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EFFECT OF ELECTROSTATIC FIELDS ON THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A FINITE TEMPERATURE MAGNETOACTIVE PLASMA

TECHNICAL REPORT NO. 96

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ELECTRON PHYSICS LABORATORY

DEPARTMENT OF ELECTRICAL ENGINEERING
THE UNIVERSITY OF MICHIGAN, ANN ARBOR

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THE UNIVERSITY OF MICHIGAN
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ABSTRACT

The dispersion relations for a finite temperature two-component plasma subjected to crossed electrostatic and magnetostatic fields have been derived using the coupled Boltzmann-Vlasov-Maxwell equations under the one-dimensional, small-signal assumptions. The derived dispersion relation is given in a form in which various characteristic modes of the system can be readily identified. Moreover it is given in a form particularly suitable for a study of the coupling between the transverse circularly polarized wave and the longitudinal wave. Inspection of the derived dispersion relation reveals that the coupling of the longitudinal mode to the transverse mode may take place in the presence of a transverse electrostatic field.

The derived dispersion relation is examined in detail for a Maxwellian plasma in the case of longitudinal propagation. As an illustration, the detailed analysis of the dispersion relation is carried out for a homogeneous, electrically neutral electron gas in which the thermal velocity of the electron is taken into account, but the ion motion is neglected. The variation of refractive index with various system parameters, such as $X \equiv (\omega_p^2/\omega^2)$, and $Y \equiv (\omega_z/\omega)$, has been studied under the conditions of low temperature and a weak applied electrostatic field. ω_p and ω_z are the plasma frequency and cyclotron frequency of the electron respectively, and ω is the angular frequency of the transverse electromagnetic wave. The plots of η vs. X, and η vs. Y, with δ and γ as parameters, are presented and discussed. $\eta \equiv (c^2/v_o^2)$ denotes the square of the refractive index, where c is the speed of light in vacuum, and v_o is the phase velocity of the transverse electromagnetic wave under consideration. $\delta \equiv (m/2KT)(E_o/B_o)^2$ and $\gamma \equiv (2KT/mv_o^2)$ in which m is the electron mass, K is the Boltzmann constant and T the electron temperature. B_o is the magnetostatic field present in the system along the direction of wave propagation and E_o is the applied electrostatic field which is perpendicular to B_o.

It is shown that the presence of an applied transverse electrostatic field E_o in the electron gas has two interesting effects upon the propagation characteristic of the transverse circularly polarized electromagnetic wave, which travels along the magnetostatic field $\vec{B}_o$:

1. The cutoff frequency of the electromagnetic wave, ω_o, shifts; e.g., an investigation of the cutoff condition reveals that, for a given B_o, an increase in E_o will cause the cutoff frequency of the left-hand circularly polarized wave to increase, while it causes that of the right-hand circularly polarized wave to decrease.

2. The longitudinal plasma oscillation may be coupled to the transverse electromagnetic wave, which is referred to as "electrostatic coupling". The temperature effect in the electrostatic coupling has been examined and it is observed that the "coupling velocity", i.e., the velocity at which the electrostatic coupling may take place, depends upon the electron temperature T; e.g., an increase in T causes the coupling velocity of the left-hand circularly polarized wave to increase and the coupling velocity of the right-hand circularly polarized wave to increase when $\omega_z > \omega_p$.

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I. INTRODUCTION

A theory of growing electromagnetic waves was advanced some years ago by Bailey[1,2] in his electromagnetoionic (EMI) theory, which is an extension of the well-known magnetoionic (MI) theory of Appleton and others. In his treatment the random motion of charged particles is taken into account by means of Maxwell's law of momentum transfer. From a detailed study of the case in which both static electric and magnetic fields are parallel to the direction of wave propagation, Bailey[1] concludes that wave amplification is possible in certain frequency ranges, and he has used the theory to explain the excess noise radiation observed in sunspots. However, Bailey's theory of amplified circularly polarized waves in an ionized medium has been criticized by Twiss[3] and Piddington[4]. Twiss points out that the growing wave which Bailey interprets as an amplified wave can only be excited by reflection and it is argued that this can explain neither the excess radiation observed from sunspots nor the excess noise observed in discharge tubes.

Piddington has examined Bailey's theory also for the case in which the ionized gas drift and the wave normal are both in the direction of the steady magnetic field. He concludes that Bailey's theory predicts spurious growing waves which do not correspond to any interchange of energy between gas and field. Piddington further points out that the presence of a steady electric field introduces no new wave forms although it modifies the existing waves.

In analyzing the dispersion relations in a finite temperature magneto-active plasma this author[5] recently found that when the externally applied static electric field is not parallel to the steady magnetic field, which is directed along the direction of wave propagation, coupling between the transverse and longitudinal waves in the plasma can take place.

The purpose of the present report is to investigate the effect of externally applied electrostatic fields upon the propagation characteristics of electromagnetic waves in a plasma pervaded by a static magnetic field. From the coupled Boltzmann-Vlasov-Maxwell equation, a small-amplitude, one-dimensional analysis is considered.

II. BASIC EQUATIONS

Consider a plasma composed of two species (positive ions and electrons) in which collision effects are negligible. The electron distribution function $f(\vec{r},\vec{v},t)$ and the ion distribution function $F(\vec{r},\vec{v},t)$ for this plasma are governed by the Boltzmann-Vlasov equation:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \frac{e}{m}\,(\vec{E} + \vec{v} \times \vec{B})\cdot\nabla_v f = 0 \quad , \tag{1a}$$

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F + \frac{e}{M}\,(\vec{E} + \vec{v} \times \vec{B})\cdot\nabla_v f = 0 \quad , \tag{1b}$$

where m and M denote the electron and ion mass respectively and e is the electronic charge which is taken as a positive quantity. The electromagnetic fields in the plasma are governed by the Maxwell equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \tag{2a}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad , \tag{2b}$$

$$\nabla \cdot \vec{D} = \rho \tag{2c}$$

and

$$\nabla \cdot \vec{B} = 0 \quad . \tag{2d}$$

The electric displacement vector $\vec{D}$ and the magnetic flux density $\vec{B}$ are, respectively, related to the electric field intensity $\vec{E}$ and the magnetic field intensity $\vec{H}$ in the following manner:

$$\vec{D} = \epsilon_o \vec{E} \tag{3a}$$

and

$$\vec{B} = \mu_o \vec{H} \quad , \tag{3b}$$

where ϵ_o and μ_o denote the dielectric constant and the permeability of vacua.

The convection current density $\vec{J}$ and the charge density ρ may be written in terms of the distribution function as

$$\vec{J} = e \int \vec{v}(F - f) d^3v \tag{4a}$$

and

$$\rho = e \int (F - f) d^3v \quad . \tag{4b}$$

Consider all quantities of interest to be composed of two parts, a time-independent part and a time-dependent part which are denoted by the subscripts 0 and 1 respectively:

$$\vec{B} = \vec{B}_o(\vec{r}) + \vec{B}_1(\vec{r},t) \quad ,$$

$$\vec{E} = \vec{E}_o(\vec{r}) + \vec{E}_1(\vec{r},t) \quad ,$$

$$\vec{J} = J_o(\vec{r}) + \vec{J}_1(\vec{r},t) \quad ,$$

$$\rho = \rho_o(\vec{r}) + \rho_1(\vec{r},t) \quad ,$$

$$f(\vec{r},\vec{v},t) = f_o(\vec{r},\vec{v}) + f_1(\vec{r},\vec{v},t)$$

and

$$F(\vec{r},v,t) = F_o(\vec{r},\vec{v}) + F_1(\vec{r},\vec{v},t) \quad . \tag{5}$$

In the present paper the following assumptions are made:

1. Small-amplitude conditions are satisfied so that the terms involving the product of time-dependent quantities are negligible.

2. A one-dimensional analysis is applicable, i.e., all quantities vary only with one spatial variable, and $\partial/\partial x = \partial/\partial y = 0$ in a rectangular Cartesian coordinate system.

3. All time-dependent quantities have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the angular frequency and the propagation constant respectively.

Based on the above assumptions, and the substitution of Eqs. 5 into Eqs. 1 through 4, the following set of equations governing the time-varying quantities is obtained:

$$j(\omega - kv_z)f_1 - \frac{e}{m}\Bigg((E_{ox} + v_y B_{oz} - v_z B_{oy})\frac{\partial f_1}{\partial v_x} + (E_{oy} + v_z B_{ox} - v_x B_{oz})\frac{\partial f_1}{\partial v_y}$$

$$+ (E_{oz} + v_x B_{oy} - v_y B_{ox})\frac{\partial f_1}{\partial v_z}\Bigg)$$

$$= \frac{e}{m}\Bigg((E_{1x} + v_y B_{1z} - v_z B_{1y})\frac{\partial f_o}{\partial v_x} + (E_{1y} + v_z B_{1x} - v_x B_{1z})\frac{\partial f_o}{\partial v_y}$$

$$+ (E_{1z} + v_x B_{1y} - v_y B_{1x})\frac{\partial f_o}{\partial v_z}\Bigg) , \qquad (6a)$$

$$j(\omega - kv_z)F_1 + \frac{e}{M}\Bigg((E_{ox} + v_y B_{oz} - v_z B_{oy})\frac{\partial F_1}{\partial v_x} + (E_{oy} + v_z B_{ox} - v_x B_{oz})\frac{\partial F_1}{\partial v_y}$$

$$+ (E_{oz} + v_x B_{oy} - v_y B_{ox})\frac{\partial F_1}{\partial v_z}\Bigg)$$

$$= -\frac{e}{M}\Bigg((E_{1x} + v_y B_{1z} - v_z B_{1y})\frac{\partial F_o}{\partial v_x} + (E_{1y} + v_z B_{1x} - v_x B_{1z})\frac{\partial F_o}{\partial v_y}$$

$$+ (E_{1z} + v_x B_{1y} - v_y B_{1x})\frac{\partial F_o}{\partial v_z}\Bigg) , \qquad (6b)$$

$$B_{1x} = -\frac{k}{\omega}E_{1y} , \quad B_{1y} = \frac{k}{\omega}E_{1x} , \quad \frac{\partial B_{1z}}{\partial z} = 0 , \qquad (6c)$$

$$\frac{\partial^2 E_{1x}}{\partial z^2} + \frac{\omega^2}{c^2}E_{1x} = j\omega\mu_o J_{1x} , \qquad (6d)$$

$$\frac{\partial^2 E_{1y}}{\partial z^2} + \frac{\omega^2}{c^2}E_{1y} = j\omega\mu_o J_{1y} , \qquad (6e)$$

$$\frac{\omega^2}{c^2} E_{1z} = j\omega\mu_o J_{1z} \quad , \tag{6f}$$

$$\vec{J}_1 = e \int \vec{v}\,(F_1 - f_1)d^3v \tag{6g}$$

and

$$\rho_1 = e \int (F_1 - f_1)d^3v \quad , \tag{6h}$$

where c is the speed of light in vacuum .

On the other hand the time-independent quantities are related to one another in the following manner:

$$v_z \frac{\partial f_o}{\partial y} - \frac{e}{m} (\vec{E}_o + \vec{v} \times \vec{B}_o) \cdot \nabla_v f_o = 0 \quad , \tag{7a}$$

$$v_z \frac{\partial F_o}{\partial z} + \frac{e}{M} (\vec{E}_o + \vec{v} \times \vec{B}_o) \cdot \nabla_v F_o = 0 \quad , \tag{7b}$$

$$\frac{\partial E_{ox}}{\partial x} = 0 \quad , \quad \frac{\partial E_{oy}}{\partial y} = 0 \quad , \quad \frac{\partial E_{oz}}{\partial z} = \frac{\rho_o(z)}{\epsilon_o} \quad , \tag{7c}$$

$$\frac{\partial B_{oy}}{\partial z} = -\mu_o J_{ox} \quad , \quad \frac{\partial B_{ox}}{\partial z} = \mu_o J_{oy} \quad , \quad \frac{\partial B_{oz}}{\partial z} = 0 \quad , \tag{7d}$$

$$\vec{J}_o = e \int \vec{v}\,(F_o - f_o)d^3v \tag{7e}$$

and

$$\rho_o = e \int (F_o - f_o)d^3v \quad . \tag{7f}$$

Now consider a transformation of velocity variable coordinates given by

$$v_x = v_\perp \cos\varphi \quad , \quad v_y = v_\perp \sin\varphi \quad \text{and} \quad v_z = v_z \quad , \tag{8}$$

and for convenience of discussion define the quantities $\vec{\omega}_c$ and $\vec{a}$ by

$$\vec{\omega}_c \equiv \left(\frac{e}{m}\vec{B}_o\right) \quad \text{and} \quad \vec{a} \equiv \left(\frac{e}{m}\vec{E}_o\right) \quad . \tag{9}$$

Then Eq. 6a can be transformed[5], with the aid of Eq. 6c, into

$$\left(j(\omega - kv_z) + \omega_z \frac{\partial}{\partial\varphi}\right) f_1$$

$$- \left[a_- \left(\frac{\partial f_1}{\partial v_\perp} + j \frac{1}{v_\perp} \frac{\partial f_1}{\partial\varphi}\right) + \omega_- \frac{v_z}{v_\perp} \frac{\partial f_1}{\partial\varphi} + j\omega_- D(f_1)\right] e^{j\varphi}$$

$$- \left[a_+ \left(\frac{\partial f_1}{\partial v_\perp} - j \frac{1}{v_\perp} \frac{\partial f_1}{\partial\varphi}\right) + \omega_+ \frac{v_z}{v_\perp} \frac{\partial f_1}{\partial\varphi} - j\omega_+ D(f_1)\right] e^{-j\varphi} - a_z \frac{\partial f_1}{\partial v_z}$$

$$= \frac{e}{m} M_-(f_o) E_- e^{j\varphi} + \frac{e}{m} M_+(f_o) E_+ e^{-j\varphi} + \frac{e}{m} E_{1z} \frac{\partial f_o}{\partial v_z} - \frac{e}{m} B_{1z} \frac{\partial f_o}{\partial\varphi} \quad , \tag{10}$$

where

$$E_\pm \equiv 1/2(E_{1x} \pm jE_{1y}) \quad ,$$

$$B_\pm \equiv 1/2(B_{1x} \pm jB_{1y}) \quad ,$$

$$\omega_\pm \equiv 1/2(\omega_x \pm j\omega_y) \quad ,$$

$$a_\pm \equiv 1/2(a_x \pm ja_y) \quad ,$$

$$\omega_{cx} \equiv \omega_x \quad , \quad \omega_{cy} \equiv \omega_y \quad , \quad \omega_{cz} \equiv \omega_z \quad ,$$

$$M_-(f_o) \equiv \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_o}{\partial v_\perp} - \frac{j}{v_\perp} \frac{\partial f_o}{\partial\varphi}\right) + \frac{kv_\perp}{\omega} \frac{\partial f_o}{\partial v_z}\right] \quad ,$$

$$M_+(f_o) \equiv \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_o}{\partial v_\perp} + \frac{j}{v_\perp} \frac{\partial f_o}{\partial\varphi}\right) + \frac{kv_\perp}{\omega} \frac{\partial f_o}{\partial v_z}\right] \quad , \tag{11}$$

and the differential operator D is defined by

$$D(\ ) \equiv \left[v_{\perp} \frac{\partial(\ )}{\partial v_z} - v_z \frac{\partial(\ )}{\partial v_{\perp}} \right] . \qquad (12)$$

It should be noted that E_- and E_+ appearing in Eq. 10 correspond to the electric field of the left-hand and right-hand circularly polarized waves respectively.

III. DISPERSION RELATIONS

Suppose that the positive z-direction is taken in the direction of the magnetostatic field $\vec{B}_o$, i.e., $B_{ox} = B_{oy} = 0$ so that $\omega_{\pm} = 0$. From Eq. 6c, B_{1z} is independent of z and it is taken to be zero in the present discussion (which is reasonable for longitudinal propagation).

Now consider that the time-varying electron distribution function f_1 is composed of three parts and may be written as:

$$f_1(z,t,v_{\perp},v_z,\varphi) = f_-(z,t,v_{\perp},v_z)e^{j\varphi} + f_+(z,t,v_{\perp},v_z)e^{-j\varphi} + g(z,t,v_{\perp},v_z) , \qquad (13)$$

where the first, second and third terms of the right-hand side can be regarded as the distribution of those electrons associated with the right-hand circularly polarized, left-hand circularly polarized and longitudinal waves respectively. Since Eq. 10 must be valid for an arbitrary value of φ, the substitution of Eq. 13 into Eq. 10 yields the following system of equations[5]:

$$j(\omega - kv_z + \omega_z)f_- - a_z \frac{\partial f_-}{\partial v_z} - a_- \frac{\partial g}{\partial v_{\perp}} = \frac{e}{m} M_-(f_o)E_- , \qquad (14a)$$

$$j(\omega - kv_z - \omega_z)f_+ - a_z \frac{\partial f_+}{\partial v_z} - a_+ \frac{\partial g}{\partial v_\perp} = \frac{e}{m} M_+ (f_o)E_+ \qquad (14b)$$

and

$$j(\omega - kv_z)g - a_z \frac{\partial g}{\partial v_z} - \frac{2a_-}{v_\perp} f_+ - \frac{2a_+}{v_\perp} f_- = \frac{e}{m} \frac{\partial f_o}{\partial v_z} E_{1z} \quad , (14c)$$

which clearly suggests that no coupling between the transverse mode and the longitudinal mode can take place when a_+ and a_- are zero, which is the case when the transverse electrostatic field is zero.

In the present investigation it is assumed that $a_z = 0$, i.e., $E_{oz} = 0$, since the effect of the transverse electrostatic field is of primary concern. This assumption is equivalent to assuming that the condition of electrical neutrality is satisfied. For this case, it is possible to solve the system of Eqs. 14 for f_-, f_+ and g explicitly in terms of E_-, E_+ and E_{1z} as follows:

$$f_- = k_{11}E_- + k_{12}E_+ + k_{13}E_{1z} \quad ,$$

$$f_+ = k_{21}E_- + k_{22}E_+ + k_{23}E_{1z} \quad ,$$

$$g = k_{31}E_- + k_{32}E_+ + k_{33}E_{1z} \quad , \qquad (15)$$

where

$$k_{11} = \frac{-j\frac{e}{m}M_-(f_o)}{b+\omega_z}, \quad k_{12} = 0, \quad k_{13} = \frac{-\frac{e}{m}a_-\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b+\omega_z)},$$

$$k_{21} = 0, \quad k_{22} = \frac{-j\frac{e}{m}M_+(f_o)}{b-\omega_z}, \quad k_{23} = \frac{-\frac{e}{m}a_+\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b-\omega_z)},$$

$$k_{31} = \frac{-2\frac{e}{m}\frac{a_+}{v_\perp}M_-(f_o)}{b(b+\omega_z)}, \quad k_{32} = \frac{-2\frac{e}{m}\frac{a_-}{v_\perp}M_+(f_o)}{b(b-\omega_z)},$$

$$k_{33} = \frac{-j\frac{e}{m}\frac{\partial f_o}{\partial v_z}}{b} + j\frac{4a_-a_+}{v_\perp}\frac{\frac{e}{m}\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_o}{\partial v_z}\right)}{b(b^2-\omega_z^2)}, \qquad (16)$$

in which $b \triangleq (\omega - kv_z)$.

Similarly the ion distribution functions may be written as

$$F_1(z,t,v_\perp,v_z\varphi) = F_-(z,t,v_\perp,v_z)e^{j\varphi} + F_+(z,t,v_\perp,v_z)e^{-j\varphi} + G(z,t,v_\perp,v_z), \qquad (17)$$

and in view of the fact that Eqs. 6a and 6b have exactly the same form, the substitution of Eq. 17 into Eq. 6b yields a system of equations governing F_-, F_+ and G which is similar to the system of Eqs. 14. By defining $\vec{\Omega}$ and $\vec{A}$ as

$$\vec{\Omega} = -\frac{e}{M}\vec{B}_o \quad \text{and} \quad \vec{A} = \frac{-e}{M}\vec{E}_o, \qquad (18)$$

F_-, F_+ and G can be expressed in terms of E_-, E_+ and E_{1z} as

$$F_- = K_{11}E_- + K_{12}E_+ + K_{13}E_{1z} \ ,$$

$$F_+ = K_{21}E_- + K_{22}E_+ + K_{23}E_{1z} \ ,$$

$$G = K_{31}E_- + K_{32}E_+ + K_{33}E_{1z} \ , \tag{19}$$

where

$$K_{11} = \frac{j\frac{e}{M}M_-(F_o)}{(b+\Omega_z)} \ , \quad K_{12} = 0 \ , \quad K_{13} = \frac{\frac{e}{M}A_- \frac{-\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b+\Omega_z)} \ ,$$

$$K_{21} = 0 \ , \quad K_{22} = \frac{j\frac{e}{M}M_+(F_o)}{(b-\Omega_z)} \ , \quad K_{23} = \frac{\frac{e}{M}A_+ \frac{\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b-\Omega_z)} \ ,$$

$$K_{31} = \frac{2\frac{e}{M}\frac{A_+}{v_\perp}M_-(F_o)}{b(b+\Omega_z)} \ , \quad K_{32} = \frac{2\frac{e}{M}\frac{A_-}{v_\perp}M_+(F_o)}{b(b-\Omega_z)} \ ,$$

$$K_{33} = \frac{j\frac{e}{M}\frac{\partial F_o}{\partial v_z}}{b} - \frac{j4A_-A_+}{v_\perp}\,\frac{\frac{e}{M}\frac{\partial}{\partial v_\perp}\left(\frac{\partial F_o}{\partial v_z}\right)}{b(b^2-\Omega_z^2)} \ , \tag{20}$$

where $A_\pm \equiv (1/2)(A_x \pm jA_y)$ and $\Omega_z \equiv [(-e/M)B_{oz}]$.

When the time-varying distribution functions are expressed explicitly in terms of the time-varying electric field, the convection current density $\vec{J}_1$ and the space-charge density ρ_1 can then be expressed in terms of the electric field with the aid of Eqs. 6g and 6h respectively. On the other hand, the electric field is related to the current density by Eqs. 6d, 6e and 6f so that it can be expressed as

$$E_{\pm} = \frac{j\left(\frac{\omega e}{\epsilon_0}\right)}{2(\omega^2 - c^2k^2)} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{2\pi} v_{\perp}^2 \, (F_1 - f_1) \, e^{\pm j\varphi} d\varphi dv_{\perp} dv_z \quad ,$$

$$E_{1z} = \frac{je}{\omega\epsilon_0} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{2\pi} v_z v_{\perp} \, (F_1 - f_1) \, d\varphi dv_{\perp} dv_z \quad . \tag{21}$$

When Eqs. 13, 15, 17 and 19 are substituted into Eqs. 21, the following set of equations is obtained:

$$\begin{aligned} E_- &= R_{11}E_- + R_{12}E_+ + R_{13}E_{1z} \quad , \\ E_+ &= R_{21}E_- + R_{22}E_+ + R_{23}E_{1z} \quad , \\ E_{1z} &= R_{31}E_- + R_{32}E_+ + R_{33}E_{1z} \quad , \end{aligned} \tag{22}$$

where

$$\begin{aligned} R_{p,q} &\equiv P\,(S_{p,q}) \quad \text{for} \quad p = 1, 2 \;;\; q = 1, 2, 3 \quad , \\ &\equiv Q\,(S_{p,q}) \quad \text{for} \quad p = 3 \;;\; q = 1, 2, 3 \quad , \end{aligned} \tag{23}$$

in which the integration operators P(S) and Q(S) are defined by

$$P(S) \equiv \frac{j\left(\frac{\omega e}{\epsilon_0}\right)}{2(\omega^2 - c^2k^2)} \int_{-\infty}^{\infty}\int_{0}^{\infty} S(v_{\perp}, v_z) v_{\perp}^2 dv_{\perp} dv_z \quad ,$$

$$Q(S) \equiv \frac{je}{\omega\epsilon_0} \int_{-\infty}^{\infty}\int_{0}^{\infty} S(v_{\perp}, v_z) v_{\perp} v_z dv_{\perp} dv_z \quad , \tag{24}$$

and the functions $S_{p,q}(v_z, v_{\perp})$ are defined by

$$S_{1\ell} \equiv \int_0^{2\pi} [(K_{1\ell} - k_{1\ell}) + (K_{2\ell} - k_{2\ell})e^{-j2\varphi} + (K_{3\ell} - k_{3\ell})e^{-j\varphi}]d\varphi \quad ,$$

$$S_{2p} \equiv \int_0^{2\pi} [(K_{1p} - k_{1p})e^{j2\varphi} + (K_{2p} - k_{2p}) + (K_{3p} - k_{3p})e^{j\varphi}]d\varphi \quad ,$$

$$S_{3q} \equiv \int_0^{2\pi} [(K_{1q} - k_{1q})e^{j\varphi} + (K_{2q} - k_{2q})e^{-j\varphi} + (K_{3q} - k_{3q})]d\varphi \quad , \tag{25}$$

for $\ell = 1, 2, 3$; $p = 1, 2, 3$; and $q = 1, 2, 3$, and where $k_{p,q}$ and $K_{p,q}$ are given in Eqs. 16 and 20 respectively.

Therefore the dispersion relation of the system is given by

$$D(\omega,k) = \begin{vmatrix} (R_{11} - 1) & R_{12} & R_{13} \\ R_{21} & (R_{22} - 1) & R_{23} \\ R_{31} & R_{32} & (R_{33} - 1) \end{vmatrix} = 0 \quad . \tag{26}$$

It should be observed that once the time-independent distribution functions f_o and F_o are known, the parameters $k_{p,q}$ and $K_{p,q}$ are specified so that the elements of the determinants, $R_{p,q}$, are determined. Thus the dispersion relation can be analyzed to obtain the information on the propagation characteristics of the waves in the system.

IV. MAXWELLIAN PLASMA

The time-independent distribution functions f_o and F_o must satisfy Eqs. 7a and 7b respectively, in which the electrostatic field $\vec{E}_o$ can be written by $\vec{E}_o = \vec{E}_s + \vec{E}_a$ where $\vec{E}_a$ represents the externally applied electrostatic field and $\vec{E}_s$ represents the space-charge field. For the present one-dimensional analysis, $\vec{E}_s$ is directed in the z-direction. By assumption $\vec{E}_a$ is perpendicular to $\vec{B}_o$, which is in the positive z-direction. Suppose that the positive y-direction is taken in the direction of applied uniform electrostatic field; then it is not difficult to show that the function $f_o(z,v_x,v_y,v_z)$ has the form (see Appendix A for details),

$$f_o = n_o e^{-\alpha\{[(v_x - u)^2 + v_y^2 + v_z^2] - (2e/m)\Phi(z)\}} , \quad (27)$$

where n_o is the normalization constant. The electronic drift velocity u, and the space-charge potential Φ are given by

$$\vec{u} = (\vec{E}_a \times \vec{B}_o)/|\vec{B}_o|^2 \quad , \quad \text{or} \quad u = (E_a/B_o) \quad \text{and} \quad (\partial\Phi/\partial z) = - E_s$$

respectively. Since it is assumed in the previous section that $E_s = E_{oz} = 0$, Φ must be independent of z. The time-independent distribution function f_o in the case of a homogeneous plasma, therefore, can be given as

$$f_o = n\left(\frac{\alpha_e}{\pi}\right)^{3/2} e^{-\alpha_e[(v_x - u)^2 + v_y^2 + v_z^2]} , \quad (28a)$$

where n is the number density of electrons, $\alpha_e \triangleq (m/2KT_e)$ with K denoting the Boltzmann constant, and T_e is the electron temperature. In view of the fact that the electronic drift velocity u depends neither on the ratio of charge to mass, nor on the initial velocities, it is the same for electrons and ions regardless of their energy. The time-independent ion distribution function F_o can be written as

$$F_o = N\left(\frac{\alpha_i}{\pi}\right)^{3/2} e^{-\alpha_i[(v_x - u)^2 + v_y^2 + v_z^2]} , \quad (28b)$$

where N is the number density of ions, $\alpha_i \equiv (M/2KT_i)$, with T_i denoting the ion temperature.

Since the form of the time-independent distribution functions f_o and F_o is specified, R_{pq} can be evaluated. After some algebraic manipulation the following expressions are obtained (see Appendix B.1 for the details):

$$R_{11} = \frac{1}{(1 - \eta)} \sum_{q=1,2} S_q \left(\frac{j\delta_q}{V_q} + \delta_q G_o(U_o) + G_o(U_{+q}) \right) , \quad (29a)$$

$$R_{12} = \frac{1}{(1 - \eta)} \sum_q S_q \left(j \frac{(2\mu_q + \delta_q)}{V_q} + \delta_q G_o(U_o) - Y_q(2\mu_q + \beta_q) G_o(U_{-q}) \right) , \quad (29b)$$

$$R_{13} = \frac{1}{(1 - \eta)} \sum_q \sqrt{\delta_q} S_q \left[j + (2 + \nu_q) V_q G_o(U_o) + \left(\frac{\beta_q}{2} - 1 \right) V_q (1 + Y_q) G_o \cdot (U_{+q}) + \left(\frac{\beta_q}{2} + \mu_q \right) V_q (1 - Y_q) G_o(U_{-q}) \right] , \quad (29c)$$

$$R_{21} = \frac{1}{(1 - \eta)} \sum_q S_q \left(j \frac{(2\mu_q + \delta_q)}{V_q} + \delta_q G_o(U_o) + Y_q(2\mu_q + \beta_q) G_o(U_{+q}) \right) , \quad (29d)$$

$$R_{22} = \frac{1}{(1 - \eta)} \sum_q S_q \left(\frac{j\delta_q}{V_q} + \delta_q G_o(U_o) + G_o(U_{-q}) \right) , \quad (29e)$$

$$R_{23} = \frac{1}{(1 - \eta)} \sum_q S_q \sqrt{\delta_q} \left[j + (2 + \nu_q) V_q G_o(U_o) + \left(\frac{\beta_q}{2} + \mu_q \right) V_q (1 + Y_q) \right.$$

$$\left. \cdot G_o(U_{+q}) + \left(\frac{\beta_q}{2} - 1 \right) V_q (1 - Y_q) G_o(U_{-q}) \right] , \qquad (29f)$$

$$R_{31} = 2 \sum_q S_q \sqrt{\delta_q} \, [j(1 + \nu_q Y_q) + V_q G_o(U_o) + V_q \nu_q Y_q (1 + Y_q) G_o(U_{+q})] , \qquad (29g)$$

$$R_{32} = 2 \sum_q S_q \sqrt{\delta_q} \, [j(1 - \nu_q Y_q) + V_q G_o(U_o) - V_q \nu_q Y_q (1 - Y_q) G_o(U_{-q})] , \qquad (29h)$$

$$R_{33} = 2 \sum_q S_q V_q \, [j + (1 - 2\lambda_q) V_q G_o(U_{oq}) + \lambda_q (1 + Y_q)^2 V_q G_o(U_{+q})$$

$$+ \lambda_q (1 - Y_q)^2 V_q G_o(U_{-q})] , \qquad (29i)$$

where

$$S_q \equiv jV_q X_q \quad , \quad V_q \equiv (\sqrt{\alpha_q}\, U_o) \quad , \quad X_q \equiv \left(\frac{\omega_{pq}}{\omega}\right)^2 \quad ,$$

$$\delta_q \equiv \alpha_q u^2 \quad , \quad \eta \equiv \left(\frac{c^2 k^2}{\omega^2}\right) \quad ,$$

$$U_o \equiv \left(\frac{\omega}{k}\right) \quad , \quad U_{+q} \equiv \left(\frac{\omega + \omega_{zq}}{k}\right) \quad , \quad U_{-q} \equiv \left(\frac{\omega - \omega_{zq}}{k}\right) \quad ,$$

$$Y_q \equiv \left(\frac{\omega_{zq}}{\omega}\right) \quad ,$$

$$\beta_q \equiv \delta_q D_q \quad ,$$

$$\mu_q \equiv (D_q - 1) \quad ,$$

$$\nu_q \equiv [1 - D_q(1 + \delta_q)] = -(\mu_q + \beta_q) \quad ,$$

$$\lambda_q \equiv \delta_q [1 - (1/2)\, D_q (1 + 2\delta_q)] \quad ,$$

$$D_q \equiv e^{-\delta_q} \sum_{n=o}^{\infty} \frac{1}{n!} \frac{1}{(n+1)} \delta_q^n \quad , \tag{30}$$

and

$$G_o(X_q) \equiv \int_{-\infty}^{\infty} \frac{e^{-\alpha_q v_z^2}}{(v_z - X_q)}\, dv_z \tag{31}$$

for q = 1 and 2, and X_q may in general be complex. The subscript q takes values of 1 or 2, and the quantities appearing in Eqs. 30 with subscript 1 denote those associated with the electron and those with subscript 2 are for the ion. The summation in Eqs. 29 is taken over both components of the plasma.

It should be noted that the integral (31) has been discussed in detail by Stix[6] and his results can be applied to the present discussion. The function $G_o(X_q)$ has an interesting asymptotic expansion property[6]. Using this property, if the condition

$$|\sqrt{\alpha_q} X_q|^4 \gg 1 \qquad (32)$$

is satisfied, then $G_o(X_q)$ may be approximated by

$$G_o(X_q) = \frac{-j}{\sqrt{\alpha_q} X_q}\left(1 + \frac{1}{2(\sqrt{\alpha_q} X_q)^2}\right) \qquad (33)$$

so that

$$G_o(U_o) = \frac{-j}{V_q}\left(1 + \frac{1}{2V_q^2}\right) ,$$

$$G_o(U_{\pm q}) = \frac{-j}{V_q(1 \pm Y_q)}\left(1 + \frac{1}{2V_q^2} \; \frac{1}{(1 \pm Y_q)^2}\right) . \qquad (34)$$

On the other hand, if the condition

$$\delta_q^2 \ll 1 \qquad (35)$$

is satisfied, then

$$D = \left(1 - \frac{\delta_q}{2}\right) , \quad \beta_q = \delta_q\left(1 - \frac{\delta_q}{2}\right) , \quad \mu_q = -\frac{\delta_q}{2} ,$$

$$\lambda_q = \frac{\delta_q}{2}\left(1 - \frac{3}{2}\delta_q\right) . \qquad (36)$$

Through use of the above approximations, Eqs. 34 and 36, Eqs. 29 become

$$R_{11} = \sum_q \frac{X_q}{(1-\eta)} (\Pi_{11} + \gamma_q \Lambda_{11}) \quad ,$$

$$R_{12} = \sum_q \frac{\delta_q X_q}{(1-\eta)} (\Pi_{12} + \gamma_q \Lambda_{12}) \quad ,$$

$$R_{13} = \sum_q \frac{\sqrt{\delta_q \gamma_q X_q}}{(1-\eta)} \Lambda_{13} \quad ,$$

$$R_{21} = \sum_q \frac{\delta_q X_q}{(1-\eta)} (\Pi_{21} + \gamma_q \Lambda_{21}) \quad ,$$

$$R_{22} = \sum_q \frac{X_q}{(1-\eta)} (\Pi_{22} + \gamma_q \Lambda_{22}) \quad ,$$

$$R_{23} = \sum_q \frac{\sqrt{\delta_q \gamma_q X_q}}{(1-\eta)} \Lambda_{23} \quad ,$$

$$R_{31} = \sum_q \sqrt{\delta_q \gamma_q X_q} \; \Lambda_{31} \quad ,$$

$$R_{32} = \sum_q \sqrt{\delta_q \gamma_q X_q} \; \Lambda_{32} \quad ,$$

$$R_{33} = \sum_q X_q (1 + \delta_q A_q) \quad , \tag{37}$$

where

$$\gamma_q \equiv \frac{1}{V_q^2} \;,\quad \Pi_{11} \equiv \frac{1}{1 + Y_q} \;,\quad \Pi_{12} \equiv \left(1 + \frac{\delta_q}{2}\,\frac{Y_q}{(1 - Y_q)}\right) \;,$$

$$\Pi_{21} \equiv \left(1 - \frac{\delta_q}{2}\,\frac{Y_q}{(1 + Y_q)}\right) \;,\quad \Pi_{22} \equiv \frac{1}{1 - Y_q} \;,\quad \Lambda_{11} \equiv \frac{1}{2}\left(\delta_q + \frac{1}{(1 + Y_q)^3}\right),$$

$$\Lambda_{12} \equiv \frac{1}{2}\left(1 + \frac{\delta_q}{2}\,\frac{Y_q}{(1 - Y_q)^3}\right) \;,\quad \Lambda_{13} \equiv \frac{1}{2}\left[\left(2 - \frac{\delta_q}{2}\right) + \frac{\left(\frac{\delta_q}{2} - 1\right)}{(1 + Y_q)^2}\right] \;,$$

$$\Lambda_{21} \equiv \frac{1}{2}\left(1 - \frac{\delta_q}{2}\,\frac{Y_q}{(1 + Y_q)^3}\right) \;,\quad \Lambda_{22} \equiv \frac{1}{2}\left(\delta_q + \frac{1}{(1 - Y_q)^3}\right) \;,$$

$$\Lambda_{23} \equiv \frac{1}{2}\left[\left(2 - \frac{\delta_q}{2}\right) + \left(\frac{\delta_q}{2} - 1\right)\frac{1}{(1 - Y_q)^2}\right] \;,\quad \Lambda_{31} \equiv \left(1 - \frac{\delta_q}{2}\,\frac{(1 - \delta_q)Y_q}{(1 + Y_q)^2}\right),$$

$$\Lambda_{32} \equiv \left(1 + \frac{\delta_q}{2}\,\frac{(1 - \delta_q)Y_q}{(1 - Y_q)^2}\right) \;,$$

$$A_q \equiv \left(\frac{3}{2}\,\delta_q - 1\right)\left(1 - \frac{1}{1 - Y_q^2}\right) = \frac{\left(1 - \frac{3}{2}\,\delta_q\right) Y_q^2}{(1 - Y_q^2)} \;. \qquad (38)$$

It should be observed that as $\delta \to 0$, $R_{pq} \to 0$ for $p \neq q$ so that the off-diagonal terms of the determinant in Eq. 26 vanish, which indicates that the coupling between the modes disappears as expected from the discussion in Section III. Equation 26 then becomes

$$(R_{11} - 1)(R_{22} - 1)(R_{33} - 1) = 0 \quad , \tag{39}$$

which implies that

$$1 = R_{11} = \sum_q \frac{X_q}{(1 - \eta)} (\Pi_{11} + \gamma\Lambda_{11}) \quad , \tag{40a}$$

$$1 = R_{22} = \sum_q \frac{X_q}{(1 - \eta)} (\Pi_{22} + \gamma\Lambda_{22}) \tag{40b}$$

and

$$1 = R_{33} = \sum_q X_q \quad , \tag{40c}$$

in which Eqs. 40a and 40b are those given by Montgomery and Tidman[8]. It is of interest to note that as $\gamma \to 0$, i.e., the plasma temperature approaches zero, Eqs. 40 are reduced to the following familiar expressions in the cold-plasma magnetoionic theory:

$$\eta = 1 - \sum_q \frac{X_q}{1 + Y_q} = 1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega + \omega_{zq})} \quad , \tag{41a}$$

$$\eta = 1 - \sum_q \frac{X_q}{1 - Y_q} = 1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega - \omega_{zq})} \tag{41b}$$

and

$$\omega^2 = \sum_q \omega_{pq}^2 \quad , \tag{41c}$$

where η is the square of the refractive index of the wave; i.e., $\eta \equiv (c^2k^2/\omega^2)$. Equations 41a and 41b are simply the dispersion equations for the left-hand and right-hand circularly polarized waves respectively, and Eq. 41c is that of the longitudinal plasma oscillation. On the other hand, in the case where $\delta \neq 0$, but $\gamma = 0$, R_{13}, R_{23}, R_{31} and R_{32} are all zero, which suggests that the plasma temperature undoubtedly has an effect on the electrostatic coupling. The term "electrostatic coupling" is introduced here to describe the phenomenon of coupling between the longitudinal wave and the transverse wave in the presence of a transverse electrostatic field. The temperature effect in the electrostatic coupling for an electron gas is considered in detail in the following section.

V. A HOMOGENEOUS NEUTRAL ELECTRON GAS

The analysis of the dispersion relation is carried out in detail here for a homogeneous neutral electron gas in which the thermal velocity of the electrons is taken into account; however the ion motion is neglected. Suppose that the applied electrostatic field is sufficiently weak so that the condition

$$\delta \ll 1 \tag{42}$$

is satisfied. Then, using the fact that $\gamma^2 \ll 1$ is assumed, which is condition (32), Eq. 26 can be expanded into the following form (see Appendix B.4 for details):

$$\Phi \frac{X^3}{(1-\eta)^2} - \Psi \frac{X^2}{(1-\eta)^2} - \Pi \frac{X^2}{(1-\eta)} + \Lambda \frac{X}{(1-\eta)} + X - 1 = 0 , \tag{43}$$

where
$$\Phi \equiv \Phi_0 + \gamma\Phi_1 \quad ,$$
$$\Psi \equiv \Psi_0 + \gamma\Psi_1 \quad ,$$
$$\Pi \equiv \Pi_0 + \gamma\Pi_1 \quad ,$$
$$\Lambda \equiv \Lambda_0 + \gamma\Lambda_1 \quad ,$$

in which

$$\Phi_0 \equiv \frac{1}{\xi}(1 + \delta^2 Y^2) \quad ,$$

$$\Phi_1 \equiv \frac{1}{\xi^3}(1 + Y^2 - \delta^2 Y^6) \quad ,$$

$$\Psi_0 \equiv \Phi_0 \quad ,$$

$$\Psi_1 \equiv \frac{1}{\xi^3}(1 + Y^2 + \delta^2 Y^6) \quad ,$$

$$\Pi_0 \equiv \frac{2}{\xi} \quad ,$$

$$\Pi_1 \equiv \frac{1}{\xi^3}(1 - 2\delta Y + 3Y^2 + 2\delta Y^3 - \delta^2 Y^5 - \delta Y^6) \quad ,$$

$$\Lambda_0 \equiv \frac{2}{\xi} \quad ,$$

$$\Lambda_1 \equiv \frac{1}{\xi^3}(1 + 3Y^2 - \delta Y^6) \quad , \qquad (44)$$

with $\xi \equiv (1 - Y^2)$.

Since Eq. 43 is a quadratic in η, it can be solved for η as follows, provided that $(\Phi X - \Psi) \neq 0$:

$$\eta = 1 - \frac{2X(\Phi X - \Psi)}{(\Pi X - \Lambda) \pm \sqrt{(\Pi X - \Lambda)^2 - 4(\Phi X - \Psi)(X - 1)}} \quad . \qquad (45)$$

It should be noted that when $\delta = 0$ from Eqs. 44, it is easily seen that $\Phi = \Psi$ and $\Pi = \Lambda$ so that Eq. 43 becomes

$$(X - 1)\left(\Phi \frac{X^2}{(1 - \eta)^2} - \Pi \frac{X}{(1 - \eta)} + 1\right) = 0 \ , \qquad (46)$$

and Eq. 45 accordingly becomes

$$\eta = 1 - \frac{2\Phi X}{\Pi \pm \sqrt{\Pi^2 - 4\Phi}} \ , \qquad (47)$$

in which

$$\Phi = \Psi = \frac{1}{\xi}\left(1 + \frac{\gamma}{\xi^2}(1 + Y^2)\right) \ ,$$

$$\Pi = \Lambda = \frac{1}{\xi}\left(2 + \frac{\gamma}{\xi^2}(1 + Y^2)\right) \ . \qquad (48)$$

On the other hand, in the case where $\delta = 0$, Eqs. 40 yield

$$1 = \frac{X}{(1 - \eta)}\left(\frac{1}{1 + Y} + \frac{\gamma/2}{(1 + Y)^3}\right) \ , \qquad (49a)$$

$$1 = \frac{X}{(1 - \eta)}\left(\frac{1}{1 - Y} + \frac{\gamma/2}{(1 - Y)^3}\right) \qquad (49b)$$

and

$$1 = X \ . \qquad (49c)$$

It is not difficult to show that upon substituting Φ and Π given by Eqs. 48 into Eq. 46, the left-hand side of Eq. 46 can be written as the product of three factors which leads to Eqs. 49 as is expected.

Based on Eq. 43, or equivalently on Eq. 45, with γ and δ as parameters, η vs. X and η vs. Y are shown in Figs. 1 and 2 respectively. Y vs. X for the case of $\eta = 0$, which corresponds to the cutoff condition, is shown in Fig. 3.

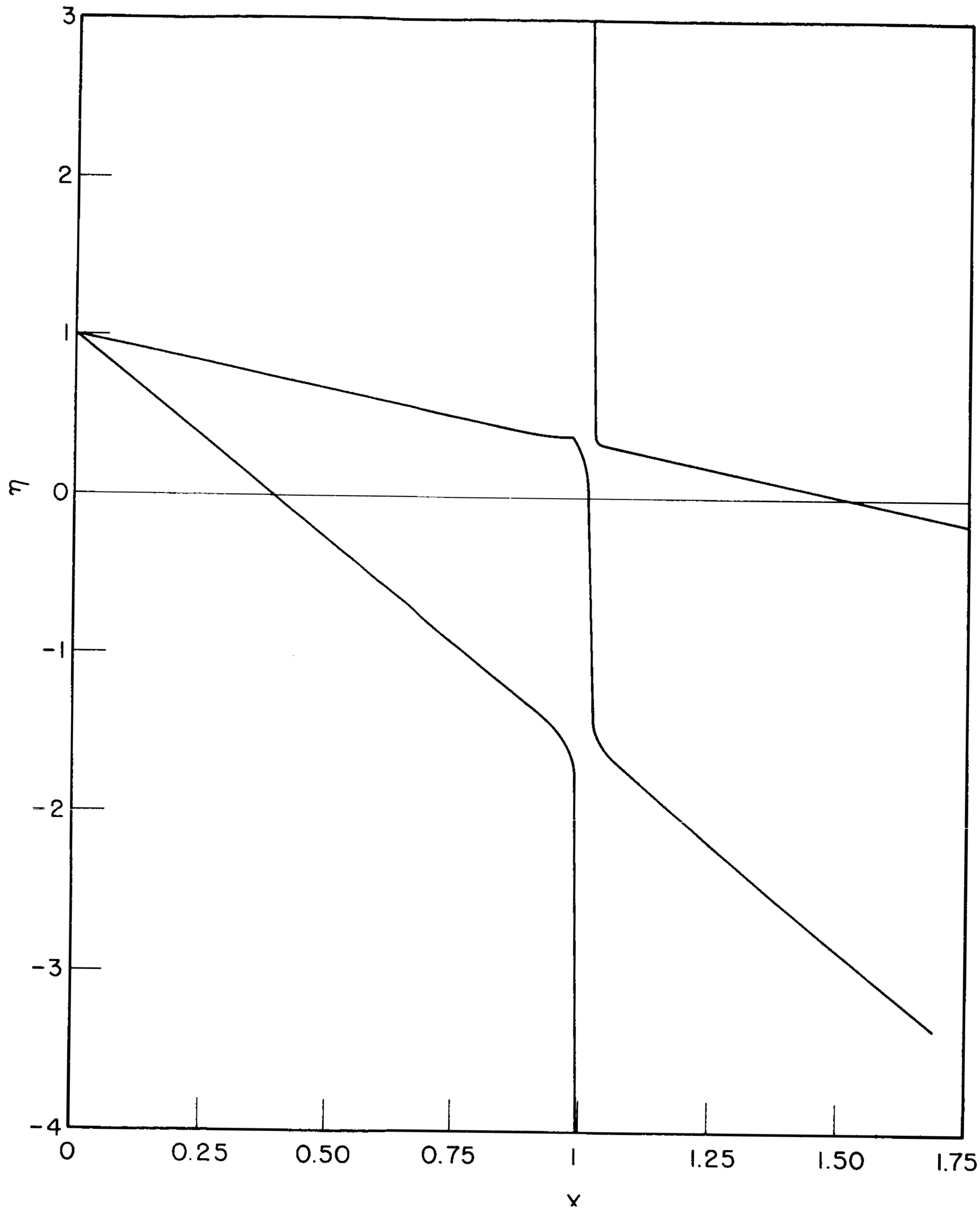

FIG. 1a THE PLOT OF η VS. X FOR $\delta = 0.01$, $\gamma = 0.1$ AND $Y = 0.5$.

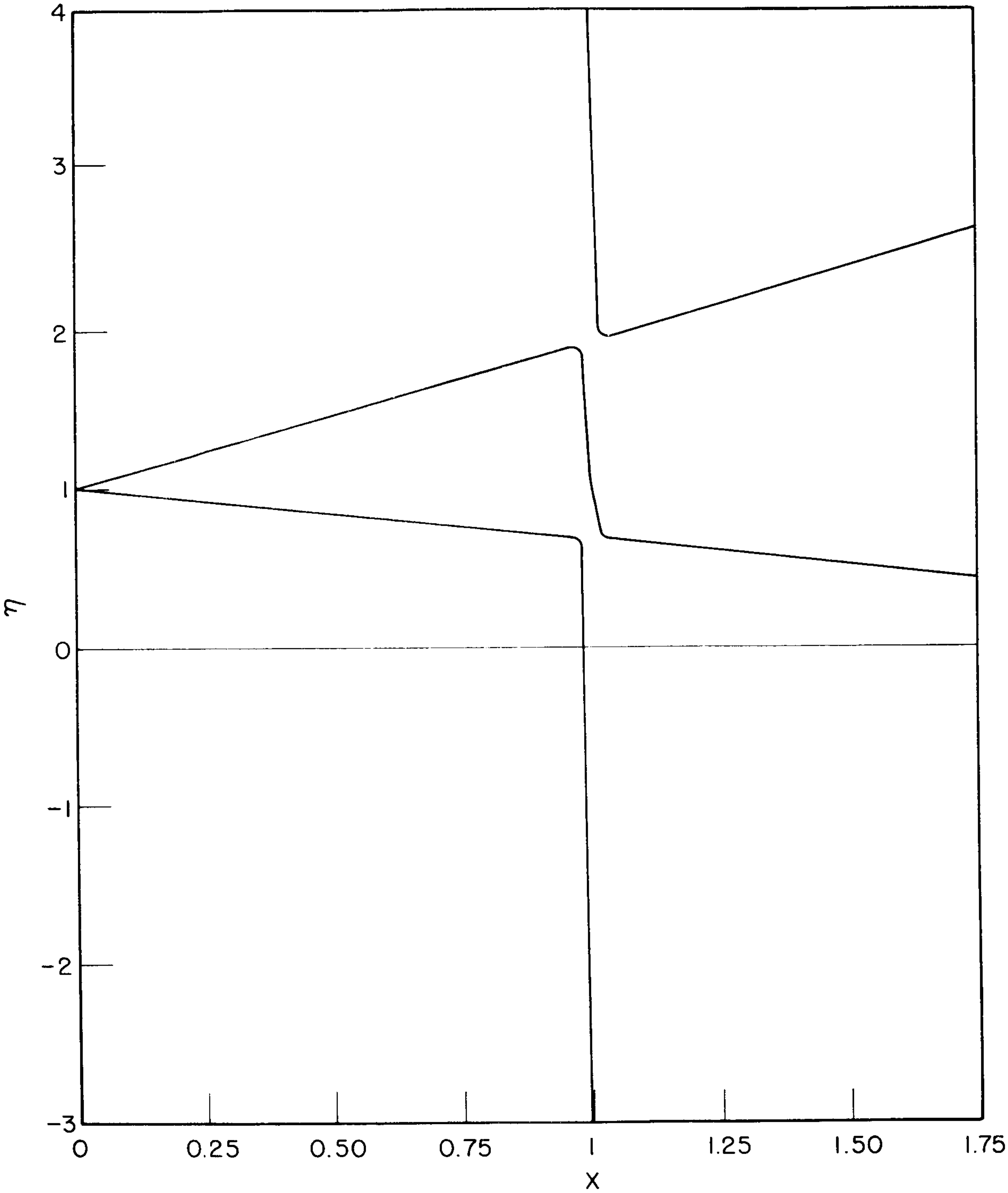

FIG. 1b THE PLOT OF η VS. X FOR $\delta = 0.01$, $\gamma = 0.1$ AND Y = 2.1.

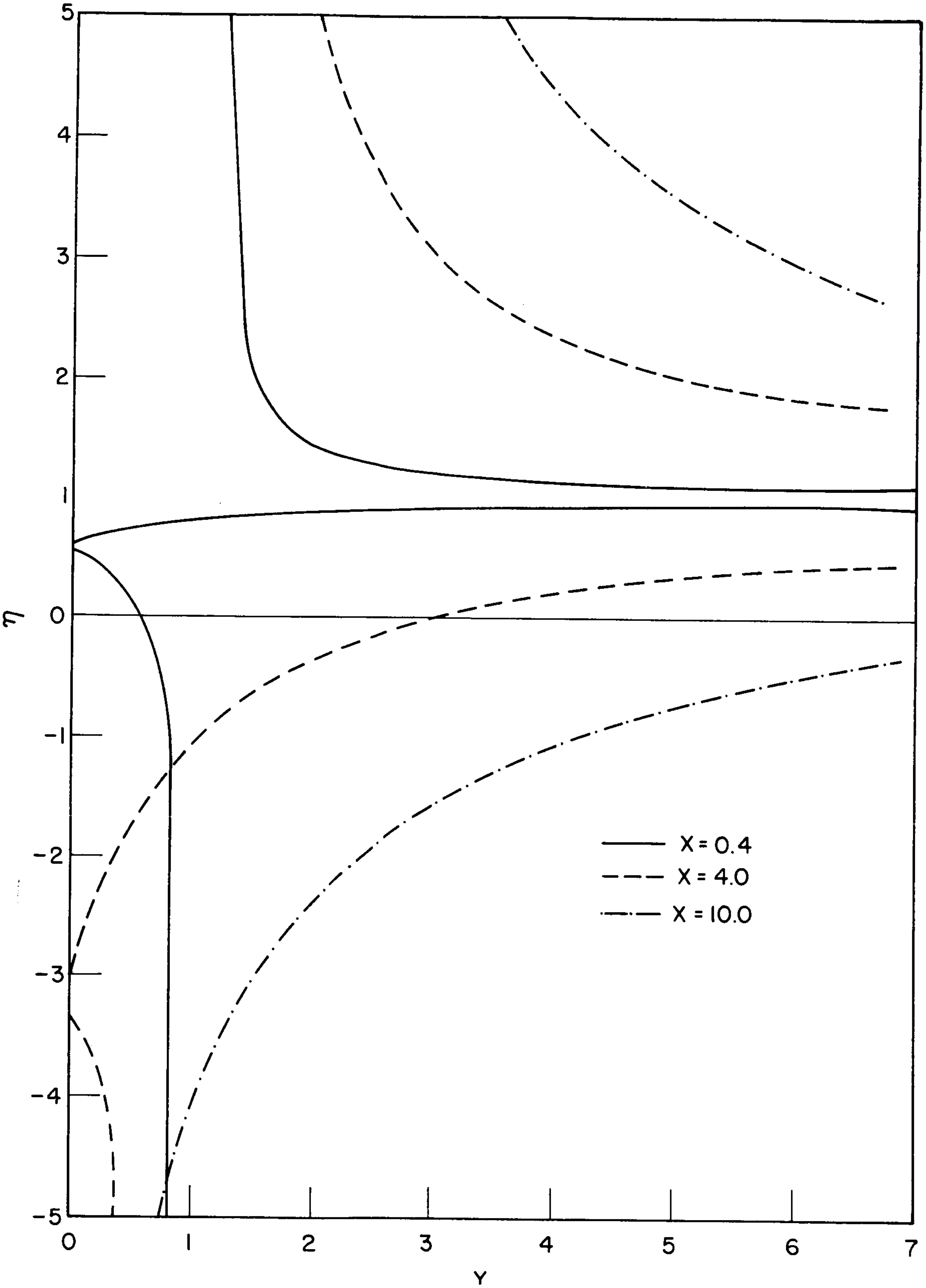

FIG. 2a THE PLOT OF η VS. Y FOR δ = 0.05, γ = 0.1 AND X = 0.4, 4.0, 10.

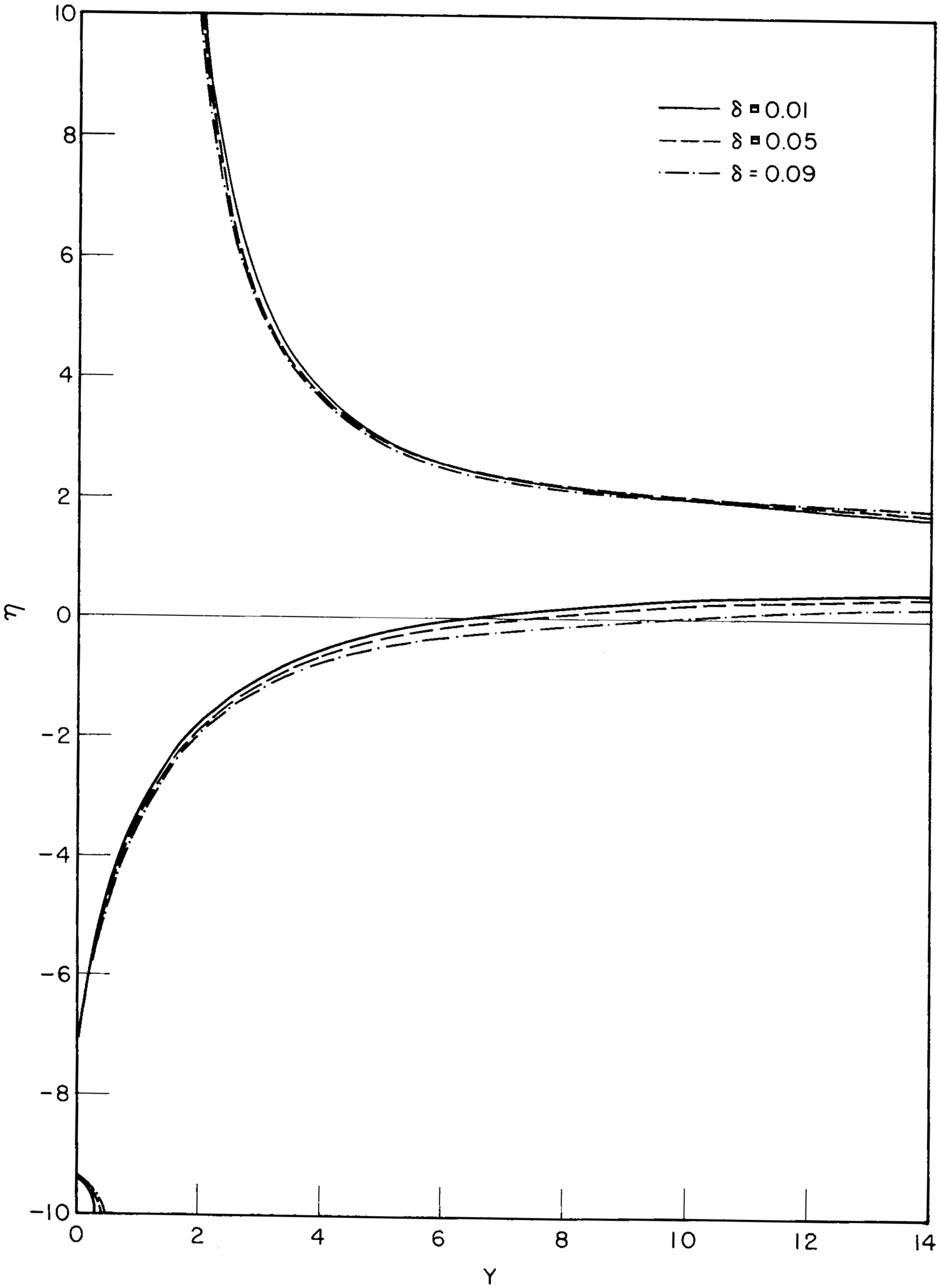

FIG. 2b THE PLOT OF η VS. Y FOR $\gamma = 0.3$, X = 8.0 AND $\delta = 0.01$, 0.05, 0.09.

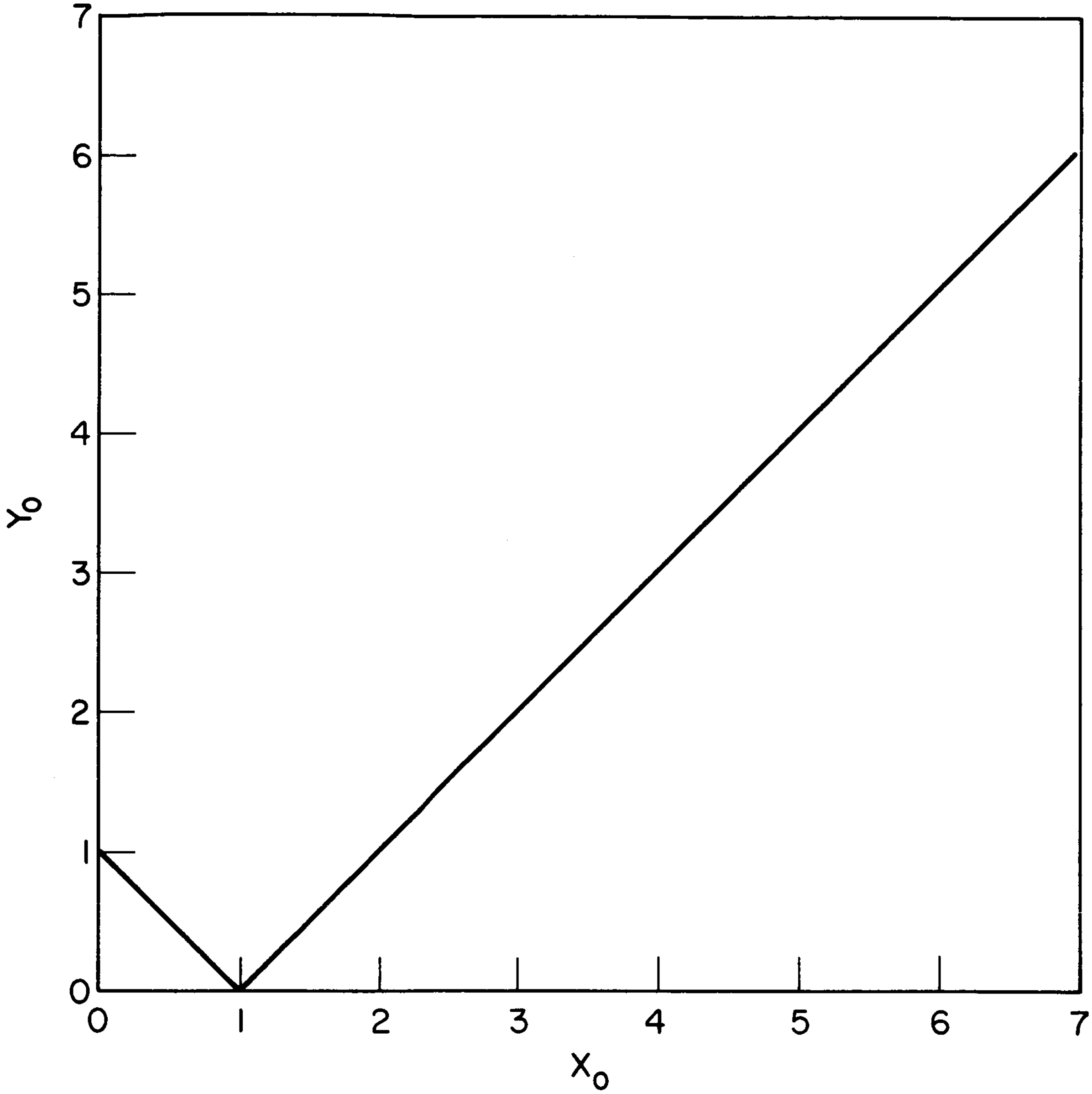

FIG. 3 THE PLOT OF Y VS. X FOR $\delta = 0$, $\gamma = 0$ AND $\eta = 0$.

It should be observed that the dispersion equations for the uncoupled modes are given in Eqs. 49 which are, respectively, for the left-hand circularly polarized wave, the right-hand circularly polarized wave and the longitudinal plasma oscillation. The plot of η vs. X based on Eqs. 49, is shown in Figs. 4.

The plots of η_{o-} vs. X and η_{o+} vs. X are shown in Fig. 4a for the case of $Y < 1$ and in Fig. 4b for the case of $Y > 1$ respectively, where η_{o-} denotes the value of η obtained from Eq. 49a, and η_{o+} denotes that obtained from Eq. 49b. The intersection point between the plot of η_{o-} vs. X and the line $X = 1$ in the η-X plane represents the "coupling point" between the left-hand circularly polarized wave and the plasma oscillation provided that $\eta > 0$. Similarly the intersection point of the plot of η_{o+} vs. X with the line $X = 1$ represents the "coupling point" between the right-hand circularly polarized wave and the longitudinal plasma oscillation. The velocity of the electromagnetic wave at which coupling between the transverse electromagnetic wave and the longitudinal plasma oscillation takes place, i.e., the "coupling velocity", can be determined from the coupling point. On the other hand, from Eqs. 49 or Figs. 4 it is not difficult to see that the coupling point depends upon the parameter γ, which in turn depends on the plasma temperature T. For example, for a given value of $Y < 1$, an increase in γ causes η_{o-} to decrease, which in turn causes an increase in the coupling velocity.

For $Y > 1$ (e.g., see Fig. 4b) an increase of γ causes η_{o-} to decrease so that the coupling velocity increases, while it causes η_{o+} to increase so that the coupling velocity decreases. In view of the fact that

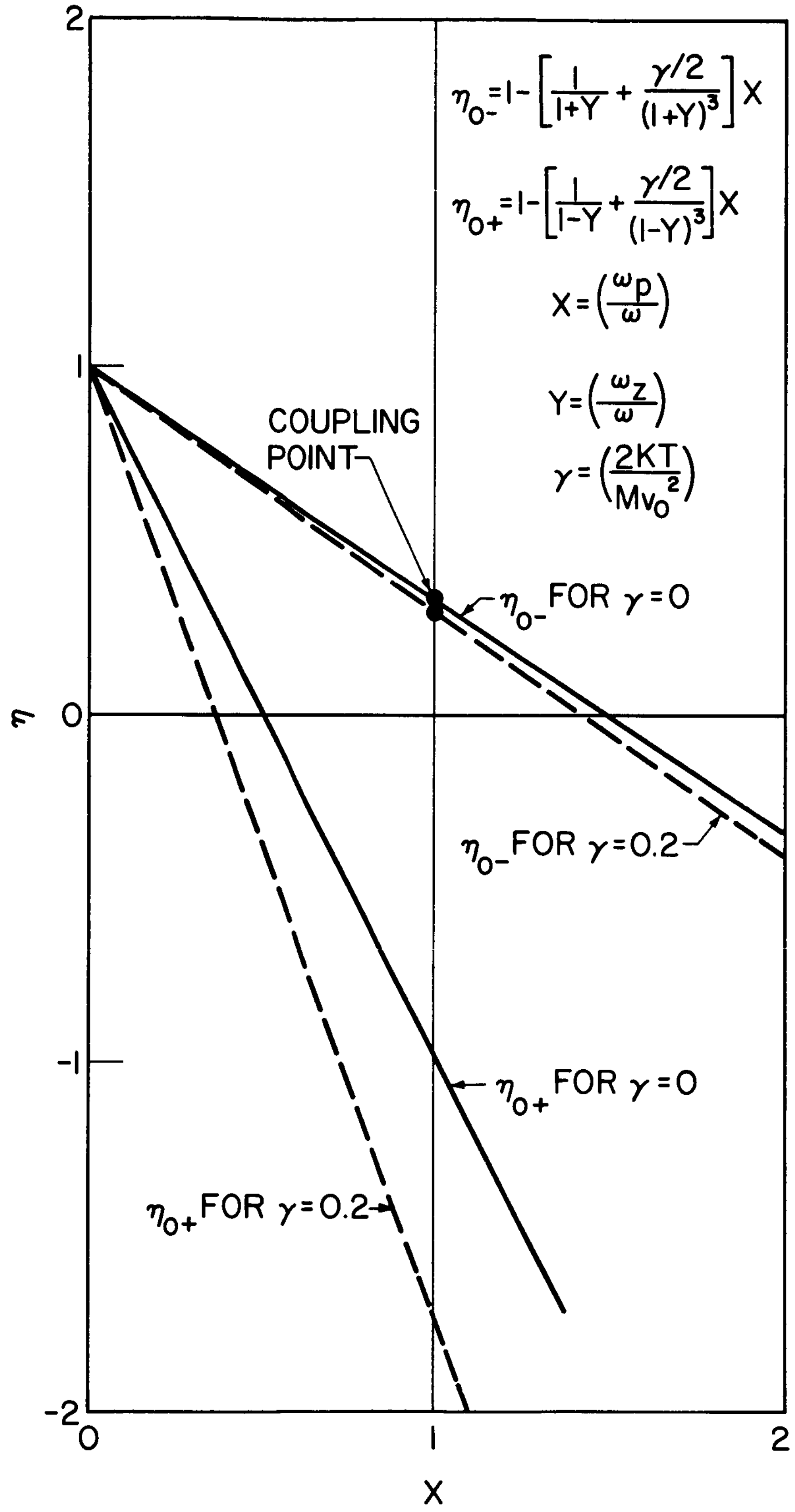

FIG. 4a THE PLOT OF η VS. X FOR $\delta = 0$, $Y = 0.5$, $\gamma = 0$ AND 0.2.

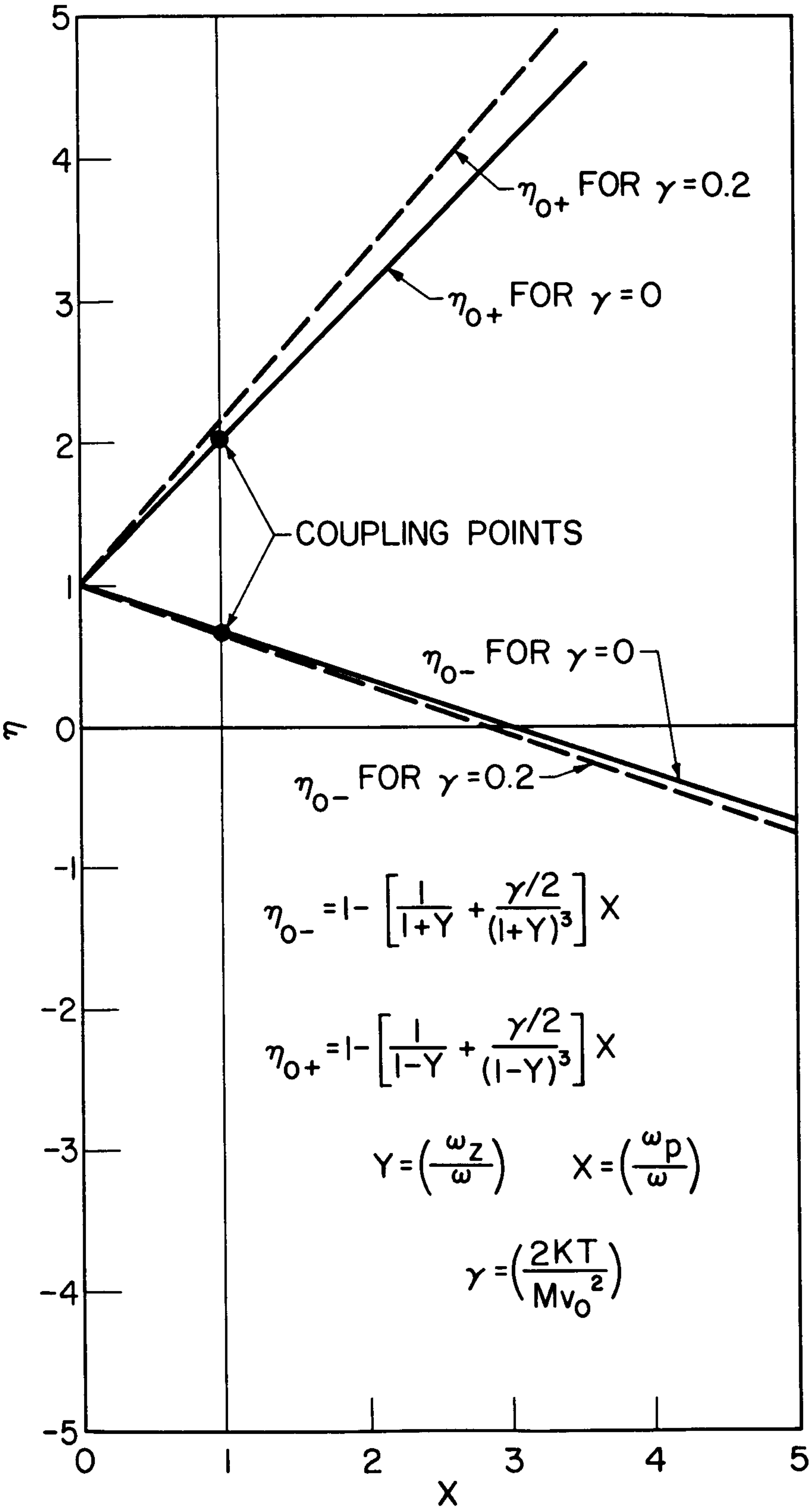

FIG. 4b THE PLOT OF η VS. X FOR $\delta = 0$, $Y = 2$, $\gamma = 0$ AND 0.2.

$\gamma \equiv (1/V^2) = (2KT/mv_o^2)$, where v_o is the phase velocity of the electromagnetic wave, Eqs. 49a and 49c give

$$c^2 = \left(1 - \frac{1}{1 + y_o}\right)v_{o-}^2 - \frac{KT}{m}\,\frac{1}{(1 + y_o)^3} \qquad (50a)$$

and Eqs. 49b and 49c give

$$c^2 = \left(1 - \frac{1}{1 - y_o}\right)v_{o+}^2 - \frac{KT}{m}\,\frac{1}{(1 - y_o)^3}\ , \qquad (50b)$$

where $y_o \equiv (\omega_z/\omega_p)$, and v_{o-} and v_{o+} denote the coupling phase velocity of the left-hand and right-hand circularly polarized waves respectively.

It should be noted that an increase in T causes v_{o-} to increase for $y_o > 0$, and y_{o+} to decrease for the case $y_o > 1$. Thus the plasma temperature appears to have an interesting effect on the coupling velocity of electromagnetic waves under the electrostatic coupling. The term "electrostatic coupling" is introduced here to describe the phenomenon of coupling between the longitudinal wave and the transverse wave in the presence of a transverse static electric field.

On the other hand, since the cutoff of an electromagnetic wave occurs when its propagation constant k becomes zero, the "cutoff condition" for the transverse mode can be obtained by setting both η and γ equal to zero in the derived dispersion relation; Eq. 26, with the aid of Eqs. 37, 38 and condition(42). This condition can be expressed in the following form:

$$\frac{\delta^2 y_o^2}{x^4} = y_o^2 - \left(x - \frac{1}{x}\right)^2\ , \qquad (51)$$

where $x \equiv (\omega_o/\omega_p)$ is the normalized cutoff frequency, and $y_o \equiv (\omega_z/\omega_p)$ is the normalized cyclotron frequency, with ω_o being the cutoff frequency. Once the values of y_o and δ are specified, Eq. 51 can be solved for x, and thus ω_o can be determined. However, the variation of ω_o with respect to δ can be easily observed with the aid of a graphical method illustrated below.

Let $F_1(x)$ be the left-hand side and $F_2(x)$ be the right-hand side of Eq. 51. If F_1 vs. x and F_2 vs. x are plotted in the same plane, as illustrated in Fig. 5, then the intersection of the two plots provides the real root of Eq. 51. Once y_o is given, the curve of $F_2(x)$ is determined, and if δ is also specified, then $F_1(x)$ is also determined. Thus the intersection point of two plots is readily determined. It should be noted that when $\delta = 0$, the F_1-curve coincides with the x-axis, and if its intersections with the F_2-curve are denoted by x_ℓ and x_r, they are given by

$$x_\ell = \frac{-y_o + \sqrt{y_o^2 + 4}}{2} \quad \text{and} \quad x_r = \frac{y_o + \sqrt{y_o^2 + 4}}{2} . \qquad (52)$$

x_ℓ determines $\omega_{o\ell}$, the cutoff frequency of the left-hand circularly polarized wave, and x_r determines ω_{or}, the cutoff frequency of the right-hand circularly polarized wave. It is easily seen from Fig. 5 that an increase of the parameter δ leads to an increase of $\omega_{o\ell}$, but to a slight decrease of ω_{or}.

VI. CONCLUDING REMARKS

In the present report the dispersion relation for a finite temperature two-component plasma subjected to crossed electrostatic and magnetostatic

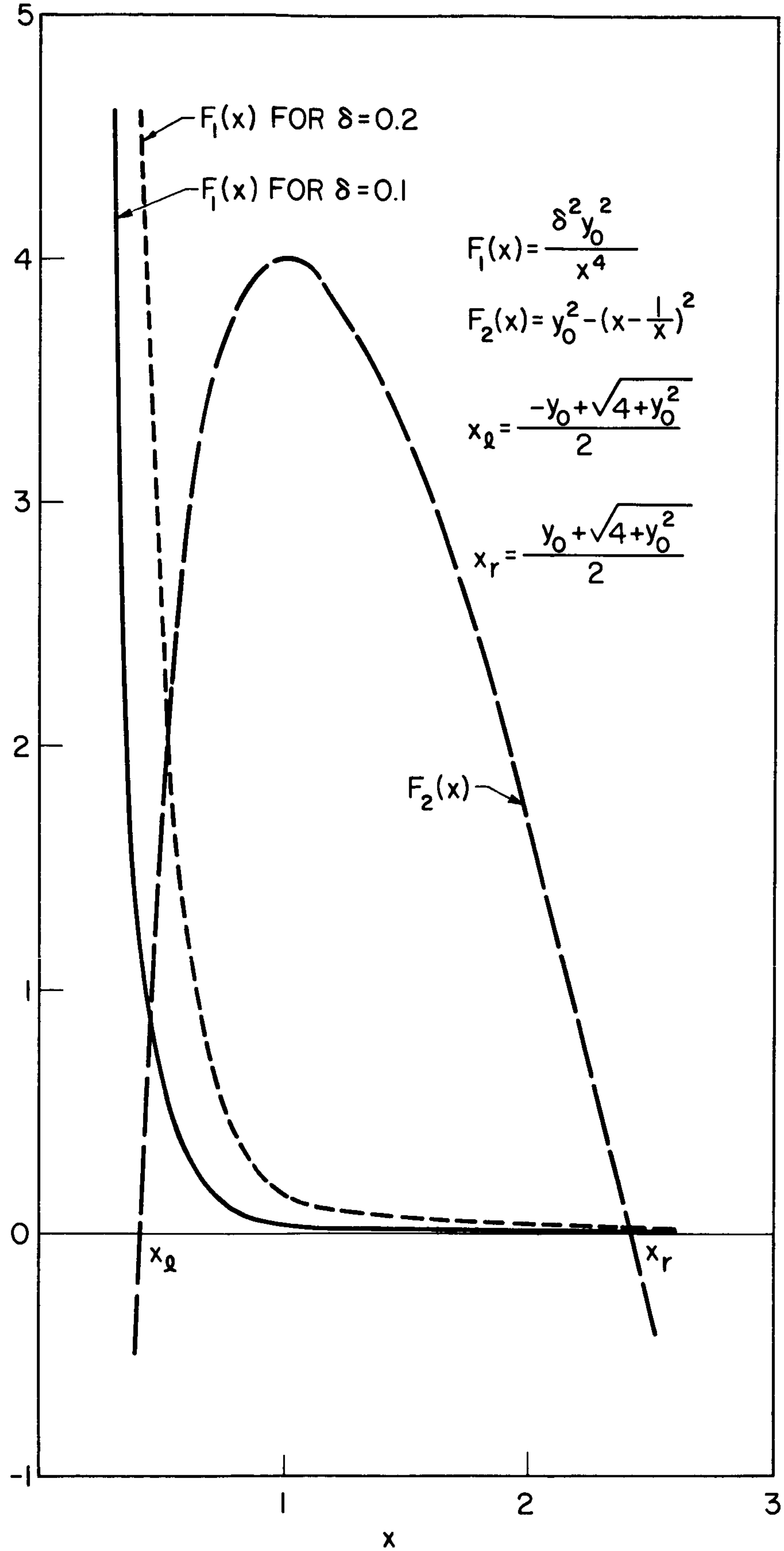

FIG. 5 ILLUSTRATION OF GRAPHICAL SOLUTION OF EQ. 51, AND THE VARIATION OF CUTOFF FREQUENCY WITH THE PARAMETER δ.

fields has been derived using the coupled Boltzmann-Vlasov-Maxwell equations assuming a one-dimensional, small-signal model. The derived dispersion relation is given in a form in which various characteristic modes of the system can be readily identified and the coupling between these characteristic modes can be studied. The investigation of the dispersion relation in Section III clearly shows the possibility of coupling the longitudinal mode to the transverse modes in the presence of transverse applied electrostatic fields.

In order to make a detailed analysis of the derived dispersion relation a knowledge of the time-independent part of the distribution functions for electrons and ions is required. A Maxwellian distribution is considered in detail for the present investigation in Section IV. As an illustration, a detailed analysis of the dispersion relation is carried out in Section V for a homogeneous, electrically neutral electron gas in which the thermal velocity of the electron is taken into account but the ion motion is neglected. In the interests of simplicity, the conditions $\delta \ll 1$ and $\gamma^2 \ll 1$ are imposed in deriving the dispersion relation Eq. 43. The desired information with regard to the propagation characteristics of the transverse electromagnetic wave is provided by Eq. 43 or equivalently by Eq. 45. Upon setting $\eta = 0$ in Eq. 43, the cutoff condition is obtained. The plots of η vs. X and η vs. Y, in general, represent a family of curves in the η-X plane and in the η-Y plane as shown in Figs. 1, 2 and 3. However, when $\delta = 0$, Eq. 45 is reduced to Eq. 47 which represents a family of straight lines for the plot of η vs. X in the η-X plane. It is shown that the presence of an applied transverse electrostatic field in the electron gas has two interesting effects upon the propagation characteristic of transverse circularly polarized electromagnetic waves:

1. It causes the cutoff frequency to shift, e.g., an increase in the parameter δ causes $\omega_{o\ell}$ to increase.

2. It causes the longitudinal plasma oscillation to be coupled to the transverse electromagnetic wave, e.g., an increase in the electron temperature T causes the coupling velocity of the circularly polarized wave to shift (see Section V).

It must be pointed out that the present investigation merely demonstrates the possibility of electrostatic coupling. In order to gain a better understanding of the mechanism of electrostatic coupling it is necessary to investigate in detail the following aspects: (1) energy conversion between the modes, and (2) effectiveness of coupling of the modes. It is intended to carry out this investigation and consider the application of the theory to ionospheric phenomena in a future report. However, it is of interest to note that if the type of coupling mechanism under consideration can be shown to be sufficiently effective, then it will provide a reasonable way of explaining phenomena such as cutoff, amplification and Landau damping of whistler propagation in the ionospheric plasma.

APPENDIX A. VERIFICATION THAT f_o GIVEN BY EQ. 27 IS A SOLUTION OF EQ. 7a.

$$f_o = n_o e^{-\alpha[\{(v_x - u)^2 + v_y^2 + v_z^2\} - \frac{2e}{m}\Phi(z)]} \qquad (27)$$

and

$$v_z \frac{\partial f_o}{\partial z} - \frac{e}{m}(\vec{E}_o + \vec{v} \times \vec{B}_o)\cdot\nabla_v f_o = 0 \;, \qquad (7a)$$

where $\vec{E}_o = \vec{E}_s + \vec{E}_a$, with $\vec{E}_s$ and $\vec{E}_a$ being the space-charge field and the externally applied electrostatic field respectively. For a one-dimensional analysis, $\vec{E}_s = \vec{k}E_s$. Suppose that $\vec{E}_a$ is taken in the positive y-direction and $\vec{B}_o$ is in the positive z-direction, i.e.,

$$\vec{E}_a = \vec{j}E_a \;, \quad \vec{B}_o = \vec{k}B_o \quad \text{and} \quad \vec{E}_s = \vec{k}E_s \;, \qquad (A.1)$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along the x-, y- and z-coordinate axes. Since

$$\nabla_v f_o = -2\alpha\vec{v}f_o + (2\alpha u f_o)\,\vec{i} \;, \quad \frac{\partial f_o}{\partial z} = \frac{2e}{m}\alpha\frac{\partial\Phi}{\partial z}f_o \;,$$

$$(\vec{v} \times \vec{B}_o)\cdot\nabla_v f_o = (2\alpha u f_o)[\vec{i}\cdot(\vec{v} \times \vec{B}_o)] = (2\alpha u f_o)(v_y B_o) \;,$$

$$\vec{E}_o\cdot\nabla_v f_o = (\vec{j}E_a + \vec{k}E_s)\cdot(-2\alpha f_o\vec{v} + 2\alpha u f_o\vec{i}) = -2\alpha f_o v_y E_a - 2\alpha f_o v_y E_s \;, \qquad (A.2)$$

Eq. 7a becomes

$$v_z \frac{\partial f_o}{\partial z} - \frac{e}{m}\left(- 2\alpha v_z f_o E_s - 2\alpha f_o v_y E_a + 2\alpha f_o v_y u B_o\right)$$

$$= v_z \left[\frac{2\alpha e}{m} f_o \left(\frac{\partial \Phi}{\partial z} + E_s\right)\right] + \frac{2\alpha e}{m} v_y f_o (E_a - uB_o) = 0$$

since $(\partial\Phi/\partial z) = - E_s$ and $E_a = uB_o$.

APPENDIX B. DERIVATION OF VARIOUS EQUATIONS

B.1 Derivation of Eqs. 29: (Determination of $R_{p,q}$)

Suppose that the time-independent distribution functions $F_o \equiv f_{o1}$ and $f_o \equiv f_{o2}$ are given as

$$f_{oq} = n_q \left(\frac{\alpha_q}{\pi}\right)^{3/2} e^{-\alpha_q[(v_x-u)^2 + v_y^2 + v_z^2]} \quad , \quad \text{for } q = 1, 2 \quad , \tag{B.1}$$

where $\alpha_q \equiv (m_q/2KT_q)$ in which $m_1 \equiv M$, $T_1 \equiv T_i$, $m_2 \equiv m$ and $T_2 \equiv T_e$ with subscripts 1 and 2 denoting the quantities associated with the ion and the electron respectively. n_q is the number density of the particle. Let

$$v_x = v_\perp \cos\varphi \quad \text{and} \quad v_y = v_\perp \sin\varphi \quad . \tag{B.2}$$

Then

$$f_{oq} = w_q \psi_q e^{\lambda_q \cos\varphi} \quad , \tag{B.3}$$

where

$$w_q \equiv n_q \left(\frac{\alpha_q}{\pi}\right)^{3/2} e^{-\alpha_q u^2} \quad ,$$

$$\psi_q \equiv e^{-\alpha_q (v_\perp^2 + v_z^2)} \quad ,$$

$$\lambda_q \equiv (2\alpha_q u v_\perp) \quad , \text{ for } q = 1, 2 \quad ,$$

$$M_\pm(f_{oq}) = \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_{oq}}{\partial v_\perp} \pm \frac{j}{v_\perp} \frac{\partial f_{oq}}{\partial \varphi}\right) + \frac{kv_\perp}{\omega} \frac{\partial f_o}{\partial v_z}\right] \quad ,$$

$$\frac{\partial f_{oq}}{\partial v_\perp} = -2\alpha_q (v_\perp - u\cos\varphi) f_{oq} \quad ,$$

$$\frac{\partial f_{oq}}{\partial \varphi} = -\lambda_q \sin\varphi f_{oq} = -(2\alpha_q u v_\perp)\sin\varphi f_{oq} \quad ,$$

$$\frac{\partial f_{oq}}{\partial v_z} = -2\alpha_q v_z f_{oq} \quad ,$$

$$M_\pm(f_{oq}) = -2\alpha_q v_\perp f_{oq} + 2\alpha_q u f_{oq}\left(1 - \frac{kv_z}{\omega}\right) e^{\mp j\varphi} \quad ,$$

$$\frac{\partial}{\partial v_\perp}\left(\frac{\partial f_{oq}}{\partial v_z}\right) = 4\alpha_q^2 v_z (v_\perp - u\cos\varphi) f_{oq} \quad . \qquad \text{(B.4)}$$

B.2 Evaluation of S_{pq} from Eqs. 25

$$S_{1\ell} = \sum_{q=1,2} \int_0^{2\pi} \left(K^q_{1\ell} + K^q_{2\ell}\, e^{-j2\varphi} + K^q_{3\ell}\, e^{-j\varphi} \right) d\varphi \quad ,$$

$$S_{2p} = \sum_{q=1,2} \int_0^{2\pi} \left(K_{1p}\, e^{j2\varphi} + K_{2p} + K_{3p}\, e^{j\varphi} \right) d\varphi \quad ,$$

$$S_{3r} = \sum_{q=1,2} \int_0^{2\pi} \left(K_{1r}\, e^{j\varphi} + K_{2r}\, e^{-j\varphi} + K_{3r} \right) d\varphi \quad . \qquad (B.5)$$

for ℓ = 1, 2, 3; p = 1, 2, 3; and r = 1, 2, 3, where

$$K^q_{11} = \frac{j\eta_q M_-(f_o q)}{(b + \omega_{zq})} \quad , \quad K^q_{12} = 0 \quad , \quad K^q_{13} = \frac{\eta_q a^q_- \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_{oq}}{\partial v_z}\right)}{b(b + \omega_{zq})} \quad ,$$

$$K^q_{21} = 0 \quad , \quad K^q_{22} = \frac{j\eta_q M_+(f_{oq})}{(b - \omega_{zq})} \quad , \quad K^q_{23} = \frac{\eta_q a^q_+ \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_{oq}}{\partial v_z}\right)}{b(b - \omega_{zq})} \quad ,$$

$$K^q_{31} = \frac{2\eta_q \frac{a^q_+}{v_\perp} M_-(f_{oq})}{b(b + \omega_{zq})} \quad , \quad K^q_{32} = \frac{2\eta_q \frac{a^q_-}{v_\perp} M_+(f_{oq})}{b(b - \omega_{zq})} \quad ,$$

$$K^q_{33} = \frac{j\eta_q \frac{\partial f_{oq}}{\partial z}}{b} - j\,\frac{4a^q_- a^q_+}{v_\perp}\,\frac{\eta_q \frac{\partial}{\partial v_\perp}\left(\frac{\partial f_{oq}}{\partial v_z}\right)}{b(b^2 - \omega^2_{zq})} \quad , \qquad (B.6)$$

in which $\eta_q \equiv e/m_q$, $\omega_{zq} \equiv eB_{oz}/m_q$, and $a^q_\pm = (1/2)\ (a^q_x \pm j a^q_y)$, i.e.,

$$\eta_1 = \frac{e}{M} , \quad \eta_2 = \frac{e}{m} , \quad \omega_{z1} = \left(\frac{-eB_o}{M}\right) \equiv \Omega_z , \quad \omega_{z2} = \left(\frac{eB_o}{m}\right) \equiv \omega_z ,$$

$$a^1_{\pm} = \frac{1}{2}(a_x \pm ja_y) , \quad a^2_{\pm} = \frac{1}{2}(A_x \pm jA_y) .$$

Summation of $\sum_q$ in Eqs. B.5 is taken over both species of the constant. Substituting Eqs. A.4 into Eqs. A.6 gives

$$K^q_{11} = (C^q_1 + D^q_1 e^{j\varphi})e^{\lambda_q \cos\varphi} , \quad K^q_{21} = 0 ,$$

$$K^q_{31} = (C^q_2 + D^q_2 e^{j\varphi})e^{\lambda_q \cos\varphi} , \quad K^q_{12} = 0 ,$$

$$K^q_{22} = (C^q_3 + D^q_3 e^{-j\varphi})e^{\lambda_q \cos\varphi} , \quad K^q_{32} = (C^q_4 + D^q_4 e^{-j\varphi})e^{\lambda_q \cos\varphi} ,$$

$$K^q_{13} = (G^q_1 + H^q_1 e^{j\varphi} + H^q_1 e^{-j\varphi})e^{\lambda_q \cos\varphi} ,$$

$$K^q_{23} = (G^q_2 + H^q_2 e^{j\varphi} + H^q_2 e^{-j\varphi})e^{\lambda_q \cos\varphi} ,$$

$$K^q_{33} = (G^q_o + G^q_3 + H^q_3 e^{j\varphi} + H^q_3 e^{-j\varphi})e^{\lambda_q \cos\varphi} , \qquad (B.7)$$

where

$$C_1^q \equiv \frac{j\eta_q C_q}{(b + \omega_{zq})} , \quad D_1^q \equiv \frac{j\eta_q D_q}{(b + \omega_{zq})} , \quad C_2^q \equiv \frac{2\eta_q \frac{a_+^q}{v_\perp} C_q}{b(b + \omega_{zq})} ,$$

$$D_2^q \equiv \frac{2\eta_q \frac{a_+^q}{v_\perp} D_q}{b(b + \omega_{zq})} , \quad C_3^q \equiv \frac{j\eta_q C_q}{(b - \omega_{zq})} , \quad D_3^q \equiv \frac{j\eta_q D_q}{(b - \omega_{zq})} ,$$

$$C_4^q \equiv \frac{2\eta_q \frac{a_-^q}{v_\perp} C_q}{b(b - \omega_{zq})} , \quad D_4^q \equiv \frac{2\eta_q \frac{a_-^q}{v_\perp} D_q}{b(b - \omega_{zq})} , \quad G_1^q \equiv \frac{\eta_q a_-^q G_q}{b(b + \omega_{zq})} ,$$

$$H_1^q \equiv \frac{\eta_q a_-^q H_q}{b(b + \omega_{zq})} , \quad G_2^q \equiv \frac{\eta_q a_+^q G_q}{b(b - \omega_{zq})} , \quad H_2^q \equiv \frac{\eta_q a_+^q H_q}{b(b - \omega_{zq})} ,$$

$$G_3^q \equiv \frac{- j4a_-^q a_+^q \eta_q G_q}{v_\perp b(b^2 - \omega_{zq}^2)} , \quad H_3^q \equiv \frac{- j4a_-^q a_+^q \eta_q H_q}{v_\perp b(b^2 - \omega_{zq}^2)} ,$$

$$G_0^q \equiv \frac{- j2\alpha_q \eta_q w_q v_z \psi_q}{b} , \tag{B.8}$$

in which $C_q \equiv - 2\alpha_q v_\perp w_q \psi_q$, $D_q \equiv 2\, \alpha_q u w_q \psi_q \left(1 - \frac{kv_z}{\omega}\right)$,

$$G_q \equiv 4\alpha_q^2 v_z v_\perp w_q \psi_q , \quad H_q \equiv - 2\alpha_q^2 v_z u w_q \psi_q .$$

Substituting Eqs. A.7 into Eqs. A.5 yields

$$S_{11} = \sum \int_0^{2\pi} [(C_1 + D_2) + D_1 e^{j\varphi} + C_2 e^{-j\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{12} = \sum \int_0^{2\pi} [C_4 e^{-j\varphi} + (C_3 + D_4) e^{-j2\varphi} + D_3 e^{-j3\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{13} = \sum \int_0^{2\pi} [(G_1 + H_3) + H_1 e^{j\varphi} + (H_1 + H_2 + G_0 + G_3) e^{-j\varphi}$$

$$+ (G_2 + H_3) e^{-j2\varphi} + H_2 e^{-j3\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{21} = \sum \int_0^{2\pi} [C_2 e^{j\varphi} + (C_1 + D_2) e^{j2\varphi} + D_1 e^{j3\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{22} = \sum \int_0^{2\pi} [(C_3 + D_4) + C_4 e^{j\varphi} + D_3 e^{-j\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{23} = \sum \int_0^{2\pi} [(G_2 + H_3) + (H_1 + H_2 + G_0 + G_3) e^{j\varphi} + H_2 e^{-j\varphi}$$

$$+ (G_1 + H_3) e^{j2\varphi} + H_1 e^{j3\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{31} = \sum \int_0^{2\pi} [C_2 + (C_1 + D_2) e^{j\varphi} + D_1 e^{j2\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

$$S_{32} = \sum \int_0^{2\pi} [C_4 + (C_3 + D_4) e^{-j\varphi} + D_3 e^{-j2\varphi}] e^{\lambda\cos\varphi} d\varphi \ ,$$

(Eqs. B.9 cont.)

$$S_{33} = \sum' \int_0^{2\pi} [(H_1 + H_2 + G_0 + G_3) + (G_1 + H_3)e^{j\varphi} + (G_2 + H_3)e^{-j\varphi}$$

$$+ H_1 e^{j2\varphi} + H_2 e^{-j2\varphi}]e^{\lambda\cos\varphi} d\varphi \quad . \qquad \text{(B.9)}$$

In the above the summation, sigma is introduced to indicate the fact that the summation is over both species (electrons and ions). The subscript q associated with the coefficients C, D, G and H is omitted here for convenience; however their dependence on the type of particles is understood.

Integration with respect to φ can be carried out with the aid of the following relation[7]:

$$e^{\lambda\cos\varphi} = \sum_{n=-\infty}^{\infty}{}' I_n(\lambda)e^{jn\varphi} \quad , \qquad \text{(B.10)}$$

where $I_n(\lambda)$ is the nth order modified Bessel function of the first kind. Furthermore using the following identities[7]:

$$I_{-n}(\lambda) = I_n(\lambda) \quad ,$$

$$\frac{2n}{\lambda} I_n(\lambda) = I_{n-1}(\lambda) - I_{n+1}(\lambda) \quad , \qquad \text{(B.11)}$$

the functions $S_{pq}(v_z,v_\perp)$, q,p = 1, 2, 3, can be expressed as

$$S_{11} = 2\pi \sum [(C_1 + D_2)I_0(\lambda) + (C_2 + D_1)I_1(\lambda)] ,$$

$$S_{12} = 2\pi \sum \left\{ \left[(C_3 + D_4) - \frac{4}{\lambda} D_3\right] I_0(\lambda) + \left[C_4 - \frac{2}{\lambda}(C_3 + D_4) + \left(1 + \frac{8}{\lambda^2}\right) D_3\right] \cdot I_1(\lambda) \right\} ,$$

$$S_{13} = 2\pi \sum \left\{ \left[(G_1 + G_2 + 2H_3) - \frac{4}{\lambda} H_2\right] I_0(\lambda) + \left[(2H_1 + H_2 + G_0 + G_3) - \frac{2}{\lambda}(G_2 + H_3) + \left(1 + \frac{8}{\lambda^2}\right) H_2\right] I_1(\lambda) \right\} ,$$

$$S_{21} = 2\pi \sum \left\{ \left((C_1 + D_2) - \frac{4}{\lambda} D_1\right) I_0(\lambda) + \left[C_2 - \frac{2}{\lambda}(C_1 + D_2) + \left(1 + \frac{8}{\lambda^2}\right) D_1\right] \cdot I_1(\lambda) \right\} ,$$

$$S_{22} = 2\pi \sum [(C_3 + D_4)I_0(\lambda) + (C_4 + D_3)I_1(\lambda)] ,$$

$$S_{23} = 2\pi \sum \left\{ \left[(G_1 + G_2 + 2H_3) - \frac{4}{\lambda} H_1\right] I_0(\lambda) + \left[(H_1 + 2H_2 + G_0 + G_3) - \frac{2}{\lambda}(G_1 + H_3) + \left(1 + \frac{8}{\lambda^2}\right) H_1\right] I_1(\lambda) \right\} ,$$

$$S_{31} = 2\pi \sum \left[(C_2 + D_1)I_0(\lambda) + \left((C_1 + D_2) - \frac{2}{\lambda} D_1\right) I_1(\lambda)\right] ,$$

(Eqs. B.12 cont.)

$$S_{32} = 2\pi \sum \left[(C_4 + D_3) I_0(\lambda) + \left((C_3 + D_4) - \frac{2}{\lambda} D_3 \right) I_1(\lambda) \right] ,$$

$$S_{33} = 2\pi \sum \left[(G_0 + G_3 + 2H_1 + 2H_2) I_0(\lambda) \right.$$

$$\left. + \left((G_1 + G_2 + 2H_3) - \frac{2}{\lambda} (H_1 + H_2) \right) I_1(\lambda) \right] . \quad \text{(B.12)}$$

The determination of R_{pq} involves the evaluation of the following integration:

$$r_{pq} \equiv \int_{-\infty}^{\infty} \int_{0}^{\infty} S_{pq} v_r^2 dv_r dv_z \quad ; \quad p = 1, 2, \quad q = 1, 2, 3 \ ,$$

$$\equiv \int_{-\infty}^{\infty} \int_{0}^{\infty} S_{pq} v_r v_z dv_r dv_z \quad ; \quad p = 3, \quad q = 1, 2, 3 \ , \quad \text{(B.13)}$$

which in turn involves the integration:

$$\tau_{pq} \equiv \int_{0}^{\infty} I_p(\lambda) v_r^q e^{-\alpha v_r^2} dv_r$$

$$\xi_q(\chi) \equiv \int_{-\infty}^{\infty} \chi v_z^q e^{-\alpha v_z^2} dv_z \ . \quad \text{(B.14)}$$

To facilitate the calculation, coefficients C, D, G and H can be written in the following more convenient form;

$$C_1 = -j(Z_+\Psi)(v_r\Psi_r) \quad ,$$

$$D_1 = ju\left(1 - \frac{kv_z}{\omega}\right)(Z_+\Psi_z)\Psi_r \quad ,$$

$$C_3 = -j(Z_-\Psi_z)(v_r\Psi_r) \quad ,$$

$$D_3 = ju\left(1 - \frac{kv_z}{\omega}\right)(Z_-\Psi_z)\Psi_r \quad ,$$

$$C_4 = ju(Z_- - Z_o)\Psi_z\Psi_r \quad ,$$

$$D_4 = -juY(Z_-\Psi_z)\left(\frac{u}{v_r}\Psi_r\right) \quad ,$$

$$G_1 = j\alpha u[(Z_+ - Z_o)v_z\Psi_z](v_r\Psi_r) \quad ,$$

$$H_1 = j\,\frac{\alpha u^2}{2}\,[(Z_o - Z_+)v_z\Psi_z]\Psi_r \quad ,$$

$$G_2 = j\alpha u[(Z_- - Z_o)v_z\Psi_z](v_r\Psi_r) \quad ,$$

$$H_2 = -j\,\frac{\alpha u^2}{2}\,[(Z_- - Z_o)v_z\Psi_z]\Psi_r \quad ,$$

$$G_3 = -j\alpha u^2[(Z_- + Z_+ - 2Z_o)v_z\Psi_z]\Psi_r \quad ,$$

$$H_3 = j\,\frac{\alpha u^2}{2}\,[(Z_- + Z_+ - 2Z_o)v_z\Psi_z]\left(\frac{u}{v_r}\Psi_r\right) \quad ,$$

$$G_o = -j(Z_o v_z\Psi_z)\Psi_r \quad , \qquad \text{(B.15)}$$

where

$$\Psi_z \equiv e^{-\alpha v_z^2} \quad , \quad \Psi_r \equiv e^{-\alpha v_r^2} \quad ,$$

$$Z_\pm(v_z) \equiv \frac{\sigma}{(b \pm \omega_z)} \quad , \quad Z_o(v_z) \equiv \frac{\sigma}{b}$$

with

$$b \equiv (\omega - kv_z) \quad , \quad \sigma \equiv 2\alpha_q \eta_q w_q$$

and the following facts are used:

$$\frac{\sigma}{b(b \mp \omega_z)} = \frac{\mp 1}{\omega_z} (Z_\pm - Z_o) \quad ,$$

$$\frac{\sigma}{b(b^2 - \omega_z^2)} = \frac{1}{2\omega_z^2} (Z_- + Z_+ - 2Z_o) \quad ,$$

$$\left(\frac{a_+}{\omega_z} \right) = j\,\frac{u}{2} \quad , \quad \left(\frac{a_-}{\omega_z} \right) = -j\,\frac{u}{2} \quad , \quad u \equiv \left(\frac{E_a}{B_o} \right) \quad , \qquad \text{(B.16)}$$

since in the present study it is considered that $E_{ox} = 0$ and $E_{oy} = E_a$. By substituting Eqs. B.15 into Eqs. B.12 and then carrying out the integration (A.13), r_{pq} can be obtained as

$$r_{11} = j2\pi \sum \Big(-u\tau_{12}\xi_o(Z_o) + (Yu^2\tau_{0\text{[illegible]}} - \tau_{03} + 2u\tau_{12})\xi_o(Z_+)$$

$$- \frac{ku}{\omega}\tau_{12}\xi_1(Z_+) \Big) \quad , \qquad \text{(B.17a)}$$

$$r_{12} = j2\pi \sum \left\{ -u\tau_{12}\xi_o(Z_o) - \left[\left(Yu^2 + \frac{2}{\alpha}\right)\tau_{o1} + \tau_{o3} - \left(\frac{Yu}{\alpha} + \frac{2}{\alpha^2 u}\right)\tau_{10} \right.\right.$$

$$\left.\left. - \left(2u + \frac{1}{\alpha u}\right)\tau_{12}\right]\xi_o(Z_-) + \frac{k}{\omega}\left(\frac{2\tau_{o1}}{\alpha} - u\tau_{12} - \frac{2\tau_{10}}{\alpha^2 u}\right)\xi_1(Z_-)\right\} , \quad \text{(B.17b)}$$

$$r_{13} = j2\pi \sum \left\{ \left[-(1 + 2\alpha u^2)u\tau_{o1} - 2\alpha u\tau_{o3} + \left(u^2 + \frac{1}{\alpha}\right)\tau_{10} \right.\right.$$

$$\left. + 4\alpha u^2\tau_{12}\right]\xi_1(Z_o) + \left(\alpha u^3\tau_{o1} + \alpha^3 u\tau_{o3} - \frac{u^2}{2}\tau_{10} - 2\alpha u^2\tau_{12}\right)\xi_1(Z_+)$$

$$\left. + \left[(1 + \alpha u^2)u\tau_{o1} + \alpha u\tau_{o3} - \left(\frac{u^2}{2} + \frac{1}{\alpha}\right)\tau_{10} - (1 + 2\alpha u^2)\tau_{12}\right]\xi_1(Z_-)\right\} ,$$

(B.17c)

$$r_{21} = j2\pi \sum \left\{ -u\tau_{12}\xi_o(Z_o) + \left[\left(Yu^2 - \frac{2}{\alpha}\right)\tau_{o1} - \tau_{o3} + \left(\frac{2}{\alpha^2 u} - \frac{Yu}{\alpha}\right)\tau_{10} \right.\right.$$

$$\left.\left. + \left(2u + \frac{1}{\alpha u}\right)\tau_{12}\right]\xi_o(Z_+) + \frac{k}{\omega}\left(\frac{2}{\alpha}\tau_{o1} - u\tau_{12} - \frac{2}{\alpha^2 u}\tau_{10}\right)\xi_1(Z_+)\right\} ,$$

(B.17d)

$$r_{22} = j2\pi \sum \left(-u\tau_{12}\xi_o(Z_o) - (Yu^2\tau_{o1} + \tau_{o3} - 2u\tau_{12})\xi_o(Z_-) \right.$$

$$\left. - \frac{k}{\omega}u\tau_{12}\xi_1(Z_-)\right) , \quad \text{(B.17e)}$$

$$r_{23} = j2\pi \sum \left\{ \left[-(2\alpha u^2 + 1)u\tau_{01} - 2\alpha u\tau_{03} + \left(\frac{1}{\alpha} + u^2\right)\tau_{10} + 4\alpha u^2\tau_{12}\right] \xi_1(Z_o) \right.$$

$$+ \left[(\alpha u^2 + 1)u\tau_{01} + \alpha u\tau_{03} - \left(\frac{u^2}{2} + \frac{1}{\alpha}\right)\tau_{10} - (2\alpha u^2 + 1)\tau_{12}\right] \xi_1(Z_+)$$

$$\left. + \left(\alpha u^3\tau_{01} + \alpha u\tau_{03} - \frac{u^2}{2}\tau_{10} - 2\alpha u^2\tau_{12}\right) \xi_1(Z_-) \right\} , \qquad \text{(B.17f)}$$

$$r_{31} = j2\pi \sum \left\{ -u\tau_{01}\xi_1(Z_o) + \left[2u\tau_{01} + \left(Yu^2 - \frac{1}{\alpha}\right)\tau_{10} - \tau_{12}\right] \xi_1(Z_+) \right.$$

$$\left. - \frac{k}{\omega}\left(u\tau_{01} - \frac{1}{\alpha}\tau_{10}\right) \xi_2(Z_+) \right\} , \qquad \text{(B.17g)}$$

$$r_{32} = j2\pi \sum \left\{ -u\tau_{01}\xi_1(Z_o) + \left[2u\tau_{01} - \left(Yu^2 + \frac{1}{\alpha}\right)\tau_{10} - \tau_{12}\right] \xi_1(Z_-) \right.$$

$$\left. - \frac{k}{\omega}\left(u\tau_{01} - \frac{\tau_{10}}{\alpha}\right) \xi_2(Z_-) \right\} , \qquad \text{(B.17h)}$$

$$r_{33} = j2\pi \sum \left\{ \left((4\alpha u^2 - 1)\tau_{01} - (2\alpha u^2 + 1)u\tau_{10} - 2\alpha u\tau_{12}\right) \xi_2(Z_o) \right.$$

$$+ \left[-2\alpha u^2\tau_{01} + \left(\alpha u^2 + \frac{1}{2}\right)u\tau_{10} + \alpha u\tau_{12}\right] \xi_2(Z_+)$$

$$\left. + \left[-2\alpha u^2\tau_{01} + \left(\alpha u^2 + \frac{1}{2}\right)u\tau_{10} + \alpha u\tau_{12}\right] \xi_2(Z_-) \right\} . \qquad \text{(B.17i)}$$

It should be noted that $\xi_q(Z_\pm)$ can be written as

$$\xi_q(Z_\pm) = j\gamma_o G_q(U_\pm) \quad , \quad q = 0, 1, 2 \; ,$$

and

$$\xi_q(Z_o) = j\gamma_o G_q(U_o) \quad , \qquad \text{(B.18)}$$

where

$$\gamma_o \equiv \frac{\sigma\sqrt{\pi}}{k} \;, \; U_\pm = \left(\frac{\omega \pm \omega_z}{k}\right) \;,$$

$$G_q(X) \equiv \frac{j}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{v_z^q \, e^{-\alpha v_z^2}}{(v_z - X)} \, dv_z \;, \qquad \text{(B.19)}$$

in which X may be complex in general. The integral (B.19) has been discussed in detail by Stix[6]. It is not difficult to show that

$$G_1(X) = \left(\frac{j}{\sqrt{\alpha}} + XG_o(X)\right) \;,$$

$$G_2(X) = \left(\frac{jX}{\sqrt{\alpha}} + X^2G_o(X)\right) \;. \qquad \text{(B.20)}$$

Furthermore, by defining the parameters Y and U_o as

$$Y \equiv \left(\frac{\omega_z}{\omega}\right) \text{ and } U_o \equiv \left(\frac{\omega}{k}\right) \;, \qquad \text{(B.21)}$$

one has

$$\left(1 - \frac{kU_-}{\omega}\right) = Y \text{ and } \left(1 - \frac{kU_+}{\omega}\right) = -Y \;. \qquad \text{(B.22)}$$

Using Eqs. B.18, B.20 and B.22, r_{pq} can be expressed in terms of $G_o(U_o)$ and $G_o(U_\pm)$ as

$$r_{11} = -2\pi\gamma_o \sum \Big\{ -j\frac{1}{\sqrt{\alpha}} \frac{k}{\omega} u\tau_{12} - u\tau_{12} G_o(U_o) + \Big[Yu^2\tau_{01} - \tau_{03}$$

$$+ (1 - Y)u\tau_{12} \Big] G_o(U_+) \Big\} \;, \qquad \text{(B.23a)}$$

$$r_{12} = -2\pi\gamma_o \sum \left\{ \frac{jk}{\sqrt{\alpha\omega}} \left(\frac{2\tau_{01}}{\alpha} - \frac{2\tau_{10}}{\alpha^2 u} - u\tau_{12} \right) - u\tau_{12} G_o(U_o) + \left[- \left(1 + \frac{2}{\alpha u^2} \right) \right. \right.$$

$$\left. \left. \cdot Yu^2\tau_{01} - \tau_{03} + \left(1 + \frac{2}{\alpha u^2} \right) \frac{Yu\tau_{10}}{\alpha} + \left(1 + Y + \frac{1}{\alpha u^2} \right) u\tau_{12} \right] G_o(U_-) \right\} ,$$

(B.23b)

$$r_{13} = -2\pi\gamma_o \sum \left\{ - \frac{j}{\sqrt{\alpha}} \tau_{12} + \left(-(1 + 2\alpha u^2) u\tau_{01} - 2\alpha u\tau_{03} + (1 + \alpha u^2) \frac{\tau_{10}}{\alpha} \right. \right.$$

$$\left. + 4\alpha u^2 \tau_{12} \right) U_o G_o(U_o) + \left(\alpha u^3 \tau_{01} + \alpha u \tau_{03} - \frac{u^2}{2} \tau_{10} - 2\alpha u^2 \tau_{12} \right) U_+ G_o(U_+)$$

$$\left. + \left[(1 + \alpha u^2) u\tau_{01} + \alpha u \tau_{03} - \frac{\tau_{10}}{\alpha} \left(1 + \frac{\alpha u^2}{2} \right) - (1 + 2\alpha u^2) \tau_{12} \right] U_- G_o(U_-) \right\} ,$$

(B.23c)

$$r_{21} = -2\pi\gamma_o \sum \left\{ \frac{jk}{\sqrt{\alpha\omega}} \left(\frac{2}{\alpha} \tau_{01} - \frac{2\tau_{10}}{\alpha^2 u} - u\tau_{12} \right) - u\tau_{12} G_o(U_o) + \left[\left(1 + \frac{2}{\alpha u^2} \right) \right. \right.$$

$$\left. \left. \cdot Yu^2\tau_{01} - \tau_{03} - \frac{Yu\tau_{10}}{\alpha} \left(1 + \frac{2}{\alpha u^2} \right) + \left(1 - Y + \frac{1}{\alpha u^2} \right) u\tau_{12} \right] G_o(v_+) \right\} ,$$

(B.23d)

$$r_{22} = -2\pi\gamma_o \sum \left[\frac{-j}{\sqrt{\alpha}} \frac{k}{\omega} u\tau_{12} - u\tau_{12} G_o(U_o) - \left(Yu^2\tau_{01} + \tau_{03} - (1 + Y) \right. \right.$$

$$\left. \left. \cdot u\tau_{12} \right) G_o(U_-) \right] , \qquad \text{(B.23e)}$$

$$r_{23} = -2\pi\gamma_o \sum \left\{ -\frac{j}{\sqrt{\alpha}} \tau_{12} + \left(-(1 + 2\alpha u^2) u\tau_{01} - 2\alpha u\tau_{03} + \frac{\tau_{10}}{\alpha}(1 + \alpha u^2) \right.\right.$$

$$\left. + 4\alpha u^2 \tau_{12} \right) U_o G_o(U_o) + \left[(1 + \alpha u^2) u\tau_{01} + \alpha u\tau_{03} - \frac{\tau_{10}}{\alpha}\left(1 + \frac{\alpha u^2}{2}\right) \right.$$

$$\left. - (1 + 2\alpha u^2)\tau_{12} \right] U_+ G_o(U_+) + \left(\alpha u^3 \tau_{01} + \alpha u\tau_{03} - \frac{u^2}{2}\tau_{10} - 2\alpha u^2 \tau_{12} \right)$$

$$\left. \cdot U_- G_o(U_-) \right\} , \qquad \text{(B.23f)}$$

$$r_{31} = -2\pi\gamma_o \sum \left[\frac{j}{\sqrt{\alpha}} \left(-Yu\tau_{01} - \tau_{12} + \frac{Y\tau_{10}}{\alpha}(1 + \alpha u^2) \right) - u\tau_{01} U_o G_o(U_o) \right.$$

$$\left. + \left((1 - Y) u\tau_{01} + \frac{Y\tau_{10}}{\alpha}(1 + \alpha u^2) - \tau_{12} \right) U_+ G_o(U_+) \right] , \qquad \text{(B.23g)}$$

$$r_{32} = -2\pi\gamma_o \sum \frac{j}{\sqrt{\alpha}} \left(Yu\tau_{01} - \tau_{12} - \frac{Y\tau_{10}}{\alpha}(1 + \alpha u^2) \right) - u\tau_{01} U_o G_o(U_o)$$

$$+ \left((1 + Y) u\tau_{01} - \tau_{12} - \frac{Y\tau_{10}}{\alpha}(1 + \alpha u^2) \right) U_- G_o(U_-) , \qquad \text{(B.23h)}$$

$$r_{33} = -2\pi\gamma_o \sum \left\{ -\frac{j}{\sqrt{\alpha}} \tau_{01} U_o - \tau_{01} U_o^2 G_o(U_o) - \left(2\alpha u^3 \tau_{01} - \frac{u\tau_{10}}{2}(1 + 2\alpha u^2) \right.\right.$$

$$\left.\left. - \alpha u\tau_{12} \right) [U_+^2 G_o(U_+) + U_-^2 G_o(U_-) - 2U_o^2 G_o(U_o)] \right\} . \qquad \text{(B.23i)}$$

By using the fact that

$$I_o(bt) = J_o(jbt) \quad \text{and} \quad I_1(bt) = \frac{1}{j} J_1(jbt) ,$$

the integrals τ_{oq} and τ_{1q} can be written as

$$\tau_{oq} = \int_0^\infty J_o(jbt)e^{-\alpha t^2} t^q dt \quad , \quad q = 1, 3 \quad ,$$

$$\tau_{1q} = \frac{1}{j}\int_0^\infty J_1(jbt)e^{-\alpha t^2} t^q dt \quad , \quad q = 1, 2 \quad , \qquad \text{(B.24)}$$

where $b \equiv (2\alpha u)$ and $t \equiv V_r$. These integrals can be evaluated by the following formula given by Watson[7]:

$$\int_0^\infty J_\nu(at)\exp(-p^2t^2)t^{\mu-1}dt$$

$$= \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu\right)\left(\frac{1}{2}\frac{a}{p}\right)^\nu}{2p^\mu\Gamma(\nu+1)} {}_1F_1\left(\frac{1}{2}\nu + \frac{1}{2}\mu, \nu+1, -\frac{a^2}{4p^2}\right) \quad , \qquad \text{(B.25)}$$

where the confluent hypergeometric function ${}_1F_1(\alpha:\rho:Z)$ is defined by

$${}_1F_1(\alpha:\rho:Z) = \sum_{n=o}^\infty \frac{(\alpha)_n}{n!(\rho)_n} Z^n$$

with

$$(\alpha)_o \equiv 1 \quad , \quad (\alpha)_n \equiv \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1) \quad ,$$

$$\tau_{01} = \frac{1}{2\alpha}\exp(\alpha u^2) \equiv \tau \quad , \quad \tau_{12} = \frac{u}{2\alpha}\exp(\alpha u^2) = u\tau \quad ,$$

$$\tau_{10} = \frac{u}{2}\,{}_1F_1(1:2:\alpha u^2) \equiv \alpha u D_1 \tau$$

$$\tau_{03} = \frac{1}{2\alpha^2}\,{}_1F_1(2:1:\alpha u^2) \equiv \frac{D_2}{\alpha}\tau \quad , \qquad \text{(B.26)}$$

where

$$D_1(\delta) \equiv e^{-\delta} {}_1F_1(1:2:\delta) = e^{-\delta} \sum_{n=o}^{\infty} \frac{1}{n!} \frac{1}{(n+1)} \delta^n ,$$

$$D_2(\delta) \equiv e^{-\delta} {}_1F_1(2:1:\delta) = e^{-\delta} \sum_{n=o}^{\infty} \frac{(n+1)}{n!} \delta^n \qquad (B.27)$$

with $\delta \equiv (\alpha u^2)$ and $\tau \equiv \tau_{o1}$. It should be noted that D_2 can also be expressed as

$$D_2 = 1 + \delta \qquad (B.28)$$

which can be verified as follows:

$$D_2 e^{\delta} = (1+\delta) \sum_{n=o}^{\infty} \frac{1}{n!} \delta^n = \sum_{n=o}^{\infty} \left(\frac{1}{n!} \delta^n + \frac{1}{n!} \delta^{n+1} \right)$$

$$= 1 + \sum_{n=o}^{\infty} \left(\frac{1}{n!} + \frac{1}{(n+1)!} \right) \delta^{n+1}$$

$$= 1 + \sum_{n=o}^{\infty} \frac{n+2}{(n+1)!} \delta^{n+1} = \sum_{\ell=o}^{\infty} \frac{(\ell+1)}{\ell!} \delta^{\ell}$$

which is nothing but Eqs. B.27. In view of the fact that R_{pq} can be expressed in terms of r_{pq} as

$$R_{pq} = \frac{j\left(\frac{\omega e}{\epsilon_o}\right)}{2(\omega^2 - c^2k^2)} r_{pq} \quad ; \quad p = 1, 2 \quad , \quad q = 1, 2, 3$$

$$= \frac{je}{\omega\epsilon_o} r_{pq} \quad ; \quad p = 3 \quad , \quad q = 1, 2, 3 \quad , \qquad \text{(B.29)}$$

upon substituting Eqs. B.26 into Eqs. B.23, R_{pq} can be obtained from Eqs. B.29 as follows:

$$R_{11} = \sum \frac{S}{(1 - \eta)} [\ell_o + \ell_1 G_o(U_o) + \ell_2 G_o(U_+)] \quad ,$$

$$R_{12} = \sum \frac{S}{(1 - \eta)} [\ell_3 + \ell_1 G_o(U_o) + \ell_5 G_o(U_-)] \quad ,$$

$$R_{13} = \sum \frac{S}{(1 - \eta)} [\ell_6 + \ell_7 G_o(U_o) + \ell_8 G_o(U_+) + \ell_9 G_o(U_-)] \quad ,$$

$$R_{21} = \sum \frac{S}{(1 - \eta)} [\ell_3 + \ell_1 G_o(U_o) - \ell_5 G_o(U_+)] \quad ,$$

$$R_{22} = \sum \frac{S}{(1 - \eta)} [\ell_o + \ell_1 G_o(U_o) + \ell_2 G_o(U_-)] \quad ,$$

$$R_{23} = \sum \frac{S}{(1 - \eta)} [\ell_6 + \ell_7 G_o(U_o) + m_8 G_o(U_+) + m_9 G_o(U_-)] \quad ,$$

$$R_{31} = \sum 2S[n_o + n_1 G_o(U_o) + n_2 G_o(U_+)] \quad ,$$

$$R_{32} = \sum 2S[n_3 + n_1 G_o(U_o) + n_5 G_o(U_-)] \quad ,$$

$$R_{33} = \sum 2S[n_6 + n_7 G_o(U_o) + n_8 G_o(U_+) + n_9 G_o(U_-)] \quad , \qquad \text{(B.30)}$$

where

$$\ell_0 \equiv j\left(\frac{\delta}{V}\right) , \quad \ell_1 \equiv \delta , \quad \ell_2 \equiv 1 ,$$

$$\ell_3 \equiv j\left(\frac{2\mu + \delta}{V}\right) , \quad \ell_5 \equiv -Y(2\mu + \beta) ,$$

$$\ell_6 \equiv j\sqrt{\delta} , \quad \ell_7 \equiv \sqrt{\delta}V(2 + \nu) , \quad \ell_8 \equiv \sqrt{\delta}V(1 + Y)\left(\frac{\beta}{2} - 1\right) ,$$

$$\ell_9 \equiv \sqrt{\delta}V(1 - Y)\left(\frac{\beta}{2} + \mu\right) ,$$

$$m_8 \equiv \sqrt{\delta}V(1 + Y)\left(\frac{\beta}{2} + \mu\right) , \quad m_9 \equiv \sqrt{\delta}V(1 - Y)\left(\frac{\beta}{2} - 1\right) ,$$

$$n_0 \equiv j\sqrt{\delta}(1 + \nu Y) , \quad n_1 \equiv \sqrt{\delta}V , \quad n_2 \equiv \sqrt{\delta}V(1 + Y)\nu Y ,$$

$$n_3 \equiv j\sqrt{\delta}(1 - \nu Y) , \quad n_5 \equiv -\sqrt{\delta}V(1 - Y)\nu Y ,$$

$$n_6 \equiv jV , \quad n_7 \equiv V^2(1 - 2\lambda) ,$$

$$n_8 \equiv V^2(1 + Y)^2\lambda , \quad n_9 \equiv V^2(1 - Y)^2\lambda \qquad \text{(B.31)}$$

in which

$$S \equiv jVX , \quad V \equiv \sqrt{\alpha}U_0 , \quad X \equiv \left(\frac{\omega_p}{\omega}\right)^2 , \quad \eta \equiv \left(\frac{c^2k^2}{\omega^2}\right) ,$$

$$\delta \equiv \alpha u^2 , \quad \beta = \delta D_1 , \quad \mu = (D_1 - 1) , \quad \nu \equiv [1 - D_1(1 + \delta)] ,$$

$$\lambda \equiv \delta\left(1 - \frac{1}{2}D_1(1 + 2\delta)\right) \qquad \text{(B.32)}$$

where D_1 is given in Eqs. B.27.

B.3 Derivation of Eqs. 37

From Eqs. 34

$$G_o(U_o) = \frac{-j}{V_q}\left(1 + \frac{1}{2}\gamma_q\right) ,$$

$$G_o(U_\pm) = \frac{-j}{V_q(1 \pm Y_q)}\left(1 + \frac{\gamma_q}{2}\ \frac{1}{(1 \pm Y_q)^2}\right) , \qquad \text{(B.33)}$$

where $\gamma_q \equiv (1/V_q^2)$, and from Eqs. 36 under condition $\delta \ll 1$

$$2\mu_q + \delta_q = 0 \ , \ \left(\frac{\beta_q}{2} - 1\right) = \left(\frac{\delta_q}{2} - 1\right) \ , \ 2\mu_q + \beta_q = -\frac{\delta_q^2}{2} \ ,$$

$$\left(\frac{\beta_q}{2} + \mu_q\right) = -\frac{\delta_q^2}{4} \ , \ (2 + \nu_q) = \left(2 - \frac{\delta_q}{2}\right) . \qquad \text{(B.34)}$$

Using the above approximation, Eqs. 29 become

$$R_{11} = \frac{1}{(1 - \eta)}\sum X_q\left[\frac{1}{(1 + Y_q)} + \frac{\gamma_q}{2}\left(\delta + \frac{1}{(1 + Y_q)^3}\right)\right] , \qquad \text{(B.35a)}$$

$$R_{12} = \frac{1}{(1 - \eta)}\sum \delta_q X_q\left\{\left[1 + \frac{\delta_q}{2}\left(\frac{Y_q}{1 - Y_q}\right)\right] + \frac{\gamma_q}{2}\left(1 + \frac{\delta_q}{2}\ \frac{Y_q}{(1 - Y_q)^3}\right)\right\} ,$$

$$\text{(B.35b)}$$

$$R_{13} = \frac{1}{(1 - \eta)}\sum \frac{1}{2}\sqrt{\delta_q}X_q V_q\gamma_q\left[\left(2 - \frac{\delta_q}{2}\right) + \left(\frac{\delta_q}{2} - 1\right)\frac{1}{(1 + Y_q)^2}\right] ,$$

$$\text{(B.35c)}$$

$$R_{21} = \frac{1}{(1 - \eta)} \sum \delta_q X_q \left[\left(1 - \frac{\delta_q}{2} \frac{Y_q}{(1 + Y_q)}\right) + \frac{\gamma_q}{2}\left(1 - \frac{\delta_q}{2} \frac{Y_q}{(1 + Y_q)^3}\right)\right] , \tag{B.35d}$$

$$R_{22} = \frac{1}{(1 - \eta)} \sum X_q \left[\frac{1}{(1 - Y_q)} + \frac{\gamma_q}{2}\left(\delta_q + \frac{1}{(1 - Y_q)^3}\right)\right] , \tag{B.35e}$$

$$R_{23} = \frac{1}{(1 - \eta)} \sum \frac{1}{2}\sqrt{\delta_q} X_q V_q \gamma_q \left[\left(2 - \frac{\delta_q}{2}\right) + \left(\frac{\delta_q}{2} - 1\right)\frac{1}{(1 - Y_q)^2}\right] , \tag{B.35f}$$

$$R_{31} = \sum \sqrt{\delta_q} X_q V_q \gamma_q \left(1 - \frac{\delta_q}{2} \frac{(1 - \delta_q)Y_q}{(1 + Y_q)^2}\right) , \tag{B.35g}$$

$$R_{32} = \sum \sqrt{\delta_q} X_q V_q \gamma_q \left(1 + \frac{\delta_q}{2} \frac{(1 - \delta_q)Y_q}{(1 - Y_q)^2}\right) , \tag{B.35h}$$

$$R_{33} = \sum X_q \left[1 + \delta_q \left(1 - \frac{3}{2} \delta_q\right)\left(\frac{1}{1 - Y_q^2} - 1\right)\right] . \tag{B.35i}$$

B.4 Derivation of Eq. 43

For a neutral homogeneous electron gas in which ion motion is negligible, Eq. 26 can be expanded into the following form:

$$\frac{X^3}{(1-\eta)^2}(\Phi_0 + \gamma\Phi_1 + \gamma^2\Phi_2) - \frac{X^2}{(1-\eta)^2}\left(\Psi_0 + \Psi_1\gamma + \gamma^2\Psi_2\right)$$

$$- \frac{X^2}{(1-\eta)}(\Pi_0 + \gamma\Pi_1) + \frac{X}{(1-\eta)}(\Lambda_0 + \gamma\Lambda_1) + (L_0 X - 1) = 0 \ , \qquad \text{(B.36)}$$

where

$$\Phi_0 \equiv L_0\Psi_0 \ ,$$

$$\Phi_1 \equiv L_0\Psi_1 + \delta(\Lambda_{31}D_0 - \Lambda_{32}C_0) \ ,$$

$$\Phi_2 \equiv L_0\Psi_2 + \delta(\Lambda_{31}D_1 - \Lambda_{32}C_1) \ ,$$

$$\Pi_0 \equiv L_0\Lambda_0 \ ,$$

$$\Pi_1 \equiv L_0\Lambda_1 - \delta(\Lambda_{13}\Lambda_{31} - \Lambda_{23}\Lambda_{32}) \ ,$$

with

$$\Psi_0 \equiv \Pi_{11}\Pi_{22} - \delta^2\Pi_{12}\Pi_{21} \ ,$$

$$\Psi_1 \equiv [(\Pi_{22}\Lambda_{11} + \Pi_{11}\Lambda_{22}) - \delta^2(\Pi_{12}\Lambda_{21} + \Pi_{21}\Lambda_{12})] \ ,$$

$$\Psi_2 \equiv (\Lambda_{11}\Lambda_{22} - \delta^2\Lambda_{12}\Lambda_{21}) \ ,$$

$$\Lambda_0 \equiv (\Pi_{11} + \Pi_{22}) \ ,$$

$$\Lambda_1 \equiv (\Lambda_{11} + \Lambda_{22}) \ ,$$

$$C_0 \equiv (\Pi_{11}\Lambda_{23} - \delta\Pi_{21}\Lambda_{13}) \ ,$$

$$C_1 \equiv (\Lambda_{11}\Lambda_{23} - \delta\Lambda_{21}\Lambda_{13}) \ ,$$

$$D_0 \equiv (\delta\Pi_{12}\Lambda_{23} - \Pi_{22}\Lambda_{13}) \ ,$$

$$D_1 \equiv (\delta\Lambda_{12}\Lambda_{23} - \Lambda_{22}\Lambda_{13}) \ ,$$

$$L_0 \equiv (1 + \delta A_q) \ ,$$

$$A_q \equiv [1 - (3/2)\delta]Y^2/(1 - Y^2) \qquad \text{(B.37)}$$

For the case in which $\delta \ll 1$, Eqs. 38 become

$$\Pi_{11} = \frac{1}{1 + Y} \ , \quad \Lambda_{11} = \frac{1}{2}\left(\delta + \frac{1}{(1 + Y)^3}\right) \ ,$$

$$\Pi_{12} = \left(1 + \frac{\delta Y}{2} \ \frac{1}{(1 - Y)}\right) \ , \quad \Lambda_{12} = \frac{1}{2}\left(1 + \frac{\delta Y}{2} \ \frac{1}{(1 - Y)^3}\right) \ ,$$

$$\Lambda_{13} = \frac{1}{2}\left(2 - \frac{1}{(1 + Y)^2}\right) \ , \quad \Pi_{21} = \left(1 - \frac{\delta Y}{2} \ \frac{1}{(1 + Y)}\right) \ ,$$

$$\Lambda_{21} = \frac{1}{2}\left(1 - \frac{\delta Y}{2} \ \frac{1}{(1 + Y)^3}\right) \ , \quad \Pi_{22} = \frac{1}{1 - Y} \ ,$$

$$\Lambda_{22} = \frac{1}{2}\left(\delta + \frac{1}{(1 - Y)^3}\right) \ , \quad \Lambda_{23} = \frac{1}{2}\left(2 - \frac{1}{(1 - Y)^2}\right) \ ,$$

$$\Lambda_{31} = \left(1 - \frac{\delta Y}{2} \ \frac{1}{(1 + Y)^2}\right) \ , \quad \Lambda_{32} = \left(1 + \frac{\delta Y}{2} \ \frac{1}{(1 - Y)^2}\right) \ ,$$

$$L_o = 1 \ . \qquad \text{(B.38)}$$

Substituting Eqs. B.38 into Eqs. B.37 yields:

$$\Phi_o = \frac{1}{\xi}(1 + \delta^2 Y^2) \quad ,$$

$$\Phi_1 = \frac{1}{\xi^3}(1 + Y^2 - \delta Y^4 - \delta^2 Y^6) \quad ,$$

$$\Phi_2 = \frac{1}{4\xi^5}(1 - 2Y^2 + Y^4 - 5\delta Y^6 + \delta^3 Y^8) \quad ,$$

$$\Pi_o = \frac{2}{\xi} \quad ,$$

$$\Pi_1 = \frac{1}{\xi^3}(1 - 2\delta Y + 3Y^2 + 2\delta Y^3 + 2\delta Y^4 - \delta^2 Y^5 - \delta Y^6) \quad ,$$

$$\Psi_o = \frac{1}{\xi}(1 + \delta^2 Y^2) \quad ,$$

$$\Psi_1 = \frac{1}{\xi^3}(1 + Y^2 + \delta Y^4 + \delta^2 Y^6) \quad ,$$

$$\Psi_2 = \frac{1}{4\xi^3}(1 + 6\delta Y^2 - \delta^3 Y^4) \quad ,$$

$$\Lambda_o = \frac{2}{\xi} \quad ,$$

$$\Lambda_1 = \frac{1}{\xi^3}(1 + 3Y^2 + 3\delta Y^4 - \delta Y^6) \quad . \qquad \text{(B.39)}$$

In view of the fact that in the present discussion $\gamma^2 \ll 1$ is assumed [i.e., condition (32)], it can be easily shown that

$$\Phi_o + \gamma^2 \Phi_2 \simeq \Phi_o \quad \text{and} \quad \Psi_o + \gamma^2 \Psi_2 \simeq \Psi_o \quad , \qquad \text{(B.40)}$$

and using the fact that $(2\delta\gamma Y^4/\xi^3) \ll (1/\xi)$, since $\gamma^2 \ll 1$ and $\delta \ll 1$, the terms involving δY^4 in the expressions Φ_1, Ψ_1, Π_1 and Λ_1 can be neglected so that Eqs. B.39 become Eqs. 44.

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