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3 SIMILARITY SOLUTION OF A JET FLOWING
OVER A CURVED SURFACE 6

by

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ABSTRACT

The flow of a laminar two-dimensional incompressible jet flowing over a curved surface is analyzed for the case of small curvature. The zeroth order solution is the same as for a jet over a flat surface. Therefore, to find the effects of curvature on the jet, the first order solution is found. The first order solution is found by the use of a similarity transformation and a perturbation of the stream function. The use of a similarity transformation restricts the type of surface curvature that may be considered, so that the allowable surface shapes are also found.

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CONTENTS

SYMBOLS

A - Constant in expression for stream function

C_1, C_2 - Constants used in finding similarity requirements

f - Velocity profile function in similarity transformation

f_0 - Velocity profile function in similarity transformation for a flat surface

f_1 - Perturbation of f to allow for effects of surface curvature

$f_{0\infty}$ - Value of f_0 evaluated at $\eta = \infty$

$f_{1\infty}$ - Value of f_1 evaluated at $\eta = \infty$

H - Quantity to describe the characteristics of a given jet, see equation (41)

h - Characteristic jet width

J - Momentum flux of jet per unit jet depth

K - Surface curvature, 1/R

$\overline{K}$ - Non-dimensional surface curvature, KL

L - Characteristic length of jet run

$\dot{m}$ - Jet mass flow rate per unit jet depth

P - Pressure

P_∞ - Atmospheric pressure

$\overline{P}$ - Non-dimensional pressure, $(P-P_\infty)/\rho U_R^2$

R - Surface radius of curvature

$\bar{R}$ - Non-dimensional surface radius of curvature, R/L

Re - Reynolds number, $U_R L/\nu$

u - Jet velocity in x-direction

$\bar{u}$ - Non-dimensional velocity in x-direction, u/U_R

v - Jet velocity in y-direction

$\bar{v}$ - Non-dimensional velocity in y-direction, v/V_R

$\bar{v}^*$ - Modified non-dimensional jet velocity in y-direction, $\left[1 + \bar{K}\bar{y}(h/L)\right]\bar{v}$

V_R - Characteristic jet velocity in y-direction

x - Distance parallel to surface

$\bar{x}$ - Non-dimensional distance parallel to surface, x/L

y - Distance normal to the surface

$\bar{y}$ - Non-dimensional distance normal to surface

α - Exponent of x-variation in stream function

β - Exponent of x-variation in η

ξ^* - Cartesian coordinate used in finding surface shapes

$\mathcal{O}()$ - Order of magnitude

μ - Absolute viscosity

ν - Kinematic viscosity, μ/ρ

ρ - Fluid density

τ_o - Shearing stress at the wall

$\bar{\psi}$ - Non-dimensional stream function

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INTRODUCTION

The flow of a laminar two-dimensional incompressible jet into a quiesient atmosphere of the same fluid is treated by Schlicting (page 164, reference 1). The problem of a jet flowing along a straight wall is treated by Glauert (reference 2). When the wall is curved, as shown in figure 1, then no known solution exists in the literature. A jet on a convex shaped curved surface can remain attached for relatively long lengths of jet run (reference 3). The tendency of the jet to remain attached to the curved surface is commonly termed the Coanda Effect.

If the jet is to remain attached to a curved surface, a pressure gradient must exist across the jet to balance the centrifugal force of the curving jet of fluid. If the surrounding atmosphere is assumed to be infinitely large, then the static pressure along the outer edge of the jet must be a constant. Therefore, the pressure along the wall is lower than atmospheric pressure (convex surface curvature), with the pressure on the wall increasing as the jet proceeds along the surface. Therefore, the jet as a whole, is flowing against an adverse pressure gradient.

The reason that the jet usually remains attached to the wall rather than immediately separating is the

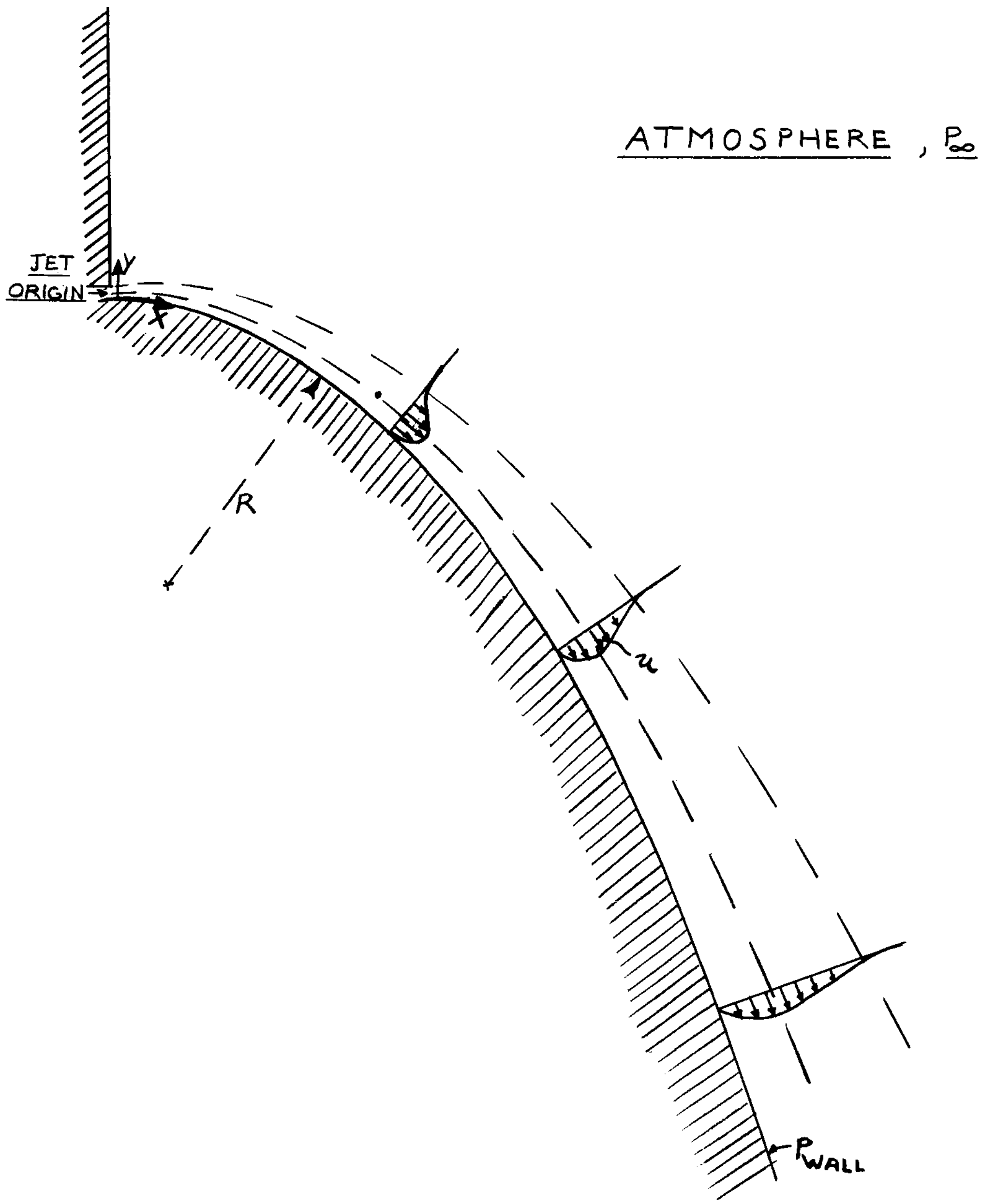

Figure 1.--Schematic diagram of a jet flowing over a curved surface.

viscous mixing or shearing action of the jet. If the jet separates, then the fluid between the jet and the wall will be acted upon by the jet. That is, the jet will entrain part of the fluid in the region between the jet and the surface. This entrained fluid is then replaced by fluid that must flow in from the atmosphere downstream. In order for this fluid to flow into the region between the jet and the wall, this region must have a pressure lower than the atmosphere. If the entrainment rate of the jet is high enough, then the pressure on the wall will be low enough to make the jet turn such that it will become attached to the surface.

This analysis is an attempt to find surface shapes that will allow similar shapes of the velocity profiles at different surface locations, and the resulting velocity profiles. Since separation of a two-dimensional flow requires a zero slope of the velocity profile at the wall, the similar shaped velocity profiles can have a velocity profile with a zero slope at the wall at some location only if it has a zero slope at all surface locations. Therefore, a solution with similar velocity profiles that has a velocity profile with a non-zero slope at the wall cannot separate at any downstream location.

This analysis does not consider the details of the generation of a pressure gradient across the jet at the jet origin. It assumes that the required pressure gradient already exists.

ANALYSIS

Flow Equations

The assumptions for the development of the simplified momentum equations for jet flows are essentially the same as for boundary layer flows. That is, the width of the jet at any location is very small compared with the length of run of the jet to the same location; and that the Reynolds number, based on a characteristic velocity and a characteristic length of jet run, is very large. When the characteristic radius of curvature is the same order of magnitude as the characteristic length of jet run, then the terms in the momentum equations containing the surface curvature are no larger than order (h/L) (ratio of characteristic jet width to characteristic run of the jet), where (h/L) 1. Therefore, to an approximation that considers only unit order terms in the momentum equations, the effects of curvature do not enter. For boundary layer flows part of the effects of curvature can enter through the external flow, so that the effects of a curved surface are considered even though they do not appear in the boundary layer equations. For the jet problems there is no external flow that can contain the effects of curvature, so that the effects of curvature must be contained in the boundary layer like equations if the curvature is to be considered at all. Therefore, the momentum equations for the jet will have to consider terms of order (h/L) as well as the unit order terms.

The complete continuity and momentum equations for steady, incompressible and two-dimensional flow over a curved surface can be written as (page 112, reference 1);

continuity equation: $$\frac{R}{R+y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{R+y} = 0 \qquad (1)$$

x-momentum equation:

$$\frac{R}{R+y} u \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{R+y} = -\frac{R}{R+y}\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\Big[\frac{R^2}{(R+y)^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{R+y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}$$
$$-\frac{u}{(R+y)^2} + \frac{2R}{(R+y)^2}\frac{\partial v}{\partial x} - \frac{R}{(R+y)^3}\frac{\partial R}{\partial x} v + \frac{Ry}{(R+y)^3}\frac{\partial R}{\partial x}\frac{\partial u}{\partial x}\Big] \qquad (2)$$

y-momentum equation:

$$\frac{R}{R+y} u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{u^2}{(R+y)} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\Big[\frac{\partial^2 v}{\partial y^2} - \frac{2R}{(R+y)^2}\frac{\partial u}{\partial x}$$
$$+\frac{1}{R+y}\frac{\partial v}{\partial y} + \frac{R^2}{(R+y)^2}\frac{\partial^2 v}{\partial x^2} - \frac{v}{(R+y)^2} + \frac{Ru}{(R+y)^3}\frac{\partial R}{\partial x} + \frac{Ry}{(R+y)^3}\frac{\partial R}{\partial x}\frac{\partial v}{\partial x}\Big] \qquad (3)$$

The two momentum equations (2 and 3) can be combined into a single equation by cross-differentiating and eliminating the pressure gradient. When terms of order $(h/L)^2$ and smaller are neglected, the following is the resulting non-dimensional equation (Appendix A).

$$\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + \bar{v}^*\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + (h/L)\Big[\bar{u}^2\frac{\partial \bar{K}}{\partial \bar{x}} + \bar{K}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{K}\bar{v}^*\frac{\partial \bar{u}}{\partial \bar{y}}\Big] =$$
$$\Big[1 + \bar{K}\bar{y}(h/L)\Big]\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + (h/L)2\bar{K}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \qquad (4)$$

The non-dimensional continuity equation can be written as (Appendix A);

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}^*}{\partial \bar{y}} = 0 \tag{5}$$

Equations (4) and (5) can be combined into a single equation by the introduction of a stream function that identically satisfies the continuity equation. That is;

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}} \quad and \quad \bar{v}^* = -\frac{\partial \bar{\psi}}{\partial \bar{x}} .$$

Equation (4) written in terms of the stream function is;

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial^3 \bar{\psi}}{\partial \bar{x} \partial \bar{y}^2} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} + (h/L)\left[-\bar{K} \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} + \bar{K} \frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y} \partial \bar{x}} + \left(\frac{\partial \bar{\psi}}{\partial \bar{y}}\right)^2 \frac{\partial \bar{K}}{\partial \bar{x}}\right] = \frac{\partial^4 \bar{\psi}}{\partial \bar{y}^4} + (h/L)\left[\bar{K} \bar{y} \frac{\partial^4 \bar{\psi}}{\partial \bar{y}^4} + 2 \bar{K} \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3}\right] \tag{6}$$

<u>Boundary Conditions</u>

If the wall is assumed to be impermeable and if the no slip condition is assumed to apply at the wall, then;

$$\bar{u} = \bar{v}^* = 0 \quad at \quad \bar{y} = 0 \tag{7}$$

The velocity of the jet is zero infinitely far away from the wall, so that;

$$\bar{u} = 0 \quad at \quad \bar{y} = \infty . \tag{8}$$

The solution of equation (4) requires one more boundary condition. The outer region of the jet is similar to the outer region of a two dimensional free jet, which has a velocity profile that goes to zero asymptotically as y goes to infinity. Therefore, it seems reasonable to assume that a suitable boundary condition would be;

$$\frac{\partial \bar{u}}{\partial \bar{y}} = 0 \quad at \quad \bar{y} = \infty . \tag{9}$$

The boundary conditions (7), (8) and (9) in terms of the stream function are;

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} = \frac{\partial \bar{\psi}}{\partial \bar{x}} = 0 \quad at \quad \bar{y} = 0 \tag{10}$$

and

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} = \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = 0 \quad at \quad \bar{y} = \infty \tag{11}$$

Similarity Transformation

The solution of equation (6) could be attempted by a number of different approaches. The approach chosen in this analysis is to use a similarity transformation. This type of transformation changes equation (6) to an ordinary differential equation. However, the type of solution obtained is rather restricted. That is, the velocity profile at different surface locations is the same as at any other location except for a scale factor. This type of transformation therefore precludes the possibility of predicting the separation point of the jet. That is, if the velocity profile is on the verge of separation at one surface location, then it is on the verge of separation at all surface locations because of the similar shape of the velocity profiles.

The similarity transformation assumed to exist for this problem is of the form;

$$\bar{\psi} = A\,\bar{x}^{\alpha} f(\eta) \quad where \quad \eta = \bar{y}/\bar{x}^{\beta} \tag{12}$$

where A is an arbitrary constant, α and β are constants to be determined later, and $f(\eta)$ denotes a function of η , which will be found from the solution of the differential equation. For the remainder of the analysis, $f(\eta)$ will just be written as f. Derivatives of f with respect to η will be denoted by primes on f.

The boundary conditions on f can be found from equation (12) and the boundary conditions on $\bar{\psi}$. They are;

$$f = f' = 0 \quad at \quad \eta = 0 \tag{13}$$

$$f' = f'' = 0 \quad at \quad \eta = \infty \tag{14}$$

When equation (12) is used in equation (6) the resulting equation is;

$$(\alpha-2\beta)f'f'' - \alpha f f''' - (h/L)\bar{K}\bar{X}^{\beta}[\alpha f f'' - (\alpha-\beta)(f')^2] + (h/L)\bar{X}^{\beta+1}(f')^2 \frac{\partial \bar{K}}{\partial \bar{X}} =$$

$$\frac{1}{A}\left[\bar{X}^{(-\alpha-\beta+1)} f'''' + (h/L)\bar{K}\,\bar{X}^{(-\alpha+1)}(f'''' + 2f''')\right] \tag{15}$$

In order for equation (15) to have η as the independent variable, the following requirements must be satisfied;

$$\bar{K}\,\bar{X}^{\beta} = C_1 \tag{16}$$

$$\bar{X}^{\beta+1} \frac{\partial \bar{K}}{\partial \bar{X}} = C_2 \tag{17}$$

$$-\alpha - \beta + 1 = 0 \quad or \quad \alpha = -\beta + 1 \tag{18}$$

Requirements (16) and (17) are not independent, since it is easily shown that $C_2 = -\beta C_1$. Using this relation and requirement (18) in equation (15) yields;

$$f'''' + (h/L)C_1[\eta f'''' + 2f''' + A\beta(f')^2 + A(1-\beta)ff'' - A(1-2\beta)(f')^2]$$

$$-A(\beta-1)ff''' + A(3\beta-1)f'f'' = 0 \tag{19}$$

The constant C_1 in equation (19) can be absorbed in the definition of the characteristic length L. If the characteristic length is taken to be the length of surface from the jet origin to the point where x=R=L or equivalentaly, the point where K=1/x=1/L, then C_1=1 as can be seen from equation (16). The constant A is an arbitrary constant, so that A can be taken as four without any loss of generality. With A=4, and C_1=1, equation (19) becomes;

$$f'''' + 4(1-\beta) f f''' + 4(3\beta-1) f' f'' + (h/L)[\eta f'''' + 2f''' + 4(1-\beta) f f'' + 4(3\beta-1)(f')^2] = 0 \qquad (20)$$

Growth Rate of the Jet

The constant β in equation (20) could be found by specifying a given surface shape (see equation (16)), if the boundary conditions for equation (20) can be satisfied for arbitrary values of β. The question of what values of β will satisfy the boundary conditions can be answered in the following manner. The unit order terms of equation (20) are the same terms that are obtained for the jet on a straight wall. For the straight wall problem, reference 2 finds that $\beta = 3/4$ is the only value that will satisfy the correct boundary conditions. This value of β was found by specifying that the velocity profile should have $f'' = 0$ at $\eta = \infty$. However, the condition that $f'' = 0$ at $\eta = \infty$ is not explicitly pointed out in reference 2.

The problem of a jet with a curved wall a priori should have a value of β near 3/4 because the effects of curvature are small. Therefore, let;

$$\beta = 3/4(1+\epsilon) \tag{21}$$

where ϵ is small, of order (h/L). However, one finds that $\epsilon \equiv 0$ as will be shown shortly. Since the effects of curvature are of order (h/L), the solution of equation (20) will be found by a perturbation method. The form of f is chosen to be;

$$f = f_0 + (h/L) f_1 \tag{22}$$

where f_0 is the solution given in reference 2 for the straight wall, and $(h/L)f_1$ is the effect of curvature on f. When equations (21) and (22) are substituted into equation (20), and terms smaller than order (h/L) are neglected, the following result is obtained;

$$f_0'''' + f_0 f_0''' + 5 f_0' f_0'' + (h/L)\left[f_1'''' + f_0 f_1''' + f_1 f_0''' + 5(f_0' f_1'' + f_1' f_0'') + \eta f_0'''' + 2 f_0''' + f_0 f_0'' + 5(f_0')^2 + \epsilon/(h/L)(-3 f_0 f_0''' + 7 f_0' f_0'')\right] = 0 \tag{23}$$

The function f_0 satisfies the equations;

$$f_0''' + f_0 f_0'' + 2(f_0')^2 = 0 \tag{24a}$$

and

$$f_0'''' + f_0 f_0''' + 5 f_0' f_0'' = 0 \tag{24b}$$

where equation (24b) is (24a) differentiated once with respect to η. Equations (24a) and (24b) can be used to simplify equation (23) with the following being the resulting expression;

$$f_1'''' + f_0 f_1''' + f_1 f_0''' + 5(f_0' f_1'' + f_1' f_0'') + \eta f_0'''' + f_0''' + 3(f_0')^2 + 3\,\epsilon/(h/L)\left[3 f_0' f_0'' - 3 f_0 f_0'''\right] = 0 \tag{25}$$

The parameter $\epsilon/(h/L)$ is the deviation of the jet growth rate from the straight wall value of 3/4.

Boundary conditions for f_0 and f_1 are as follows;

$$f_0 = f_0' = f_1 = f_1' = 0 \quad \text{at } \eta = 0 \tag{26}$$

$$f_0' = f_1' = 0 \quad \text{at } \eta = \infty \tag{27}$$

$$f_0''' = f_1''' = 0 \quad \text{at } \eta = \infty$$

$$\text{or} \quad f_0'' = f_1'' = 0 \quad \text{at } \eta = \infty \tag{28}$$

The solution to equation (24b) given in reference 2, yields the following results that will be useful in finding the solution to equation (25).

$$f_0' = 2/3\,(f_0^{1/2} - f_0^{3/2}) \tag{29}$$

$$\eta = \log\left[\frac{\sqrt{1 + f_0^{1/2} + f_0}}{1 - f_0^{1/2}}\right] + \sqrt{3}\,\tan^{-1}\left[\frac{\sqrt{3 f_0}}{2 + f_0^{1/2}}\right] \tag{30}$$

and
$$f_{0(\eta=\infty)} = f_{0\infty} = 1 \tag{31}$$

When solving the differential equation for f_1, terms that contain η and η^2 multiplying derivatives of f_0, will have to be evaluated at $\eta = \infty$. For very large η, f_0 is very close to unity, so that the dominant term in equation (30) is log $(1 - f_0^{1/2})$. This term yields that

$f_o = (1 - ce^{-\eta})$ for very large η. All derivatives of f_o then go to zero as $e^{-\eta}$ for very large η, so that terms such as $\eta e^{-\eta}$ and $\eta^2 e^{-\eta}$ go to zero as η goes to ∞.

Equation (25) can be integrated from η to ∞ to yield;

$$f_1''' + f_o f_1'' + f_1 f_o'' + 4 f_o' f_1' + \eta f_o''' + 3 \int_\infty^\eta (f_o')^2 d\eta$$
$$+ 3\epsilon/(h/L) \left[3(f_o')^2 - 3 f_o f_o'' \right] = 0 \qquad (32)$$

where the conditions $f_1'' = 0$ and $f_1''' = 0$ at $\eta = \infty$ were both used, because as mentioned earlier, one implies the other. If equation (32) is multiplied by f_o and integrated from 0 to η then;

$$f_o f_1'' + f_1 f_o'' - f_o' f_1' + f_o^2 f_1' + 2 f_o f_o' f_1 + \eta f_o f_o'' - \eta (f_o')^2/2$$
$$+ 3/2 \int_0^\eta (f_o')^2 d\eta - f_o f_o' + 3 \int_0^\eta f_o \int_\infty^\eta (f_o')^2 d\eta\, d\eta + 3\epsilon/(h/L) \Big[$$
$$9 \int_0^\eta f_o (f_o')^2 d\eta - 3 f_o^2 f_o' \Big] = 0 \qquad (33)$$

The integrals in equation (33) can be evaluated by using equations (29) and (31).

$$\int_0^\eta (f_o')^2 d\eta = 2/9 \left[2 f_o^{3/2} - f_o^3 \right]$$
$$\int_0^\eta f_o \int_\infty^\eta (f_o')^2 d\eta\, d\eta = 1/3 \left[1/2 f_o^2 - 2 f_o^{1/2} \right]$$
$$\int_0^\eta f_o (f_o')^2 d\eta = 2/3 \left[2/5 f_o^{5/2} - 1/4 f_o^4 \right]$$

Using these expressions for the integrals in equation (33) yields;

$$f_0 f_1'' + f_1 f_0'' - f_0' f_1' + f_0^2 f_1' + 2 f_0 f_0' f_1 + \eta f_0 f_0'' - \eta (f_0')^2/2$$
$$- f_0 f_0' + 3\,\epsilon/(h/L)\left[\tfrac{12}{5} f_0^{5/2} - 3/2\, f_0^4\right] = 0 \qquad (34)$$

If equation (34) is evaluated at $\eta = \infty$, the following result is obtained;

$$3\,\epsilon/(h/L)\left[\tfrac{12}{5} - 3/2\right] = 0$$

$$\text{or,} \quad \frac{27}{10}\,\epsilon/(h/L) = 0$$

Therefore, the only value of $\epsilon/(h/L)$ that will satisfy the boundary conditions is $\epsilon/(h/L) = 0$. This means that the jet has just one value of the growth rate, $\beta = 3/4$, for all small curvatures. This also means that there is one shape of surface that varies in the magnitude of the curvature.

Equation (34) can be integrated twice to yield the following solution for f_1 (Appendix B).

$$f_1 = -\eta^2 f_0'/2 - 6\eta f_0 + 4 f_0^{3/2} - 4 f_0 + 4 f_0' \log\left[\frac{\sqrt{1 + f_0^{1/2} + f_0}}{1 - f_0^{1/2}}\right]$$
$$+ f_{1\infty}(\eta f_0' + f_0) \qquad (35)$$

The value of $f_{1\infty}$ is an arbitrary constant. That is, equation (34) can be satisfied for any value of $f_{1\infty}$. The dimensional quantities such as u, $(\partial u/\partial y)_{y=0}$ and the pressure are independent of the choice of $f_{1\infty}$. The independence of u on $f_{1\infty}$ will be shown in Appendix C after the expressions for the momentum flux and mass flow are derived.

Dimensional Quantities

The expressions for the velocities, pressure and other physical quantities have been treated as non-dimensional quantities, whose characteristic or reference quantities were not explicitly given. Once a given surface is specified, then the characteristic length L is specified. The characteristic velocity U_R, or the characteristic length h must be specified to describe particular jet characteristics. Only U_R or h needs to be specified because the two are related by the relation;

$$Re = \frac{U_R L}{\nu} = (L/h)^2$$

where L is given by the prescribed surface curvature.

The most convenient quantity to specify for a given jet is neither U_R nor h, but a quantity that contains the jet mass flow and momentum flux, which will in turn yield a given value U_R or h.

The dimensional u velocity can be found from equation (12) and the definition of the non-dimensional quantities.

$$u = 4\, U_R\, L^{1/2} f' / X^{1/2} \tag{36}$$

The mass flow rate of the jet per unit depth can be written as;

$$\dot{m} = \rho \int_0^\infty u\,dy = 4\rho U_R h X^{1/4} f_\infty / L^{1/4}$$

The momentum flux per unit depth can be written as;

$$J = \rho \int_0^\infty u^2 dy = 16 \rho U_R^2 h L^{1/4} \int_0^\infty (f')^2 d\eta / X^{1/4}$$

The product of the mass flow and the momentum flux is independent of the distance x and has a clearer meaning than either U_R or h, therefore, a term containing the product of the mass flow and momentum flux was chosen as the quantity to describe a given jet. The product of the mass flow and momentum flux can be written as;

$$J\dot{m} = 64 \rho^2 U_R^3 h^2 f_\infty \int_0^\infty (f')^2 d\eta$$

The reference velocity U_R can be eliminated by the definition of the Reynolds number and the relation $Re = (L/h)^2$, or

$$U_R = \nu/L \, (L/h)^2$$

When this is substituted into the expression for $J\dot{m}$, the result becomes;

$$J\dot{m} = \frac{64\rho^2 \nu^3}{L} (L/h)^4 f_\infty \int_0^\infty (f')^2 d\eta \tag{37}$$

Using the perturbation form of f, and neglecting the term of order $(h/L)^2$ compared to the unit order term, the term $f_\infty \int_0^\infty (f')^2 d\eta$ in equation (37) can be rewritten as;

$$f_\infty \int_0^\infty (f')^2 d\eta = f_{0\infty} \int_0^\infty (f_0')^2 d\eta + (h/L)\left[f_{1\infty} \int_0^\infty (f_0')^2 d\eta + 2 f_{0\infty} \int_0^\infty f_0' f_1' d\eta\right]$$

or, $$f_\infty \int_0^\infty (f')^2 d\eta = 2/9 + (h/L)\left[2/9\, f_{1\infty} + 2\int_0^\infty f_0' f_1' d\eta\right] \tag{38}$$

Equation (37) can be written in the following form using equation (38);

$$J\dot{m} = \frac{128\rho^2\nu^3}{9L}(L/h)^2\left[1+(h/L)(f_{1\infty}+9\int_0^\infty f_0' f_1' d\eta\right] \qquad (39)$$

A convenient choice of $f_{1\infty}$ is to choose $f_{1\infty}$ such that,

$$f_{1\infty} = -9\int_0^\infty f_0' f_1' d\eta$$

Appendix C shows that the dimensional velocity u is independent of $f_{1\infty}$. Then, solving equation (39) for L/h yields;

$$(L/h) = \left[\frac{9J\dot{m}}{128\rho^2\nu^3}\right]^{1/4} L^{1/4} = H^{1/4}L^{1/4} \qquad (40)$$

where

$$H = \frac{9J\dot{m}}{128\rho^2\nu^3} \qquad (41)$$

The quantity H will be used as the parameter to describe a given jet flow. The expression for the dimensional u velocity (equation (36))can be written in terms of the quantity H as follows;

$$u = \frac{4\nu}{L^{1/2}}(L/h)^2 f'/x^{1/2} = \frac{4\nu H^{1/2}}{x^{1/2}}\left[f_0' + (HL)^{-1/4} f_1'\right] \qquad (42)$$

The shear stress along the wall for laminar flow is defined as;

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

which can be written as;

$$\tau_0 = \frac{4\rho\nu^2 H^{3/4}}{x^{5/4}}\left[f_0'' + (HL)^{-1/4} f_1''\right]_{\eta=0} \qquad (43)$$

The non-dimensional pressure can be found from equation (8A).

$$\bar{P} - \bar{P}_\infty = -(h/L)\bar{K}\int_{\bar{y}}^\infty \bar{u}^2 d\bar{y}$$

or, $$P - P_\infty = -(h/L)\,\frac{16\,\rho U_R^2 L}{X}\int_\eta^\infty (f')^2 d\eta\,.$$

Using the quantity H and the perturbation form of f in the expression for the pressure yields;

$$P - P_\infty = -\frac{16\,\rho\,\nu^2 H^{3/4}}{X L^{1/4}}\int_\eta^\infty (f_0')^2 d\eta \qquad (44)$$

Only the unit order term of $(f')^2$ was retained because the pressure difference $(P-P_\infty)$ is of order (h/L). The pressure along the wall can be written as,

$$P_{WALL} = P_\infty - \frac{32\,\rho\,\nu^2 H^{3/4}}{9 X L^{1/4}} \qquad (45)$$

Curves of the quantities f_0, f_0', $\int_\eta^\infty (f_0')^2 d\eta$, f_1 and f_1' are plotted against η in figures 2 and 3.

Surface Shapes

The equation for the non-dimensional surface curvature was presented as equation (16) where $\beta = 3/4$ and $C_1 = 1$. The form of the dimensional surface curvature can be written as;

$$K = \frac{1}{L^{1/4} X^{3/4}} \qquad (46)$$

where L is parameter. The type of surface that this equation describes is found in Appendix D, and several such shapes are shown in figure 4. All of the surface shapes are the same general shape (spirals) where the parameter L just determines the size of the spiral.

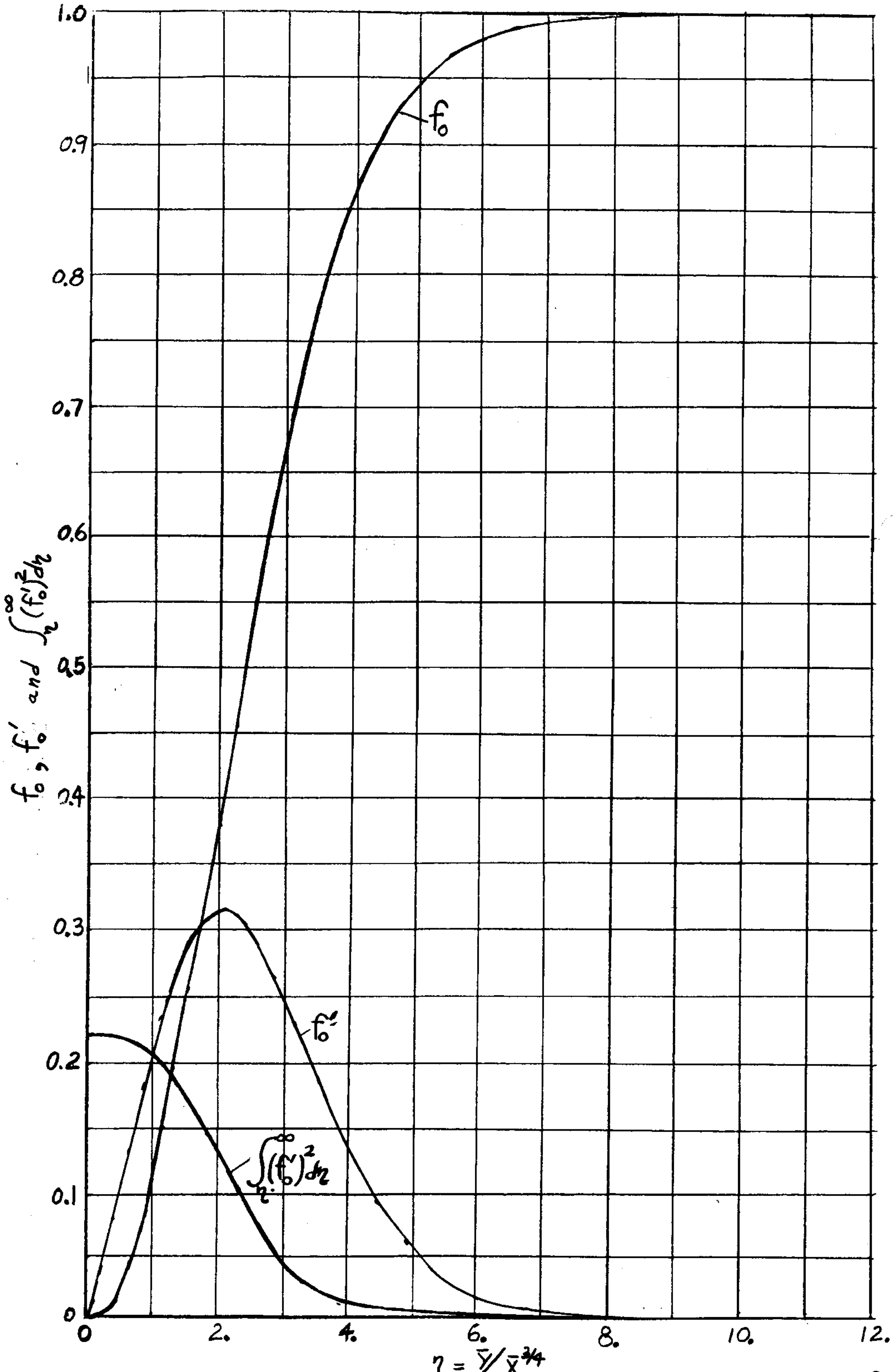

Figure 2 - Straight surface functions f_o, f_o' and $\int_\eta^\infty (f_o')^2 d\eta$ versus η.

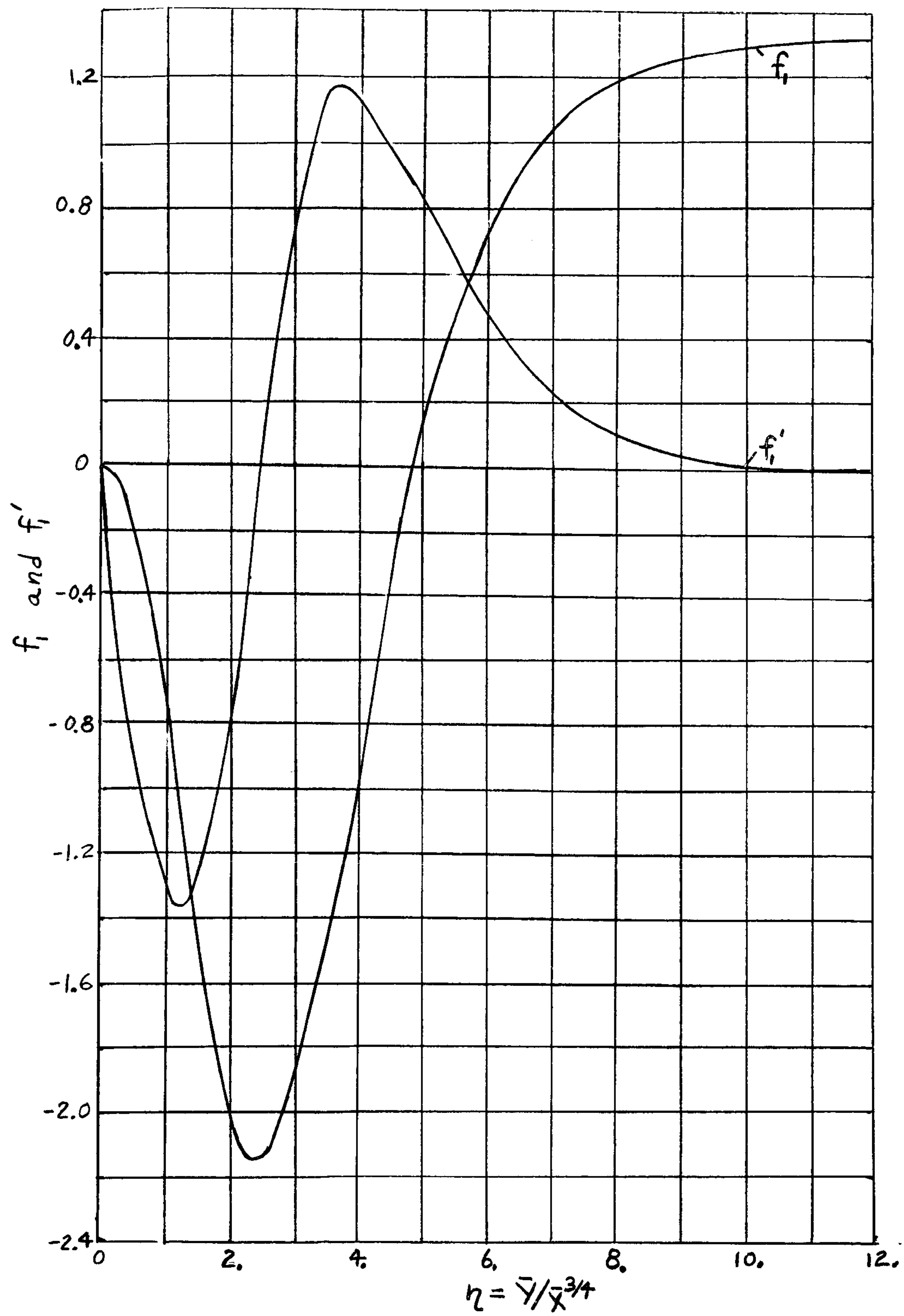

Figure 3 - Functions f_1 and f_1' versus η.

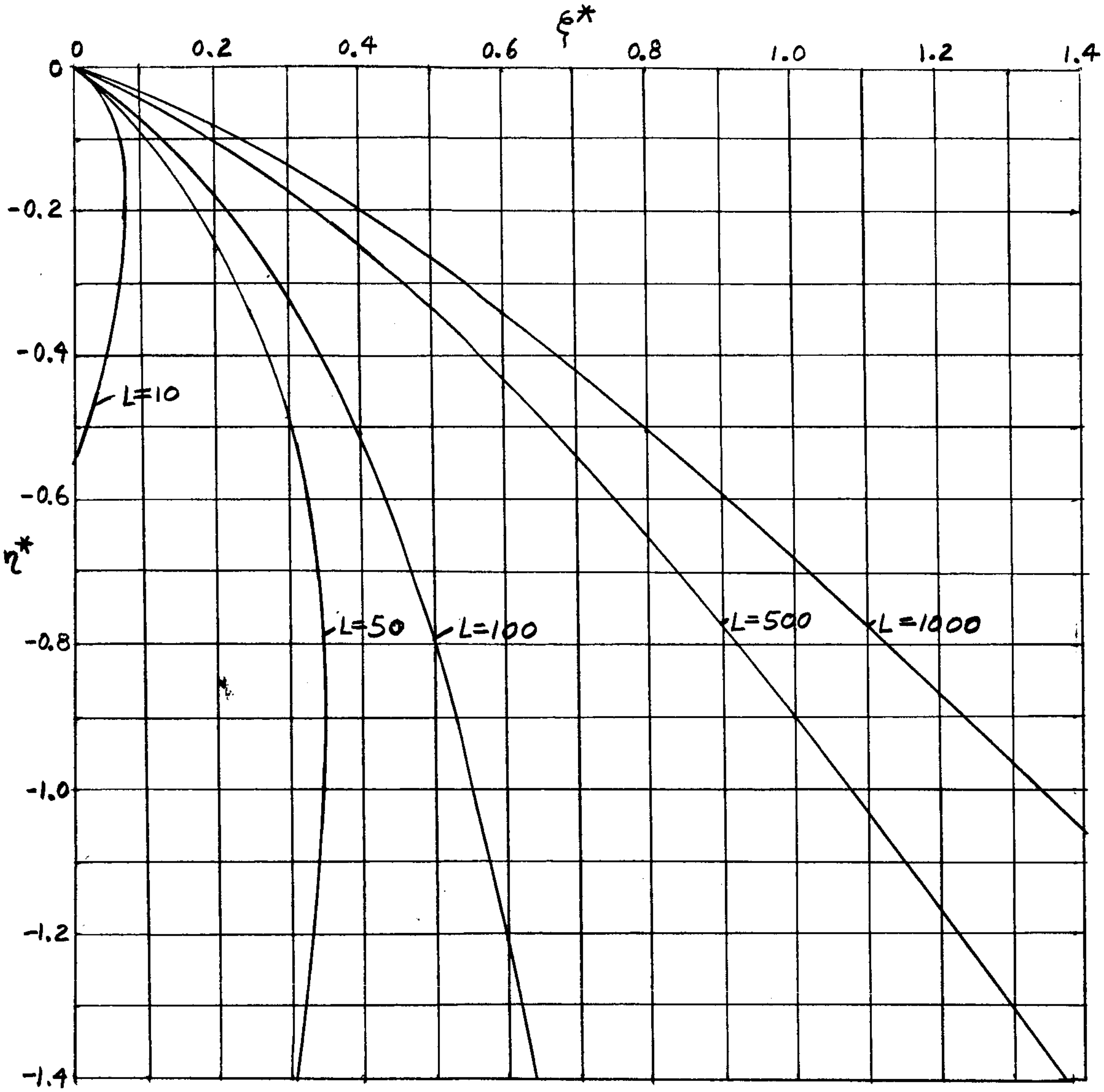

Figure 4 - Sample shapes of surfaces that allow similarity solutions.

CONCLUDING REMARKS AND SUMMARY OF RESULTS

The solutions found in this analysis have been obtained by a perturbation of the results for a jet over a flat plate. The values of H and L that can be considered valid for this analysis are values such that the perturbation scheme remains within acceptable limits of accuracy. That is, $f_1/(HL)^{1/4}$ must remain small compared with f_0. This is necessary not only to assure good accuracy of the perturbation forms, but also to keep the ordering procedure of the flow equations valid. If $f_1/(HL)^{1/4}$ is not small compared with f_0, then the assumption that the effects of curvature are small is no longer a valid assumption. If the effects of curvature are relatively large, then the flow equations would have to be considered for the case of the characteristic radius of curvature being the same order of magnitude as the characteristic jet width, not the characteristic length of jet run.

The effect on the velocity profile of the perturbation due to surface curvature is to decrease the jet velocity near the wall, and increase the jet velocity away from the wall. The jet profile for a curved surface then has the appearance of being wider than the profile for the straight surface. The curved surface velocity profile also has a reduced skin friction when compared with a comparable jet on a straight surface.

the results of the pressure distribution yield that the pressure is lowest at the surface, and highest at the jet outer edge. This agrees with what seems physically reasonable because the curving jet requires a low pressure near the wall to balance the centrifugal force of the curving jet.

The solutions for this jet problem have a singularity at the jet origin. However, this is not too disturbing because the ordered flow equations are not really valid near the jet origin anyway. The region near the jet origin would require a different type of analysis because of the non-zero width of an actual physical jet at the origin, and the different initial velocity profile of the actual jet. Therefore, a separate analysis would have to be performed for the region near the jet origin, and then the solution near the origin and the solution found in this analysis could be matched at some downstream location.

REFERENCES

1. Schlicting, Hermann. Boundary Layer Theory. New York: McGraw-Hill Book Company, 1960.

2. Glauert, M. B. "The Wall Jet." JFM Vol. I, part 6, 1956, p. 625.

3. Newman, B. G. "The Deflection of Plane Jets by Adjacent Boundaries, Coanda Effect." Boundary Layer and Flow Control, Its Principles and Applications, Vol. I, ed. G. V. Lachmann. New York: Pergamon Press, 1961.

4. Murphy, James S. "Extensions of the Falkner-Skan Similar Solutions to Flows with Surface Curvature," AIAA Journal, Vol. 3, No. 11, November, 1965.

APPENDIX A

NON-DIMENSIONALIZATION AND ORDER OF MAGNITUDE ANALYSIS

As a convenience, the local surface radius of curvature R, will be replaced by the local surface curvature K, which is equal to the inverse of the radius of curvature. The non-dimensional quantities used in the order of magnitude analysis are as follows;

$$\bar{u} = u/U_R \; ; \quad \bar{v} = v/V_R \; ; \quad \bar{x} = x/L \; ;$$
$$\bar{y} = y/h \; ; \quad \bar{K} = KL \; ; \quad \bar{P} = (P-P_\infty)/\rho U_R^2 \; .$$

All of the non-dimensional quantities except the non-dimensional pressure, are assumed to be of unit order of magnitude. The order of magnitude of the non-dimensional pressure will be determined from the y-momentum equation.

Substituting the non-dimensional quantities into the continuity equation (equation (1)) yields;

$$\frac{1}{1+\bar{K}\bar{y}(h/L)}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{V_R}{U_R}\left(L/h\right)\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\bar{K}\bar{v}}{1+\bar{K}\bar{y}(h/L)}\left(\frac{V_R}{U_R}\right) = 0$$

or,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{V_R}{U_R}(L/h)\frac{\partial}{\partial \bar{y}}\left\{\left[1+\bar{K}\bar{y}(h/L)\right]\bar{v}\right\} = 0$$

If a new velocity $\bar{v}^*$ is defined such that;

$$\bar{v}^* = \left[1+\bar{K}\bar{y}(h/L)\right]\bar{v} \tag{1A}$$

then the non-dimensional continuity equation becomes;

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{V_R}{U_R}\right)\left(\frac{L}{h}\right)\frac{\partial \bar{v}^*}{\partial \bar{y}} = 0 \tag{2A}$$

Since $\partial\bar{u}/\partial\bar{x}$ and $\partial\bar{v}^*/\partial\bar{y}$ are both of unit order, then $\left(\frac{V_R}{U_R}\right)\left(\frac{L}{h}\right)$ must also be of unit order. Without any loss of generality, let

$$V_R/U_R = h/L. \tag{3A}$$

The continuity then becomes;

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}^*}{\partial \bar{y}} = 0 \tag{4A}$$

The x-momentum equation (equation (2)) can be written in the following non-dimensional form;

$$\frac{1}{1+\bar{K}\bar{y}(h/L)}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{V_R}{U_R}\right)\left(\frac{L}{h}\right)\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \left(\frac{V_R}{U_R}\right)\frac{\bar{K}\bar{u}\bar{v}}{1+\bar{K}\bar{y}(h/L)} = -\frac{\partial \bar{P}}{\partial \bar{x}} + (L/h)^2\frac{1}{Re}\Bigg\{ \frac{1}{[1+\bar{K}\bar{y}(h/L)]^2}(h/L)^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{1}{[1+\bar{K}\bar{y}(h/L)]}(h/L)\frac{\partial \bar{u}}{\partial \bar{y}} - \frac{\bar{u}^2\bar{K}^2}{[1+\bar{K}\bar{y}(h/L)]^2}(h/L)^2 + \frac{2\bar{K}}{[1+\bar{K}\bar{y}(h/L)]^2}\left(\frac{V_R}{U_R}\right)(h/L)^2 - \frac{1}{[1+\bar{K}\bar{y}(h/L)]^3}\left[\bar{v}\frac{\partial \bar{K}}{\partial \bar{x}}\left(\frac{V_R}{U_R}\right)\left(\frac{h}{L}\right)^2 + \bar{y}\frac{\partial \bar{K}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{x}}\left(\frac{h}{L}\right)^3\right]\Bigg\}$$

If the characteristics of the jet are assumed to be such that the width of the jet is small compared with the length of run of the jet, then $h/L \ll 1$. If the largest inertia terms are assumed to be the same order of magnitude as the largest viscous terms in the x-momentum equation, then $(L/h)^2/Re = \mathcal{O}(1)$. Without any loss of generality, let

$$(L/h)^2/Re = 1 \tag{5A}$$

Using the fact that $h/L \ll 1$, equation (3A) and equation (5A) to determine the size of the terms in the x-momentum equation and retaining only unit order and order (h/L) terms, the following result is obtained;

$$\frac{1}{1+\bar{K}\bar{y}(h/L)}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}+(h/L)\bar{K}\bar{u}\bar{v}=-\frac{\partial \bar{P}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+(h/L)\bar{K}\frac{\partial \bar{u}}{\partial \bar{y}} \tag{6A}$$

When $\bar{v}$ is replaced by $\bar{v}^*$, the x-momentum equation becomes;

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}^*\frac{\partial \bar{u}}{\partial \bar{y}}+(h/L)\bar{K}\bar{u}\bar{v}^*=-\frac{\partial \bar{P}}{\partial \bar{x}}+[1+\bar{K}\bar{y}(h/L)]\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+(h/L)\bar{K}\frac{\partial \bar{u}}{\partial \bar{y}} \tag{7A}$$

The equation similar to equation (6A) presented in Murphy (reference 4) for boundary layers on curved surfaces gives the first term in his equation as just $u\frac{\partial u}{\partial x}$ not $\frac{1}{1+Ky}u\frac{\partial u}{\partial x}$. While the contribution of Ky is small compared to 1, it is still the same order of magnitude as the other order (h/L) terms appearing in equation (6A). Therefore, it appears that Murphy over-looked the contribution of this term, or else he had a reason for discarding it that is not apparent from the order of magnitude analysis.

The y-momentum equation (equation (3)) can be written in the following non-dimensional form.

$$\frac{1}{1+\bar{K}\bar{y}(h/L)}(h/L)(V_R/V_R)\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+(V_R/U_R)^2\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}-(h/L)\frac{\bar{K}\bar{u}^2}{1+\bar{K}\bar{y}(h/L)}=-\partial \bar{P}/\partial \bar{y}$$

$$+(L/h)^2/Re\left\{(h/L)(V_R/U_R)\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}-\frac{2\bar{K}}{[1+\bar{K}\bar{y}(h/L)]^2}(h/L)^3\frac{\partial \bar{u}}{\partial \bar{x}}\right.$$

$$+\frac{\bar{K}}{[1+\bar{K}\bar{y}(h/L)]}(V_R/U_R)(h/L)^2\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{1}{[1+\bar{K}\bar{y}(h/L)]^2}(V_R/U_R)(h/L)^3\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}$$

$$-\frac{\bar{K}^2\bar{v}}{[1+\bar{K}\bar{y}(h/L)]}(V_R/U_R)(h/L)^3-\frac{1}{[1+\bar{K}\bar{y}(h/L)]^3}\left[(h/L)^3\bar{u}\frac{\partial \bar{K}}{\partial \bar{x}}-(V_R/U_R)(h/L)^4\bar{y}\frac{\partial \bar{K}}{\partial \bar{x}}\frac{\partial \bar{v}}{\partial \bar{x}}\right]$$

Retaining only terms of order unity and order (h/L) as in the x-momentum equation, and using equations (3A) and (4A) yields the following result,

$$\frac{\partial \bar{P}}{\partial \bar{y}} = (h/L)\,\bar{K}\,\bar{u}^2 \tag{8A}$$

The magnitude of the non-dimensional pressure can be estimated from equation (8A). Since $\bar{K}\bar{u}^2$ and $\bar{y}$ are of unit order, then $\partial\bar{P}/\partial\bar{y}$ is of order (h/L), and $\bar{P}$ is of order (h/L).

The pressure can be eliminated from equations (7A) and (8A) by cross-differentiating and adding. The resultant equation is;

$$\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial \bar{v}^*}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{v}^*\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + (h/L)\Big[\bar{K}\frac{\partial \bar{u}}{\partial \bar{y}}\bar{v}^* + \bar{K}\bar{u}\frac{\partial \bar{v}^*}{\partial \bar{y}} + 2\bar{K}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u}^2\frac{\partial \bar{K}}{\partial \bar{x}}\Big] = \big[1+\bar{K}\bar{y}(h/L)\big]\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + 2(h/L)\bar{K}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{9A}$$

By using the continuity equation (equation (4A)) equation (9A) can be simplified to the following form;

$$\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + \bar{v}^*\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + (h/L)\Big[\bar{K}\bar{v}^*\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{K}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u}^2\frac{\partial \bar{K}}{\partial \bar{x}}\Big] = \frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + (h/L)\Big[\bar{K}\bar{y}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + 2\bar{K}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\Big] \tag{10A}$$

Equations (4A) and (10A) are the final equations to be solved for $\bar{v}^*$ and $\bar{u}$ as a function of $\bar{x}$ and $\bar{y}$.

APPENDIX B

SOLUTION OF THE DIFFERENTIAL EQUATION FOR f_1

The equation to be solved for f_1 is equation (34) with $\epsilon/(h/L) = 0$. First, multiply equation (34) by $f_o^{-3/2}$ which yields;

$$f_1''/f_o^{1/2} + f_1 f_o''/f_o^{3/2} - f_1' f_o'/f_o^{3/2} + f_1' f_o^{1/2} + 2f_1 f_o' = -\eta f_o''/f_o^{1/2} + \tfrac{1}{2}\eta (f_o')^2/f_o^{3/2} + f_o'/f_o^{1/2} \qquad \text{(1B)}$$

The terms in equation (1B) can be regrouped to yield;

$$\frac{f_1''}{f_o^{1/2}} - \frac{f_1' f_o'}{2f_o^{3/2}} - \frac{f_1' f_o'}{2f_o^{3/2}} - \frac{f_1 f_o''}{2f_o^{3/2}} + \frac{3f_1(f_o')^2}{4f_o^{5/2}} + f_1' f_o^{1/2} + \frac{f_1 f_o'}{2f_o^{1/2}}$$
$$+ \frac{3f_1 f_o'}{2f_o^{1/2}} + \frac{3f_1 f_o''}{2f_o^{3/2}} - \frac{3f_1(f_o')^2}{4f_o^{5/2}} = \frac{-\eta f_o''}{f_o^{1/2}} + \frac{\eta (f_o')^2}{2f_o^{3/2}} + \frac{f_o'}{f_o^{1/2}} \qquad \text{(2B)}$$

where $\frac{3}{4} f_1 (f_o')^2/f_o^{5/2}$ and $f_1 f_o''/f_o^{3/2}$ were added and subtracted from the left hand side of equation (1B).

The terms on the left hand side form the following differentials;

$$f_1''/f_o^{1/2} - f_1' f_o'/f_o^{3/2} = \frac{d}{d\eta}\left(f_1'/f_o^{1/2}\right)$$

$$\tfrac{1}{2} f_1' f_o'/f_o^{3/2} + \tfrac{1}{2} f_1 f_o''/f_o^{3/2} - \tfrac{3}{4} f_1 (f_o')^2/f_o^{5/2} = \frac{d}{d\eta}\left(\tfrac{1}{2} f_1 f_o'/f_o^{3/2}\right)$$

$$f_1' f_o^{1/2} + \tfrac{1}{2} f_1 f_o'/f_o^{1/2} = \frac{d}{d\eta}\left(f_1 f_o^{1/2}\right)$$

$$\tfrac{3}{2} f_1 f_o'/f_o^{1/2} + \tfrac{3}{2} f_1 f_o''/f_o^{3/2} - \tfrac{3}{4} f_1 (f_o')^2/f_o^{5/2} = \tfrac{3}{2} f_1 \left(f_o'/f_o^{1/2} + f_o''/f_o^{3/2} - \tfrac{1}{2}(f_o')^2/f_o^{5/2}\right)$$

The term f_o'' can be replaced by; $f_o'' = \frac{1}{2}(f_o')^2/f_o - f_o f_o'$

which was obtained from the expression for f_o'.

Then, $\frac{3f_1f_o'}{2f_o^{1/2}} + \frac{3f_1f_o''}{2f_o^{3/2}} - \frac{3f_1(f_o')^2}{4f_o^{5/2}} = \frac{3}{2}f_1(0) = 0$

Equation (2B) can then be written as;

$$\frac{d}{d\eta}\left(f_1'/f_o^{1/2}\right) - \frac{d}{d\eta}\left(\tfrac{1}{2}f_1f_o'/f_o^{3/2}\right) + \frac{d}{d\eta}\left(f_1f_o^{1/2}\right) = -\eta f_o''/f_o^{1/2} + \tfrac{1}{2}\eta(f_o')^2/f_o^{3/2} + f_o'/f_o^{1/2} \qquad (3B)$$

This expression can be integrated from η to ∞ to yield;

$$f_1'/f_o^{1/2} - \tfrac{1}{2}f_1f_o'/f_o^{3/2} + f_1f_o^{1/2} - f_{1\infty}f_{o\infty}^{1/2} = -\eta f_o'/f_o^{1/2} + 4f_o^{1/2} - 4f_{o\infty}^{1/2} \qquad (4B)$$

Equation (4B) can be integrated once more if it is first divided by $(1-f_o^{3/2})$. When $f_{o\infty}$ is replaced by 1, and equation (4B) is divided by $(1-f_o^{3/2})$, the resultant expression becomes;

$$\frac{f_1'}{f_o^{1/2}(1-f_o^{3/2})} - \frac{f_1f_o'}{2f_o^{3/2}(1-f_o^{3/2})} + \frac{f_1f_o^{1/2}}{(1-f_o^{3/2})} = -\frac{\eta f_o'}{f_o^{1/2}(1-f_o^{3/2})} + \frac{4f_o^{1/2}}{(1-f_o^{3/2})} - \frac{4-f_{1\infty}}{(1-f_o^{3/2})} \qquad (5B)$$

Equation (5B) can be integrated from o to η to yield;

$$\frac{f_1}{f_o^{1/2}(1-f_o^{3/2})} - \left[\frac{f_1}{f_o^{1/2}(1-f_o^{3/2})}\right]_{\eta=0} = \tfrac{1}{3}\eta^2 + 4\int_0^\eta \frac{f_o^{1/2}}{1-f_o^{3/2}}d\eta - (4-f_{1\infty})\int_0^\eta \frac{d\eta}{1-f_o^{3/2}} \qquad (6B)$$

The term $\int_0^\eta \frac{f_o^{1/2}d\eta}{1-f_o^{3/2}}$ can be integrated in the following way;

$$\int_0^\eta \frac{f_o^{1/2}}{1-f_o^{3/2}}d\eta = 3/2\int_0^{f_o}\frac{df_o}{(1-f_o^{3/2})^2}$$

The expression $\int_0^\eta \frac{df_o}{(1-f_o^{3/2})^2}$ can be integrated to give;

$$\int_0^{f_o}\frac{df_o}{(1-f_o^{3/2})^2} = \frac{2f_o}{3(1-f_o^{3/2})} + 2/9\left[\log\left(\frac{\sqrt{1+f_o^{1/2}+f_o}}{1-f_o^{1/2}}\right) - \sqrt{3}\tan^{-1}\left(\frac{\sqrt{3f_o}}{2+f_o^{1/2}}\right)\right]$$

Using the expression for η given by equation (30) yields;

$$\int_0^\eta \frac{f_o^{1/2}}{(1-f_o^{3/2})}d\eta = \frac{f_o}{1-f_o^{3/2}} + 2/3\log\left(\frac{\sqrt{1+f_o^{1/2}+f_o}}{1-f_o^{1/2}}\right) - \eta/3 \qquad (7B)$$

The integral $\int_0^\eta \frac{d\eta}{(1-f_o^{3/2})}$ can be integrated to give;

$$\int_0^\eta \frac{d\eta}{(1-f_0^{3/2})} = \frac{3}{2}\int_0^{f_0} \frac{df_0}{f_0^{1/2}(1-f_0^{3/2})^2} = \frac{f_0^{1/2}}{(1-f_0^{3/2})} + \frac{2}{3}\log\left(\frac{\sqrt{1+f_0^{1/2}+f_0}}{1-f_0^{1/2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3f_0}}{2+f_0^{1/2}}\right)$$

Again making use of equation (30) yields;

$$\int_0^\eta \frac{d\eta}{(1-f_0^{3/2})} = \frac{f_0^{1/2}}{(1-f_0^{3/2})} + \frac{2}{3}\eta \tag{8B}$$

Substituting equations (7B) and (8B) into equation (6B) yields;

$$\frac{f_1}{f_0^{1/2}(1-f_0^{3/2})} - \left[\frac{f_1}{f_0^{1/2}(1-f_0^{3/2})}\right]_{\eta=0} = -\frac{1}{3}\eta^2 - 4\eta + \frac{2}{3}f_{1\infty}\eta + \frac{4f_0^{1/2}}{1-f_0^{3/2}} - \frac{(4-f_{1\infty})}{1-f_0^{3/2}} + \frac{8}{3}\log\left(\frac{\sqrt{1+f_0^{1/2}+f_0}}{1-f_0^{1/2}}\right) \tag{9B}$$

The expression $\left[\frac{f_1}{f_0^{1/2}(1-f_0^{3/2})}\right]_{\eta=0}$ is the indeterminate form 0/0, however, one application of L' Hosiptal's rule gives the form 0/(1/3). Therefore, equation (9B) can be rearranged to yield;

$$f_1 = -\frac{1}{2}\eta^2 f_0' - 6\eta f_0' + 4f_0^{3/2} + 4f_0' \log\left(\frac{\sqrt{1+f_0^{1/2}+f_0}}{1-f_0^{1/2}}\right) + f_{1\infty}(\eta f_0' + f_0) \tag{10B}$$

where the expression $f_0' = \frac{2}{3}f_0^{1/2}(1-f_0^{3/2})$ was used in simplifying some of the terms. Equation (10B) then is the final form of the equation for f_1.

APPENDIX C

EFFECT OF ARBITRARY $f_{1\infty}$ ON u

If the value of $f_{1\infty}$ is to be an arbitrary value, then the values of the dimensional velocity u at a given x and y location must not be affected by changes in $f_{1\infty}$. A direct expression for the velocity u does not explicitly yield that the velocity u is unchanged by changes in $f_{1\infty}$. Therefore, an alternative approach is used. If the derivative of the velocity u with respect to $f_{1\infty}$ is zero, then this yields the result that u is not dependent on $f_{1\infty}$.

The expression for the dimensional velocity u can be written as (equation (36));

$$u = 4 U_R L^{1/2} f'/X^{1/2}$$

or

$$u = \frac{4(L/h)^2 \nu}{L^{1/2} X^{1/2}} \left[f_0' + (h/L) f_1' \right] \tag{1C}$$

The expression for L/h when $f_{1\infty}$ is arbitrary can be obtained from equation (39). Equation (39) yields;

$$(L/h)^4 \left[1 + (h/L)\left(f_{1\infty} + 9\int_0^\infty f_0' f_1' d\eta \right) \right] = HL \tag{2C}$$

To solve equation (2C) for L/h, expand equation (2C) in a series to yield;

$$(L/h) = (HL)^{1/4}/\left[1+(h/L)\left(f_{1\infty}+9\int_0^\infty f_0' f_1' d\eta\right)\right]^{1/4} = (HL)^{1/4}\Big[1-$$

$$\tfrac{1}{4}(h/L)\left(f_{1\infty}+9\int_0^\infty f_0' f_1' d\eta\right) \text{ + terms order } (h/L)^2 \text{ and smaller}\Big] \qquad (3C)$$

Neglecting the terms marked as order $(h/L)^2$ in equation (3C) gives the following result for (L/h) from equation (3C);

$$L/h = (HL)^{1/4} - \frac{f_{1\infty} + 9\int_0^\infty f_0' f_1' d\eta}{4}$$

The term $\int_0^\infty f_0' f_1' d\eta$ can be expressed in terms of $f_{1\infty}$ by integrating equation (34) from $\eta = 0$ to $\eta = \infty$. This integration gives;

$$\int_0^\infty f_0' f_1' d\eta = 1/3\, f_{1\infty} - 7/12$$

The expression for (L/h) then becomes;

$$(L/h) = \left[(HL)^{1/4} + 21/16\right] - f_{1\infty} \qquad (4C)$$

where $(HL)^{1/4}$ is of order (L/h) and $f_{1\infty}$ is of order (1).

The quantity f_1 can be split into two components, g_1 and $f_{1\infty}(\eta f_0' + f_0)$.

That is,

$$f_1 = g_1 + f_{1\infty}(\eta f_0' + f_0)$$

where $g_1 = -\tfrac{1}{2}\eta^2 f_0' - 6\eta f_0' + 4 f_0^{3/2} - 4 f_0 + 4 f_0' \log\left[\frac{\sqrt{1+f_0^{1/2}+f_0}}{1-f_0^{1/2}}\right]$

The expression for the velocity u (equation (1C)) can then be written as;

$$u = \frac{4\nu}{L\,x^{1/2}}\left[(L/h)^2 f_0' + (L/h) g_1' + (L/h) f_{1\infty}(\eta f_0'' + 2 f_0')\right] \qquad (5C)$$

If equation (5C) is differentiated with respect to $f_{1\infty}$, then terms such as $\partial\eta/\partial f_{1\infty}$ and $\partial(L/h)/\partial f_{1\infty}$ will be required. The term $\partial(L/h)/\partial f_{1\infty}$ can be obtained from equation (4C).

$$\partial(L/h)/\partial f_{1\infty} = -1 \tag{6C}$$

The expression for η is defined as (equation (12));

$$\eta = \bar{Y}/\bar{X}^{3/4} = (L/h)\, Y/L^{1/4}X^{3/4}$$

Then, for given values of y, L, and x;

$$\left(\frac{\partial\eta}{\partial f_{1\infty}}\right) = \left(Y/L^{1/4}X^{3/4}\right)\frac{\partial(L/h)}{\partial f_{1\infty}} = (h/L)\,\eta\,\frac{\partial(L/h)}{\partial f_{1\infty}} = -(h/L)\,\eta \tag{7C}$$

Equation (5C) differentiated with respect to $f_{1\infty}$ then becomes;

$$\frac{du}{df_{1\infty}} = \frac{4\nu}{L\,X^{1/2}}\Big[-2(L/h)f_0' - (L/h)\,\eta f_0'' - g_1' - \eta g_1'' - f_{1\infty}(\eta f_0''+2f_0')$$
$$+ (L/h)(\eta f_0''+2f_0') - \eta f_{1\infty}(\eta f_0''+2f_0')\Big]$$

The three terms multiplied by (L/h) cancel, so the resultant expression for $du/df_{1\infty}$ becomes;

$$\frac{du}{df_{1\infty}} = \frac{4\nu}{L X^{1/2}}\left[-g_1' - \eta g_1'' - f_{1\infty}(\eta f_0''+2f_0') - \eta f_{1\infty}(\eta f_0''+2f_0')\right] \tag{8C}$$

when equation (8C) is compared with the expression for u given by equation (5C), it can be seen that $du/df_{1\infty}$ is of order $(h/L)^2$ small than u. Since terms of order $(h/L)^2$ have been consistently neglected in this analysis, it must be concluded that changes in $f_{1\infty}$ do not affect the velocity u to an order of accuracy consistent with this analysis.

APPENDIX D

SURFACE SHAPES

The equation for the shape of the permissible surfaces found in this analysis is given by;

$$K = \frac{1}{L^{1/4} X^{3/4}} \tag{46}$$

or,

$$R = L^{1/4} X^{3/4} \tag{1D}$$

To find the shape of a surface given by equation (1D), use the cartesian coordinates η^* and ξ^*.

From the definition of the radius of curvature,

$$R = \frac{\left[1 + (d\eta^*/d\xi^*)^2\right]^{3/2}}{\left|d^2\eta^*/d\xi^{*2}\right|} \tag{2D}$$

A length of surface dx can be expressed as;

$$dx = d\xi^* \sqrt{1 + (d\eta^*/d\xi^*)^2} \tag{3D}$$

Differentiating equation (3D) with respect to ξ^* and solving for $d^2\eta^*/d\xi^{*2}$ yields;

$$\frac{d^2\eta^*}{d\xi^{*2}} = \frac{(dX/d\xi^*)(d^2X/d\xi^{*2})}{(d\eta^*/d\xi^*)} = \frac{(dX/d\xi^*)(d^2X/d\xi^{*2})}{\sqrt{(dX/d\xi^*)^2 - 1}} \tag{4D}$$

Combining equations (2D) and (4D) yields;

$$R = (dX/d\xi^*)^2 \sqrt{(dX/d\xi^*)^2 - 1} \Big/ (d^2X/d\xi^{*2})$$

or,

$$\frac{d^2X}{d\xi^{*2}} = (dX/d\xi^*)^2 \sqrt{(dX/d\xi^*)^2 - 1} \Big/ R \tag{5D}$$

Substituting equation (1D) for R in equation (5D) yields;

$$\frac{d^2X}{d\xi^{*2}} = \frac{(dX/d\xi^*)^2\sqrt{(dX/d\xi^*)^2-1}}{L^{1/4}\,X^{3/4}}$$

This expression can be integrated once to yield;

$$\frac{dX}{d\xi^*} = 1/\cos\left(4X^{1/4}/L^{1/4}\right) \tag{6D}$$

where the boundary condition that $(dx/d\xi^*) = 1$ at $x = 0$ was used. Integrating equation (6D) again yields;

$$\xi^* = \frac{L}{64}\left[6 + \alpha^3\sin\alpha + 3\alpha^2\cos\alpha - 6\cos\alpha - 6\alpha\sin\alpha\right] \tag{7D}$$

where

$$\alpha = 4X/L^{1/4} \tag{8D}$$

The boundary condition was $\xi^* = 0$ at $x = 0$.

The expression for η^* can be found by using the relation;

$$d\eta^*/d\xi^* = (d\eta^*/dX)(dX/d\xi^*)$$

or

$$\frac{d\eta^*}{dX} = \frac{(d\eta^*/d\xi^*)}{(dX/d\xi^*)} = -\sqrt{1-(d\xi^*/dX)^2} = -\sin\left(4X^{1/4}/L^{1/4}\right)$$

This expression can be integrated to yield;

$$\eta^* = \frac{L}{64}\left[\alpha^3\cos\alpha - 3\alpha^2\sin\alpha + 6\sin\alpha - 6\alpha\cos\alpha\right] \tag{9D}$$

where α is given by equation (8D). The boundary condition was taken to be $\eta^* = 0$ at $x = 0$.

Equations (7D) and (9D) then give the surface co-ordinates in terms of the parameter L and the surface length x. The length x could be eliminated from equations (7D) and (9D), giving η^* as a function of ξ^* with the parameter L remaining. However, this would be very difficult because of the complicated expressions for η^* and ξ^*, so that the intermediate variable x will not be eliminated.