma+1}v_o^2\right)\frac{\partial u_o}{\partial z} + \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1}u_ou_1\right)\frac{\partial v_o}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_ou_1 + v_o^2\right)\frac{\omega v_o}{r} \tag{5-25}$$

$$\begin{aligned}\phi_2 = {} & \left(2u_ou_2 + u_1^2 + 2\frac{\gamma-1}{\gamma+1}v_ov_1\right)\frac{\partial u_o}{\partial z} + \left[2v_ov_1 + \frac{\gamma-1}{\gamma+1}\left(2u_ou_2 + u_1^2\right)\right]\frac{\partial v_o}{\partial r} \\ & + \frac{\gamma-1}{\gamma+1}\left(2u_ou_2 + u_1^2 + 2v_ov_1\right)\frac{\omega v_o}{r} + \left(2u_ou_1 + \frac{\gamma-1}{\gamma+1}v_o^2\right)\frac{\partial u_1}{\partial z} \\ & + \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1}u_ou_1\right)\frac{\partial v_1}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_ou_1 + v_o^2\right)\frac{\omega v_1}{r} + \frac{4}{\gamma+1}\left(u_ov_1 + u_1v_o\right)\frac{\partial u_1}{\partial r}\end{aligned} \tag{5-26}$$

...

and

$$\left(1 - \bar{u}_o^2\right)\frac{\partial \bar{u}_o}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}\bar{u}_o^2\right)\left(\frac{\partial \bar{v}_o}{\partial r} + \frac{\omega \bar{v}_o}{r}\right) = 0 \tag{5-27}$$

$$\frac{\partial \bar{u}_o}{\partial r} = 0 \tag{5-28}$$

$$\left(1 - \bar{u}_o^2\right)\frac{\partial \bar{u}_n}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}\bar{u}_o^2\right)\left(\frac{\partial \bar{v}_n}{\partial r} + \frac{\omega \bar{v}_n}{r}\right) - \frac{4}{\gamma+1}\bar{u}_o\bar{v}_o\frac{\partial \bar{u}_n}{\partial r} = \bar{\phi}_n, \quad n \geq 1 \tag{5-29}$$

$$\frac{\partial \bar{v}_{n-1}}{\partial z} - \frac{\partial \bar{u}_n}{\partial r} = 0, \quad n \geq 1 \tag{5-30}$$

in the outer zone where

$$\bar{\phi}_1 = \left(2\bar{u}_o\bar{u}_1 + \frac{\gamma-1}{\gamma+1}\bar{v}_o^2\right)\frac{\partial \bar{u}_o}{\partial z} + \left(\bar{v}_o^2 + 2\frac{\gamma-1}{\gamma+1}\bar{u}_o\bar{u}_1\right)\frac{\partial \bar{v}_o}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2\bar{u}_o\bar{u}_1 + \bar{v}_o^2\right)\frac{\omega \bar{v}_o}{r} \tag{5-31}$$

$$\bar{\phi}_2 = \left(2\bar{u}_o\bar{u}_2 + \bar{u}_1^2 + 2\frac{\gamma-1}{\gamma+1}\bar{v}_o\bar{v}_1\right)\frac{\partial \bar{u}_o}{\partial z} + \left[2\bar{v}_o\bar{v}_1 + \frac{\gamma-1}{\gamma+1}\left(2\bar{u}_o\bar{u}_2 + \bar{u}_1^2\right)\right]\frac{\partial \bar{v}_o}{\partial r}$$

$$+ \frac{\gamma-1}{\gamma+1}\left(2\bar{u}_o\bar{u}_2 + \bar{u}_1^2 + 2\bar{v}_o\bar{v}_1\right)\frac{\omega \bar{v}_o}{r} + \left(2\bar{u}_o\bar{u}_1 + \frac{\gamma-1}{\gamma+1}\bar{v}_o^2\right)\frac{\partial \bar{u}_1}{\partial z} + \left(\bar{v}_o^2 + 2\frac{\gamma-1}{\gamma+1}\bar{u}_o\bar{u}_1\right)\frac{\partial \bar{v}_1}{\partial r}$$

$$+ \frac{\gamma-1}{\gamma+1}\left(2\bar{u}_o\bar{u}_1 + \bar{v}_o^2\right)\frac{\omega \bar{v}_1}{r} + \frac{4}{\gamma+1}\left(\bar{u}_o\bar{v}_1 + \bar{u}_1\bar{v}_o\right)\frac{\partial \bar{u}_1}{\partial r} \tag{5-32}$$

. . . .

The above system of equations are identical to those governing a uniform expansion. Thus, the solution for the inner zone is identical to the solution previously derived except that the dividing streamline boundary condition replaces the wall boundary condition. As was shown earlier [equation (2-23)], the complete solution of the above equations for v_o is of the form

$$v_o(r,z) = a_1(z)r + \omega a_3(z)r^{-1} + (1-\omega)\, a_5(z) \tag{5-33}$$

where the functions $a_3(z)$ and $a_5(z)$ are identically zero in uniform expansions and in the inner zone. In the outer zone, however, $a_3(z)$ and $a_5(z)$ are not identically zero but are determined from the dividing streamline boundary conditions. Thus, the outer zone solutions contain additional terms dependent on $a_3(z)$ and $a_5(z)$ which do not appear in the inner zone solutions. Since planar two-zone expansions were not of interest in the present study, only the axisymmetric solution is given below.

Since the solution in both zones is obtained as power series in inverse powers of R, it is convenient to reduce the boundary conditions to a series of conditions for the various order solutions. The axis and wall boundary conditions are

$$v_n(o,z) = 0 \tag{5-34}$$

$$\bar{v}_n(r_w,z) = \bar{u}_n(r_w,z)\frac{dr_w}{dz} \tag{5-35}$$

By expanding the position of the dividing streamline (r_s) in inverse powers of R, i.e.,

$$r_s(z) = r_{s1}(z) + \frac{r_{s1}(z)}{R} + \frac{r_{s2}(z)}{R^2} + \ldots \tag{5-36}$$

and noting that

$$u(r_s,z) = u(r_{so},z) + \frac{1}{R}\frac{\partial u}{\partial r}\bigg|_{r_{so},z} r_{s1}(z) + \frac{1}{R^2}\left[\frac{\partial u}{\partial r}\bigg|_{r_{so},z} r_{s2}(z) + \frac{1}{2}\frac{\partial^2 u}{\partial r^2}\bigg|_{r_{so},z} r_{s1}(z)^2\right] + \ldots$$

$$= u_o(r_{so},z) + \frac{1}{R}\left[u_1(r_{so},z) + \frac{\partial u_o}{\partial r}\bigg|_{r_{so},z} r_{s1}(z)\right] + \frac{1}{R^2}\left[u_2(r_{so},z) + \frac{\partial u_1}{\partial r}\bigg|_{r_{so},z} r_{s1}(z) + \frac{\partial u_o}{\partial r}\bigg|_{r_{so},z} r_{s2}(z) + \frac{\partial^2 u_o}{\partial r^2}\bigg|_{r_{so},z} \frac{r_{s1}(z)^2}{2}\right] + \ldots \tag{5-37}$$

(and similarly for the other velocity components), the first dividing streamline boundary condition [equation (5-15)] can be rewritten in terms of the various order solutions as

$$v_o(r_{so},z) = u_o(r_{so},z)\frac{dr_{so}}{dz} \tag{5-38}$$

$$v_1(r_{so},z) + \left.\frac{\partial v_o}{\partial r}\right|_{r_{so},z} r_{s1}(z) = \left[u_1(r_{so},z) + \left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\frac{dr_{so}}{dz}$$

$$+ u_o(r_{so},z)\frac{dr_{s1}}{dz} \tag{5-39}$$

$$v_2(r_{so},z) + \left.\frac{\partial v_1}{\partial r}\right|_{r_{so},z} r_{s1}(z) + \left.\frac{\partial v_o}{\partial r}\right|_{r_{so},z} r_{s2}(z) + \frac{1}{2}\left.\frac{\partial^2 v_o}{\partial z^2}\right|_{r_{so},z} r_{s1}(z)^2$$

$$= \left[u_2(r_{so},z) + \left.\frac{\partial u_1}{\partial r}\right|_{r_{so},z} r_{s1}(z) + \left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s2}(z)\right.$$

$$\left. + \frac{1}{2}\left.\frac{\partial^2 u_o}{\partial r^2}\right|_{r_{so},z} r_{s1}(z)^2\right] + \left[u_1(r_{so},z) + \left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\frac{dr_{s1}}{dz}$$

$$+ u_o(r_{so},z)\frac{dr_{s2}}{dz} \tag{5-40}$$

.....

$$\bar{v}_o(r_{so},z) = \bar{u}_o(r_{so},z)\frac{dr_{so}}{dz} \tag{5-41}$$

$$\bar{v}_1(r_{so},z) + \left.\frac{\partial \bar{v}_o}{\partial r}\right|_{r_{so},z} r_{s1}(z) = \left[\bar{u}_1(r_{so},z) + \left.\frac{\partial \bar{u}_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\frac{dr_{so}}{dz}$$

$$+ \bar{u}_o(r_s,z)\frac{dr_{s1}}{dz} \tag{5-42}$$

$$\bar{v}_2(r_{so},z) + \frac{\partial \bar{v}_1}{\partial r}\Big|_{r_{so},z} r_{s1}(z) + \frac{\partial \bar{v}_o}{\partial r}\Big|_{r_{so},z} r_{s2}(z) + \frac{1}{2}\frac{\partial^2 \bar{v}_o}{\partial r^2}\Big|_{r_{so},z} r_{s1}(z)^2$$

$$= \Big[\bar{u}_2(r_{so},z) + \frac{\partial \bar{u}_1}{\partial r}\Big|_{r_{so},z} r_{s1}(z) + \frac{\partial \bar{u}_o}{\partial r}\Big|_{r_{so},z} r_{s2}(z)$$

$$+ \frac{1}{2}\frac{\partial^2 \bar{u}_o}{\partial r^2}\Big|_{r_{so},z} r_{s1}(z)^2\Big] \frac{dr_{so}}{dz} + \Big[\bar{u}_1(r_{so},z) + \frac{\partial \bar{u}_o}{\partial r}\Big|_{r_{so},z} r_{s1}(z)\Big]\frac{dr_{s1}}{dz}$$

$$+ \bar{u}_o(r_{so},z)\frac{dr_{s2}}{dz} \tag{5-43}$$

...

by equating inverse powers of R. Similarly, by expanding as a power series and equating inverse powers of R, the second dividing streamline boundary condition [equation (5-16)] can be rewritten in terms of the various order solutions as

$$\Big\{\frac{\gamma+1}{2}\Big[1 - \frac{\gamma-1}{\gamma+1} u_o(r_{so},z)^2\Big]\Big\}^{\frac{\gamma}{\gamma-1}} = \Big\{\frac{\bar{\gamma}+1}{2}\Big[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}_o(r_{so},z)^2\Big]\Big\}^{\frac{\bar{\gamma}}{\bar{\gamma}-1}} \tag{5-44}$$

$$\frac{\gamma}{\gamma+1}\Big[1 - \frac{\gamma-1}{\gamma+1} u_o(r_{so},z)^2\Big]^{-1} \Big\{2u_o(r_{so},z)\Big[u_1(r_{so},z) + \frac{\partial u_o}{\partial r}\Big|_{r_{so},z} r_{s1}(z)\Big]$$

$$+ v_o(r_{so},z)^2\Big\}$$

$$= \frac{\bar{\gamma}}{\bar{\gamma}+1}\Big[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}_o(r_{so},z)^2\Big]^{-1} \Big\{2\bar{u}_o(r_{so},z)\Big[\bar{u}_1(r_{so},z + \frac{\partial \bar{u}_o}{\partial r}\Big|_{r_{so},z} r_{s1}(z)\Big]$$

$$+ \bar{v}_o(r_{so},z)^2\Big\} \tag{5-45}$$

$$
\frac{\gamma}{(\gamma+1)^2}\left[1-\frac{\gamma-1}{\gamma+1}u_o(r_{so},z)^2\right]^{-2}\left\{2u_o(r_{so},z)\left[u_1(r_{so},z)+\left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\right.
$$

$$
\left.+\,v_o(r_{so},z)^2\right\}^2-\frac{\gamma}{\gamma+1}\left[1-\frac{\gamma-1}{\gamma+1}u_o(r_{so},z)^2\right]^{-1}
$$

$$
\left\{2u_o(r_{so},z)\left[u_2(r_{so},z)+\left.\frac{\partial u_1}{\partial r}\right|_{r_{so},z} r_{s1}(z)+\left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s2}(z)\right.\right.
$$

$$
\left.+\,\frac{1}{2}\left.\frac{\partial^2 u_o}{\partial r^2}\right|_{r_{so},z} r_{s1}(z)^2\right]+\left[u_1(r_{so},z)+\left.\frac{\partial u_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]^2
$$

$$
\left.+\,2v_o(r_{so},z)\left[v_1(r_{so},z)+\left.\frac{\partial v_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\right\}
$$

$$
=\frac{\bar{\gamma}}{(\bar{\gamma}+1)^2}\left[1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}_o(r_{so},z)^2\right]^{-2}\left\{2\bar{u}_o(r_{so},z)\left[\bar{u}_1(r_{so},z)\right.\right.
$$

$$
\left.\left.+\left.\frac{\partial \bar{u}_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]+\bar{v}_o(r_{so},z)^2\right\}^2
$$

$$
-\frac{\bar{\gamma}}{\bar{\gamma}+1}\left[1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}_o(r_{so},z)^2\right]^{-1}\left\{2\bar{u}_o(r_{so},z)\left[\bar{u}_2(r_{so},z)+\left.\frac{\partial \bar{u}_1}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right.\right.
$$

$$
\left.+\left.\frac{\partial \bar{u}_o}{\partial r}\right|_{r_{so},z} r_{s2}(z)+\frac{1}{2}\left.\frac{\partial^2 \bar{u}_o}{\partial r^2}\right|_{r_{so},z} r_{s1}(z)^2\right]
$$

$$
+\left[\bar{u}_1(r_{so},z)+\left.\frac{\partial \bar{u}_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]^2+2\bar{v}_o(r_{so},z)\left[\bar{v}_1(r_{so},z)\right.
$$

$$
\left.\left.+\left.\frac{\partial \bar{v}_o}{\partial r}\right|_{r_{so},z} r_{s1}(z)\right]\right\} \tag{5-46}
$$

The boundary condition on the position of the dividing streamline is that the ratio of mass flows through the two zones be constant at the throat. This condition can be expressed as

$$\frac{\int_{r_o(o)}^{1} 2\pi r\bar{\rho}(r,o)\bar{u}(r,o)dr}{\int_{0}^{r_s(o)} 2\pi r\rho(r,o)u(r,o)dr} = \frac{\bar{\rho}^*\bar{a}^*}{\rho^*a^*}\left\{\frac{1-r_{so}(o)^2}{r_{so}(o)^2} - \frac{1}{R^2}\left[\frac{\bar{\gamma}+1}{2}\int_{r_{so}(o)}^{1} 2\pi r\bar{u}_1(r,o)dr\right.\right.$$

$$\left.\left. - \frac{\gamma+1}{2}\int_{0}^{r_{so}(o)} 2\pi ru_1(r,o)dr\right] + \ldots\right\}$$

$$= \text{constant} \tag{5-47}$$

by expanding the integrals as a power series in R^{-1}. Substituting for $r_s(0)$ [equation (5-36)] and equating powers of R^{-1} to zero yields

$$r_{so}(0) = \left[1 + \frac{\bar{x}}{1-\bar{x}}\frac{\rho^*a^*}{\bar{\rho}^*\bar{a}^*}\right]^{-1/2} \tag{5-48}$$

$$r_{s1}(0) = 0 \tag{5-49}$$

$$r_{s2}(0) = \frac{r_{so}[1-r_{so}(o)^2]}{2}\left[\frac{\gamma+1}{2}\int_{0}^{r_{so}(o)} 2\pi ru_1(r,o)dr\right.$$

$$\left. - \frac{\bar{\gamma}+1}{2}\int_{r_{so}(o)}^{1} 2\pi r\bar{u}_1(r,o)dr\right] \tag{5-50}$$

where $\bar{x}$ is the fraction of the nozzle mass flow in the outer zone.

Equations (5-22) and (5-28) show that $u_o(r,z)$ and $\bar{u}_o(r,z)$ are functions of z alone. Thus,

$$u_o(r,z) = a_o(z) \tag{5-51}$$

$$\bar{u}_o(r,z) = \bar{a}_o(z) \tag{5-52}$$

Equations (5-21), (5-27) and (5-34) are satisifed if $v_o(r,z)$ and $\bar{v}_o(r,z)$ are of the form

$$v_o(r,z) = a_1(z)r \tag{5-53}$$

$$\bar{v}_o(r,z) = \bar{a}_1(z)r + \bar{a}_3(z)r^{-1} \tag{5-54}$$

From the remaining boundary conditions [equations (5-35), (5-38), (5-41), (5-44) and (5-48)], it is easily shown that

$$a_1 = \frac{a_o}{r_{so}}\frac{dr_{so}}{dz} \tag{5-55}$$

$$\bar{a}_1 = \frac{a_o}{r_w^2 - r_{so}^2}\left(r_w \frac{dr_w}{dz} - r_{so}\frac{dr_{so}}{dz}\right) \tag{5-56}$$

$$\bar{a}_3 = \frac{a_o r_w r_{so}}{r_w^2 - r_{so}^2}\left(r_w \frac{dr_{so}}{dz} - r_{so}\frac{dr_w}{dz}\right) \tag{5-57}$$

$$\left\{\frac{\gamma+1}{2}\left[1 - \frac{\gamma-1}{\gamma+1}u_o^2\right]\right\}^{\frac{\gamma}{\gamma-1}} = \left\{\frac{\gamma+1}{2}\left[1 - \frac{\gamma-1}{\gamma+1}u_o^2\right]\right\}^{\frac{\gamma}{\gamma-1}} \tag{5-58}$$

$$r_{so}(0) = \left[1 + \frac{\bar{x}}{1-\bar{x}}\frac{\rho^* a^*}{\bar{\rho}^*\bar{a}^*}\right]^{-1/2} \tag{5-59}$$

Substituting the above results into equations (5-21) and (5-27) yields

$$\left(1 - a_o^2\right)\frac{da_o}{dz} + 2\left(1 - \frac{\gamma-1}{\gamma+1}a_o^2\right)\frac{a_o}{r_{so}}\frac{dr_{so}}{dz} = 0 \tag{5-60}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{a}_o}{dz} + 2\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\frac{\bar{a}_o}{r_w^2 - r_{so}^2}\left(r_w\frac{dr_w}{dz} - r_{so}\frac{dr_{so}}{dz}\right) = 0 \tag{5-61}$$

which are the one-dimensional channel flow equations. The solution of the above equations defines the one-dimensional velocity distribution (u_o, v_o, $\bar{u}_o$ and $\bar{v}_o$)

and dividing streamline location (r_{so}) through the nozzle. Since the one-dimensional solution is valid for all (subsonic, transonic and supersonic) nozzle flow regimes, the present solution will also be valid for all nozzle flow regimes.

The first order equations are

$$\left(1 - u_o^2\right)\frac{\partial u_1}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1}\,u_o^2\right)\left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r}\right) - \frac{4}{\gamma+1}\,u_o v_o\,\frac{\partial u_1}{\partial r}$$

$$= \left(2u_o u_1 + \frac{\gamma-1}{\gamma+1}\,v_o^2\right)\frac{\partial u_o}{\partial z} + \left(v_o^2 + 2\,\frac{\gamma-1}{\gamma+1}\,u_o u_1\right)\frac{\partial v_o}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_o u_1 + v_o^2\right)\frac{v_o}{r} \qquad (5\text{-}62)$$

$$\frac{\partial v_o}{\partial z} - \frac{\partial u_1}{\partial r} = 0 \qquad (5\text{-}63)$$

$$\left(1 - \bar{u}_o^2\right)\frac{\partial \bar{u}_1}{\partial z} + \left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{u}_o^2\right)\left(\frac{\partial \bar{v}_1}{\partial r} + \frac{\bar{v}_1}{r}\right) - \frac{4}{\bar{\gamma}+1}\,\bar{u}_o \bar{v}_o\,\frac{\partial \bar{u}_1}{\partial r}$$

$$= \left(2\bar{u}_o \bar{u}_1 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{v}_o^2\right)\frac{\partial \bar{u}_o}{\partial z} + \left(\bar{v}_o^2 + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{u}_o \bar{u}_1\right)\frac{\partial \bar{v}_o}{\partial r} + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\left(2\bar{u}_o \bar{u}_1 + \bar{v}_o^2\right)\frac{\bar{v}_o}{r} \qquad (5\text{-}64)$$

$$\frac{\partial \bar{v}_o}{\partial z} - \frac{\partial \bar{u}_1}{\partial r} = 0 \qquad (5\text{-}65)$$

From equations (5-53), (5-54), (5-63) and (5-65), it is easily shown that

$$u_1 = b_o(z) + b_2(z)r^2 \qquad (5\text{-}66)$$

$$\bar{u}_1 = \bar{b}_o(z) + \bar{b}_2(z)r^2 + \bar{b}_4(z)\,\ln r \qquad (5\text{-}67)$$

where

$$b_2 = \frac{1}{2}\frac{da_1}{dz} \qquad (5\text{-}68)$$

$$\bar{b}_2 = \frac{1}{2}\frac{d\bar{a}_1}{dz} \qquad (5\text{-}69)$$

$$\bar{b}_4 = \frac{d\bar{a}_3}{dz} \qquad (5\text{-}70)$$

From equations (5-34), (5-62), (5-64), (5-66) and (5-67), it can be shown that

$$v_1 = b_1(z)r + b_3(z)r^3 \tag{5-71}$$

$$\bar{v}_1 = \bar{b}_1(z)r + \bar{b}_3(z)r^3 + \bar{b}_5(z)\ r\ \ell n\ r + \bar{b}_7(z)r^{-1}$$

$$+ \bar{b}_9(z)r^{-3} + \bar{b}_{11}(z)r^{-1}\ \ell n\ r \tag{5-72}$$

where

$$\left(1 - a_o^2\right)\frac{db_o}{dz} + 2\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)b_1 = 2a_o b_o\left[\frac{da_o}{dz} + 2\frac{\gamma-1}{\gamma+1} a_1\right] \tag{5-73}$$

$$\left(1 - a_o^2\right)\frac{db_2}{dz} + 4\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)b_3 - \frac{8}{\gamma+1} a_o a_1 b_2$$

$$= 2a_o b_2\left[\frac{da_o}{dz} + 2\frac{\gamma-1}{\gamma+1} a_1\right] + a_1^2\left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \frac{2\gamma}{\gamma+1} a_1\right] \tag{5-74}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{b}_o}{dz} + \left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)(2\bar{b}_1 + \bar{b}_5) = 2\bar{a}_o\bar{b}_o\left[\frac{d\bar{a}_o}{dz} + 2\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_1\right]$$

$$+ 2\bar{a}_1\bar{a}_3\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] - \frac{2}{\bar{\gamma}+1}\bar{a}_1^2\bar{a}_3 + \frac{4}{\bar{\gamma}+1}\bar{a}_o\bar{a}_1\frac{d\bar{a}_3}{dz} + \frac{4}{\bar{\gamma}+1}\bar{a}_o\bar{a}_3\frac{d\bar{a}_1}{dz} \tag{5-75}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{b}_2}{dz} + 4\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{b}_3 - \frac{8}{\bar{\gamma}+1}\bar{a}_o\bar{a}_1\bar{b}_2$$

$$= 2\bar{a}_o\bar{b}_2\left[\frac{d\bar{a}_o}{dz} + 2\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right] + \bar{a}_1^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] \tag{5-76}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{b}_4}{dz} + 2\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{b}_5 = 2\bar{a}_o\bar{b}_4\left[\frac{d\bar{a}_o}{dz} + 2\frac{\bar{\gamma}-1}{\bar{\gamma}+1} a_1\right] \tag{5-77}$$

$$2\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{b}_9 = \frac{2}{\bar{\gamma}+1}\bar{a}_3^3 \tag{5-78}$$

$$\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{b}_{11} = \bar{a}_3^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] + \frac{4}{\bar{\gamma}+1}\bar{a}_o\bar{a}_3\frac{d\bar{a}_3}{dz} - \frac{4}{\bar{\gamma}+1}\bar{a}_1\bar{a}_3^2 \qquad (5\text{-}79)$$

From the remaining boundary conditions [equations (5-35), (5-39), (5-42), (5-45) and (5-49)], it can be shown that

$$\bar{b}_1 r_w + \bar{b}_3 r_w^3 + \bar{b}_5 r_w \ell n\, r_w + \bar{b}_7 r_w^{-1} + \bar{b}_9 r_w^{-3} + \bar{b}_{11} r_w^{-1} \ell n\, r_w$$

$$= (\bar{b}_o + \bar{b}_2 r_w^2 + \bar{b}_4 \ell n\, r_w)\frac{dr_w}{dz} \qquad (5\text{-}80)$$

$$b_1 r_{so} + b_3 r_{so}^3 + a_1 r_{s1} = (b_o + b_2 r_{so}^2)\frac{dr_{so}}{dz} + a_o\frac{dr_{s1}}{dz} \qquad (5\text{-}81)$$

$$\bar{b}_1 r_{so} + \bar{b}_3 r_{so}^3 + \bar{b}_5 r_{so} \ell n\, r_{so} + \bar{b}_7 r_{so}^{-1} + \bar{b}_9 r_{so}^{-3} + \bar{b}_{11}(z) r_{so}^{-1} \ell n\, r_{so}$$

$$+ (\bar{a}_1 - \bar{a}_3 r_{so}^{-2}) r_{s1} = (\bar{b}_o + \bar{b}_2 r_{so}^2 + \bar{b}_4 \ell n\, r_{so})\frac{dr_{so}}{dz} + \bar{a}_o\frac{dr_{s1}}{dz} \qquad (5\text{-}82)$$

$$\frac{\gamma}{\gamma+1}\left[1 - \frac{\gamma-1}{\gamma+1}a_o^2\right]^{-1}\left[2a_o(b_o + b_2 r_{so}^2) + a_1^2 r_{so}^2\right].$$

$$= \frac{\bar{\gamma}}{\bar{\gamma}+1}\left[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right]^{-1}\left[2\bar{a}_o(\bar{b}_o + \bar{b}_2 r_{so}^2 + \bar{b}_4 \ell n\, r_{so})\right.$$

$$\left.+ \bar{a}_1^2 r_{so}^2 + 2\bar{a}_1\bar{a}_3 + \bar{a}_3^2 r_{so}^{-2}\right] \qquad (5\text{-}83)$$

$$r_{s1}(0) = 0 \qquad (5\text{-}84)$$

The solution of the above equations defines the first order velocity components (u_1, u_1, $\bar{u}_1$ and $\bar{v}_1$) and dividing streamline location (r_{s1}) through the nozzle.

Examination of equations (5-73) through (5-77) reveals that they are singular at the nozzle throat (where $a_o = \bar{a}_o = 1$). Thus, the above equations are algebraic at the throat and can be solved directly for the throat conditions as in the uniform expansion case. Thus,

$$b_o(0) = B_o \tag{5-85}$$

$$b_1(0) = \frac{1}{2}(\gamma+1)B_oB_1 \tag{5-86}$$

$$b_2(0) = \frac{1}{4}(\gamma+1){B_1}^2 \tag{5-87}$$

$$b_3(0) = \frac{1}{16}(\gamma+1)^2\ {B_1}^3 \tag{5-88}$$

$$\bar{b}_o(0) = \bar{B}_o \tag{5-89}$$

$$\bar{b}_1(0) = \frac{1}{2}(\bar{\gamma}+1)\bar{B}_1\ (\bar{B}_o - \frac{1}{2}\bar{C}_1) \tag{5-90}$$

$$\bar{b}_2(0) = \frac{1}{4}(\bar{\gamma}+1)\bar{B}_1^2 \tag{5-91}$$

$$\bar{b}_3(0) = \frac{1}{16}(\bar{\gamma}+1)^2\ \bar{B}_1^3 \tag{5-92}$$

$$\bar{b}_4(0) = \bar{C}_1 \tag{5-93}$$

$$\bar{b}_5(0) = \frac{1}{2}(\bar{\gamma}+1)\bar{B}_1\ \bar{C}_1 \tag{5-94}$$

$$\bar{b}_7(0) = \bar{C}_2 \tag{5-95}$$

$$\bar{b}_9(0) = 0 \tag{5-96}$$

$$\bar{b}_{11}(0) = 0 \tag{5-97}$$

where

$$B_1 = [\frac{1}{2}(\gamma+1){r_{so}}^2 + \frac{1}{2}\left(\frac{\gamma}{\bar{\gamma}}\right)^2(\bar{\gamma}+1)(1 - {r_{so}}^2)]^{-1/2} \tag{5-98}$$

$$\bar{B}_1 = \left(\frac{\gamma}{\bar{\gamma}}\right)B_1 \tag{5-99}$$

$$\bar{C}_1 = 1 - \frac{1}{2}(\bar{\gamma}+1)\bar{B}_1^2 \tag{5-100}$$

$$B_o = -B_1\,\overline{B}_1(\overline{\gamma}+1)\,[\tfrac{1}{4}\,\overline{C}_1(r_{so}^{\;2} - 1 - 2\ln r_{so}) + \tfrac{1}{16}(\overline{\gamma}+1)\overline{B}_1^2(1 - r_{so}^{\;2})^2]$$

$$- (\gamma+1)B_1^{\;2}\, r_{so}^{\;2}\,[\tfrac{1}{8}(\overline{\gamma}+1)\overline{B}_1^2(1 - r_{so}^{\;2}) + \tfrac{1}{16}(\gamma+1)B_1^{\;2}\, r_{so}^{\;2}] \tag{5-101}$$

$$\overline{B}_o = -\tfrac{1}{4}(\overline{\gamma}+1)\overline{B}_1^2\, r_{so}^{\;2} - \overline{C}_1\,\ln r_{so} + \left(\frac{\gamma}{\overline{\gamma}}\right)[\tfrac{1}{4}(\gamma+1)B_1^2\, r_{so}^{\;2} + B_o] \tag{5-102}$$

$$\overline{C}_2 = -(\overline{\gamma}+1)\overline{B}_1\,[\tfrac{1}{16}(\overline{\gamma}+1)\overline{B}_1^2 - \tfrac{1}{4}\,\overline{C}_1 + \tfrac{1}{2}\,\overline{B}_o] \tag{5-103}$$

The above first order two-zone throat conditions are identical to those which would be obtained by a Sauer[2] or Hall[1] type transonic analysis for this case[7]. As discussed previously, the present solution and such a transonic solution will differ away from the throat plane, however.

Examination of equations (5-23) and (5-29) reveals that they are also singular at the nozzle throat (where $u_o = \overline{u}_o = 1$). Thus, the boundary conditions for all orders are set at the nozzle throat and the various order throat conditions can be determined directly.

Examination of equations (5-56), (5-57), (5-69), (5-70), (5-76) and (5-77) shows that $\overline{b}_2$ and $\overline{b}_4$ depend on $\dfrac{d^2 r_w}{dz^2}$ and $\overline{b}_3$ and $\overline{b}_5$ depend on $\dfrac{d^3 r_w}{dz^3}$. Thus, if $\dfrac{d^3 r_w}{dz^3}$ is discontinuous, the first order two-zone solution will also be discontinuous. Thus, in general, if the wall derivative $\dfrac{d^{2n+1} r_w}{dz^{2n+1}}$ is nonanalytic, the nth order solutions for the above equations will be discontinuous. The complete solution of the above equations will be analytic only if the wall is analytic.

The second order equations are

$$\left(1 - u_o^2\right)\frac{\partial u_2}{\partial z} + \left(1 - \frac{\gamma-1}{\gamma+1} u_o^2\right)\left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r}\right) - \frac{4}{\gamma+1} u_o v_o \frac{\partial u_2}{\partial r}$$

$$= \left(2u_o u_2 + u_1^2 + 2\frac{\gamma-1}{\gamma+1} v_o v_1\right)\frac{\partial u_o}{\partial z} + \left[2v_o v_1 + \frac{\gamma-1}{\gamma+1}(2u_o u_2 + u_1^2)\right]\frac{\partial v_o}{\partial r}$$

$$+ \frac{\gamma-1}{\gamma+1}\left(2u_o u_2 + u_1^2 + 2v_o v_1\right)\frac{v_o}{r} + \left(2u_o u_1 + \frac{\gamma-1}{\gamma+1} v_o^2\right)\frac{\partial u_1}{\partial z}$$

$$+ \left(v_o^2 + 2\frac{\gamma-1}{\gamma+1} u_o u_1\right)\frac{\partial v_1}{\partial r} + \frac{\gamma-1}{\gamma+1}\left(2u_o u_1 + v_o^2\right)\frac{v_1}{r} + \frac{4}{\gamma+1}\left(u_o v_1 + u_1 v_o\right)\frac{\partial u_1}{\partial r} \qquad (5\text{-}104)$$

$$\frac{\partial v_1}{\partial z} - \frac{\partial u_2}{\partial r} = 0 \qquad (5\text{-}105)$$

$$\left(1 - \bar{u}_o^2\right)\frac{\partial \bar{u}_2}{\partial z} + \left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{u}_o^2\right)\left(\frac{\partial \bar{v}_2}{\partial r} + \frac{\bar{v}_2}{r}\right) - \frac{4}{\bar{\gamma}+1} \bar{u}_o \bar{v}_o \frac{\partial \bar{u}_2}{\partial r}$$

$$= \left(2\bar{u}_o \bar{u}_2 + \bar{u}_1^2 + 2\frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{v}_o \bar{v}_1\right)\frac{\partial \bar{u}_o}{\partial z} + \left[2\bar{v}_o \bar{v}_1 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}(2\bar{u}_o \bar{u}_2 + \bar{u}_1^2)\right]\frac{\partial \bar{v}_o}{\partial r}$$

$$+ \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\left(2\bar{u}_o \bar{u}_2 + \bar{u}_1^2 + 2\bar{v}_o \bar{v}_1\right)\frac{\bar{v}_o}{r} + \left(2\bar{u}_o \bar{u}_1 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{v}_o^2\right)\frac{\partial \bar{u}_1}{\partial z}$$

$$+ \left(\bar{v}_o^2 + 2\frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{u}_o \bar{u}_1\right)\frac{\partial \bar{v}_1}{\partial r} + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\left(2\bar{u}_o \bar{u}_1 + \bar{v}_o^2\right)\frac{\bar{v}_1}{r} + \frac{4}{\bar{\gamma}+1}\left(\bar{u}_o \bar{v}_1 + \bar{u}_1 \bar{v}_o\right)\frac{\partial \bar{u}_1}{\partial r} \qquad (5\text{-}106)$$

$$\frac{\partial \bar{v}_1}{\partial z} - \frac{\partial \bar{u}_2}{\partial r} = 0 \qquad (5\text{-}107)$$

From equations (5-71), (5-72), (5-105), and (5-107), it is easily shown that

$$u_2 = c_o(z) + c_2(z)r^2 + c_4(z)r^4 \qquad (5\text{-}108)$$

$$\bar{u}_2 = \bar{c}_o(z) + \bar{c}_2(z)r^2 + \bar{c}_4(z)r^4 + \bar{c}_6(z)\ln r$$

$$+ \bar{c}_8(z)r^2\ln r + \bar{c}_{10}(z)(\ln r)^2 + \bar{c}_{12}(z)r^{-2} \qquad (5\text{-}109)$$

where

$$c_2 = \frac{1}{2}\frac{db_1}{dz} \tag{5-110}$$

$$c_4 = \frac{1}{4}\frac{db_3}{dz} \tag{5-111}$$

$$\bar{c}_2 = \frac{1}{2}\frac{d\bar{b}_1}{dz} - \frac{1}{4}\frac{d\bar{b}_5}{dz} \tag{5-112}$$

$$\bar{c}_4 = \frac{1}{4}\frac{d\bar{b}_3}{dz} \tag{5-113}$$

$$\bar{c}_6 = \frac{d\bar{b}_7}{dz} \tag{5-114}$$

$$\bar{c}_8 = \frac{1}{2}\frac{d\bar{b}_5}{dz} \tag{5-115}$$

$$\bar{c}_{10} = \frac{1}{2}\frac{d\bar{b}_{11}}{dz} \tag{5-116}$$

$$\bar{c}_{12} = -\frac{1}{2}\frac{d\bar{b}_9}{dz} \tag{5-117}$$

From equations (5-104), (5-106), (5-108), and (5-109), it can be shown that

$$v_2 = c_1(z)r + c_3(z)r^3 + c_5(z)r^5 \tag{5-118}$$

$$\begin{aligned}\bar{v}_2 = {} & \bar{c}_1(z)r + \bar{c}_3(z)r^3 + \bar{c}_5(z)r^5 + \bar{c}_7(z)r\ell n\ r + \bar{c}_9(z)r(\ell n\ r)^2 \\ & + \bar{c}_{11}(z)r^3\ell n\ r + \bar{c}_{13}(z)r^{-1} + \bar{c}_{15}(z)r^{-3} + \bar{c}_{17}(z)r^{-5} \\ & + \bar{c}_{19}(z)r^{-1}\ \ell n\ r + \bar{c}_{21}(z)r^{-1}(\ell n\ r)^2 + \bar{c}_{23}(z)r^{-3}\ell n\ r\end{aligned} \tag{5-119}$$

where

$$\begin{aligned}\left(1 - a_o^2\right)\frac{dc_o}{dz} + 2\left(1 - \frac{\gamma-1}{\gamma+1}a_o^2\right)c_1 = {} & (2a_oc_o + b_o^2)\left[\frac{da_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,a_1\right] \\ & + 2a_ob_o\left[\frac{db_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,b_1\right]\end{aligned} \tag{5-120}$$

$$\left(1 - a_o^2\right)\frac{dc_2}{dz} + 4\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)c_3 = 2(a_o c_2 + b_o b_2)\left[\frac{da_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,a_1\right]$$

$$+ 2a_1 b_1\left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \frac{2\gamma}{\gamma+1}\,a_1\right] + \frac{4}{\gamma+1}\left(a_o b_1 + a_1 b_o\right)\frac{da_1}{dz}$$

$$+ 2a_o b_o\left[\frac{db_2}{dz} + 4\,\frac{\gamma-1}{\gamma+1}\,b_3\right] + 2a_o b_2\left[\frac{db_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,b_1\right]$$

$$+ a_1^2\left[\frac{\gamma-1}{\gamma+1}\frac{db_o}{dz} + \frac{2\gamma}{\gamma+1}\,b_1\right] + \frac{4}{\gamma+1}\,a_o a_1\,\frac{db_1}{dz} \qquad \text{(5-121)}$$

$$\left(1 - a_o^2\right)\frac{dc_4}{dz} + 4\left(1 - \frac{\gamma-1}{\gamma+1} a_o^2\right)c_5 = (2a_o c_4 + b_2^2)\left[\frac{da_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,a_1\right]$$

$$+ 2a_1 b_3\left[\frac{\gamma-1}{\gamma+1}\frac{da_o}{dz} + \frac{2\gamma}{\gamma+1}\,a_1\right] + \frac{4}{\gamma+1}\left(a_o b_3 + a_1 b_2\right)\frac{da_1}{dz}$$

$$+ 2a_o b_2\left[\frac{db_2}{dz} + 4\,\frac{\gamma-1}{\gamma+1}\,b_3\right] + a_1^2\left[\frac{\gamma-1}{\gamma+1}\frac{db_2}{dz} + \left(3 + \frac{\gamma-1}{\gamma+1}\right)b_3\right] + \frac{4}{\gamma+1}\,a_o a_1\,\frac{db_3}{dz} \qquad \text{(5-122)}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_o}{dz} + \left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o^2\right)(2\bar{c}_1 + \bar{c}_7) = (2\bar{a}_o\bar{c}_o + \bar{b}_o^2)\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_1\right]$$

$$+ 2(\bar{a}_1\bar{b}_7 + \bar{a}_3\bar{b}_1)\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\,\bar{a}_1\right] + \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_7 + \bar{a}_3\bar{b}_o\right)\frac{d\bar{a}_1}{dz}$$

$$+ \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_1 + \bar{a}_1\bar{b}_o + \bar{a}_3\bar{b}_2\right)\frac{d\bar{a}_3}{dz} - \frac{4}{\bar{\gamma}+1}\left(\bar{a}_1\bar{b}_1 + \bar{a}_3\bar{b}_3\right)\bar{a}_3$$

$$+ 2\bar{a}_o\bar{b}_o\left[\frac{d\bar{b}_o}{dz} + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}(2\bar{b}_1 + \bar{b}_5)\right] + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_2\bar{b}_{11} + 2\bar{a}_1\bar{a}_3\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{b}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\,\bar{b}_1 + \bar{b}_5\right]$$

$$+ \bar{a}_1^2\left(-\frac{2}{\bar{\gamma}+1}\,\bar{b}_7 + \bar{b}_{11}\right) + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_7}{dz} + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_1}{dz}$$

$$+ \bar{a}_3^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{b}_2}{dz} + \left(3 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right)\bar{b}_3\right] \qquad \text{(5-123)}$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_2}{dz} + \left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o^2\right)(4\bar{c}_3 + \bar{c}_{11}) = (2\bar{a}_o\bar{c}_o + \bar{b}_o^2)\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_1\right]$$

$$+ 2(\bar{a}_1\bar{b}_1 + \bar{a}_3\bar{b}_3)\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\,\bar{a}_1\right] - \frac{4}{\bar{\gamma}+1}\,\bar{a}_1\bar{a}_3\bar{b}_3$$

$$+ 2\bar{a}_o\bar{b}_o\left[\frac{d\bar{b}_2}{dz} + 4\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{b}_3\right] + 2\bar{a}_o\bar{b}_2\left[\frac{d\bar{b}_o}{dz} + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}(2\bar{b}_1 + \bar{b}_5)\right]$$

$$+ 2\bar{a}_1\bar{a}_3\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\frac{d\bar{b}_2}{dz} + \left(3 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right)\bar{b}_3\right] + \bar{a}_1^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\frac{d\bar{b}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\,\bar{b}_1 + \bar{b}_5\right]$$

$$+ \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_1}{dz} + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_3}{dz} + \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_1 + \bar{a}_1\bar{b}_o + \bar{a}_3\bar{b}_2\right)\frac{d\bar{a}_1}{dz}$$

$$+ \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_3 + \bar{a}_1\bar{b}_2\right)\frac{d\bar{a}_3}{dz} \qquad (5\text{-}124)$$

$$\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_4}{dz} + 6\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o^2\right)\bar{c}_5 = (2\bar{a}_o\bar{c}_4 + \bar{b}_2^2)\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_1\right]$$

$$+ 2\bar{a}_1\bar{b}_3\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\,\bar{a}_1\right] + 2\bar{a}_o\bar{b}_2\left[\frac{d\bar{b}_2}{dz} + 4\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{b}_3\right]$$

$$+ \bar{a}_1^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\frac{d\bar{b}_2}{dz} + \left(3 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right)\bar{b}_3\right] + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_3}{dz}$$

$$+ \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_1 + \bar{a}_1\bar{b}_o + \bar{a}_3\bar{b}_2\right)\frac{d\bar{a}_1}{dz} \qquad (5\text{-}125)$$

$$
\begin{aligned}
\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_6}{dz} + 2\left(1 - \frac{\gamma-1}{\gamma+1}\,\bar{a}_o^2\right)(\bar{c}_7 + \bar{c}_9) \;=\;& 2(\bar{a}_o\bar{c}_o + \bar{b}_o\bar{b}_4)\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,\bar{a}_1\right] \\
&+ 2(\bar{a}_1\bar{b}_{11} + \bar{a}_3\bar{b}_5)\left[\frac{\gamma-1}{\gamma+1}\,\frac{d\bar{a}_o}{dz} + \frac{2\gamma}{\gamma+1}\,\bar{a}_1\right] + 2\bar{a}_o\bar{b}_o\left[\frac{d\bar{b}_4}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,\bar{b}_5\right] \\
&+ 2\bar{a}_o\bar{b}_4\left[\frac{d\bar{b}_o}{dz} + \frac{\gamma-1}{\gamma+1}(2\bar{b}_1 + \bar{b}_5)\right] + 2\bar{a}_1\bar{a}_3\left[\frac{2\gamma}{\gamma+1}\,\bar{b}_5 + \frac{\gamma-1}{\gamma+1}\,\frac{d\bar{b}_4}{dz}\right] \\
&+ \frac{4}{\gamma+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_{11}}{dz} + \frac{4}{\gamma+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_5}{dz} + \frac{4}{\gamma+1}\left(\bar{a}_o\bar{b}_5 + \bar{a}_1\bar{b}_4\right)\frac{d\bar{a}_3}{dz} \\
&+ \frac{4}{\gamma+1}\left(\bar{a}_o\bar{b}_{11} + \bar{a}_3\bar{b}_4\right)\frac{d\bar{a}_1}{dz} - \frac{4}{\gamma+1}\,\bar{a}_1\bar{a}_3\bar{b}_5 - \frac{2}{\gamma+1}\,\bar{a}_1^2\,\bar{b}_{11}
\end{aligned}
\tag{5-126}
$$

$$
\begin{aligned}
\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_8}{dz} + 4\left(1 - \frac{\gamma-1}{\gamma+1}\,\bar{a}_o^2\right)\bar{c}_{11} \;=\;& 2(\bar{a}_o\bar{c}_8 + \bar{b}_2\bar{b}_4)\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,\bar{a}_1\right] \\
&+ 2\bar{a}_1\bar{b}_5\left[\frac{\gamma-1}{\gamma+1}\,\frac{d\bar{a}_o}{dz} + \frac{2\gamma}{\gamma+1}\,\bar{a}_1\right] + 2\bar{a}_o\bar{b}_2\left[\frac{d\bar{b}_4}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,\bar{b}_5\right] \\
&+ 2\bar{a}_o\bar{b}_4\left[\frac{d\bar{b}_2}{dz} + 4\,\frac{\gamma-1}{\gamma+1}\,\bar{b}_3\right] + \bar{a}_1^2\left[\frac{2\gamma}{\gamma+1}\,\bar{b}_5 + \frac{\gamma-1}{\gamma+1}\,\frac{d\bar{b}_4}{dz}\right] \\
&+ \frac{4}{\gamma+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_5}{dz} + \frac{4}{\gamma+1}\left(\bar{a}_o\bar{b}_5 + \bar{a}_1\bar{b}_4\right)\frac{d\bar{a}_1}{dz}
\end{aligned}
\tag{5-127}
$$

$$
\begin{aligned}
\left(1 - \bar{a}_o^2\right)\frac{d\bar{c}_{10}}{dz} + 2\left(1 - \frac{\gamma-1}{\gamma+1}\,\bar{a}_o^2\right)\bar{c}_9 \;=\;& (2\bar{a}_o\bar{c}_{10} + \bar{b}_4^2)\left[\frac{d\bar{a}_o}{dz} + \frac{\gamma-1}{\gamma+1}\,\bar{a}_1\right] \\
&+ 2\bar{a}_o\bar{b}_4\left[\frac{d\bar{b}_4}{dz} + 2\,\frac{\gamma-1}{\gamma+1}\,\bar{b}_5\right]
\end{aligned}
\tag{5-128}
$$

$$\begin{aligned}\left(1-\bar{a}_o^2\right)\frac{d\bar{c}_{12}}{dz} + \left(1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{c}_{19} &= 2\bar{a}_o\bar{c}_{12}\left[\frac{d\bar{a}_o}{dz} + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_1\right] \\ &- \frac{4}{\bar{\gamma}+1}\left(\bar{a}_1\bar{b}_7 + \bar{a}_3\bar{b}_7\right)\bar{a}_3 + 2\left(\bar{a}_1\bar{b}_9 + \bar{a}_3\bar{b}_7\right)\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] \\ &+ 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_o\bar{b}_{11} - 4\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_2\bar{b}_9 + 2\bar{a}_1\bar{a}_3\left(-\frac{2}{\bar{\gamma}+1}\bar{b}_7 + \bar{b}_{11}\right) \\ &+ \bar{a}_1^2\left(-3 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right)\bar{b}_9 + \bar{a}_3^2\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{b}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{b}_1 + \bar{b}_5\right] + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_1\,\frac{d\bar{b}_9}{dz} \\ &+ \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_7}{dz} + \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_7 + \bar{a}_3\bar{b}_o\right)\frac{d\bar{a}_3}{dz} + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_9\,\frac{d\bar{a}_1}{dz}\end{aligned} \tag{5-129}$$

$$\begin{aligned}\left(1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\left(-2\bar{c}_{15} + \bar{c}_{23}\right) &= -\frac{4}{\bar{\gamma}+1}\left(\bar{a}_1\bar{b}_9 + \bar{a}_3\bar{b}_7\right)\bar{a}_3 + 2\bar{a}_3\bar{b}_9\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] \\ &- 4\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_o\bar{b}_9 + 2\left(-3 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right)\bar{a}_1\bar{a}_3\bar{b}_9 + \bar{a}_3^2\left(-\frac{2}{\bar{\gamma}+1}\bar{b}_7 + \bar{b}_{11}\right) \\ &+ \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_9}{dz} + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_9\,\frac{d\bar{a}_3}{dz}\end{aligned} \tag{5-130}$$

$$4\left(1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{c}_{17} = \left(\frac{5}{\bar{\gamma}+1} - 3\right)\bar{a}_3^2\,\bar{b}_9 \tag{5-131}$$

$$\begin{aligned}2\left(1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{c}_{21} &= -\frac{4}{\bar{\gamma}+1}\left(\bar{a}_1\bar{b}_{11} + \bar{a}_3\bar{b}_5\right)\bar{a}_3 + 2\bar{a}_3\bar{b}_{11}\left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{a}_o}{dz} + \frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{a}_1\right] \\ &+ 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_4\bar{b}_{11} + \bar{a}_3^2\left[\frac{2\bar{\gamma}}{\bar{\gamma}+1}\bar{b}_5 + \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\frac{d\bar{b}_4}{dz}\right] + \frac{4}{\bar{\gamma}+1}\,\bar{a}_o\bar{a}_3\,\frac{d\bar{b}_{11}}{dz} \\ &+ \frac{4}{\bar{\gamma}+1}\left(\bar{a}_o\bar{b}_{11} + \bar{a}_3\bar{b}_4\right)\frac{d\bar{a}_3}{dz}\end{aligned} \tag{5-132}$$

$$\left(1-\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{a}_o^2\right)\bar{c}_{23} = -\frac{1}{\bar{\gamma}+1}\,\bar{a}_3^2\,\bar{b}_{11} + 2\,\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o\bar{b}_4\bar{b}_9 \tag{5-133}$$

From the remaining boundary conditions [equations (5-34), (5-40), (5-43), (5-46) and (5-50)], it is found that

$$\bar{c}_1 r_w + \bar{c}_3 r_w^{\,2} + \bar{c}_5 r_w^{\,5} + \bar{c}_7 r_w \ln r_w + \bar{c}_9 r_w (\ln r_w)^2 + \bar{c}_{11} r_w^{\,3} \ln r_w$$

$$+ \bar{c}_{13} r_w^{-1} + \bar{c}_{15} r_w^{-3} + \bar{c}_{17} r_w^{-5} + \bar{c}_{19} r_w^{-1} \ln r_w + \bar{c}_{21} r_w^{-1} (\ln r_w)^2$$

$$+ \bar{c}_{23} r_w^{-2} \ln r_w + [\bar{b}_1 + 3\bar{b}_3 r_w^{\,2} + \bar{b}_5 (1 + \ln r_w) - \bar{b}_7 r_w^{-2} - 3\bar{b}_9 r_w^{-4}$$

$$+ \bar{b}_{11} (1 - \ln r_w) r_w^{-2}] r_w = [\bar{c}_o + \bar{c}_2 r_w^{\,2} + \bar{c}_4 r_w^{\,4} + \bar{c}_6 \ln r_w + \bar{c}_8 r_w^{\,2} \ln r_w$$

$$+ \bar{c}_{10} (\ln r_w)^2 + \bar{c}_{12} r_w^{-2}] \frac{dr_w}{dz} \qquad (5\text{-}134)$$

$$\bar{c}_1 r_{so} + \bar{c}_3 r_{so}^{\,3} + \bar{c}_5 r_{so}^{\,5} + \bar{c}_7 r_{so} \ln r_{so} + \bar{c}_9 r_{so} (\ln r_{so})^2 + \bar{c}_{11} r_{so}^{\,3} \ln r_{so}$$

$$+ \bar{c}_{13} r_{so}^{-1} + \bar{c}_{15} r_{so}^{-3} + \bar{c}_{17} r_{so}^{-5} + \bar{c}_{19} r_{so}^{-1} \ln r_{so} + \bar{c}_{21} r_{so}^{-1} (\ln r_{so})^2$$

$$+ \bar{c}_{23} r_{so}^{-3} \ln r_{so} + [\bar{b}_1 + 3\bar{b}_3 r_{so}^{\,2} + \bar{b}_5 (1 + \ln r_{so}) - \bar{b}_7 r_{so}^{-2} - 3\bar{b}_9 r_{so}^{-4}$$

$$+ \bar{b}_{11} (1 - \ln r_{so}) r_{so}^{-2}] r_{s1} + [\bar{a}_1 - \bar{a}_3 r_{so}^{-2}] r_{s2} + a_3 r_{so}^{-3} r_{s1}^{\,2}$$

$$= \frac{dr_{s2}}{dz} + [\bar{b}_o + \bar{b}_2 r_{so}^{\,2} + \bar{b}_4 \ln r_{so}] \frac{dr_{s1}}{dz} + [\bar{c}_o + \bar{c}_2 r_{so}^{\,2} + \bar{c}_4 r_{so}^{\,4}$$

$$+ \bar{c}_6 \ln r_{so} + \bar{c}_8 r_{so}^{\,2} \ln r_{so} + \bar{c}_{10} (\ln r_{so})^2 + \bar{c}_{12} r_{so}^{-2}$$

$$+ (2\bar{b}_2 r_{so} + \bar{b}_4 r_{so}^{-1}) r_{s1}] \frac{dr_{so}}{dz} \qquad (5\text{-}135)$$

$$c_1 r_{so} + c_3 r_{so}^{\,3} + c_5 r_{so}^{\,5} + [b_1 + 3b_3 r_{so}^{\,2}] r_{s1} + a_1 r_{s2} = a_o \frac{dr_{s2}}{dz}$$

$$+ [b_o + b_2 r_{so}^{\,2}] \frac{dr_{s1}}{dz} + [c_o + c_2 r_{so}^{\,2} + c_4 r_{so}^{\,4} + 2b_2 r_{s1}] \frac{dr_{so}}{dz} \qquad (5\text{-}136)$$

$$
\frac{\gamma}{(\gamma+1)^2}\,[1 - \frac{\gamma-1}{\gamma+1}\,a_1^{\,2}]^{-2}\;[2a_o b_o + 2a_o b_2 r_{so}^{\;2} + a_1^{\,2} r_{so}^{\;2}]^2 - \frac{\gamma}{\gamma+1}\,[1 - \frac{\gamma-1}{\gamma+1}\,a_o^{\,2}]^{-1}
$$

$$
[2a_o c_o + 2a_o c_2 r_{so}^{\;2} + 2a_o c_4 r_{so}^{\;4} + 4a_o b_2 r_{so} r_{s1} + b_o^{\,2} + 2b_o b_2 r_{so}^{\;2}
$$

$$
+ b_2^{\,2} r_{so}^{\;4} + 2a_1 b_1 r_{so}^{\;2} + 2a_1 b_3 r_{so}^{\;4} + 2a_1^{\,2} r_{so} r_{s1}]
$$

$$
= \frac{\bar{\gamma}}{(\bar{\gamma}+1)^2}\,[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o^2]^{-2}\;[2\bar{a}_o\bar{b}_o + 2\bar{a}_o\bar{b}_2 r_{so}^{\;2} + 2\bar{a}_o\bar{b}_4\,\ell n\, r_{so}
$$

$$
+ \bar{a}_1^2\, r_{so}^{\;2} + 2\bar{a}_1\bar{a}_3 + a_3^{\,2} r_{so}^{\;-2}]^2 - \frac{\bar{\gamma}}{\bar{\gamma}+1}\,[1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\,\bar{a}_o^2]^{-1}\;[2\bar{a}_o\bar{c}_o + 2\bar{a}_o\bar{c}_2 r_{so}^{\;2}
$$

$$
+ 2\bar{a}_o\bar{c}_4 r_{so}^{\;4} + 2\bar{a}_o\bar{c}_6\,\ell n\, r_{so} + 2\bar{a}_o\bar{c}_8 r_{so}^{\;2}\,\ell n\, r_{so} + 2\bar{a}_o\bar{c}_{10}(\ell n\, r_{so})^2
$$

$$
+ 2\bar{a}_o\bar{c}_{12} r_{so}^{\;-2} + 4\bar{a}_o\bar{b}_2 r_{so} r_{s1} + 2\bar{a}_o\bar{b}_4 r_{so}^{\;-1} r_{s1} + \bar{b}_o^2 + 2\bar{b}_o\bar{b}_2 r_{so}^{\;2}
$$

$$
+ 2\bar{b}_o\bar{b}_4\,\ell n\, r_{so} + \bar{b}_2^2\, r_{so}^{\;2} + 2\bar{b}_2\bar{b}_4 r_{so}^{\;2}\,\ell n\, r_{so} + \bar{b}_4^2(\ell n\, r_{so})^2
$$

$$
+ 2\bar{a}_1\bar{b}_1 r_{so}^{\;2} + 2\bar{a}_1\bar{b}_3 r_{so}^{\;4} + 2\bar{a}_1\bar{b}_5 r_{so}^{\;2}\,\ell n\, r_{so} + 2\bar{a}_1\bar{b}_7 + 2\bar{a}_1\bar{b}_9 r_{so}^{\;-2}
$$

$$
+ 2\bar{a}_1\bar{b}_{11}\,\ell n\, r_{so} + 2\bar{a}_3\bar{b}_1 + 2\bar{a}_3\bar{b}_3 r_{so}^{\;2} + 2\bar{a}_3\bar{b}_5\,\ell n\, r_{so} + 2\bar{a}_3\bar{b}_7 r_{so}^{\;-2}
$$

$$
+ 2\bar{a}_3\bar{b}_9 r_{so}^{\;-4} + 2\bar{a}_3\bar{b}_{11} r_{so}^{\;-2}\,\ell n\, r_{so} + 2\bar{a}_1^2\, r_{so} r_{s1} - 2\bar{a}_3^2\, r_{so}^{\;-3}\, r_{s1}] \qquad (5\text{-}137)
$$

$$
\begin{aligned}
r_{s2}(0) = \frac{r_{so}^{2}(1 - r_{so}^{2})}{2} \Big\{ & \frac{\gamma+1}{2}\left[b_o^{2} + b_o b_2 r_{so}^{2} + \frac{1}{3} b_2^{2} r_{so}^{4}\right] \\
& - \frac{\bar{\gamma}+1}{2}\Big[\bar{b}_o^{2} + \bar{b}_o \bar{b}_2 \left(\frac{1-r_{so}^{4}}{1-r_{so}^{2}}\right) + \frac{1}{3}\bar{b}_2^{2}\left(\frac{1-r_{so}^{6}}{1-r_{so}^{2}}\right) - 4\bar{b}_o \bar{b}_4 \left(\frac{1-r_{so} + r_{so}\ell n\, r_{so}}{1-r_{so}^{2}}\right) \\
& - \frac{4}{9}\bar{b}_2 \bar{b}_4 \left(\frac{1-r_{so}^{3} + 3r_{so}^{3}\,\ell n\, r_{so}}{1-r_{so}^{2}}\right) + 2\bar{b}_4^{2}\left(\frac{2-2r_{so} - r_{so}(\ell n\, r_{so})^{2} + 2r_{so}\ell n\, r_{so}}{1-r_{so}^{2}}\right)\Big]\Big\}
\end{aligned}
\tag{5-138}
$$

where the terms on the right-hand side of equation (5-138) are evaluated at the throat $(z = 0)$.

The solution of the above equations defines the second order velocity components (u_2, v_2, $\bar{u}_2$ and $\bar{v}_2$) and dividing streamline location (r_{s2}) through the nozzle.

As previously discussed, the above equations are singular at the nozzle throat and can in principle be solved directly to determine the second order throat conditions. This has been done numerically in the present study, since direct solution of the above equations for the second order throat conditions requires the solution of twenty-six linear algebraic equations.

It is noted that both the first and second order two-zone throat conditions are independent of the nozzle shape and are thus universally applicable to all two-zone nozzle flows. The solution away from the throat depends on the nozzle shape for all orders, however. The third and higher order two-zone throat conditions depend on the nozzle shape as in the uniform expansion case.

Figures 5-2 and 5-3 show the second order results of the present two-zone analysis for a typical 'barrier' cooled rocket engine having a hyperbolic nozzle with a normalized throat wall radius of curvature of 5. The inner and outer zone properties were chosen as representative of an ablative engine operating with Aerozine - $50/N_2O_4$ at an overall engine mixture ratio of 1.6, in which twenty percent of the propellant mass flow is discharged through a barrier zone at a mixture ratio of 0.8 and recovery temperature of 3900 °K in order to minimize dimensional erosion. Examination of the figures shows that while the constant pressure lines are continuous through the nozzle (as required by the dividing streamline boundary condition), the constant Mach lines are discontinuous across the dividing streamline, except the sonic (M = 1) line, which is continuous. Figures 5-4 and 5-5 show the second-order constant pressure and constant Mach number lines for the same engine operating without a barrier zone (uniform 1.6 mixture ratio throughout). Comparison of the two sets of figures graphically illustrates the difference between a two-zone and uniform mixture expansion through a typical rocket engine. Although the pressure distribution is similar in the two cases, the Mach number distributions are quite different except near the sonic surfaces, which are nearly identical. The performance losses associated with 'barrier' cooling will be discussed in a later report.

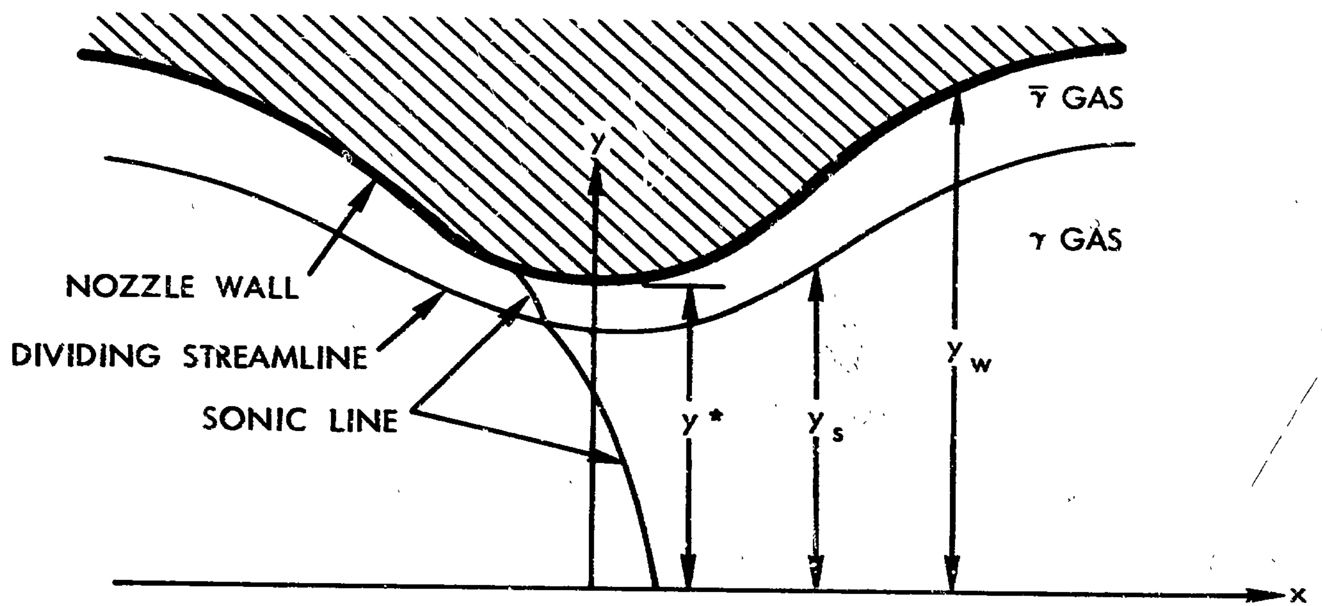

Figure 5-1. Two-Zone Expansion in Nozzle.

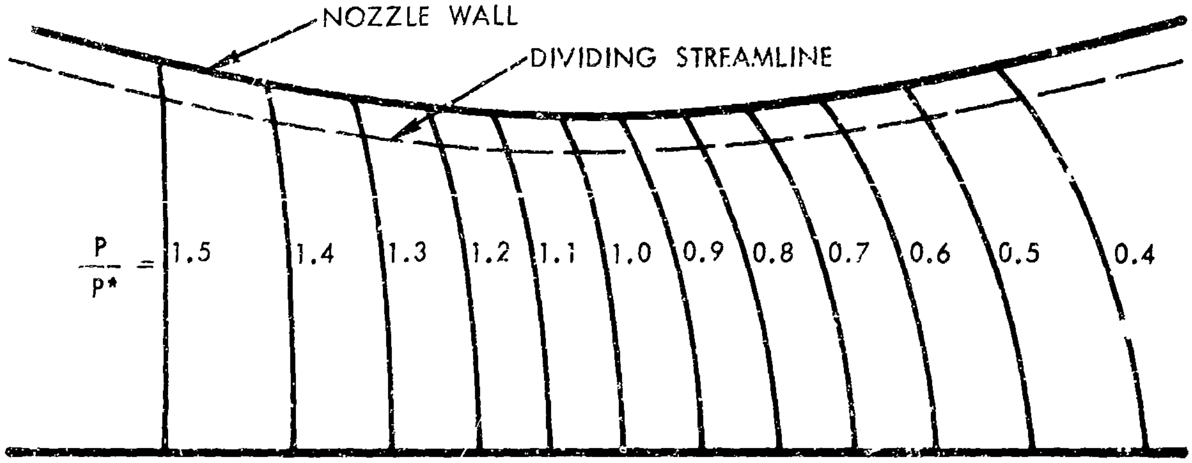

Figure 5-2. Contours of Constant Pressure in Two-Zone Nozzle Expansion.

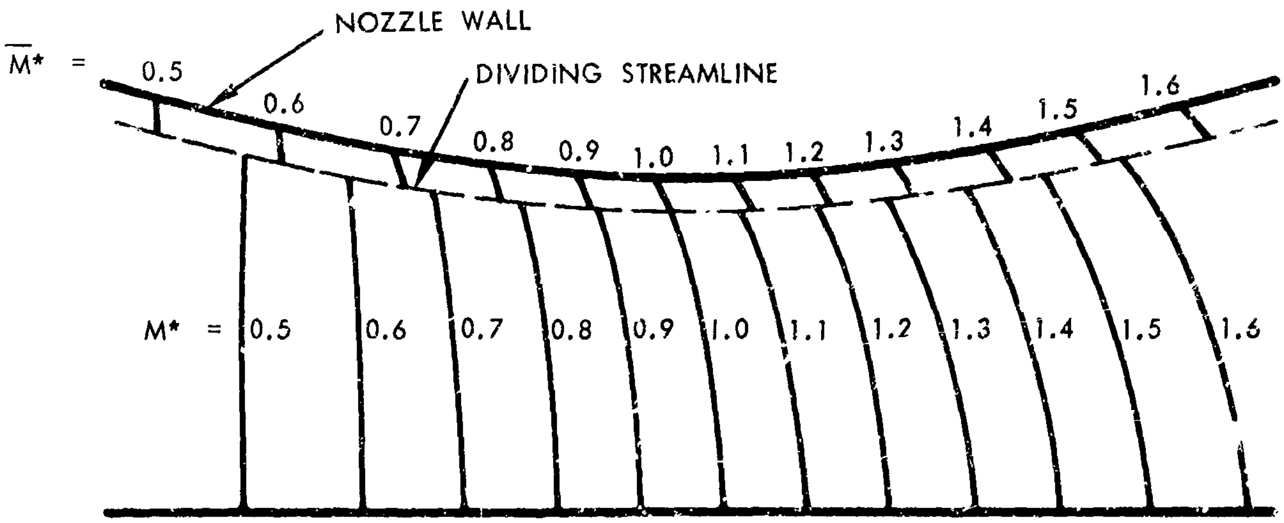

Figure 5-3. Contours of Constant Speed in Two-Zone Nozzle Expansion.

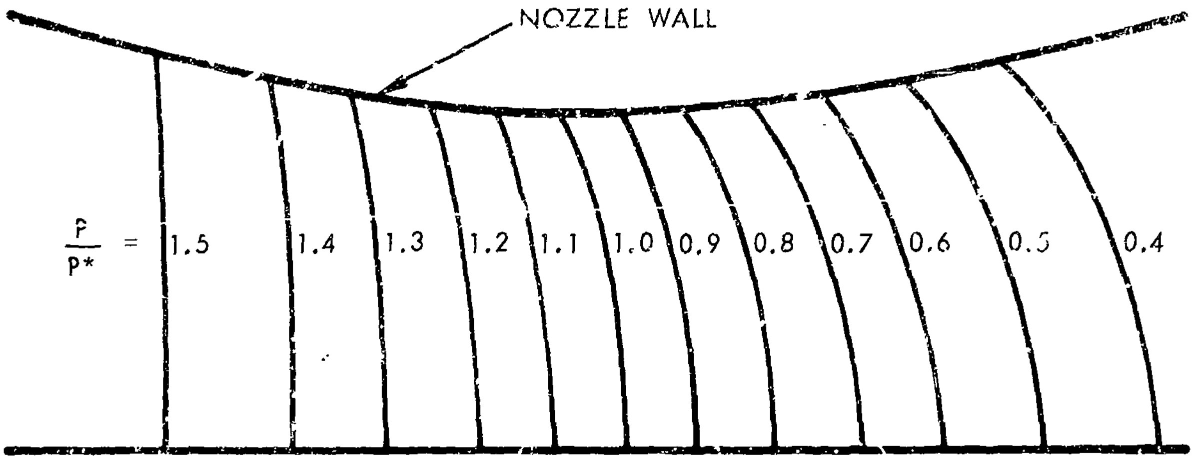

Figure 5-4. Contours of Constant Pressure in Uniform Nozzle Expansion.

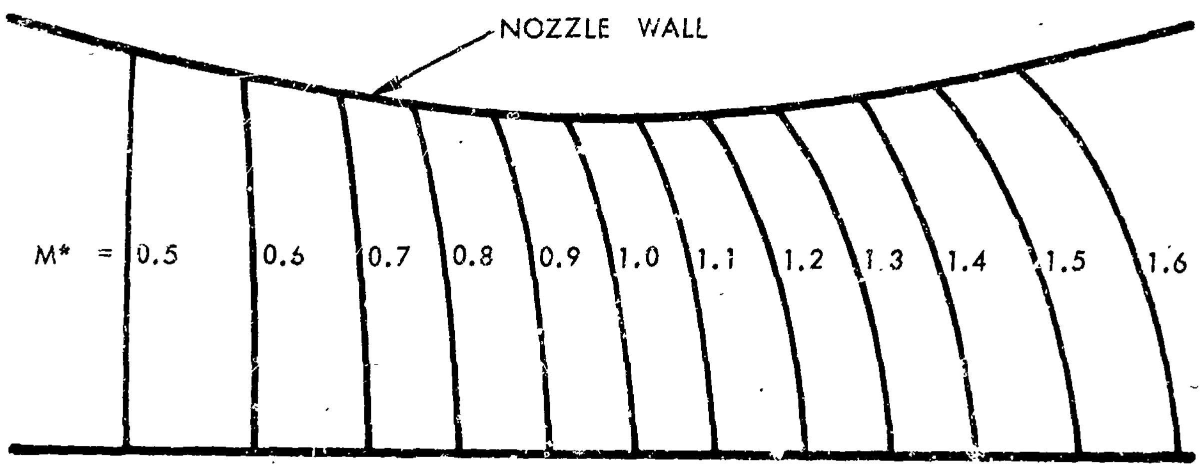

Figure 5-5. Contours of Constant Speed in Uniform Nozzle Expansion.

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APPENDIX A. ONE-DIMENSIONAL CHANNEL FLOW EQUATIONS

A.1. Uniform Expansions

The one-dimensional channel flow equation governing the inviscid isentropic expansion of a perfect gas through a nozzle is

$$\left(1 - u^2\right) \frac{du}{dx} + (\omega + 1) \left(1 - \frac{\gamma - 1}{\gamma + 1} u^2\right) \frac{u}{y_w} \frac{dy_w}{dx} = 0 \tag{A-1}$$

where the velocity has been normalized with respect to the throat sonic velocity and ω equals 0 or 1 depending on whether the nozzle is planar or axisymmetric. At the nozzle throat,

$$y_w = y^* \left[1 + \frac{x^2}{2Ry^{*2}} + \ldots \right] \tag{A-2}$$

$$\frac{dy_w}{dx} = \frac{x}{Ry^*} + \ldots \tag{A-3}$$

By substituting the above expressions into equation (A-1), expanding u as a power series in x and equating powers of x, it can be shown that

$$u = 1 + \sqrt{\frac{\omega + 1}{\gamma + 1} \frac{1}{R}} \frac{x}{y^*} + \ldots \tag{A-4}$$

for choked flows and

$$u = u^* - \frac{\omega + 1}{2} \left(1 - \frac{\gamma - 1}{\gamma + 1} u^{*2}\right) \frac{u^*}{1 - u^{*2}} \frac{x^2}{Ry^{*2}} + \ldots \tag{A-5}$$

for unchoked flows ($u^* < 1$). The above equations are equations (2-3) and (3-1) in the text.

A.2. Multistream Expansions

Consider a rocket engine in which two fixed quantities of propellant of different mixture ratio are injected into a finite contraction ratio (ε_c) chamber in such a manner that they burn and expand through the nozzle without mixing. The one-dimensional channel flow relationships governing the inviscid isentropic expansion of two perfect gas streams through a nozzle are

$$m = A\,P_o \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{1}{RT_o}\left(\frac{P}{P_o}\right)^{2/\gamma}\left[1-\left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{A-6}$$

and

$$\bar{m} = \bar{A}\,\bar{P}_o \sqrt{\frac{2\bar{\gamma}}{\bar{\gamma}-1}\,\frac{1}{\bar{R}\bar{T}_o}\left(\frac{P}{\bar{P}_o}\right)^{2/\bar{\gamma}}\left[1-\left(\frac{P}{\bar{P}_o}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]} \tag{A-7}$$

where the pressure in both streams is equal throughout the nozzle. Applying the above mass flow relationships at the chamber and throat, it is found that

$$m = A_c P_o \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{1}{RT_o}\left(\frac{P_c}{P_o}\right)^{2/\gamma}\left[1-\left(\frac{P_c}{P_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{A-8}$$

$$\bar{m} = \bar{A}_c \bar{P}_o \sqrt{\frac{2\bar{\gamma}}{\bar{\gamma}-1}\,\frac{1}{\bar{R}\bar{T}_o}\left(\frac{P_c}{\bar{P}_o}\right)^{2/\bar{\gamma}}\left[1-\left(\frac{P_c}{\bar{P}_o}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]} \tag{A-9}$$

$$m = A_t P_o \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{1}{RT_o}\left(\frac{P_t}{P_o}\right)^{2/\gamma}\left[1-\left(\frac{P_t}{P_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{A-10}$$

$$\bar{m} = \bar{A}_t \bar{P}_o \sqrt{\frac{2\bar{\gamma}}{\bar{\gamma}-1}\,\frac{1}{\bar{R}\bar{T}_o}\left(\frac{P_t}{\bar{P}_o}\right)^{2/\bar{\gamma}}\left[1-\left(\frac{P_t}{\bar{P}_o}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]} \tag{A-11}$$

The pressure ratio function appearing on the right-hand side of the above equations monotonically increases as the pressure ratio decreases, reaching a maximum at the sonic pressure ratio. Thus for fixed mass flows through the two streams, equations (A-10) and (A-11) show that $A_t P_o$ and $\bar{A}_t \bar{P}_o$ are minimum when $\frac{P_t}{P_o}$ and $\frac{P_t}{\bar{P}_o}$ are the sonic pressure ratios, $\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$ and $\left(\frac{2}{\bar{\gamma}+1}\right)^{\frac{\bar{\gamma}}{\bar{\gamma}-1}}$, respectively. Since for fixed mass flows through the two streams, A_t and $\bar{A}_t$ are a minimum when $\frac{P_t}{P_o}$ and $\frac{P_t}{\bar{P}_o}$ are the sonic pressure ratios, then P_o and $\bar{P}_o$ are also a minimum when $\frac{P_t}{P_o}$ and $\frac{P_t}{\bar{P}_o}$ are the sonic pressure ratio.

Since the total flow area equals the nozzle area,

$$A_c + \bar{A}_c = \varepsilon_c A^* \tag{A-12}$$

equations (A-8), (A-9) and (A-12) may be solved for the contraction ratio, yielding

$$\varepsilon_c = \frac{m}{A^* P_o \sqrt{\frac{2\gamma}{\gamma-1} \frac{1}{RT_o} \left(\frac{P_c}{P_o}\right)^{2/\gamma} \left[1 - \left(\frac{P_t}{P_o}\right)^{\frac{\gamma-1}{\gamma}}\right]}}$$

$$+ \frac{\bar{m}}{A^* \bar{P}_o \sqrt{\frac{2\bar{\gamma}}{\bar{\gamma}-1} \frac{1}{\bar{R}T_o} \left(\frac{P_c}{\bar{P}_o}\right)^{2/\bar{\gamma}} \left[1 - \left(\frac{P_c}{\bar{P}_o}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]}} \tag{A-13}$$

Examination of this equation reveals that the pressure ratio functions on the right-hand side of the equation will be maximum when P_o and $\bar{P}_o$ are minimum for fixed mass flows through the two streams and fixed engine geometry (contraction ratio). Since these functions monotonically increase as the pressure ratios $\frac{P_c}{P_o}$ and $\frac{P_c}{\bar{P}_o}$ decrease, it is concluded that the engine (static) chamber pressure, P_c, will be a minimum when P_o and $\bar{P}_o$ are a minimum. Since the engine will operate at minimum chamber pressure in the absence of external influences on the flow through the nozzle (such as secondary injection ahead of the throat, etc.), it is concluded that at the throat, $\frac{P_t}{P_o}$ and $\frac{P_t}{\bar{P}_o}$ are the sonic pressure ratios for fixed mass flows through the two streams. Thus the sonic points in the two streams coincide and are located at the nozzle throat. Since the sonic pressures in the two streams are equal, the total pressures in the two streams are unequal unless the two streams are identical ($\gamma = \bar{\gamma}$), their ratio being

$$\frac{P_o}{\bar{P}_o} = \frac{P_t}{\bar{P}_o}\frac{P_o}{P_t} = \left(\frac{2}{\bar{\gamma}+1}\right)^{\frac{\bar{\gamma}}{\bar{\gamma}-1}} \Bigg/ \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \tag{A-14}$$

Generalizing the above analysis to multistream flows, it is concluded that:

- The sonic pressure of each stream is equal.
- The sonic point of each stream coincides with the nozzle throat (for one-dimensional flows).
- The total pressure of each stream is different (unless the streams are identical).
- There does not exist a common engine (stream) stagnation pressure for performance reference.
- The proper performance reference pressure is the sonic pressure for multistream nozzle flows since it is the only reference pressure common to all streams.

Thus Wrobel's* analysis of multistream rocket nozzle flows is incorrect, since it is based on the assumption that the total pressure in each stream is equal and that there exists a common stream (engine) stagnation pressure for performance reference.

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*Wrobel, J. R., Some Effects of Gas Stratification on Choked Nozzle Flows, AIAA paper No. 64-266, presented at the first annual AIAA meeting, Washington, D. C., 29 June to 2 July, 1965.

A.3. Two-Zone Expansions

The one-dimensional channel flow equations governing the inviscid two-zone isentropic expansion of two perfect gases through a nozzle are

$$\left(1 - u^2\right)\frac{du}{dx} + (\omega + 1)\left(1 - \frac{\gamma-1}{\gamma+1}u^2\right)\frac{u}{y_s}\frac{dy_s}{dx} = 0 \tag{A-15}$$

in the inner zone and

$$\left(1 - \bar{u}^2\right)\frac{d\bar{u}}{dx} + (\omega + 1)\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1}\bar{u}^2\right)\frac{\bar{u}}{y_w^{\omega+1} - y_s^{\omega+1}} \cdot$$

$$\left(y_w^{\omega}\frac{dy_w}{dx} - y_s^{\omega}\frac{dy_s}{dx}\right) = 0 \tag{A-16}$$

in the outer zone where the velocities have been normalized with respect to the appropriate throat sonic velocity and ω equals 0 or 1 depending on whether the nozzle and the two zones are planar or axisymmetric. Since the sonic points of both streams coincide with the nozzle throat,

$$y_s = y_s^*\left[1 + \frac{x^2}{2R_s y_s^{*2}} + \ldots\right] \tag{A-17}$$

$$\frac{dy_s}{dx} = \frac{x}{R_s y^*} + \ldots \tag{A-18}$$

By substituting the above expressions and equations (A-2) and (A-3) into equations (A-15) and (A-16), expanding u and $\bar{u}$ as power series in x and equating powers of x, it can be shown that

$$u = 1 + \sqrt{\frac{\omega+1}{\gamma+1}\frac{y^*}{y_s^*}\frac{1}{R_s}}\;\frac{x}{y^*} + \ldots \tag{A-19}$$

$$\bar{u} = 1 + \sqrt{\frac{\omega+1}{\bar{\gamma}+1}\frac{y^*}{y^{*\omega+1} - y_s^{*\omega+1}}\left[\frac{y^{*\omega}}{R} - \frac{y_s^{*\omega}}{R_s}\right]}\;\frac{x}{y^*} + \ldots \tag{A-20}$$

Since the pressure and sonic pressures are equal in both zones of the nozzle,

$$\frac{p}{p*} = \left[\frac{\gamma+1}{2}\left(1 - \frac{\gamma-1}{\gamma+1} u^2\right)\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{\bar{\gamma}+1}{2}\left(1 - \frac{\bar{\gamma}-1}{\bar{\gamma}+1} \bar{u}^2\right)\right]^{\frac{\bar{\gamma}}{\bar{\gamma}-1}} \qquad \text{(A-21)}$$

Differentiating this expression it is found that

$$\left.\frac{du}{dx}\right|_* = \left.\frac{d\bar{u}}{dx}\right|_* \qquad \text{(A-22)}$$

Using this relationship and solving equations (A-19) and (A-20) for R_s, it is found that

$$R_s = \left[\frac{\gamma^2}{\bar{\gamma}^2}\frac{\bar{\gamma}+1}{\gamma+1}\frac{y*^{\omega+1} - y_s*^{\omega+1}}{y*^{\omega+1}} + \frac{y_s*^{\omega+1}}{y*^{\omega+1}}\right]\frac{y*}{y_s*} R \qquad \text{(A-23)}$$

Substituting the above expression into equations (A-19) and (A-20) it is found that

$$u = 1 + \sqrt{\frac{\omega+1}{\gamma+1}\frac{k}{R}\frac{x}{y*}} + \ldots \qquad \text{(A-24)}$$

$$\bar{u} = 1 + \frac{\gamma}{\bar{\gamma}}\sqrt{\frac{\omega+1}{\gamma+1}\frac{k}{R}\frac{x}{y*}} + \ldots \qquad \text{(A-25)}$$

where

$$k = \left[\frac{\gamma^2}{\bar{\gamma}^2}\frac{\bar{\gamma}+1}{\gamma+1}\frac{y*^{\omega+1} - y_s*^{\omega+1}}{y_s*^{\omega+1}} + \frac{y_s*^{\omega+1}}{y*^{\omega+1}}\right]^{-1} \qquad \text{(A-26)}$$

It is noted that k is a dimensionless constant which varies between $\frac{\bar{\gamma}^2}{\gamma^2}\frac{\gamma+1}{\bar{\gamma}+1}$ and 1 and is thus of order one.

APPENDIX B. RATIONAL FRACTION APPROXIMATIONS

Van Dyke* discusses the use of various transformations to improve the convergence of perturbation expansions. Of particular interest in the current analysis is the use of rational fraction approximations of such series. Consider the series

$$u(0,1) = 1 + \frac{1}{4R} - \frac{14\gamma + 15}{288R^2} \tag{B-1}$$

which is the second order solution for the throat wall velocity in axisymmetric nozzles. Since the neglected terms in the series are $O\left(\frac{1}{R^3}\right)$, alternate representations of the above series may be considered which match the indicated terms for large R but whose behavior for small R (for which the above series is ill-behaved) is a better representation of the true solution.

Following Van Dyke*, the above series can also be represented as

$$u(0,1) = \frac{1 + \frac{14\gamma + 33}{72R}}{1 + \frac{14\gamma + 15}{72R}} \tag{B-2}$$

which, when expanded in inverse powers of R, matches the first three terms of the above series for large R. The advantage of the above representation can be seen by comparing the behavior of the two expressions as functions of R as shown in Figure B-1. Examination of the figure shows that the throat wall velocity given by the first expression maximizes for R approximately one , and indicates that the throat wall velocity is subsonic for R less than approximately one-half, which is physically impossible. In the limit as R goes to zero, the first expression goes to negative infinity. Thus, the first expression is clearly not a good representation of the wall velocity for small R.

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*Van Dyke, M., Perturbation Methods in Fluid Mechanics, Academic Press, New York, 1964.

Examination of the behavior of the second expression as a function of R indicates that the throat wall velocity monotonically increases as R decreases, reaching the limit $\frac{14\gamma + 33}{14\gamma + 15}$ as R goes to zero. Physically, the wall velocity is known to behave in this manner. Thus the rational fraction representation of the throat wall velocity is probably closely representative of its true behavior for all values of R and can be used to approximately determine the accuracy of various order solutions.

In a similar fashion, it can be shown that the rational fraction approximation for the throat axis velocity is

$$u(0,0) = \frac{1 + \frac{10\gamma + 39}{72R}}{1 + \frac{10\gamma + 57}{72R}} \qquad \text{(B-3)}$$

in axisymmetric nozzles. The above rational fractions were used for estimating the accuracy of the various order solutions in Section 2.

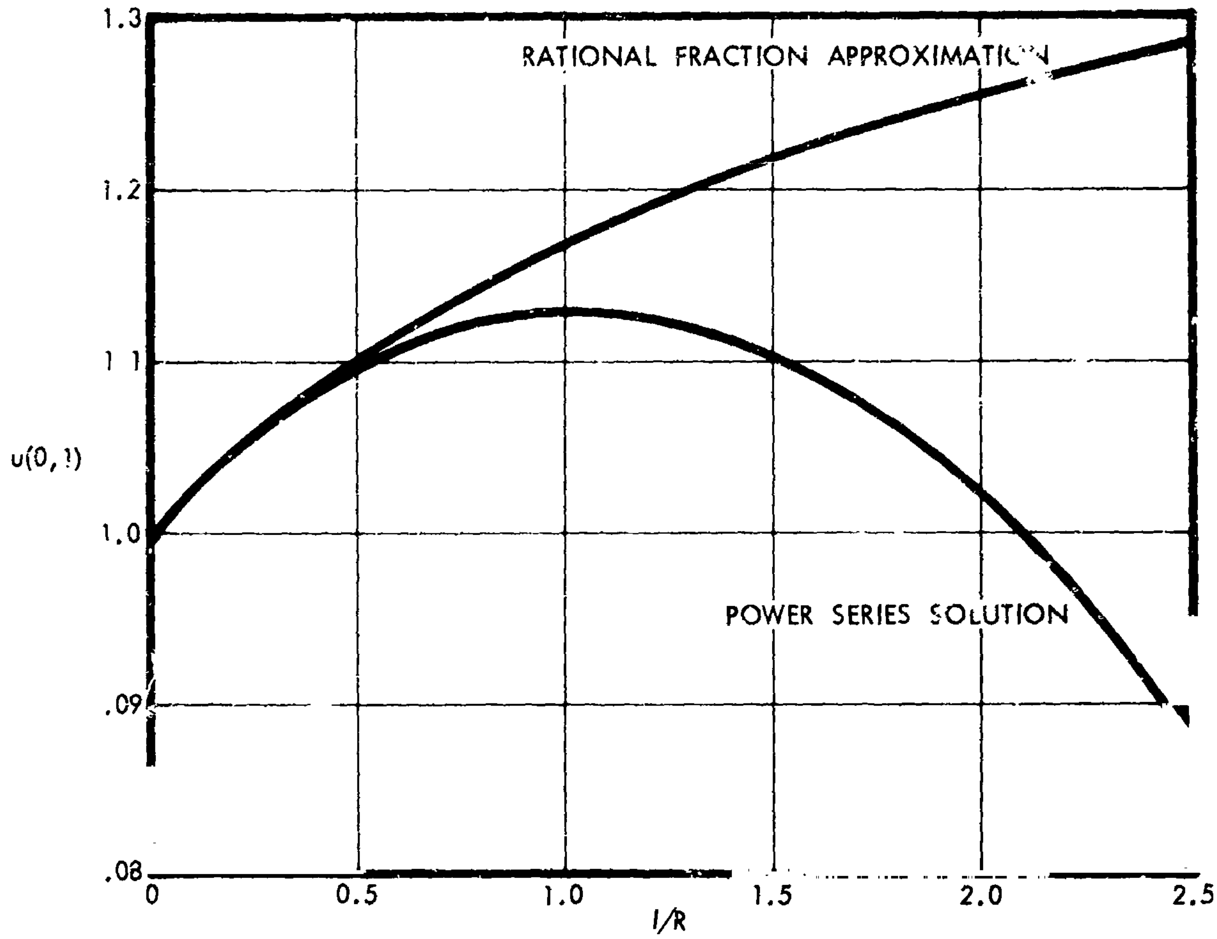

Figure B-1. Comparison of Power Series and Rational Fraction Representation of the Throat Wall Velocity as a Function of the Inverse Normalized Throat Wall Radius of Curvature, $\gamma = 1.4$.