ON THE USE OF A PRIORI STATISTICS IN PROBLEMS OF PLASMA TURBULENCE

Ronald C. Davidson

Department of Physics, University of California, Berkeley, California

ABSTRACT

The usual procedure in problems of uniform plasma turbulence is to perform an a posteriori statistical averaging (the random phase approximation) on perturbation solutions to the Vlasov Equation. A more direct formulation in which ensemble averaging is done a priori is advocated in this article. Using the Vlasov equation as the dynamical equation for an individual system, equations are constructed for the time evolution of correlations in a spatially homogeneous ensemble of such plasmas. This hierarchy, which is identical to the BBGKY hierarchy without the effects of single particle encounters, automatically embodies the random phase approximation. Upon ordering the system of equations in the weak turbulence parameter E^2/nmv_{av}^2, the questions of closure and time validity are examined.

I. INTRODUCTION

In recent years considerable research has been devoted to the study of spatially uniform turbulence in a slightly unstable (in velocity space) plasma.[1-9] With few exceptions the dynamical equation used has been the collisionless Vlasov equation with self consistent electric field,[10] i.e.,

$$\frac{\partial}{\partial t} f_{a_1}(1)+\underset{\sim}{v}_1\cdot \frac{\partial}{\partial \underset{\sim}{x}_1} f_{a_1}(1) = -\frac{e_{a_1}}{m_{a_1}} \underset{\sim}{E}(\underset{\sim}{x}_1,t)\cdot \frac{\partial}{\partial \underset{\sim}{v}_1} f_{a_1}(1) \tag{1}$$

where

$$e_{a_1}\underset{\sim}{E}(\underset{\sim}{x}_1,t) = -\sum_{a_2} n_{a_2}\int \frac{\partial \phi_{a_1 a_2}}{\partial \underset{\sim}{x}_1} f_{a_2}(2)d(2) ,$$

$\phi_{a_1 a_2}$ is the coulomb potential $(e_{a_1} e_{a_2}/|\underset{\sim}{x}_1-\underset{\sim}{x}_2|)$, n_{a_i} is the number of i'th species particles per unit volume, (1),(2),... denote the phase space coordinates $(\underset{\sim}{x}_1,\underset{\sim}{v}_1),(\underset{\sim}{x}_2,\underset{\sim}{v}_2),\ldots$, and $n_{a_i} f_{a_i}(i)$ is the (smooth) number density of the a_i'th species at $(\underset{\sim}{x}_i,\underset{\sim}{v}_i)$ at time t. The general procedure has been to Fourier analyse Eq. (1) with respect to the position variable and obtain the solution in a perturbation expansion

$$f_{a_1}(1) \cong f_{a_1}^{(0)}(\underset{\sim}{v}_1,t)+f_{a_1}^{(1)}(1)+f_{a_1}^{(2)}(1)+f_{a_1}^{(3)}(1)+\ldots,$$

and correspondingly

$$\underset{\sim}{E}(\underset{\sim}{k},t) \cong \underset{\sim}{E}^{(1)}(\underset{\sim}{k},t)+\underset{\sim}{E}^{(2)}(\underset{\sim}{k},t)+\underset{\sim}{E}^{(3)}(\underset{\sim}{k},t)+\ldots , \tag{2}$$

where the expansion parameter is essentially

$$\delta \sim |[f_{a_1}(x_1,v_1,0)-f_{a_1}^{(0)}(v_1,0)] /f_{a_1}^{(0)}(v_1,0)| \ll 1 .$$

It is assumed that $f_{a_1}^{(0)}(\underset{\sim}{v}_1,0)$ is such as to give weak instability in a Landau analysis. Once the solution to Eq. (1) is obtained to a given order, say δ^3, appropriate statistical averages (see for example Ref. 8) are then performed a posteriori over a spatially uniform ensemble in order to obtain kinetic equations for the energy density in the turbulent electric field spectrum and for the background distribution $f^0_{a_1}(\underset{\sim}{v}_1,t)$. The averaging technique, usually referred to as the random phase approximation, consists of averaging over the phases of the first order fields $E^{(1)}(\underset{\sim}{k},t)$ $(\underset{\sim}{E}(\underset{\sim}{k},t) = \frac{\underset{\sim}{k}}{|\underset{\sim}{k}|} E(\underset{\sim}{k},t))$ assuming these phases are random. For example

$$\langle E^{(1)}(\underset{\sim}{k}_1,t)E^{(1)}(\underset{\sim}{k}_2,t)\rangle = \mathcal{E}(\underset{\sim}{k}_1,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2) \tag{3}$$

$$\langle E^{(1)}(\underset{\sim}{k}_1,t)E^{(1)}(\underset{\sim}{k}_2,t)E^{(1)}(\underset{\sim}{k}_3,t)\rangle = T(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3) \tag{4}$$

It is usually assumed that T is zero initially (viewed effectively as the product of three random numbers); that it remains zero in the time scales of interest must in general be proven. As for the average of four first order field amplitudes the procedure is to consider products in all possible pairs, i.e.,

$$\begin{aligned}\langle E^{(1)}(\underset{\sim}{k}_1,t)\ldots E^{(1)}(\underset{\sim}{k}_4,t)\rangle &= \mathcal{E}(\underset{\sim}{k}_1,t)\,\mathcal{E}(\underset{\sim}{k}_3,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\delta(\underset{\sim}{k}_3+\underset{\sim}{k}_4)\\ &+\mathcal{E}(\underset{\sim}{k}_1,t)\,\mathcal{E}(\underset{\sim}{k}_3,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_4)\delta(\underset{\sim}{k}_3+\underset{\sim}{k}_2)\\ &+\mathcal{E}(\underset{\sim}{k}_1,t)\,\mathcal{E}(\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_3)\delta(\underset{\sim}{k}_2+\underset{\sim}{k}_4)\ .\end{aligned} \tag{5}$$

The averages 3-5 are clearly compatible with the spatial homogeneity of the ensemble; however the entire procedure of solving the Vlasov Equation order by order and then performing a statistical average inherently entails much more information (and presumably algebra) in the early stages of the analysis than is required to describe the turbulent ensemble. A more direct approach would be to carry out the statistical averaging a priori rather than a posteriori and obtain dynamical equations for ensemble quantities at the outset.

Utilizing Eq. (1) as the dynamical equation for an individual system, in Sec. II a hierarchy of equations is constructed for the correlations in a spatially homogeneous ensemble of Vlasov plasmas.[2,11] The formalism automatically embodies the averaging procedure given in Eqs. (3)-(5). In addition it is demonstrated that this Vlasov hierarchy is identical to the BBGKY hierarchy for a spatially homogeneous ensemble if the effects of single particle encounters are deleted from the BBGKY formalism. In Sec. III the Vlasov hierarchy is ordered in the weak turbulence parameter $\lambda \sim E^2/nmv_{av}^2$ and the questions of closure and time validity of the formalism are examined.

II. THE VLASOV HIERARCHY

Working within the Vlasov framework we rewrite Eq. (1) as

$$\frac{\partial f_{a_1}}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} f_{a_1} = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int \frac{\partial \phi_{a_1 a_2}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} (f_{a_1}(1) f_{a_2}(2)) d(2). \tag{6}$$

From Eq. (6) the following chain of equations can be simply constructed advancing $f_{a_1} f_{a_2}$ in terms of $f_{a_1} f_{a_2} f_{a_3}$, and $f_{a_1} f_{a_2} f_{a_3}$ in terms of $f_{a_1} f_{a_2} f_{a_3} f_{a_4}$, etc.,

$$\frac{\partial}{\partial t} (f_{a_1} f_{a_2}) + \left\{ \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} (f_{a_1} f_{a_2}) + (1 \leftrightarrow 2) \right\}$$

$$= \left\{ \frac{1}{m_{a_1}} \sum_{a_3} n_{a_3} \int \frac{\partial \phi_{a_1 a_3}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} f_{a_1} f_{a_2} f_{a_3} d(3) + (1 \leftrightarrow 2) \right\} , \tag{7}$$

$$\frac{\partial}{\partial t}\left(f_{a_1} f_{a_2} \cdots f_{a_s}\right) + \sum_{s'=2}^{s}\left\{\left(\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} f_{a_1} \cdots f_{a_s}\right) + (s' \leftrightarrow 1)\right\}$$

$$= \sum_{s'=2}^{s}\left\{\left(\frac{1}{m_{a_1}} \sum_{a_{s+1}} n_{a_{s+1}} \int \frac{\partial \phi_{a_1 a_{s+1}}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} f_{a_1} f_{a_2} \cdots f_{a_{s+1}} \, d(s+1)\right) + (s' \leftrightarrow 1)\right\} \qquad (8_s)$$

$$\vdots$$

Equation (6) is to be interpreted as the dynamical equation for an individual system. We now imagine a collection of such Vlasov systems and average Eqs. (6)-(8_s) over this ensemble. The ensemble average, $\langle f_{a_1} \cdots f_{a_n} \rangle$, may be viewed as the arithmetic mean of $f_{a_1} \cdots f_{a_n}$ taken over a large number of systems, or as an average over a (probability) distribution of systems. With the problem of spatially uniform turbulence in mind, it is assumed that the various averages are invariant under translation.[12] The analysis for a spatially inhomogeneous ensemble is a simple extension of the technique described here and the results are quoted in the concluding section of this article. In the discussion which follows it is convenient to write f_{a_1} in terms of its ensemble average plus a fluctuation, i.e.,

$$f_{a_1} = \langle f_{a_1} \rangle + \delta f_{a_1} \, , \qquad \langle \delta f_{a_1} \rangle = 0 \, . \qquad (9)$$

Averaging Eqs. (6)-(8_s) and utilizing the macroscopic charge neutrality condition $\sum_a n_a e_a = 0$, give the following chain of equations for $\langle f_{a_1} \rangle$, $\langle \delta f_{a_1} \delta f_{a_2} \rangle$, ...

$$\frac{\partial}{\partial t} \langle f_{a_1} \rangle = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int d(2) \, \frac{\partial \phi_{a_1 a_2}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} \langle \delta f_{a_1} \delta f_{a_2} \rangle \, , \qquad (10)$$

$$\frac{\partial}{\partial t} \langle \delta f_{a_1} \delta f_{a_2} \rangle + \left\{\left(v_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_1} \delta f_{a_2} \rangle\right) + (1 \leftrightarrow 2)\right\}$$

$$- \left\{\left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} \langle f_{a_1} \rangle \cdot \sum_{a_3} n_{a_3} \int d(3) \, \frac{\partial \phi_{a_1 a_3}}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_2} \delta f_{a_3} \rangle\right) + (1 \leftrightarrow 2)\right\}$$

$$= \left\{ \frac{1}{m_{a_1}} \sum_{a_3} n_{a_3} \int d(3) \frac{\partial\phi_{a_1a_3}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} \langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \rangle + (1 \leftrightarrow 2) \right\} , \qquad (11)$$

$$\frac{\partial}{\partial t} \langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \rangle + \sum_{s'=2}^{3} \left\{ \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \rangle + (s' \leftrightarrow 1) \right\}$$

$$- \sum_{s'=2}^{3} \left\{ \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} \langle f_{a_1} \rangle \cdot \sum_{a_4} n_{a_4} \int d(4) \frac{\partial\phi_{a_1a_4}}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_2} \delta f_{a_3} \delta f_{a_4} \rangle \right) + (s' \leftrightarrow 1) \right\}$$

$$= \sum_{s'=2}^{3} \left\{ \left(\frac{1}{m_{a_1}} \sum_{a_4} n_{a_4} \int d(4) \frac{\partial\phi_{a_1a_4}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} [\langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \delta f_{a_4} \rangle \right.\right.$$

$$\left.\left. - \langle \delta f_{a_1} \delta f_{a_4} \rangle \langle \delta f_{a_2} \delta f_{a_3} \rangle] \right) + (s' \leftrightarrow 1) \right\} , \qquad (12)$$

$$\vdots$$

$$\frac{\partial}{\partial t} \langle \delta f_{a_1} \cdots \delta f_{a_s} \rangle + \sum_{s'=2}^{s} \left\{ \left(\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_1} \cdots \delta f_{a_s} \rangle \right) + (1 \leftrightarrow s') \right\}$$

$$- \sum_{s'=2}^{s} \left\{ \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} \langle f_{a_1} \rangle \cdot \sum_{a_{s+1}} n_{a_{s+1}} \int d(s+1) \frac{\partial\phi_{a_1a_{s+1}}}{\partial \underset{\sim}{x}_1} \langle \delta f_{a_2} \cdots \delta f_{a_{s+1}} \rangle \right) + (1 \leftrightarrow s') \right\}$$

$$= \sum_{s'=2}^{s} \left\{ \left(\frac{1}{m_{a_1}} \sum_{a_{s+1}} n_{a_{s+1}} \int d(s+1) \frac{\partial\phi_{a_1a_{s+1}}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} [\langle \delta f_{a_1} \delta f_{a_2} \cdots \delta f_{a_{s+1}} \rangle \right.\right.$$

$$\left.\left. - \langle \delta f_{a_1} \delta f_{a_{s+1}} \rangle \langle \delta f_{a_2} \cdots \delta f_{a_s} \rangle] \right) + (1 \leftrightarrow s') \right\} \qquad (13_s)$$

$$\vdots$$

By virtue of the assumed spatial uniformity of the ensemble

$\langle f_{a_1} \rangle$ is independent of $\underset{\sim}{x}_1$,

$\langle \delta f_{a_1} \delta f_{a_2} \rangle$ depends on the difference $\underset{\sim}{x}_1 - \underset{\sim}{x}_2$,

$$\vdots$$

and similarly $\langle \delta f_{a_1} \delta f_{a_2} \cdots \delta f_{a_s} \rangle$ may be taken as a function of the differences $\underset{\sim}{x}_1 - \underset{\sim}{x}_2, \underset{\sim}{x}_1 - \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_1 - \underset{\sim}{x}_s$ to ensure invariance under translation.

The system of equations, (10)-(13_s), will be practical to use only if the ensemble averages, $\langle \delta f_{a_1} \ldots \delta f_{a_s} \rangle$, in some sense become small as s increases (as is expected in problems of weak plasma turbulence), and closure can be obtained at some level. This is examined at a later point. An alternate hierarchy can be constructed from Eqs. (10)-(13_s). We define the s-irreducible correlation function, ${}_sg_{a_1 \ldots a_s}$, by subtracting from $\langle \delta f_{a_1} \delta f_{a_2} \ldots \delta f_{a_s} \rangle$ all irreducible correlations of lower order. Keeping in mind $\langle \delta f_{a_i} \rangle = 0$, it is evident that

$$
{}_2g_{a_1a_2} \equiv \langle \delta f_{a_1} \delta f_{a_2} \rangle \, , \tag{14}
$$

$$
{}_3g_{a_1a_2a_3} \equiv \langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \rangle \, , \tag{15}
$$

$$
\begin{aligned}
{}_4g_{a_1a_2a_3a_4} \equiv {} & \langle \delta f_{a_1} \delta f_{a_2} \delta f_{a_3} \delta f_{a_4} \rangle - {}_2g_{a_1a_4} \, {}_2g_{a_2a_3} \\
& - {}_2g_{a_1a_2} \, {}_2g_{a_3a_4} - {}_2g_{a_1a_3} \, {}_2g_{a_2a_4} \, ,
\end{aligned} \tag{16}
$$

$$\vdots$$

$$
\begin{aligned}
{}_sg_{a_1a_2 \ldots a_s} \equiv {} & \langle \delta f_{a_1} \delta f_{a_2} \ldots \delta f_{a_s} \rangle \\
& - \sum_{s'=2}^{s_M'} \sum_{\{1,2,\ldots s\}} {}_{s'}g_{a_1 \ldots a_{s'}} \, {}_{s-s'}g_{a_{s'+1} \ldots a_s} \, .
\end{aligned} \tag{17_s}
$$

$$\vdots$$

The sum $\sum_{\{1,2,\ldots,s\}}$ is over permutations of $\{1,2,\ldots,s\}$, and $s_M' = (s-1)/2$ if s is odd, and s/2 if s is even. Utilizing definitions (14)-(17_s) in Eqs. (10)-(13_s), we have

$$
\frac{\partial \langle f_{a_1} \rangle}{\partial t} = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int d(2) \, \frac{\partial \phi_{a_1a_2}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} \, {}_2g_{a_1a_2} \, , \tag{18}
$$

$$\frac{\partial}{\partial t}\, {}_2g_{a_1a_2} + \left\{ \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1}\, {}_2g_{a_1a_2} + (1 \leftrightarrow 2) \right\}$$

$$- \left\{ \frac{1}{m_{a_1}} \frac{\partial \langle f_{a_1} \rangle}{\partial \underset{\sim}{v}_1} \cdot \sum_{a_3} n_{a_3} \int d(3)\, \frac{\partial \phi_{a_1a_3}}{\partial \underset{\sim}{x}_1}\, {}_2g_{a_2a_3} + (1 \leftrightarrow 2) \right\}$$

$$= \left\{ \frac{1}{m_{a_1}} \sum_{a_3} n_{a_3} \int d(3)\, \frac{\partial \phi_{a_1a_3}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1}\, {}_3g_{a_1a_2a_3} + (1 \leftrightarrow 2) \right\}, \qquad (19)$$

$$\frac{\partial}{\partial t}\, {}_3g_{a_1a_2a_3} + \sum_{s'=2}^{3} \left\{ \left(\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1}\, {}_3g_{a_1a_2a_3} \right) + (1 \leftrightarrow s') \right\}$$

$$- \sum_{s'=2}^{3} \left\{ \frac{1}{m_{a_1}} \frac{\partial \langle f_{a_1} \rangle}{\partial \underset{\sim}{v}_1} \cdot \sum_{a_4} n_{a_4} \int d(4)\, \frac{\partial \phi_{a_1a_4}}{\partial \underset{\sim}{x}_1}\, {}_3g_{a_2a_3a_4} + (1 \leftrightarrow s') \right\}$$

$$= \sum_{s'=2}^{3} \left\{ \left(\frac{1}{m_{a_1}} \sum_{a_4} n_{a_4} \int d(4)\, \frac{\partial \phi_{a_1a_4}}{\partial \underset{\sim}{x}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} [\, {}_4g_{a_1a_2a_3a_4} \right. \right.$$

$$\left. + {}_2g_{a_1a_2}\, {}_2g_{a_3a_4} + {}_2g_{a_1a_3}\, {}_2g_{a_2a_4}] \right)$$

$$\left. + (s' \leftrightarrow 1) \right\}, \qquad (20)$$

$$\vdots$$

and

$$\frac{\partial}{\partial t}\, {}_sg_{a_1 \ldots a_s} + \sum_{s'=2}^{s} \left\{ \left(\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1}\, {}_sg_{a_1 \ldots a_s} \right) + (s' \leftrightarrow 1) \right\}$$

$$- \sum_{s'=2}^{s} \left\{ \left(\frac{1}{m_{a_1}} \frac{\partial \langle f_{a_1} \rangle}{\partial \underset{\sim}{v}_1} \cdot \sum_{a_{s+1}} n_{a_{s+1}} \int d(s+1)\, \frac{\partial \phi_{a_1a_{s+1}}}{\partial \underset{\sim}{x}_1}\, {}_sg_{a_2a_3 \ldots a_{s+1}} \right) + (s' \leftrightarrow 1) \right\}$$

$$= \sum_{s'=2}^{s} \left\{ \left(\frac{1}{m_{a_1}} \sum_{a_{s+1}} n_{a_{s+1}} \int d(s+1)\, \frac{\partial}{\partial \underset{\sim}{x}_1} \phi_{a_1a_{s+1}} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} [\, {}_{s+1}g_{a_1a_2 \ldots a_{s+1}} \right. \right.$$

$$\left. \left. + \sum_{s''=2}^{s''_M} \sum_{\{1,2,\ldots,s\}} {}_{s''}g_{a_1 \ldots a_{s''}}\, {}_{s+1-s''}g_{a_{s''+1} \ldots a_s a_{s+1}}] \right) + (s' \leftrightarrow 1) \right\}. \qquad (21_s)$$

$$\vdots$$

Equations (18)-(21_s) form an interconnected chain in which $\langle f_{a_1}\rangle$ is driven by two-correlations, ${}_2g$; ${}_2g$ in turn is driven by three-correlations, ${}_3g$, etc. The differential-integral operator acting on the s-correlation ${}_sg_{a_1 \ldots a_s}$ in Eq. (21_s) is clearly a time derivative plus s Landau operators acting on particles 1,2,...,s, respectively. Since the original dynamical (Vlasov) equation is void of the effects of single particle encounters, the only many-body interactions described by the hierarchy (18)-(21_s) are collective and may be roughly classified as wave-particle, multiwave-particle and multiwave. As such, a valid description via Eqs. (18)-(21_s) can be expected only for times less than the relaxation time due to single particle encounters.

We reiterate that utilizing the Vlasov hierarchy (Eqs. (18)-(21_s)) in problems of plasma turbulence represents a more direct approach than the usual procedure (see for example Ref. 8) since the statistical averaging is done a priori rather than a posteriori. This formalism automatically embodies the averaging process described in Eqs. (3)-(5), since relations (14)-(16), the spatial homogeneity of the ensemble, and

$$\delta E(\underset{\sim}{k}_1,t) = \sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}}{i|k_1|} \int \delta f_{a_1}(\underset{\sim}{k}_1,\underset{\sim}{v}_1,t)d\underset{\sim}{v}_1 ,$$

imply

$$\langle \delta E(\underset{\sim}{k}_1,t)\delta E(\underset{\sim}{k}_2,t)\rangle = {}_2\Psi(\underset{\sim}{k}_1,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2) , \tag{22}$$

$$\langle \delta E(\underset{\sim}{k}_1,t)\delta E(\underset{\sim}{k}_2,t)\delta E(\underset{\sim}{k}_3,t)\rangle = {}_3\Psi(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3) , \tag{23}$$

$$\langle \delta E(\underset{\sim}{k}_1,t)\delta E(\underset{\sim}{k}_2,t)\delta E(\underset{\sim}{k}_3,t)\delta E(\underset{\sim}{k}_4,t)\rangle$$

$$= {}_4\Psi(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3+\underset{\sim}{k}_4)$$

$$+ {}_2\psi(\underset{\sim}{k}_1,t)\,{}_2\psi(\underset{\sim}{k}_3,t)\,\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\,\delta(\underset{\sim}{k}_3+\underset{\sim}{k}_4)$$

$$+ {}_2\psi(\underset{\sim}{k}_1,t)\,{}_2\psi(\underset{\sim}{k}_3,t)\,\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_4)\,\delta(\underset{\sim}{k}_3+\underset{\sim}{k}_2)$$

$$+ {}_2\psi(\underset{\sim}{k}_1,t)\,{}_2\psi(\underset{\sim}{k}_2,t)\,\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_3)\,\delta(\underset{\sim}{k}_2+\underset{\sim}{k}_4)\ , \qquad (24)$$

$$\vdots$$

where

$${}_s\psi(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\ldots,\underset{\sim}{k}_{s-1},t)$$

$$\equiv \sum_{a_1,\ldots,a_s} \left(\frac{4\pi n_{a_1} e_{a_1}}{i|\underset{\sim}{k}_1|}\right)\left(\frac{4\pi n_{a_2} e_{a_2}}{i|\underset{\sim}{k}_2|}\right)\cdots \int d\underset{\sim}{v}_1 \cdots d\underset{\sim}{v}_s \; {}_s g_{a_1\ldots a_s}\ . \qquad (25)$$

Equations (22)-(24) are identical in form to Eqs. (3)-(5) if the irreducible correlations, ${}_s g$ $s \geq 4$, are omitted. Conditions for neglecting higher correlations in this manner must be examined in the context of Eqs. (18)-(21_s). This and related problems of closure are examined at a later point where an explicit ordering of the Vlasov hierarchy is performed.

It is evident that the hierarchy (18)-(21_s) is identical to the BBGKY hierarchy[13] for a spatially homogeneous ensemble of plasmas if all terms associated with the discreteness of matter[14] are deleted. These discreteness terms, which are displayed in Eqs. (A-1) and (A-2), may formally be removed from the BBGKY formalism by subdividing the charged particles into smaller and smaller units such that

$$e_\alpha \to 0\ , \quad n_\alpha e_\alpha = \text{constant},$$
$$m_\alpha \to 0\ , \quad n_\alpha m_\alpha = \text{constant}\ .$$
$$1/n_\alpha \to 0\ ,$$

In this limit collective effects are retained as the plasma frequency, ω_p, and the Debye length, λ_D, remain constant; however, the plasma parameter of smallness,

$\epsilon_p \sim (1/n\lambda_D^3)$, tends to zero thus removing the effects of single particle encounters. The Vlasov hierarchy (18)-(21_s) is then identical to the $\epsilon_p \to 0$ limit of the BBGKY hierarchy. This identity should in fact be the case if Eqs. (18)-(21_s) are to represent an acceptable plasma description excluding the effects of single particle encounters. One arrives at the same conclusion in the $\epsilon_p \to 0$ version of the Klimontovich formalism[15] (since the BBGKY hierarchy may be obtained from the Klimontovich framework). Alternatively the similarity may be demonstrated directly by taking advantage of the similar form of the dynamical equation describing the phase function

$$\rho_{a_1}(x_1, v_1, t) = \frac{1}{(N_{a_1}/V)} \sum_{i_{a_1}}^{N_{a_1}} \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_{i_{a_1}}(t))\delta(\underset{\sim}{v}_1 - \underset{\sim}{v}_{i_{a_1}}(t)) ,$$

and Eq. (6). This is shown in Appendix B.

III. WEAK TURBULENCE ORDERING

As mentioned earlier the hierarchy (18)-(21_s) (or the $\epsilon_p \to 0$ version of the BBGKY formalism) will be a practical description to use only if the correlations ${}_s g$ become small as s increases and closure can be obtained at some level. For purposes of estimating the relative magnitudes of terms in Eqs. (18)-(21_s) we introduce

$\langle \phi \rangle \sim$ characteristic strength of potential $\sim e^2/r_0$, where

$r_0 \sim$ effective range of potential,

$v_{av} \sim$ characteristic particle speed,

$\tau \sim$ typical time scale of interest.

For simplicity, a single species of interacting electrons in a fixed background of neutralizing ions is assumed. Taking r_0 to be of order of the Debye length, i.e.,

$$r_0^{\,2} \sim (mv_{av}^{\,2}/4\pi ne^2) ,$$

then

$$\left(\frac{\langle \phi \rangle}{mv_{av}^{\,2}}\right)(nr_0^{\,3}) \sim 1 .$$

In addition

$$f \sim (1/v_{av}^{\,3}) ,$$

from the normalization condition $\int f d\underset{\sim}{v} = 1$. On the short time scale $\tau_0 \sim (r_0/v_{av}) \sim (1/\omega_p)$, the terms in Eqs. (18)-(21_s) then stand in the ratio

$$\dot{f} : {}_2gv_{av}^{\,3} , \tag{26}$$

$${}_2\dot{g} : {}_2g : {}_2g : {}_3gv_{av}^{\,3} , \tag{27}$$

$${}_3\dot{g} : {}_3g : {}_3g : {}_4gv_{av}^{\,3} : {}_2g_2gv_{av}^{\,3} , \tag{28}$$

$$\vdots$$

$${}_s\dot{g} : {}_sg : {}_sg : {}_{s+1}gv_{av}^{\,3} : {}_2g_{s-1}gv_{av}^{\,3} : {}_3g_{s-2}gv_{av}^{\,3} : \dots , \tag{29_s}$$

respectively.

The basic parameter of smallness in weak plasma turbulence is the ratio of the energy density in the fluctuating electric fields to the kinetic energy density of the plasma particles, i.e.,

$$\lambda \sim \frac{\langle \delta\underset{\sim}{E}(\underset{\sim}{x}_1)\cdot\delta\underset{\sim}{E}(\underset{\sim}{x}_1)\rangle}{nmv_{av}^{\,2}} \ll 1 . \tag{30}$$

It is also implicitly assumed in utilizing Eqs. (18)-(21_s), that

$$\epsilon_p \ll \lambda . \tag{31}$$

That is to say the energy density in the turbulent, fluctuating fields is assumed large compared to the energy density in the fluctuating fields that would exist in thermal equilibrium since

$$\epsilon_p \sim \frac{1}{nmv_{av}^2} \langle \delta\underset{\sim}{E}\cdot\delta\underset{\sim}{E}\rangle_{\text{THERM. EQ.}} \quad .$$

For a single species of interacting electrons, $\langle \delta\underset{\sim}{E}\cdot\delta\underset{\sim}{E}\rangle$ may be written

$$\langle \delta\underset{\sim}{E}(\underset{\sim}{x}_1)\cdot\delta\underset{\sim}{E}(\underset{\sim}{x}_1)\rangle = \frac{n^2}{e^2}\iint \frac{\partial\phi_{12'}}{\partial\underset{\sim}{x}_1}\cdot\frac{\partial\phi_{13'}}{\partial\underset{\sim}{x}_1}\langle \delta f(2')\delta f(3')\rangle d(2')d(3')$$

$$\equiv \frac{n^2}{e^2}\iint \frac{\partial\phi_{12'}}{\partial\underset{\sim}{x}_1}\cdot\frac{\partial\phi_{13'}}{\partial\underset{\sim}{x}_1}\, {}_2g((2'),(3'))d(2')d(3') \ . \qquad (32)$$

Making order of magnitude estimates as before, Relation 30 becomes

$$\lambda \sim {}_2g v_{av}^{\ 6} \ll 1 \ . \qquad (33)$$

With ${}_2g \sim \lambda$ and noting that ${}_3g$ has ${}_2g_3g$ driving terms, we assume

$${}_3g \sim \lambda^2 \text{ to leading order} \ . \qquad (34)$$

Similarly ${}_4g$ has ${}_2g_3g$ driving terms giving

$${}_4g \sim \lambda^3 \text{ to leading order} \ , \qquad (35)$$

and in general

$${}_sg \sim \lambda^{s-1} \ . \qquad (36_s)$$

The level of sophistication with which we describe the evolution of $\langle f\rangle$ (Eq. (18)) and the energy density of the electric field (Eq. (32)) depends vitally on the accuracy of description of ${}_2g$. With this in mind, the following remarks can be made regarding closure of the hierarchy (18)-(21_s) in the context of the estimates (26)-(29_s) and (33)-(36_s). In order to calculate ${}_2g$ to order λ^2 and describe the leading order ${}_2g$ for times $t \sim \tau_0/\lambda$, ${}_4g$ and higher correlations may be neglected. This follows upon noting that ${}_3g$ is needed only to an accuracy λ^2. Similarly, to calculate ${}_2g$ to order λ^n and describe the leading order ${}_2g$ for times $t \sim \tau_0/\lambda^{n-1}$, ${}_{n+2}g$ and higher correlations may be omitted. It must be emphasized that inherent in the estimates and associated comments of this section is the assumption that the instability (initially) driving the fluctuating fields is sufficiently weak that the ordering

$${}_s g \sim \lambda^{s-1}$$

is not worsened (to ${}_s g \sim \lambda^{s-2}$, say) during the course of time.

In the case of a real plasma, the plasma parameter ϵ_p although small, is not zero as is implicitly assumed in utilizing the hierarchy (18)-(21_s). Consequently Eqs. (18)-(21_s) will not hold indefinitely, as the effects of single particle encounters will ultimately play a role in the time evolution of the plasma. In order to obtain some estimate of the regime of validity of the Vlasov hierarchy, the additional terms relating to single particle encounters are explicitly displayed within the BBGKY framework in Appendix A. There it is shown that the dominant contribution in the equation for ${}_2g$ is of order $\epsilon_p ff$, and of order $\epsilon_p f_{s-1}g$ in the equation for ${}_s g$. These effects are negligible in leading order in the context of the estimate ${}_s g \sim \lambda^{s-1}$, and the inequality $\lambda \gg \epsilon_p$; however they do become important in times t where

$$t \sim \frac{\lambda}{\epsilon_p} \tau_0 \; . \tag{37}$$

Depending on the smallness of ϵ_p relative to λ, the ensemble may evolve for an appreciable length of time through the collective effects of wave-particle, multiwave and multiwave-particle interactions as described by the hierarchy (18)-(21_s). However, for times longer than the estimate given in Eq. (37) the effects of single particle encounters, A-1 and A-2 must be included in Eqs. (18)-(21_s). The maximal ordering including single particle encounters but consistent with $\epsilon_p \ll \lambda$, is clearly $\epsilon_p \sim \lambda^2$.

CONCLUDING REMARKS

In problems of weak plasma turbulence the motivation is strong to use the Vlasov or BBGKY hierarchies where appropriate, with closure at a certain level. The usual "random phase approximation" is automatically embedded in these formalisms and no a posteriori averages need be performed.

In conclusion we state without proof that if the assumption of spatial uniformity of the ensemble is removed in the analysis of Sec. I the net result is to add to the left hand side of Eq. (18)

$$\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_1} \langle f_{a_1} \rangle - \frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} \langle f_{a_1} \rangle \cdot \sum_{a_2} n_{a_2} \int d(2) \frac{\partial \phi_{a_1 a_2}}{\partial \underset{\sim}{x}_1} \langle f_{a_2} \rangle ,$$

and to the left side of Eq. (21_s) ($s = 2,3,\ldots$)

$$-\sum_{s'=2}^{s} \left\{ \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} {}_s g_{a_1 \ldots a_s} \cdot \sum_{a_{s+1}} n_{a_{s+1}} \int d(s+1) \frac{\partial}{\partial \underset{\sim}{x}_1} \phi_{a_1 a_{s+1}} \langle \overline{f}_{a_{s+1}} \rangle \right) + (s' \leftrightarrow 1) \right\} .$$

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APPENDIX A

Single Particle Encounters

In the BBGKY formalism the inclusion of single particle encounters associated with discreteness of matter add to the right hand side of Eq. (19) the terms

$$\frac{\partial \phi_{a_1 a_2}}{\partial \underset{\sim}{x}_1} \cdot \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{1}{m_{a_2}} \frac{\partial}{\partial \underset{\sim}{v}_2} \right) (\langle f_{a_1} \rangle \langle f_{a_2} \rangle + {}_2 g_{a_1 a_2}) \ . \qquad \text{(A-1)}$$

Similarly, the addition to the right side of Eq. (21_s) for ${}_s g_{a_1 a_2 \ldots a_s}$ is

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{s,s} \frac{1}{m_{a_i}} \frac{\partial \phi_{a_i a_j}}{\partial \underset{\sim}{x}_i} \cdot \frac{\partial}{\partial \underset{\sim}{v}_i} \Bigg(\langle f_{a_i} \rangle \, {}_{s-1} g_{\ldots a_j \ldots} + {}_s g_{\ldots a_i \ldots a_j \ldots} + \sum_{s'=2}^{s_M'} \sum_{\{1, \ldots s\}} {}_{s'} g \ {}_{s-s'} g \Bigg) \ . \qquad \text{(A-2)}$$

Making order of magnitude estimates of (A-1) and (A-2) with

$$\frac{\partial \phi}{\partial x} \sim \frac{\phi}{r_0} \ ; \quad r_0 \sim \lambda_D \sim \left(\frac{m v_{av}^2}{4\pi n e^2} \right)^{1/2} \ ,$$

and

$$\frac{\partial}{\partial v} \sim \frac{1}{v_{av}} \ ,$$

we have

$$\frac{(v_{av}/r_0)}{(n r_0^3)} \quad (ff : {}_2 g) \ , \qquad \text{(A-3)}$$

and

$$\frac{(v_{av}/r_0)}{(n r_0^3)} \quad (f \, {}_{s-1} g : {}_s g : {}_2 g \, {}_{s-2} g : {}_3 g \, {}_{s-3} g : \ldots) \ , \qquad \text{(A-4)}$$

respectively. Scaling with respect to the short time scale $\tau_0 \sim r_0/v_{av} \sim 1/\omega_p$, estimates (A-3) and (A-4) may be rewritten as

$$\epsilon_p \, (ff:{}_2g) \; , \tag{A-5}$$

$$\epsilon_p \, (f_{s-1}g:{}_sg:{}_2g_{s-2}g:{}_3g_{s-3}g:\dots) \; , \tag{A-6}$$

where

$$\epsilon_p \sim \frac{1}{n\lambda_D^{\;3}}$$

is the usual plasma parameter of smallness. Assuming as in the text that to leading order

$$_sg \sim \lambda^{s-1} \; , \tag{A-7}$$

it is clear from the above estimates that the dominant effect of single particle encounters in the equation for $_sg$ is of order

$$\epsilon_p \lambda^{s-2} \; . \tag{A-8}$$

For the hierarchy (18)-(21_s) to represent a valid description in leading orders on the short time scale τ_0, we thus require that

$$\lambda \gg \epsilon_p \; . \tag{A-9}$$

This is just the condition that the energy density in the turbulent fluctuating fields be large compared to the energy density that would exist in the fluctuating fields in thermal equilibrium.

The estimates (A-3) and (A-4) indicate that the system of equations (18)-(21_s) holds only for times t such that

$$t \lesssim \frac{\lambda}{\epsilon_p} \tau_0 \tag{A-10}$$

For times longer than this the effects of single particle encounters, (A-1) and (A-2), must be included. The time estimate, (A-10), may be lengthened by strengthening the inequality (A-9) through an increase of the plasma mean kinetic energy. This decreases ϵ_p and stretches out the relaxationtime due to single particle encounters, thus allowing the plasma to evolve for a considerable length of time through collective interactions as described by Eqs. (18)-(21_s).

APPENDIX B

The Klimontovich Formalism

The phase function

$$\rho_{a_1}(\underset{\sim}{x}_1,\underset{\sim}{v}_1,t) \equiv \frac{1}{(N_{a_1}/V)} \sum_{i_{a_1}=1}^{N_{a_1}} \delta(\underset{\sim}{x}_1-\underset{\sim}{x}_{i_{a_1}}(t))\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_{i_{a_1}}(t)) \qquad \text{(B-1)}$$

satisfies the equation[15]

$$\frac{\partial\rho_{a_1}}{\partial t} + \underset{\sim}{v}_1\cdot \frac{\partial}{\partial \underset{\sim}{x}_1} \rho_{a_1} = \frac{1}{m_{a_1}} \sum_{a_2} \frac{N_{a_2}}{V} \int \frac{\partial\phi_{a_1a_2}}{\partial \underset{\sim}{x}_1} (|\underset{\sim}{x}_1-\underset{\sim}{x}'|)\cdot \frac{\partial}{\partial \underset{\sim}{v}_1} \rho_{a_1}(\underset{\sim}{x}_1,\underset{\sim}{v}_1,t)$$

$$\times \left\{\rho_{a_2}(\underset{\sim}{x}',\underset{\sim}{v}',t) - \frac{1}{N_{a_2}/V} \delta(\underset{\sim}{x}_1-\underset{\sim}{x}')\delta(\underset{\sim}{v}_1-\underset{\sim}{v}')\right\} d\underset{\sim}{x}'d\underset{\sim}{v}' \qquad \text{(B-2)}$$

by virtue of the classical equations of motion

$$\frac{d}{dt}\underset{\sim}{x}_i(t) = \underset{\sim}{v}_i(t) \ ; \quad m_i \frac{d}{dt}\underset{\sim}{v}_i(t) = -\sum_{\substack{j \\ j\neq i}} \frac{\partial\phi_{ij}}{\partial \underset{\sim}{x}_i} .$$

With the exception of the δ-function term appearing in the integral, Eq. (B-2) is identical in form to Eq. (6). Using the explicit form of ρ given in (B-1), the definition of the s-particle reduced distribution

$$F_{a_1\ldots a_s}(1,2,\ldots,s) \equiv V^s \int f_{N_{a_1}N_{a_2}\ldots} d(s+1)\ldots \qquad \text{(B-3)}$$

where $f_{N_{a_1}N_{a_2}\ldots}$ is the Liouville distribution, and the normalization condition

$$1 = \int f_{N_{a_1}N_{a_2}\ldots} d(1)\ldots ,$$

it is straightforward to show in the N-V limit, ($N \to \infty$, $V \to \infty$, $N/V \to n$, a constant), by averaging over $f_{N_{a_1}N_{a_2}\ldots}$ (denote by superbar) that

$$F_{a_1}(1) = \overline{\rho_{a_1}(1)} \qquad \text{(B-4)}$$

$$F_{a_1a_2}(1,2) = \overline{\rho_{a_1}(1)(\rho_{a_2}(2) - \frac{\delta_{a_1a_2}}{n_{a_2}}\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_2))} \qquad \text{(B-5)}$$

$$\vdots$$

$$F_{a_1a_2\ldots a_s}(1,2,\ldots,s) = \overline{\rho_{a_1}(1)(\rho_{a_2}(2) - \frac{\delta_{a_1a_2}}{n_{a_2}}\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_2))}$$

$$\overline{\mathrm{x}\ldots\mathrm{x}(\rho_{a_s}(s) - \frac{\delta_{a_1a_s}}{n_{a_s}}\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_s)\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_s) - \ldots - \frac{\delta_{a_{s-1}a_s}}{n_{a_s}}\delta(\underset{\sim}{x}_{s-1}-\underset{\sim}{x}_s)\delta(\underset{\sim}{v}_{s-1}\underset{\sim}{v}_s))} \qquad \text{(B-6)}$$

In the limit in which discreteness effects are deleted, we have that

$$F_{a_1\ldots a_s}(1,2,\ldots,s) \cong \overline{\rho_{a_1}(1)\rho_{a_2}(2)\ldots\rho_{a_s}(s)}$$

and that the equation for $\overline{\rho_{a_1}(1)\ldots\rho_{a_s}(s)}$ is identical in form to the equation for $\langle f_{a_1}(1)\ldots f_{a_s}(s)\rangle$ obtained by averaging Eq. (9_s). The equations for $F_{a_1}(1)$ and for the irreducible correlations, ${}_sG_{a_1\ldots a_s}$, are then trivially the same as Eqs. (18) and (21_s) for $\langle f_{a_1}(1)\rangle$ and ${}_sg_{a_1\ldots a_s}$.

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