Report No. S-67-1

A FURTHER STUDY OF ELASTIC FOUNDATION MODELS

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Office of Grants and Research Contracts

National Aeronautics and Space Administration

under grant

NGR-33-016-067

January 1967

New York University
School of Engineering and Science
University Heights, New York, N.Y. 10453

Report No. S-67-1

NEW YORK UNIVERSITY
New York, N.Y.

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Arnold D. Kerr* and Warren J. Rhines**

SUMMARY

In the present paper the study of foundation models initiated in Ref. [1] is continued. At first it is shown that the response expression of the Pasternak model can be derived from the continuum point of view when the assumption is made that the in-plane stresses and the in-plane displacements are zero throughout the foundation layer. It is also shown that, by expanding formally the exact pressure-deflection relationship for the upper surface of an elastic layer which rests on a rigid base, the response expressions of a number of models are obtained, the type depending upon the number of terms retained in the expansion.

INTRODUCTION AND STATEMENT OF PROBLEM

In a recent paper A. D. Kerr [1] discussed a number of foundation models which have been published in the literature. This study showed that, in order to construct foundation models, one may proceed either by introducing simplifying assumptions about some of the expected displacements and/or stresses in the equations of a continuum [2-4] or by starting with the Winkler foundation and, in order to bring it closer to reality, assuming some kind of interactions between the spring elements [5-8].

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1) Part of the material contained in the present paper is taken from the Ph.D. Dissertation submitted by one of the authors (W.J.R.) to New York University.

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A third approach to construct response expressions is the formal mathematical procedure. An example is the approach used by J. Ratzersdorfer who expanded formally Wieghardt's integral relation for load and deflections and retained terms of low order [9, 10]. Another expansion technique was used by Favre [11, 12] who obtained expressions of the Pasternak type. The approach by W. J. van der Eb and A. D. de Pater [13] which is closely related to some of the derivations by K. Wieghardt [14] should also be mentioned.

In the present paper the study of foundation models is continued. At first it is shown that the response expression of the Pasternak model can be derived from the continuum point of view, thus establishing a "continuum" model for the Pasternak foundation. Then it is shown that by expanding formally the exact pressure-deflection relationship for the upper surface of an elastic layer which rests on a rigid base, the response expressions of a number of models discussed in [1] are obtained, the type depending upon the number of terms retained in the expansion.

A "CONTINUUM" MODEL FOR THE PASTERNAK FOUNDATION

We consider a weightless elastic layer of thickness H which rests on a rigid base and is subjected to loads on its free surface ($z = H$) as shown in Fig. 1.

E. Reissner [4] by assuming that throughout the layer the in-plane stresses are zero, i.e.,

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{1}$$

as well as that at the upper and lower surface the in-plane displacements are zero, i.e.,

$$u = v = 0 \text{ at } z = 0 \text{ and } z = H \tag{2}$$

obtained the following response equation at the surface $z = H$

$$p - \left(\frac{H^2 G}{12E}\right)\nabla^2 p = \left(\frac{E}{H}\right)W - \left(\frac{HG}{3}\right)\nabla^2 W \tag{3}$$

where

$$\begin{aligned} W &= -w(x,y,H) \\ p &= -\sigma_z(x,y,H) \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned} \tag{4}$$

and E and G are parameters of the foundation material.

Using the same notation, the response expression of the Pasternak foundation is

$$p = k_p W - g_p \nabla^2 W \tag{5}$$

where k_p and g_p are foundation parameters.

Derivations of Filonenko-Borodich in connection with his membrane model [5] suggest that the assumption that throughout the layer

$$\begin{aligned} \sigma_x = \sigma_y = \tau_{xy} = 0 \\ u = v = 0 \end{aligned} \tag{6}$$

will yield the response expression of the Pasternak foundation.

To show this we substitute the assumptions listed under (6) into the equilibrium equations (Ref. [15] p. 229) the stress-strain equations (Ref. [15] p. 7 and p. 9), and the strain-displacement relations (Ref. [15] p. 6) and obtain

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \tag{7a}$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \tag{7b}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{7c}$$

$$\epsilon_x = -\frac{\nu}{E}\sigma_z \; ; \qquad \epsilon_y = -\frac{\nu}{E}\sigma_z \; ; \qquad \epsilon_z = \frac{\sigma_z}{E} \tag{8a}$$

$$\gamma_{xy} = 0 \; ; \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \; ; \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \tag{8b}$$

$$\epsilon_x = 0 \; ; \qquad \epsilon_y = 0 \; ; \qquad \epsilon_z = \frac{\partial w}{\partial z} \tag{9a}$$

$$\gamma_{xy} = 0 \; ; \qquad \gamma_{xz} = \frac{\partial w}{\partial x} \; ; \qquad \gamma_{yz} = \frac{\partial w}{\partial y} \tag{9b}$$

The first two equations in (8a) and (9a) indicate that ν has to be assumed equal to zero.

From (7a) and (7b) it follows that, as in the Reissner foundation, the shear stresses are independent of z, i.e.,

$$\tau_{xz} = \tau_{xz}(x,y) \tag{10}$$

$$\tau_{yz} = \tau_{yz}(x,y) \tag{11}$$

an unrealistic result particularly for thick foundation layers. However, since foundation models are introduced to study the response of the foundation surface to applied loads and not to study the stresses and displacements within the foundation material, this deficiency, as well as another which will be pointed out later, may in general be of no serious consequence.

Integrating (7c) with respect to z, noting (11), we obtain

$$\sigma_z = - z\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}\right) + f(x,y) \tag{12}$$

From the boundary condition

$$\sigma_z(x,y,H) = - p(x,y) \tag{13}$$

it follows that

$$f(x,y) = H\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}\right) - p(x,y) \tag{14}$$

and thus

$$\sigma_z = - p(x,y) + (H-z)\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}\right) \tag{15}$$

From the third equation in (8a) and (9a) it follows that

$$\frac{\partial w}{\partial z} = \frac{\sigma_z}{E} \tag{16}$$

Substituting (15) into (16), and then integrating the resulting equation with respect to z, we obtain

$$Ew = - zp(x,y) + z(H-\frac{z}{2})\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}\right) + g(x,y) \tag{17}$$

From the boundary condition

$$w(x,y,0) \equiv 0 \tag{18}$$

it follows that

$$g(x,y) \equiv 0 \tag{19}$$

Noting the second and third equations in (8b) and (9b), Eq. (17) assumes the form

$$Ew(x,y,z) = - zp(x,y) + Gz(H-\frac{z}{2})\ \nabla^2 w \tag{20}$$

At z = H Eq. (20) becomes, setting w(x,y,H) = - W(x,y,),

$$p = \left(\frac{E}{H}\right)W - \left(\frac{GH}{2}\right)\nabla^2 W \tag{21}$$

This expression is identical to the response equation of the Pasternak foundation given in (5) if we set

$$k_p = \frac{E}{H} \qquad g_p = \frac{GH}{2} \tag{22}$$

It should be noted that boundary condition (18) implies that at z = 0 $\frac{\partial w}{\partial x} = 0$ and $\frac{\partial w}{\partial y} = 0$, which, in view of Eqs. (7a), (7b), (8b) and (9b) leads to a contradiction.

DERIVATION OF THE RESPONSE OF VARIOUS FOUNDATION MODELS FROM A FORMAL EXPANSION OF AN EXACT RESPONSE EXPRESSION FOR AN ELASTIC LAYER RESTING ON A RIGID BASE

The mechanical foundation models which were introduced in [2-8] were all arrived at by making various, sometimes arbitrary, simplifying assumptions concerning the mechanical behavior of the supporting foundation. In this section an attempt is made to provide a systematic basis for the generation of foundation models from an analytical point of view.

For this purpose, let us consider a foundation layer of depth H subjected to a normal pressure at the top surface which rests on a rigid frictionless base as a plane strain problem (Fig. 2). The derivation of the relationship between the contact pressure and vertical deflection at the upper surface is similar to the one given by Bosson [16] for the case of generalized plane stress, and hence only the major steps will be repeated here.

From plane elasticity it is known that when a function $\Phi(x,y)$ satisfies the biharmonic equation

$$\nabla^4 \Phi = 0 \tag{23}$$

and the proper boundary conditions then the stresses in the body may be determined from

$$\sigma_x = \frac{\partial^2 \Phi}{\partial z^2} \; ; \qquad \sigma_z = \frac{\partial^2 \Phi}{\partial x^2} \; ; \qquad \tau_{xz} = - \frac{\partial^2 \Phi}{\partial x \partial z} \tag{24}$$

Further if we define a displacement function, $\Psi(x,y)$, which satisfies

$$\nabla^2 \Psi = 0 \text{ and } \frac{\partial^2 \Psi}{\partial x \partial z} = \nabla^2 \Phi \tag{25}$$

then it can be verified that the displacements are given by

$$Eu = (1-\nu^2)\frac{\partial \psi}{\partial z} - (1+\nu)\frac{\partial \Phi}{\partial x} \tag{26}$$

$$Ew = (1-\nu^2)\frac{\partial \psi}{\partial x} - (1+\nu)\frac{\partial \Phi}{\partial z} \tag{27}$$

Following Bosson, who employed the procedure devised by W. M. Shepherd [17], we assume the stress function for the foundation layer in the form

$$\Phi(x,z) = \sum_{n=0}^{\infty} X_n(x)\frac{z^n}{n!} \tag{28}$$

Substituting (28) into (23) we obtain, after some reductions, the symbolic form

$$\Phi(x,z) = [\cos(zD) + \tfrac{1}{2}zD\sin(zD)]X_o + [\frac{3}{2D}\sin(zD) - \tfrac{1}{2}z\cos(zD)]X_1 + \\ + \frac{z}{2D}\sin(zD)X_2 + \tfrac{1}{2}\left[\frac{\sin(zD)}{D^3} - \frac{z}{D^2}\cos(zD)\right]X_3 \tag{29}$$

where

$$D^n = \frac{d^n}{dx^n} \qquad n = 1, 2, \ldots \tag{30}$$

The corresponding displacement function, ψ, according to Bosson is

$$\psi(x,z) = \sin(zD)X_o - \frac{1}{D}\cos(zD)X_1 + \frac{1}{D^2}\sin(zD)X_2 - \frac{1}{D^3}\cos(zD)X_3 \tag{31}$$

After satisfying the boundary conditions at $z = -H$

$$\left.\begin{aligned} \tau_{xz}(x,-H) &= 0 \\ v(x,-H) &= 0 \end{aligned}\right\} \quad (32)$$

and one boundary condition at the interface z = 0

$$\tau_{xz}(x,0) = 0 \quad (33)$$

we obtain, denoting

$$\left.\begin{aligned} \sigma_z(x,0) &= -p(x) \\ w(x,0) &= -W(x) \end{aligned}\right\}, \quad (34)$$

the following relationship between the pressure and deflection at the interface z = 0

$$\left\{2\sin^2(HD)\right\}p = \frac{E}{1-\nu^2}\left\{\left[\frac{1}{D}\sin(HD)\cos(HD) + H\right]D^2\right\}W \quad (35)$$

Bosson, points out that by expanding the operators in his equation, which is equivalent to (35), in ascending powers of (HD), assuming H to be small, and then retaining only the leading terms in the expansions, one obtains the Winkler response. This argument and some of the relevant derivations are also reproduced in Ref. [18].

At this point it is of interest to investigate what response equations are obtained from (35) when higher order terms are retained in the expansions.

To study this problem we expand the operators in (35) as indicated above, noting that

$$\left.\begin{aligned} \sin^2\alpha &= \alpha^2 - \frac{\alpha^4}{3} + \frac{2\alpha^6}{45} - \frac{\alpha^8}{315} + \dots \\ \sin\alpha\cos\alpha &= \alpha - \frac{2\alpha^3}{3} + \frac{2\alpha^5}{15} - \frac{4\alpha^7}{315} + \dots \end{aligned}\right\} \quad (36)$$

and obtain, after dividing the resulting equation by $2HE/(1-\nu^2)$ and setting $E/(1-\nu^2) = E'$,

$$\frac{H}{E'}\frac{d^2p}{dx^2} - \frac{H^3}{3E'}\frac{d^4p}{dx^4} + \frac{2H^5}{45E'}\frac{d^6p}{dx^6} - \frac{H^7}{315E'}\frac{d^8p}{dx^8} + \ldots =$$

$$= \frac{d^2W}{dx^2} - \frac{H^2}{3}\frac{d^4W}{dx^4} + \frac{H^4}{15}\frac{d^6W}{dx^6} - \frac{2H^6}{315}\frac{d^8W}{dx^8} + \ldots \qquad (37)$$

It can be seen that by retaining only the leading term on each side of Eq. (35), it reduces to

$$\frac{d^2p}{dx^2} = \frac{E'}{H}\frac{d^2W}{dx^2} \qquad (38)$$

Integration of (38) yields $p = (E'/H)\,W+ax+b$. Since for $W \equiv 0$ it is expected that $p \equiv 0$, it follows that $a = b = 0$. Thus Eq. (38) is of the same form as the response of the Winkler foundation

$$p = kW \qquad (39)$$

which consists of a layer of closely spaced independent linear springs as shown in Fig. 3.

Retaining terms whose coefficients contain powers of H up to and including H^2 yields

$$\frac{d^2p}{dx^2} = \frac{E'}{H}\frac{d^2W}{dx^2} - \frac{E'H}{3}\frac{d^4W}{dx^4} \qquad (40)$$

which, by a similar argument as above, reduces to the same form as the response of the Pasternak foundation

$$p = k_pW - g_p\frac{d^2W}{dx^2} \qquad (41)$$

which consists of a spring layer with shear interactions as shown in Fig. 4, or expression (21) derived in the present paper.

Retaining terms up to and including H^3 yields

$$\frac{d^2p}{dx^2} - \frac{H^2}{3}\frac{d^4p}{dx^4} = \frac{E'}{H}\frac{d^2W}{dx^2} - \frac{E'H}{3}\frac{d^4W}{dx^4} \qquad (42)$$

which is of the same form as the response of the foundation model discussed recently in [19]

$$(1 + \frac{k}{c})p - \frac{g}{c}\frac{d^2p}{dx^2} = kW - g\frac{d^2W}{dx^2} \qquad (43)$$

which consists of an upper and lower spring layer interconnected by a shear layer as shown in Fig. 5, or expression (3) of the Reissner foundation model.

Retaining terms up to and including H^4 yields

$$\frac{d^2p}{dx^2} - \frac{H^2}{3}\frac{d^4p}{dx^4} = \frac{E'}{H}\frac{d^2W}{dx^2} - \frac{E'H}{3}\frac{d^4W}{dx^4} + \frac{E'H^3}{15}\frac{d^6W}{dx^6} \tag{44}$$

which is of the same form as the response of the foundation model shown in Fig. 6.

$$(1 + \frac{k}{c})p - \frac{g_L}{c}\frac{d^2p}{dx^2} = kW - g_u\ (1 + \frac{k}{c} + \frac{g_L}{g_u})\frac{d^2W}{dx^2} + (\frac{g_u g_L}{c})\frac{d^4W}{dx^4} \tag{45}$$

Retaining terms up to and including H^5 yields

$$\frac{d^2p}{dx^2} - \frac{H^2}{3}\frac{d^4p}{dx^4} + \frac{2H^4}{45}\frac{d^6p}{dx^6} = \frac{E'}{H}\frac{d^2W}{dx^2} - \frac{E'H}{3}\frac{d^4W}{dx^4} + \frac{E'H^3}{15}\frac{d^6W}{dx^6} \tag{46}$$

which is of the same form as the response of the mechanical model shown in Fig. 7

$$(1 + \frac{k}{c})p - \frac{g}{c}\frac{d^2p}{dx^2} + \frac{D}{c}\frac{d^4p}{dx^4} = kW - g\frac{d^2W}{dx^2} + D\frac{d^4W}{dx^4} \tag{47}$$

where D is the flexural rigidity of the bending layer.

It may be of interest to point out that the model shown in Fig. 8 yields the response expression

$$(1 + \frac{k}{c} + \frac{k}{s})p - \left[\frac{g_L}{c} + \frac{g_L + (1 + \frac{k}{c})g_u}{s}\right]\frac{d^2p}{dx^2} + \frac{g_L g_u}{cs}\frac{d^4p}{dx^4} =$$

$$= kw - \left[g_L + (1 + \frac{k}{c})g_u\right]\frac{d^2w}{dx^2} + \frac{g_L g_u}{c}\frac{d^4w}{dx^4} \tag{48}$$

which, apart from the constant coefficients is identical to (47). Thus, the model which corresponds to a higher order response expression is not unique.

It is of interest to find out what models will correspond to even higher order approximations and in particular, assuming that the expansions in Eq. (35) converge, to determine what mechanical model will yield a response expression like the continuous elastic layer. In Fig. 9 foundation models which consist of spring, shear, and bending layers, and which correspond to retained terms up to and including H^{10} are tabulated, and they indicate an answer to this question. The non-uniqueness of the presentation of higher order models should be noted.

REMARKS AND CONCLUSIONS

At this point it is of interest to note that the foundation model suggested by Hetényi [7] consists of an upper and lower spring layer interconnected by a bending layer (like the model in Fig. 7 but without the shear layer) and its response equation is

$$(1 + \frac{k}{c})p + \frac{D}{c}\frac{d^4 p}{dx^4} = kW + D\frac{d^4 W}{dx^4} \tag{49}$$

Comparing (49) with (47) it appears that, from the point of view of the formal expansion discussed above, the Hetényi model retains terms of higher order than those it omits.

The derivations of the response expression of the Pasternak model by generalizing the Winkler model, or by retaining terms in the expansion for a continuous layer as performed in the present paper, show that the physical significance of the second term is, at least, the effect of shear interactions. Therefore, the association of this term with only an in-plane force field, which is sometimes encountered in the literature, may be misleading from the point of view of the real mechanical behavior of the foundation.

The method of formal expansion shows that the first order approximation represents the compressibility of the foundation layer in the Winkler sense. The next order term represents the effect of shear interactions in the Pasternak sense, etc. Although the procedure is formal in that the convergence of the final series has not been proven, the agreement between the obtained response expressions and those of a number of mechanical models is noteworthy. It appears that the models and their order of presentation as shown in Fig. 9 could be used, if necessary, as a guide for the construction of foundation models of higher order.

ACKNOWLEDGEMENT

This publication is the result of research sponsored in part by the National Aeronautics and Space Administration under Research Grant NGR-33-016-067 to New York University and in part by the Bell Telephone Laboratories under its Doctoral Support plan.

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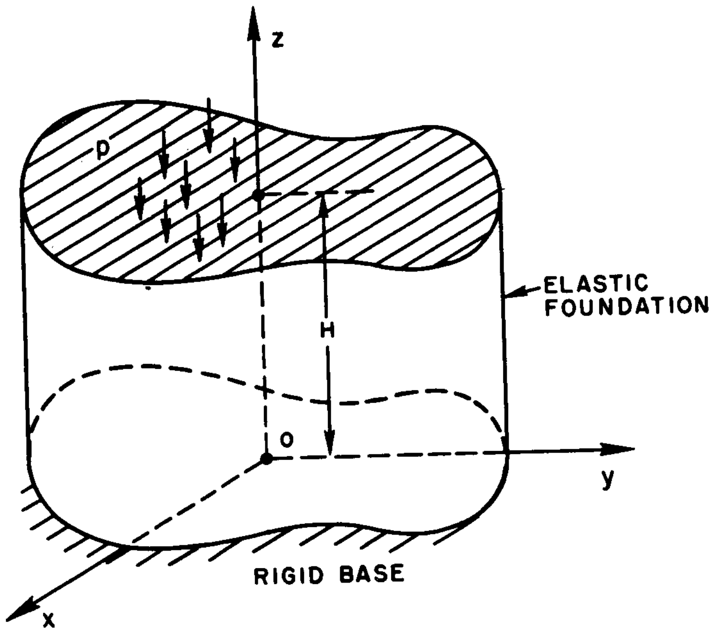

Fig. 1

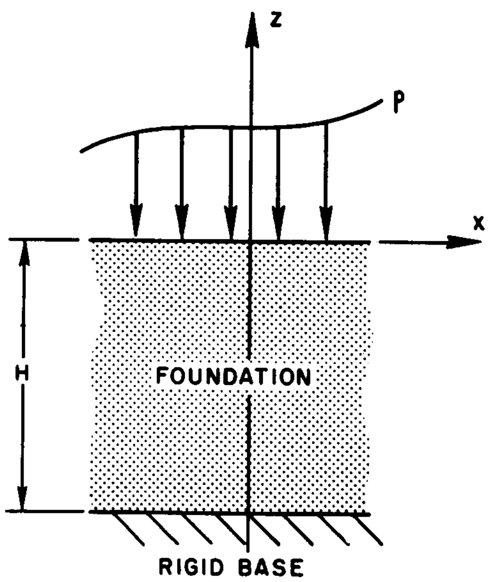

Fig. 2

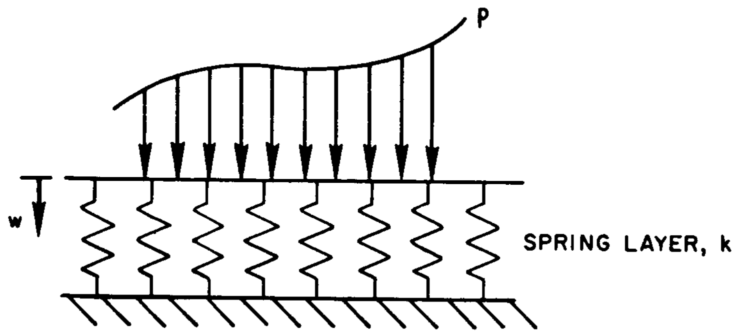

Fig. 3

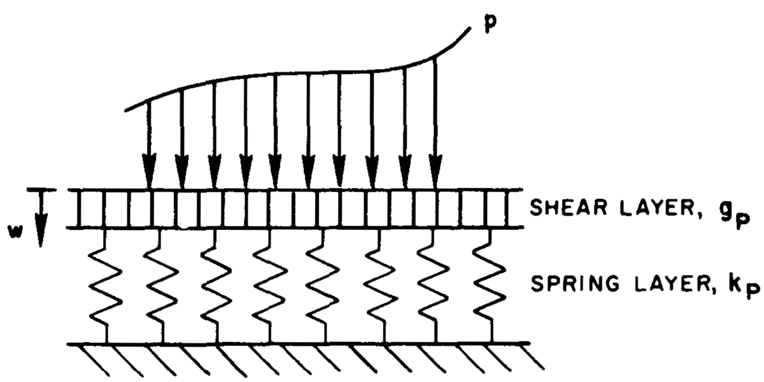

Fig. 4

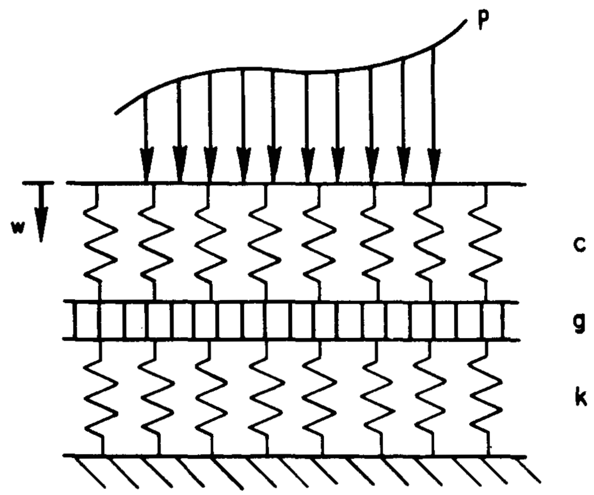

Fig. 5

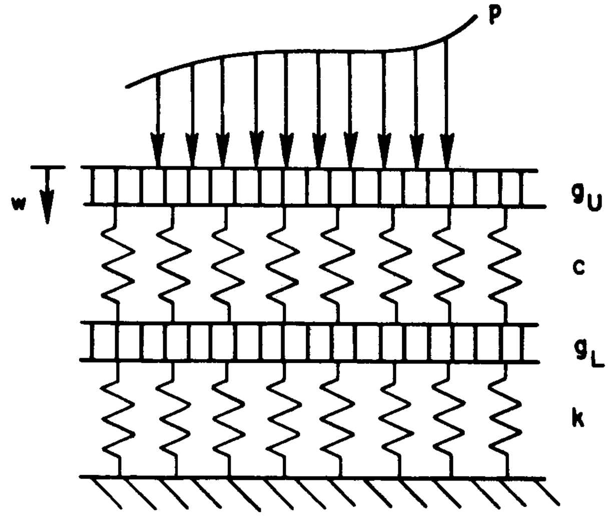

Fig. 6

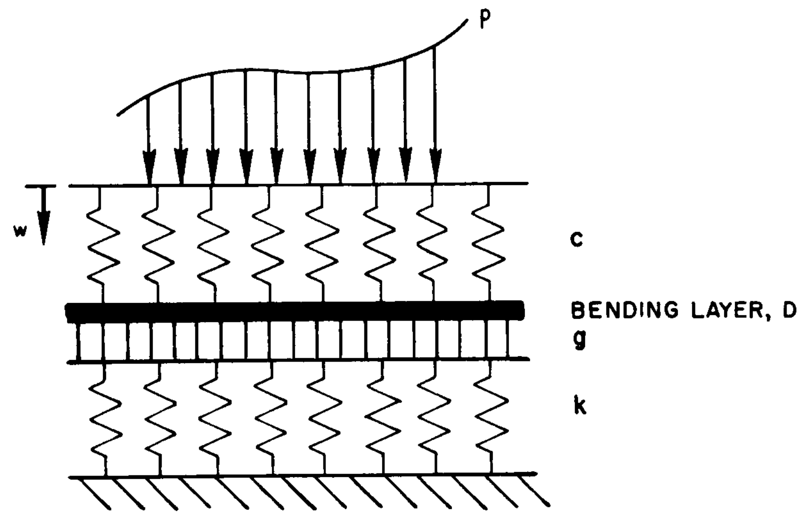

Fig. 7

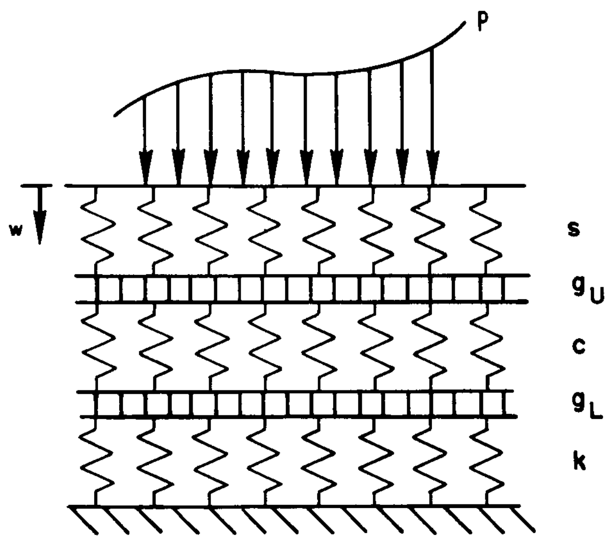

Fig. 8

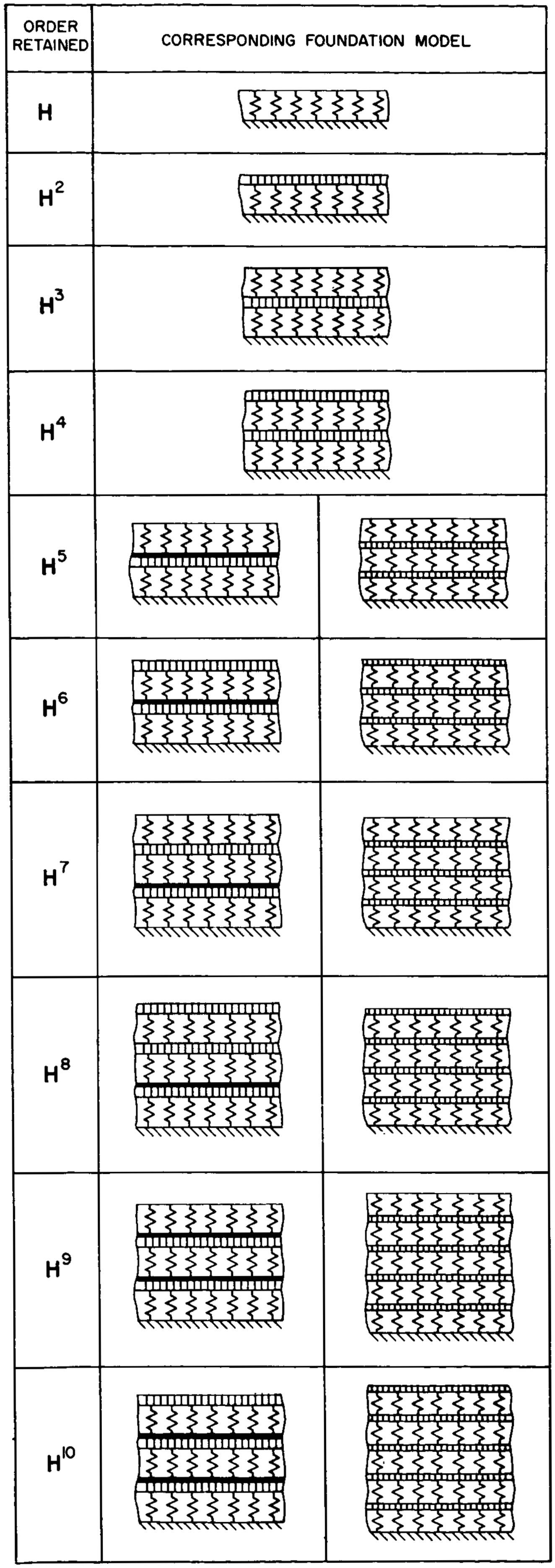
ORDER RETAINED
CORRESPONDING FOUNDATION MODEL
H
H2
H3
H4
H5
H6
H7
H8
H9
H10

Fig. 9