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THE STABILITY THEORY OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS: A BIBLIOGRAPHICAL SURVEY

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1. Summary

A survey has been made of recent contributions to the stability theory of solutions to partial differential equations. Although some results employing approximate methods are mentioned, the main emphasis is on the use of Lyapunov's Direct Method. The number of applications employing this powerful method is limited. This is mainly due to the lack of systematic procedures for applying this method to partial differential equations. Some closely related results on the existence and uniqueness of solutions to partial differential equations employing the theory of distributions or generalized functions are also given.

2. Introduction

Since many physical systems must be represented by partial differential equations (PDE) the study of properties of solutions to these equations is very important. One of these properties is the stability or instability of certain solutions. Over the years a number of methods have been developed for investigating the stability properties of solutions to PDE. Since most of these methods require linearization, truncation or other approximations of the original equations, the results obtained might not be sufficient for stability. A method that appears most promising is Lyapunov's Direct Method, which is well-established in the stability theory of ordinary differential equations. When applied properly, this method can offer many advantages-- mathematical rigor, a minimum knowledge concerning the solutions, convenient introduction of nonlinearities, and meaningful interpretation of the results. One of the main drawbacks of the method is the difficulty in constructing a suitable Lyapunov function. The emphasis in this survey will be on contributions to the stability theory via Lyapunov's Direct Method.

The concept of stability can be interpreted in many different ways. In the following, stability will be referred to in the sense of Lyapunov, , that is: for a sufficiently small perturbation the system will remain close to the original solution for all future time.

In virtually all of the literature reviewed, the existence and uniqueness of solutions to the PDE in question have not been investigated. In some references it is assumed beforehand that the PDE defines a dynamical system. Many of the stability investigations have implicitly imposed conditions on the solutions to PDE which are virtually the same as requiring the PDE to act as if they are ordinary differential equations defined in some n-dimensional Euclidean space which is obviously not the case. Hence any rigorous investigation must take into account the peculiarities of solutions to PDE. Section 5 is devoted to a review of some of the literature which is concerned with establishing the properties of solutions to PDE.

3. Approximate Methods

The approximate methods are based on the principle of reducing the partial differential equations to a system of ordinary differential equations. This can be done by either approximating the model by one having a finite number of degrees of freedom via spatial discretization or by assuming a harmonic time dependence. The first case allows the application of the well-known techniques for analyzing the stability of ordinary differential equations, in particular for infinitesimally small perturbations, which is presented as a justification for the system linearization.

In the second case a modal analysis is in general necessary. To achieve this, use is made of the Galerkin process which is based on a truncation of the modal expansion. A linearization limits again the amount of work involved. The use of these methods is wide spread and well published. Bolotin's [1,2] are the most recently published books on the stability analysis of elastic systems.

Eckhaus [3] develops a theory for analyzing the stability properties of the solutions of nonlinear partial differential equations common to the field of fluid mechanics. This theory is based on asymptotic expansions with respect to suitably defined small parameters and series expansions in terms of eigenfunctions. The method becomes overly complicated for more extensive systems.

Lax and Richtmyer [4] and Lax [5] consider the general aspects of the stability of difference equations as derived from partial differential equations with constant and variable coefficients respectively.

Although far from exhaustive, these few examples of the literature available, give an indication of some of the approximate methods which have been applied. The next section deals with the literature presently (December, 1966) available to the authors on the application of Lyapunov Stability Theory to PDE.

4. Lyapunov's Direct Method

In recent years Lyapunov's Direct Method has occupied a prominent place in stability investigations of solutions to ordinary differential equations. The original work of Lyapunov [6] has generated thousands of contributions to the stability theory of solutions to ordinary differential equations and applications. Two of the more significant ones are the paper by Kalman and Bertram [7] and the book by LaSalle and Lefschetz [8]. However its application to the stability problem of the solutions to partial differential equations has been limited.

The application of Lyapunov's Direct Method for the stability analysis of solutions to partial differential equations requires a generalization of the method to function spaces in which a metric ρ is defined.

Consequently the concepts of stability are also defined in terms of this metric. A general stability theory now based on the existence of a Lyapunov functional is established by Zubov [9] for the invariant sets of dynamic systems in general metric spaces. Zubov [9, chapter 5] employs this general theory to derive results for the system of partial differential equations:

$$\frac{\partial \underline{u}}{\partial t} = \underline{f}(\underline{x}, \underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}}) \tag{1}$$

where $\underline{u}$ is an n-vector and $\underline{x}$ a k-vector.

It is furthermore assumed that the right hand side of (1) satisfies sufficient conditions for the existence, uniqueness and continuity of solutions to (1). In order to apply the earlier derived stability theorems (1) must define a dynamical system. This can be done by defining a metric on the general n-dimensional space Θ of functions $\underline{\theta}(\underline{x})$ and assigning to each $\underline{\theta}(\underline{x}) \in \Theta$ a solution of $\underline{u} = \underline{u}(t, \underline{\theta})$ of (1) having $\underline{u} = 0$ as invariant set.

Next Zubov compares the stability properties of the trivial solutions of the system

$$\frac{\partial \underline{u}}{\partial t} = \underline{f}(\underline{u}) + \underline{b}\frac{\partial \underline{u}}{\partial \underline{x}} \tag{2}$$

with $\underline{b}$ a constant n-vector, and

$$\frac{d\underline{u}}{dt} = \underline{f}(\underline{u}) \tag{3}$$

He shows that the asymptotic stability of the trivial solution of (3) assures the asymptotic stability of the trivial solution of (2). A similar result relates the stability behavior of the equilibrium of the system of partial differential equations of higher order

$$\frac{\partial \underline{u}}{\partial t} = \sum_{\sum_{j=1}^{k} \alpha_j = 0}^{M} \underline{A}_{\alpha_1, \ldots \alpha_k} \frac{\partial^{\alpha_1 + \ldots + \alpha_k} \underline{u}}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} \tag{4}$$

to the stability of the equilibrium of the system

$$\frac{d\underline{u}}{dt} = \underline{A}\ \underline{u} \tag{5}$$

The nature of the derived results is very theoretical and often difficult to implement in practical applications. However a direct application can be found in the work of Blodgett [10]. He considers a system of perturbed motion in one space variable of the form:

$$\frac{\partial \underline{u}}{\partial t} = \underline{D}(x)\ \frac{\partial \underline{u}}{\partial x} + \underline{B}(x)\underline{u} + \underline{\varepsilon}(x,\underline{u}) \tag{6}$$

with $0 \leq x \leq L$. $\underline{\varepsilon}(x,\underline{u})$ contains the higher order terms. For a continuous vector function $\underline{y}(x)$ on the interval $0 \leq x \leq L$, the norm of $\underline{y}$ is defined as

$$||y|| = [\int_0^L f(x)\underline{y}'\ \underline{y}\ dx]^{1/2}$$

where $f(x)$ satisfies the conditions $f(x) > 0$, $0 < x < L$ and $f(0)$, $f(L) \geq 0$. Zubov's stability theorem is used to find conditions on $\underline{F}(x)$ so that the functional

$$V(\underline{u}) = \int_0^L \underline{u}'\ \underline{F}\ \underline{u}\ dx$$

is a Lyapunov functional. An example pertaining to a chemical reactor has been worked out.

Early work on extending Lyapunov's Direct Method to partial differential equations was carried out by Volkov [11], see also Hahn [12]. He selects a certain family of solutions from among all solutions of a hyperbolic partial differential equation and considers an operator J. This operator associates each solution $u(x,t)$ from this family with a functional $J(u)$ depending on t. Integral inequalities are then used to define the concept of definiteness. The stability of the trivial solution is defined correspondingly.

However most of the results are not as general as the above ones. These other results reflect more direct applications to specific problems, thus allowing certain simplifications in deriving the stability conditions. It should be noted that in general not sufficient attention is paid to the question of existence and uniqueness of the solution to the specific problem.

Thus Movchan [13] considered the equation

$$\frac{\partial^4 u}{\partial x^4} - a\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0 \qquad (7)$$

with the boundary conditions

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x = 0 \text{ and } x = 1.$$

By defining the metric ρ in a suitable manner he is able to verify results from the theory of vibrations of plates by taking as Lyapunov functional:

$$V(u) = \int_0^1 (u_{xx}^2 + au_x^2 + u_t^2)\, dx$$

Similarly Movchan [14] verifies classical stability results for a system of hinged rectangular plates under compression the deflection of which, $u(x,y,t)$, is given by the dimensionless equation:

$$a_1 \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + a_2 \frac{\partial^4 u}{\partial y^4} - \pi^2 a_3 \frac{\partial^2 u}{\partial x^2} - \pi^2 a_4 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2} = 0 \tag{8}$$

with the boundary conditions

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ at } x = 0,\ x = 1 \qquad u = \frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0,\ y = 1$$

An important inequality, used repeatedly in deriving stability results is:

For a function $u(x)$ twice continuously differentiable and satisfying $u(0) = u(1) = 0$, the following inequality holds:

$$\int_0^1 u_{xx}^2 dx \geq \pi^2 \int_0^1 u_x^2 dx \geq \pi^4 \int_0^1 u^2 dx \tag{9}$$

Wang [15] uses a similar result to study the stability of a simplified flexible vehicle. The dimensionless equation of perturbed motion about its equilibrium state is given by

$$m(x) v_o^2 \ell^2 \frac{\partial^2 u(t,x)}{\partial t^2} + v_o \ell^3 k_d(t,x) \frac{\partial\, u(t,x)}{\partial t} = - \frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 u(t,x)}{\partial x}\right) \tag{10}$$

and the boundary conditions:

$$u(t,0) = 0; \left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = 0; \left. EI(x) \frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=1} = 0$$

and $\left.\frac{\partial}{\partial x} EI(x) \frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=1} = 2\pi\, \rho_a v_o^2 \ell^2 ab \left.\left[\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x}\right]\right|_{x=1}$

Taken as a Lyapunov functional is:

$$V = 1/2 \int_0^1 [m(x)v_o^2\ell^2u_t^2 + 2c_o v_o^2\ell^2 m(x)u_t u_x + 2\pi\rho_a v_o^2\ell^2 ab u_x^2 + EI(x)u_{xx}^2]dx \quad (11)$$

Parks [16] applies Lyapunov's Direct Method to the panel flutter problem. The equation in nondimensional form is given by:

$$\mu\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + d\frac{\partial^4 u}{\partial x^4} - f\frac{\partial^2 u}{\partial x^2} + M\frac{\partial u}{\partial x} = 0 \quad (12)$$

and boundary conditions

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x = 0 \text{ and } x = 1.$$

His final Lyapunov functional is

$$V(u) = \int_0^1 (\mu\, u_t^2 + u_x u_t + \frac{1}{2\mu}u^2 + f\, u_x^2 + du_{xx}^2)dx \quad (13)$$

The conditions obtained for the stability of the equilibrium are compared with those obtained with the Galerkin method.

Even though the system equations are linear, the selection of the Lyapunov functionals is not a trivial task in most of these applications. As distinct from the stability theory of systems of linear ordinary differential equations, no systematic way seems to be available for constructing Lyapunov functionals for linear partial differential equations. Another deficiency is the general interpretation of the obtained conditions for stability.

In order to talk about stability in a meaningful sense it is often necessary to put restrictions on the initial states. Although Volkov [11] implied this already in his method, the idea of introducing a second metric for this purpose seems to have been originated by Movchan [17]. As such stability is defined in terms of the two metrics, rather than one. Slobodkin

[18,19,20] applies this approach to systems with an infinite number of degrees of freedom. Wang [21] uses the same idea in a stability analysis of elastic an aeroelastic systems. But again he encounters difficulties in constructing Lyapunov functionals and in interpreting the obtained results.

Some contributions concern the extension of specific results for ordinary differential equations to partial equations. In [22] Lakshmikantham obtains theorems for the stability of solutions to parabolic partial differential equations. These results are based on majorizing Lyapunov like functions. However the selection of these Lyapunov functions remains as an apparently insurmountable task. Chou [23] extends Lyapunov's stability theorems to continuous mechanics. In particular stability theorems are discussed concerning laminar fluid flows between two parallel plates.

Most of the physical problems investigated so far are problems in mechanics. Thus it is quite natural that some attempts have been made to link Lyapunov's theorems with existing theorems in this field. The work of Fronteau [24] falls in this category. He links Lyapunov's problem of stability with the original form of Liouville's theorem.

Brayton and Miranker [25] use a theorem by Massera to establish stability conditions for a nonlinear system representing an electrical circuit. The constructed Lyapunov functionals are based on energy considerations. The peculiarity of their method is that they make the boundary conditions components of the general system state vector.

Wei [26] has studied the stability of a system of partial differential equations describing the first-order chemical reaction in the presence of a catalyst. The system reduces to a pair of identical partial differential equations of the form

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - \phi^2 \, y \exp \frac{\beta y(1-y)}{1+\beta(1-y)} \tag{14}$$

with $0 \leq x \leq 1$ and boundary conditions:

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad y(1) = 1.$$

After linearizing (14) the Euclidean metric has been taken as Lyapunov functional.

Brand [27] applies the Lyapunov stability theorems to the Navier-Stokes equations. He takes the vorticity as Lyapunov function. The given results are tentative. The same can be said about the results obtained by Hsu and Bailey [28] in a stability analysis of nonlinear reactor systems.

Pringle [29,30] explores the use of the Hamiltonian function as Lyapunov function for stability investigations of bodies with connected moving parts and damped mechanical systems. Rumiantsev [31] uses energy like functions as Lyapunov functions and bases his results for the stability of motion of solid bodies with liquid-filled cavities on the work of Chetaev.

In most of the references little attention is paid to the question of existence and uniqueness of solutions to the particular partial differential equation considered. Except in the cases where the system represents a dynamic system these assumptions are not immediate. In the next section some references will be given concerning recent results on the existence and uniqueness of solutions to partial differential equations describing important physical systems.

5. The Existence and Uniqueness of Solutions to Partial Differential Equations.

Many of the results available today concerning this subject are due to the introduction of generalized derivatives following the work of Sobolev in the 1930's on PDE. He introduced the Sobolev spaces, the $W^{m,p}(\Omega)$ spaces [32,33,34]. Essentially there are two methods of dealing with differential operators. The first method, L. Schwartz's theory of distributions [35] is based on the general theory of continuous operators. The second method is the development and application of a general theory of unbounded linear operators within the Banach space structure (Browder, [36]).

It is beyond the scope of this survey to give all the references concerning this subject, however much can be found in the book by Goldberg [37].

The wave equation is among the most important partial differential equations in physics. Especially the nonlinear wave equation occupies an important place in the quantum theory.

Jörgen's first results [38] concerning the existence and uniqueness of solutions in the large to the nonlinear wave equation:

$$u_{tt} - \Delta u + u^3 = f(t)$$

applied only to the case of one space variable. He subsequently extended these results to a three dimensional Euclidean space [39]. Sather [40] obtained also the existence of a global classical solution to the initial-boundary value problem for (15).

Lions and Strauss [41] consider the more general nonlinear evolution equation

$$A(t)\, u(t) + u''(t) + \beta(t;\, u(t),\, u'(t)) = f(t),\quad 0 \leq t \leq T \tag{16}$$

where A(t) is a general unbounded formally self-adjoint linear operator. They show the existence and uniqueness of a weak solution to (16) and as such they extend also Jörgens' results to more than three space variables. In [42] Lions obtained similar results for certain nonlinear parabolic partial differential equations of the form:

$$\frac{\partial u}{\partial t} + A(u) = f \qquad 0 \leq t \leq T \tag{17}$$

where A is a specific nonlinear elliptic operator. Many results on nonlinear partial differential equations are due to Browder and can be found in his papers on this subject.

Some results on the scattering problem of the quantum theory are given by Wilcox [43], who gives also a generalization of Sobolev's imbedding theorem.

One striking feature of the literature referenced is its contemporary character. Most of the results on solutions to PDE have been obtained using distributions or generalized functions, which in itself is an area of recent development. This emphasizes the importance of implementing and propagating now a basic understanding of these methods among engineers so that these techniques may be used for real engineering problems.

6. Conclusions

This survey of contributions to the stability of solutions to PDE shows that the impetus given to the stability theory of ordinary differential equations by Lyapunov's Direct Method has not yet materialized for PDE. The encouraging results together with the many advantages of this method emphasize a strong need for further investigations.

Due to the properties of the differential operators it is apparent that there has to be a much closer correlation between the system and the Lyapunov functional than for systems of ordinary differential equations. This in turn requires a thorough understanding of the solutions of partial differential equations.

The success of the application of this method to engineering problems will to a large extent depend on results from functional analysis and the development of the sophisticated mathematical theory of generalized functions.

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