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THEORETICAL FOUNDATION OF SOME CRITERIA OF EQUILIBRIUM STABILITY OF PLATES

a translation of

Obosnovanie Nekotorykh Kriteriev Ustoichivosti Ravnovesiia Plastin

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Notice

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The problem of Lyapunov's stable equilibrium is formulated for orthotropic rectangular plates which are hinged over the contour and are compressed in two directions by constant loads. It is shown, that the stability criteria given for the problems considered in [1] and [2] follow from conditions in which the energy integral possesses the properties of a Lyapunov functional. The condition of positive definiteness of the energy integral is established by means of estimates related to the completeness of certain specified systems of functions [3].

1. The equations of small deflections w(x,y,t) of orthotropic plane, rectangular plates, hinged over the contour and compressed or stretched in two directions by constant loads, can, after introduction of dimensionless variables be brought into the form

$$a_1 \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + a_2 \frac{\partial^4 w}{\partial y^4} - \pi^2 a_3 \frac{\partial^2 w}{\partial x^2} - \pi^2 a_4 \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1.1)$$

$$w\Big|_{\substack{x=0\\x=1}} = \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=1}} = w\Big|_{\substack{y=0\\y=1}} = \frac{\partial^2 w}{\partial y^2}\Big|_{\substack{y=0\\y=1}} = 0 \qquad (1.2)$$

Here $a_1 > 0$ and $a_2 > 0$ are dimensionless rigidities, a_3 and a_4 are dimensionless loads, positive for tensions. If one adds on the left hand side of (1.1) the terms:

$$a_5 \frac{\partial^2 w}{\partial x \partial t} \quad \text{and} \quad a_6 \frac{\partial^2 w}{\partial y \partial t} \qquad (1.3)$$

then by the selection of the parameters a_3, a_4, a_5 and a_6 one can also take into account the effect of a liquid flowing along the middle surface with constant velocity in the direction of the x or y axis [4] on the plates. The dimensionless coordinates x and y lie in the square $K(0 \le x \le 1, 0 \le y \le 1)$ and the time t is in some interval, $t_o \le t < t_1$ belonging to the real axis.

The equations (1.1) and (1.2) allow the solution

$$w(x,y,t) \equiv 0 \tag{1.4}$$

corresponding to the equilibrium of the plates. For the investigation of the stability of this equilibrium choose as metric the perturbed functional

$$\rho(z) = \int_0^1 \int_0^1 [(\frac{\partial w}{\partial t})^2 + (\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial x \partial y})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + (\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2] dxdy + \sup_K w^2 \tag{1.5}$$

determined in the "points"

$$z = \{w(x,y,t), \frac{\partial w(x,y,t)}{\partial t}\}_{t = \text{constant}}$$

where the function w(x,y,t) belongs to the class W of sufficiently smooth solutions of the problem (1.1), (1.2).

Let us denote the "static" part of the metric by

$$\rho_s(z) = \int_0^1 \int_0^1 [(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial x \partial y})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + (\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2] dxdy + \sup_K w^2 \tag{1.6}$$

The equilibrium (1.4) is considered stable if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all solutions $w(x,y,t) \varepsilon W$ satisfying at the initial time t_o the condition

$$\rho(z(t_o)) < \delta \tag{1.7}$$

for all $t \geq t_o$ the condition

$$\rho(z(t)) \leq \varepsilon \tag{1.8}$$

is satisfied in the region where the solution $w(x,y,t)$ is defined.

In order to finish the formulation of the problem of stability let us define more accurately the solutions $w(x,y,t)$ of class W: In the region where the solutions $w(x,y,t)$ are defined they must be continuous in the variables x,y and t together with their derivatives which occur in obtaining from the equations (1.1) and (1.2) the relations

$$\frac{dH}{dt} = 0 \tag{1.9}$$

$$H(z) = H_s(z) + \int_0^1 \int_0^1 \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy \tag{1.10}$$

$$H_s(z) = \int_0^1 \int_0^1 \left[a_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + a_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \pi^2 a_3\left(\frac{\partial w}{\partial x}\right)^2 + \pi^2 a_4\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy \tag{1.11}$$

It is obvious that for any solution $w(x,y,t) \varepsilon W$ the right hand side of (1.5) is a continuous function of the time t.

Below it is shown, that if the condition

$$a_1 m^4 + 2\, m^2 n^2 + a_2 n^4 + a_3 m^2 + a_4 n^2 > 0 \quad (m,n = 1,2,\ldots) \tag{1.12}$$

is fulfilled, the functional H(z) possesses the following property:

(1) For any $\varepsilon > 0$ one can find a number $\mu(\varepsilon) > 0$ such that the condition

$$H(z) \geq \mu \text{ for } \rho(z) \geq \varepsilon \tag{1.13}$$

is fulfilled.

Apart from this, it is evident from (1.5), (1.9) - (1.11) that the functional H(z) possesses the following properties:

(2) For any solution $w(x,y,t) \varepsilon W$ the functional H does not increase with time t.

(3) For each $\mu > 0$ one can find a $\delta(\mu) > 0$ such that the condition

$$|H(z)| < \mu \text{ for } \rho(z) < \delta$$

is satisfied.

Theorems 5.2 [5,6] states accordingly that the properties (1), (2) and (3) are sufficient for stability in the sense indicated by the inequalities (1.7) and (1.8). Therefore the stability of the equilibrium (1.4), satisfying conditions (1.12) is proved if it can be shown that property (1) defines a positive functional H(z) with respect to the metric $\rho(z)$.

2. Every orthonormal system of functions

$$\phi_m(x) = \sqrt{2} \sin m\pi x \qquad m = 1,2,.. \tag{2.1}$$

$$\psi_m(x) = \begin{cases} 1 & m = 0 \\ \sqrt{2} \cos m\pi x & m = 1,2,.. \end{cases} \tag{2.2}$$

is complete in the interval $0 \leq x \leq 1$.

From [7,8] follows that the following orthonormal system of functions is complete in the rectangle K:

$$\phi_{mn}(x,y) = \phi_m(x)\phi_n(y). \qquad m,n = 1,2,... \tag{2.3}$$

$$\psi_{mn}(x,y) = \psi_m(x)\psi_n(y) \qquad m,n = 1,2,.. \tag{2.4}$$

$$\chi_{mn}(x,y) = \phi_m(x)\psi_n(y) \qquad m = 1,2,..; \ n = 0,1,.. \tag{2.5}$$

$$\omega_{mn}(x,y) = \psi_m(x)\phi_n(y) \qquad m = 0,1,..; \ n = 1,2,... \tag{2.6}$$

Let us next examine the Fourier coefficients of the deflection w(x,y,t) with respect to the system (2.3):

$$\int_0^1 \int_0^1 w(x,y,t)\ \phi_{mn}(x,y)\ dxdy = a_{mn} \tag{2.7}$$

Then, making use of the boundary conditions (1.2) and the properties of the functions (2.1) - (2.6) we obtain:

$$\int_0^1 \int_0^1 \frac{\partial^2 w}{\partial x^2} \phi_{mn}\ dx\ dy = -\pi^2 m^2 a_{mn} \qquad m,n = 1,2,...$$

$$\int_0^1 \int_0^1 \frac{\partial^2 w}{\partial y^2} \phi_{mn}\ dx\ dy = -\pi^2 n^2 a_{mn} \qquad m,n = 1,2,...$$

$$\int_0^1 \int_0^1 \frac{\partial^2 w}{\partial x \partial y} \psi_{mn}\ dx\ dy = \begin{cases} 0 & m = 0 \\ \pi^2 mn\ a_{mn} & m,n = 1,2,... \end{cases} \tag{2.8}$$

$$\int_0^1 \int_0^1 \frac{\partial w}{\partial x} \omega_{mn}\ dx\ dy = \begin{cases} 0 & m = 0 \\ \pi m\ a_{mn} & m,n = 1,2,... \end{cases}$$

$$\int_0^1 \int_0^1 \frac{\partial w}{\partial y} \chi_{mn}\ dx\ dy = \begin{cases} 0 & n = 0 \\ \pi n\ a_{mn} & m,n = 1,2,... \end{cases}$$

Under the made assumptions of relative smoothness of the deflection $w(x,y,t)$ all the functions, for which the Fourier coefficients are given by the formulas (2.8), are square integrable over K. Therefore from (2.8) and the completeness conditions of the system (2.3) - (2.6) one finds:

$$\begin{aligned}
&\int_0^1\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,dy = \pi^4 \sum_{m,n=1}^{\infty} a_{mn}^2 m^4 \\
&\int_0^1\int_0^1 \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}\, dx\,dy = \pi^4 \sum_{m,n=1}^{\infty} a_{mn}^2 m^2 n^2, \\
&\int_0^1\int_0^1 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \pi^4 \sum_{m,n=1}^{\infty} a_{mn}^2 n^4 \\
&\int_0^1\int_0^1 \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \pi^4 \sum_{m,n=1}^{\infty} a_{mn}^2 m^2 n^2 \\
&\int_0^1\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\,dy = \pi^2 \sum_{m,n=1}^{\infty} a_{mn}^2 m^2 \\
&\int_0^1\int_0^1 \left(\frac{\partial w}{\partial y}\right)^2 dx\,dy = \pi^2 \sum_{m,n=1}^{\infty} a_{mn}^2 n^2
\end{aligned} \tag{2.9}$$

With (2.9) follows for the functionals of (1.6) and (1.11) the representations

$$\rho_s(z) = \pi^2 \sum_{m,n=1}^{\infty} a_{mn}^2 [\pi^2(m^4 + m^2n^2 + n^4) + m^2 + n^2] + \sup_K w^2 \tag{2.10}$$

$$H_s(z) = \pi^4 \sum_{m,n}^{\infty} a_{mn}^2 (m^4 + n^4) \cdot f\ (m,n), \tag{2.11}$$

$$f(m,n) = \frac{a_1 m^4 + 2m^2n^2 + a^2 n^4}{m^4 + n^4} + \frac{a_3 m^2 + a_4 n^2}{m^4 + n^4}$$

Taking into account the initial conditions (1,2) one obtains also

$$w(x,y,t) = \int_0^x \int_0^y \frac{\partial^2 w}{\partial x \partial y}\, dx\, dy.$$

Hence, applying the Buniakovskii-Schwartz inequality one finds

$$w^2 \leq \int_0^1 \int_0^1 \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dx\, dy$$

$$\sup_K w^2 \leq \int_0^1 \int_0^1 \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dx\, dy = \pi^4 \sum_{m,n=1}^{\infty} a_{mn}^2 m^2 n^2 \tag{2.12}$$

Let for the given meaning of the parameters of the problem, a_1, a_2, a_3 and a_4 the condition (1.12) be fulfilled: then the condition

$$f(m,n) = \frac{a_1 m^4 + 2m^2 n^2 + a_2 n^4}{m^4 + n^4} + \frac{a_3 m^2 + a_4 n^2}{m^4 + n^4} > 0,\ m,\ n = 1,\ 2\ldots \tag{2.13}$$

is also satisfied.

The first term of (2.13) exceeds for any n, m = 1, 2,... the positive number $\min(a_1, a_2) \equiv \gamma_1$; therefore it follows from (2.13) that for any considered pair of numbers (m,n) lying in a bounded square $1 \leq m \leq M$; $1 \leq n \leq M$, where M is a sufficiently large number the inequality $f(m,n) > \gamma_1 > 0$ is satisfied.

Among the **ultimate** number of positive numbers f(m,n), m, n = 1, 2, ... M. There is going to be a smallest number $\gamma_2 > 0$. Therefore denote $\min(1,\gamma_1,\gamma_2) \equiv 4\gamma > 0$, from (2.13) one has

$$f(m,n) \geq 4\gamma > 0 \quad m,\ n = 1,2,.. \tag{2.14}$$

and from (2.10), (2.11), (2.12), (2.14) one obtains

$$H_s(z) \geq \gamma\, \rho_s(z) \quad 0 < \gamma < 1$$

Hence taking into account the structure of the functional (1.5), the estimate for (1.10) becomes

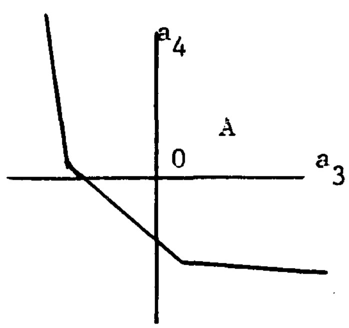
$$H(z) \geq \gamma\, \rho(z) \qquad \gamma > 0,$$

from which follows the condition of positive definiteness of (1.13) for $\mu = \gamma\, \varepsilon$.

The stability of the equilibrium (1.4) is thus demonstrated for condition (1.12) being satisfied.

Further it should be noted that the addition of the gyroscopic terms (1.3) to (1.1) does not change the relationships (1.9) - (1.11) and the resulting sufficient condition for stability (1.12).

3. For fixed values of the parameters a_1, a_2 the stability condition (1.12) determines in the plane of parameters a_3, a_4 a convex region of stability, A, formed by the intersections of an infinite number of half planes corresponding to the various values of the parameters m, n = 1, 2,... .



The figure gives a representation of the kind of region of stability A with its boundary consisting of short pieces of straight line. (see figure 175 [1]).

It can be verified that for the problem considered (1.1), (1.2) the equilibrium under violation of condition (1.12) will not be stable. Actually in that case at least for one pair of numbers m, n the left side of (1.12) will be either zero or negative and the equations (1.1) and (1.2) will allow a solution

$$w(x,y,t) = a_{mn}(t - t_o) \sin m \pi x \sin n \pi y,$$

$$w(x,y,t) = a_{mn} e^{p(t-t_o)} \sin m \pi x \sin n \pi y, \; p > 0$$

respectively which will not satisfy the condition (1.8) for all $t \geq t_o$. Therefore it is clear, that each point in the $a_3 - a_4$ parameter plane can be found on the boundary of the region A, or outside this region corresponding to an unstable condition of the equilibrium (1.4).

In this article the question of the stability properties of the equilibrium (1.4) for violation of condition (1.12) for the case with the terms (1.3) present has not been considered.

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