# EXPLICIT INTEGRATION OF DIFFERENTIAL EQUATIONS

by

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ABSTRACT

A polynomial $Y_c(x)$ is found which approximates the solution to the differential equation

$$Y' = F(x,y) \quad , \quad Y(x_0) = A$$

in the interval $x_0 \leq x \leq x_0 + h$. The procedure employed is somewhat similar to Runge-Kutta methods and may be a special case thereof.



## 1. Introduction:

Axelsson[1] and Sims and O'Hara[2] have suggested methods of solving the differential equation

$$Y' = F(x,y) \quad , \quad Y(x_0) = A$$

by constructing a polynomial $Y_c(x)$ satisfying the conditions

$$(1) \qquad Y_c(x_0) = Y(x_0), \text{ and}$$

$$Y'(x_i) - F(x_i, Y(x_i)) = 0 \quad 0 \leq i < n$$

where $x_0, x_1 \cdots x_{n-1}$ are a set of quadrature points. In each of the previous instances construction of the desired polynomial requires an iterative solution to a system of equations. The purpose here is to describe a method of satisfying conditions (1) which does not involve iteration. Construction of the required polynomial by explicit methods is achieved at the cost of accuracy.

It is possible that the best use of the explicit method here described will be to determine starting values for the more accurate iterative schemes.

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## 2. The Estimated Solution:

Let $\phi_n(x) = P_{n-1}(x) - P_n(x)$ where $P_n(x)$ is the Legendre Polynomial orthogonal over $0 \le x \le 1$. The zeroes $x_1, x_2, x_3 \cdots x_n$ of $\phi_n(x)$ are the Radau quadrature points.

Let $Q_k(x) = \int_0^x \ell_k(u)\ \Pi_{k-1}(u)du.$ $(0 \le k \le n)$

Where $\ell_k(x)$ the Lagrange interpolant $\dfrac{Q_n(x)}{Q_n'(x_h)(x - x_k)}$ and $\Pi_k(x)$ is a polynomial of degree (k-1) satisfying the conditions $\Pi_k(x_k) = 1$ and $Q_k(x_j) = 0$ $(0 < j \le k)$. The existence of such polynomials is readily deduced from the non-vanishing of the Vandermonde determinant. It easily follows that

$$Q_k'(x_j) = \delta_{jk} \text{ and } Q_k(x_j) = 0 \quad (0 \le j \le k).$$

If $Y_c(x)$ is defined by

$$Y_c(x_0 + hx) = Y(x_0) + h \sum_{j=0}^{n} Q_k(x)\ F(x_0 + hx_j,\ Y_c(x_0 + hx_j))$$

Then $Y_c'(x_0 + hx_j) - \sum_{j=0}^{n} F(x_0 + hx_j,\ Y_c(x_0 + hx_j)) = 0.$ $(0 \le j \le n)$

## 3. Error of the Estimate:

If $F(x,y)$ has sufficiently many partial derivatives then

$$\begin{aligned} Y'(x_0 + hx) - Y_c'(x_0 + hx) &= F(x_0 + hx,\ Y(x_0 + hx)) \\ &- F(x_0 + hx,\ Y_c(x_0 + hx)) + F(x_0 + hx,\ Y_c(x_0 + hx) \\ &- Y_c'(x_0 + hx). \end{aligned}$$

and

$$Y'(x_0 + hx) - Y_c'(x_0 + hx) - H(x_0 + hx)\{Y(x_0 + hx) - Y_c(x_0 + hx)\} =$$
$$F(x_0 + hx,\ Y_c(x_0 + hx)) - Y_c'(x_0 + hx).$$

Where $H(x_0 + hx) = \dfrac{F(x_0 + hx, Y(x_0 + hx) - F(x_0 + hx\ Y_c(x_0 + hx))}{Y(x_0 + hx) - Y_c(x_0 + hx)}$ .

Thus $Y(x_0 + hx) - Y_c(x_0 + hx) =$

$$h \int_0^x \exp(h \int_v^x H(x_0 + hu)du)\{F(x_0 + hv, Y_c(x_0 + hv) - Y_c'(x_0 + hv)\}dv$$

$$= h\{\exp h \int_\theta^x H(x_0 + hu)du\} \int_0^x E(x_0 + hv)dv \text{ where } 0 < \theta < x, \text{ and}$$

$E(x_0 + hv) = F(x_0 + hv, Y_c(x_0 + hv)) - Y_c'(x_0 + hv)$. But $E(x_0 + hxj) = 0$ for $0 \leq j \leq n$. Hence by applying the formula for interpolating error it is seen that the difference $Y(x) - Y_c(x)$ is of order $h^n$.

## 4. Conclusion and Results:

Order of error is all that is established. In order to use the process it first is necessary to compute $G_k(xj)$, $j \geq k$. The results of this computation constitute a data package for implementing the scheme.

It is a simple matter to create high order methods of this type, but such testing as has been done to date seems to suggest that the scheme is not competitive with established methods. It appears that it will be a good predictor for use in a more elaborate method of solving differential equations.

## REFERENCES


1. Axelsson, O., *Global Integration Differential Equations Through Lobatto Quadrature*, BIT 4, 2(1964), 69-86.

2. Sims, S. A. and O'Hara, M. T., *A Picard Iterative Method for Estimating Solutions to Certain Differential Equations*, NASA Report, January, 1966.