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Computation Laboratory

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ABSTRACT

A coding scheme for additive Gaussian channel is developed using a noisy feedback link and D-dimensional elementary signals with no band width constraint. This allows error free transmission at a rate $R < R_c$ where R_c is slightly less than the channel capacity C. When there is no noise in the Feedback channel, the coding scheme reduces to a D-dimensional generalization of the coding scheme of Schalkwijk and Kailath. In addition, the expression for the probability of error is determined when T, the time of Transmission rate is finite. Our scheme is also compared with the best codes which use only the forward channel.

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SEQUENTIAL CODING SCHEMES FOR AN ADDITIVE NOISE CHANNEL WITH A NOISY FEEDBACK LINK

I. INTRODUCTION

We are interested in the transmission of messages over a noisy channel using a noisy feedback channel which will be used to convey the state of the uncertainty of the receiver to the transmitter. This model is suitable for many of the communication problems. The only constraint on the transmitted signals is their limited average power. The starting point of our paper is the classical result due to Shannon [1] who showed the existence of a coding scheme in which the probability of error can be made arbitrarily small for a nonzero transmission rate. Specifically, suppose that one has to transmit one of M messages during a period of T seconds across a Gaussian channel with double sided spectral density $\frac{N_o}{2}$ with P_{av} being the average power constraint on the transmitted signals. Let $R \triangleq$ transmission rate $= ((\ln M)/T)$ nats/sec and $P_{e,\,opt}(M, T, \frac{P_{AV}}{N_o}) =$ Probability of error using the optimal coding scheme with a signal-to-noise ratio P_{AV}/N_o. Then

$$\lim_{T\to\infty} P_{e,\,opt}\left(M, T, \frac{P_{AV}}{N_o}\right) = \begin{cases} 0 & \text{if } R < C \\ 1 & \phantom{\text{if }} R > C \end{cases}$$

where C = channel capacity $= P_{AV}/N_o$. A number of authors [2-6] have investigated the transmission of messages over noisy channels using a noiseless feedback link. Shannon [2] showed that the existence of a noiseless feedback link will not result in an increase in the channel capacity in the forward direction. The recent striking contribution is due to Schalkwijk and Kailath [6] who developed a coding scheme using noiseless feedback link and one dimensional elementary signals which realizes a transmission rate equal to the channel capacity and demonstrated its superiority over best known conventional codes like the simplex codes [7]. However, if there is noise in the feedback link, their coding scheme implies zero transmission rate if we insist on zero probability of error.

The coding scheme of this paper is obtained by considering the problem of information transmission on one of recursive estimation problem both at the transmitter and the receiver. The main result is that messages using D-dimensional elementary signals can be transmitted over a noisy channel using a noisy feedback channel with zero probability of error at any rate less than the critical rate R_c which is only slightly less than channel capacity C of the forward channel

$$R_c \approx \left(\frac{1}{1 + \sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\frac{N_b}{N_o}} \right) \frac{P_{AV}}{N_o}$$

where $\frac{N_b}{2}$ in the two-sided spectral density of the additive Gaussian noise in the feedback link, and other symbols have been defined earlier. In particular, when the noise in the feedback link is absent, the coding scheme becomes a generalization of the coding scheme in reference [6] for D-dimensional signals. In addition, if the time of transmission T is finite, expression will be derived for the probability of error and this scheme will be compared with the best codes (simplex codes) which use only the forward channel.

II. CODING SCHEME

We will first convert the continuous time Gaussian channel into a discrete time Gaussian channel and describe the coding scheme in terms of the latter.

1. Transformation of a continuous time Gaussian channel into a discrete time channel.

Suppose one has to transmit one of the M messages belonging to the set

$$\{m^{(j)}\} \triangleq \{m^{(1)}, m^{(2)}, \ldots, m^{(M)}\}$$

over a time T seconds. Let us assume a set of orthogonal elementary signals $\phi_1(t), \ldots, \phi_D(t)$ are available which satisfy the relation

$$\int \phi_i(t-k\Delta)\, \phi_j(t-\ell\Delta)\, dt = \delta_{ij}\, \delta_{k\ell}\,, \quad \begin{matrix} i, j = 1, \ldots, D \\ k, \ell = 1, \ldots, N \end{matrix} \tag{1}$$

where Δ is the discretization interval. The actual signal transmitted is $s(t)$

$$s(t) = \sum_{i=1}^{N} u^T(i)\, \phi(t-i\Delta)$$

where

$$\phi^T(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_D(t))$$

$$u^T(i) = (u_1(i), \ldots\ldots, u_D(i))$$

N = Largest integer less than or equal to (T/Δ). The vectors $u(i)$, $i = 1, \ldots, N$ are yet unknown and depend on the particular message to be transmitted.

Let the received signal be $r_1(t)$ and the additive white Gaussian noise with spectral density $\frac{N_o}{2}$.

$$r_1(t) = s(t) + n_1(t) \tag{2}$$

$$\begin{cases} E\,[n_1(t)\,] = 0 \\ E\,[n_1(t_1)\; n_1^T\,(t_2)\,] = \frac{N_o}{2}\,\delta\,(t_1 - t_2) \end{cases} \tag{3}$$

The receiver computes a signal $s_2(t)$ on the basis of its measurements and sends it back to the transmitter. Let the noise in the feedback channel be $n_2(t)$ which is white Gaussian and additive with spectral density $\frac{N_b}{2}$.

$$s_2(t) = \sum_{i=1}^{N} v^T(i)\,\phi(t-i\,\Delta) \tag{4}$$

$$r_2(t) = s_2(t) + n_2(t) \tag{5}$$

$$\begin{aligned} &E\,[n_2(t)\,] = 0 \\ &E\,[n_2(t_1)\, n_2(t_2)\,] = \frac{N_b}{2}\,\delta\,(t_1 - t_2) \end{aligned} \tag{6}$$

If we define the following vectors of dimension D

$$y(i) \triangleq \int r_1(t)\,\phi\,(t-i\Delta)\,dt$$

$$\eta(i) \triangleq \int n_1(t)\,\phi\,(t-i\Delta)\,dt$$

$$z(i) \triangleq \int r_2(t)\,\phi\,(t-i\Delta)\,dt$$

$$\xi(i) \triangleq \int n_2(t)\,\phi\,(t-i\Delta)\,dt$$

then the continuous time model represented by Figure 1 and equations (2), (3), (5), (6) can be replaced by the discrete time model represented in Figure 2 and equations (7)-(9), with discretization interval Δ [7]

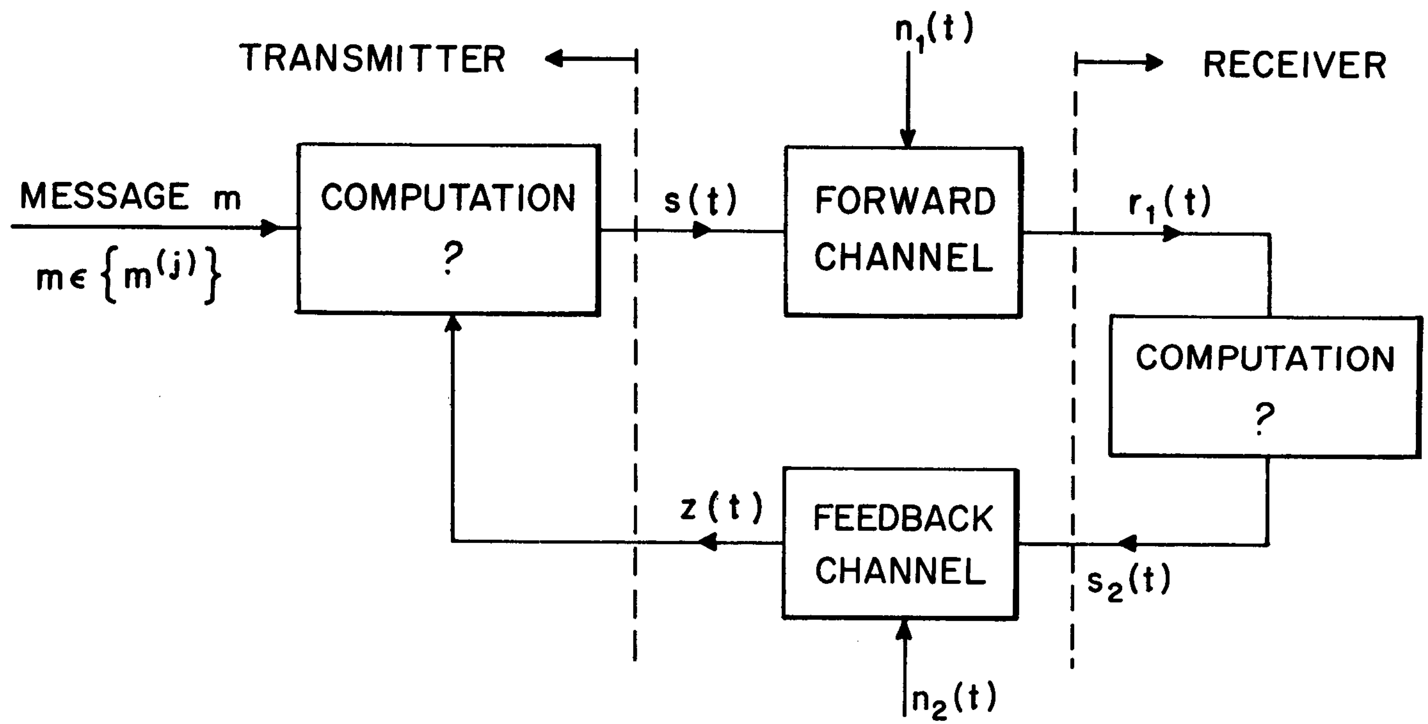

FIG. 1 CONTINUOUS TIME FEEDBACK COMMUNICATION MODEL

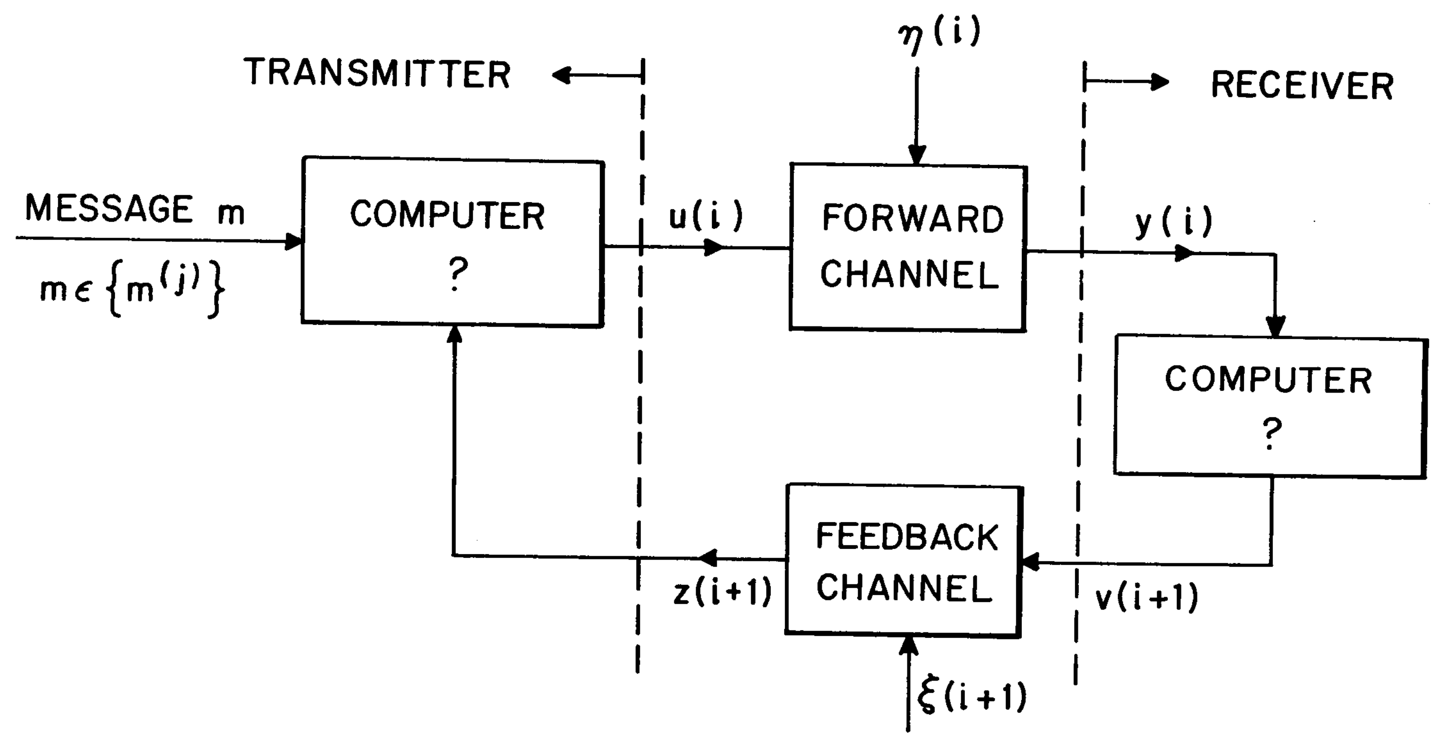

FIG. 2 DISCRETE TIME FEEDBACK COMMUNICATION MODEL

$$y(i) = u(i) + \eta(i) \tag{7}$$

$$z(i) = v(i) + \xi(i) \tag{8}$$

$\eta(i)$, $\xi(i)$, $i=1, \ldots, N$ are white Gaussian random vectors with

$$\begin{cases} E(\eta(i)) = E(\xi(i)) = 0 \\ E(\eta(i)\,\eta^T(j)) = \frac{N_o}{2}\,\delta_{ij}\, I \\ E(\xi(i)\,\xi^T(j)) = \frac{N_b}{2}\,\delta_{ij}\, I \\ E(\eta(i)\,\xi^T(j)) = 0 \end{cases} \tag{9}$$

Let $\sigma_\eta^2 \triangleq \frac{N_o}{2}$ and $\sigma_\xi^2 \triangleq \frac{N_b}{2}$

The problem is to determine the vectors $u(i)$, $i=1, \ldots, N$ that are to be sent at the transmitter and the vectors $v(i)$, $i=1, \ldots, N$ that are to be sent at the receiver so that error free transmission is possible at a nonzero transmission rate. Of course, the vectors $u(i)$, $v(i)$, $i=1, 2, \ldots, N$, will depend on the particular message that has to be sent to the receiver.

2. Description of coding scheme CS-1

Let us assume that the number $M = M_1^D$ where M_1 is an integer. Let us represent the messages of set $\{m^{(j)}\}$ by M equispaced points in a D-dimensional typercube centered about the origin. Figure 3 illustrates this for the case $M = 3^2$ and $D = 2$.

We will associate the j^{th} message $m^{(j)}$ with the D-dimensional vector $x^{(j)}$ joining the origin to the j^{th} point on the lattice. The coding scheme CS-1 can be described briefly as follows:

(A) Let $x = x^{(j)}$ if the message $m^{(j)}$ is to be transmitted to the receiver. set $i = 1$.

(B) At the i^{th} step, the transmitter sends the vector $u(i) \triangleq \alpha(x-\bar{x}(i))$ where α is a scalar constant and $\bar{x}(i)$ will be described later.

(C) The receiver has a measurement $y(i)$ (D-vector)

$$y(i) = \alpha(x-\bar{x}(i)) + \eta(i) \tag{10}$$

Using this measurement, the receiver recursively computes the vector $\hat{x}(i+1)$ to be described later. It sends back to the transmitter the vector $\alpha\hat{x}(i+1)$.

(D) The transmitter receives a measurement $z(i+1)$ (D-vector)

$$z(i+1) = \alpha\hat{x}(i+1) + \xi(i+1) \tag{11}$$

Using this measurement, the transmitter recursively computes $\bar{x}(i+1)$ and hence $\alpha(x-\bar{x}(i+1))$

(E) Increment i by one and go back to step (B)

$\hat{x}(i+1)$ = Maximum likelihood estimates of the vector parameter x at the receiver on the i^{th} stage based on all the available measurements till that stage i.e., $y(1), y(2), \ldots, y(i)$.

$\bar{x}(i+1)$ = Maximum likelihood estimate (Kalman estimator) of the random vector $\hat{x}(i+1)$ by the transmitter based on all the measurements $z(1), \ldots, z(i+1)$ and the parameter x.

$= E[\hat{x}(i+1) / z(1), \ldots, z(i+1); \ x]$

The recursive equations for $\hat{x}(i)$ and $\bar{x}(i)$ are given below and a block diagram of the coding scheme is in Figure 4. It should be noted that $\zeta(i)$, $\theta(i)$, $p(i)$, $q(i)$, and $r(i)$ are all scalars

$$\hat{x}(i+1) = \hat{x}(i) + \zeta(i)\, y(i) \tag{12}$$

$$\zeta(i) = (\alpha^2 p(i) + \sigma_\eta^2 + \alpha^2 q(i) + 2\alpha^2 r(i))^{-1} (q(i)+r(i))\alpha \tag{14}$$

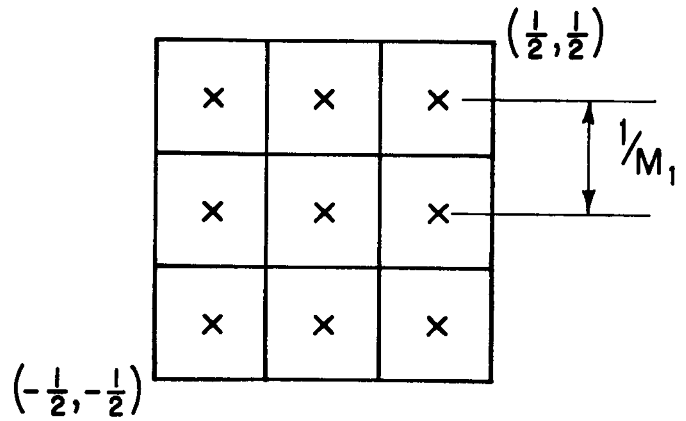

FIG. 3 REPRESENTATION OF THE $M = 3^2$ MESSAGES ON A 2-CUBE. CROSSES DENOTE MESSAGES.

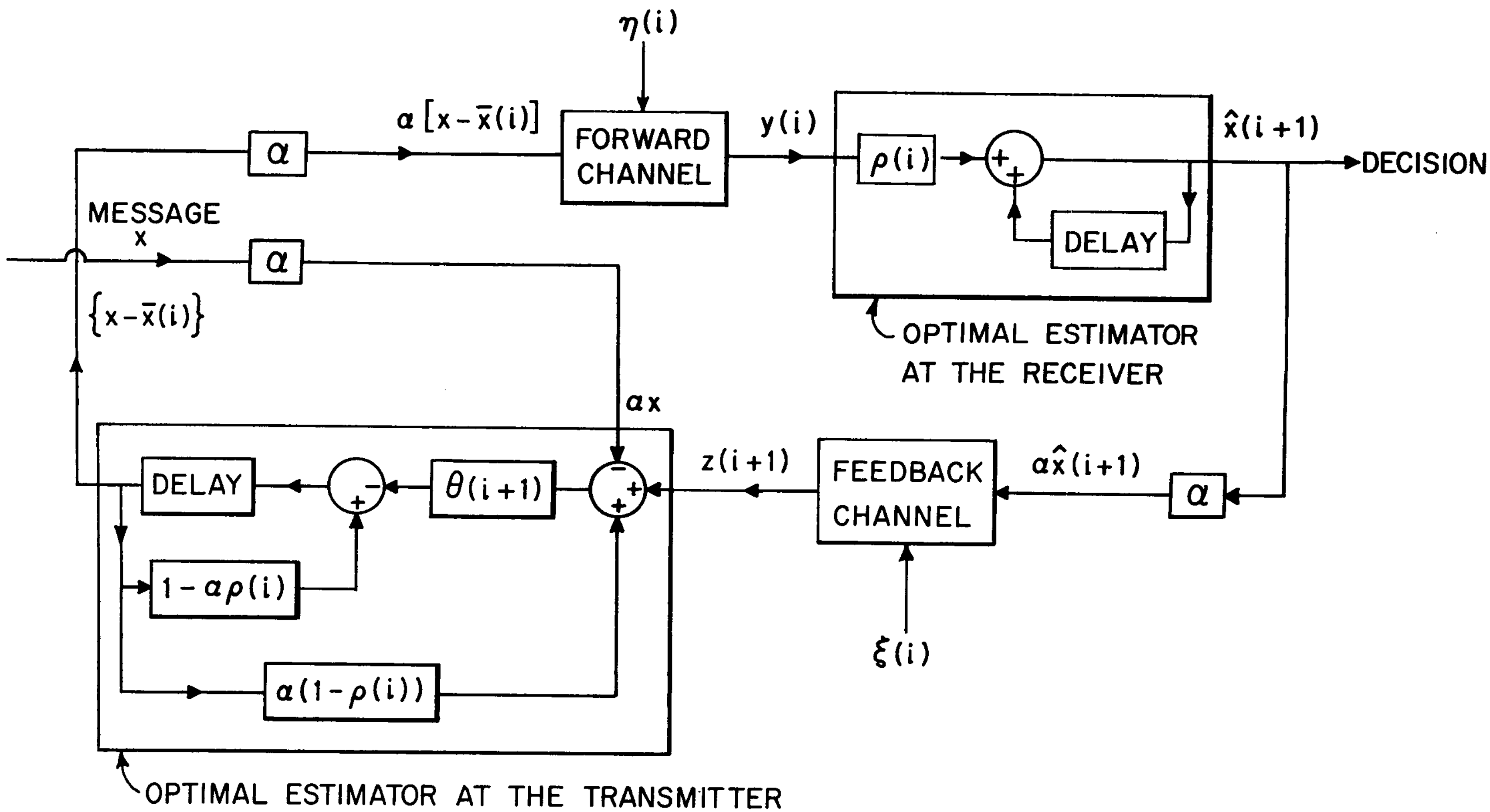

FIG. 4 FEEDBACK CODING SCHEME CS - 1

$$\bar{x}(i+1) = \bar{x}(i) + \zeta(i)\,\alpha\,(x-\bar{x}(i)) + \theta(i+1)\{z(i+1)-\alpha\bar{x}(i) - \alpha\zeta(i)(x-\bar{x}(i))\} \quad (13)$$

$$\theta(i) = p(i)\,\alpha / \sigma_\xi^2 \quad (15)$$

$$q(i+1) = \{1-\alpha\,\zeta(i)\}^2 q(i) + \zeta^2(i)\{\alpha^2 p(i) + \sigma_\eta^2\} - 2\alpha\,\zeta(i) r(i)\{1-\alpha\zeta(i)\} \quad (16)$$

$$p(i+1) = \frac{(p(i) + \zeta^2(i)\,\sigma_\eta^2)\,\sigma_\xi^2}{\sigma_\xi^2 + \alpha^2\{p(i) + \zeta^2(i)\,\sigma_\eta^2\}} \quad (17)$$

$$\begin{aligned} r(i+1) = [\,&(1-\alpha\,\zeta(i))(1-\alpha\theta(i+1))\,r(i) - \alpha\zeta(i)p(i)(1-\alpha\,\theta(i+1)) \\ &- \sigma_\eta^2\,\zeta^2(i)(1-\alpha\,\theta(i+1)\,]\end{aligned} \quad (18)$$

It should be noted that (12) is stored at the receiver, (13) at the transmitter and the deterministic difference equations (16)-(18) and the definitions (14)-(15) are stored both at transmitter and the receiver.

III. ANALYSIS OF THE CODING SCHEME

Before we demonstrate the possibility of error free transmission, we will analyse the ML (Maximum Likelihood) estimators at the transmitter and the receiver, more closely. It should be emphasized here that the ML estimator at the receiver and the ML (Kalman) estimator at the transmitter are intimately related to each other, even though they are treated separately here.

1. <u>ML estimator of x at the receiver.</u>

Define $\quad \tilde{x}(i) \triangleq \hat{x}(i) - \bar{x}(i)$

= error in the optimal estimator at the transmitter.

Then the equation for $y(i)$ can be rewritten as:

$$y(i) = \alpha(x-\hat{x}(i)) + \{\alpha\,\tilde{x}(i) + \eta(i)\} \quad (19)$$

Let $\hat{x}(i)$ = ML estimate of the parameter x at the receiver based on the measurements $y(1), \ldots, y(i-1)$

Let us try to compute $\hat{x}(i+1)$ from $\hat{x}(i)$ and $y(i)$

Let

$$\left\{\begin{array}{l} \mathrm{Cov}\,[x-\hat{x}(i)] \triangleq q(i)\, I \\ \mathrm{Cov}[\tilde{x}(i)] \triangleq p(i)\, I \\ E\,[(x-\hat{x}(i))\,(\tilde{x}(i))^T] \triangleq r(i)\, I \end{array}\right. \tag{20}$$

But the noise $\alpha\,\tilde{x}(i)$ occuring in (19) is <u>not</u> white, though Gaussian. Therefore, while computing the MLE $\hat{x}(i+1)$, the correlation between $(x-\hat{x}(i))$ and $\tilde{x}(i)$ has to be considered.

By definition, $\hat{x}(i+1)$ is obtained by minimizing J w.r.t u

$$J = \left\| (u-\hat{x}(i)),\ (\alpha\tilde{x}(i) + \eta(i)) \right\|^2_{\begin{bmatrix} q(i)\, I & \alpha r(i)\, I \\ \alpha r(i)\, I & (\alpha^2 p(i) + \sigma_\eta^2)\, I \end{bmatrix}^{-1}}$$

$$= \left\| (u-\hat{x}(i)),\ (y(i)-\alpha u+\alpha\hat{x}(i)) \right\|^2_{\begin{bmatrix} q(i)\, I & \alpha r(i)\, I \\ \alpha r(i)\, I & (\alpha^2 p(i) + \sigma_\eta^2)\, I \end{bmatrix}^{-1}}$$

By the straightforward minimization, we obtain

$$\hat{x}(i+1) = \hat{x}(i) + \zeta(i)\, y(i) \tag{21}$$

where the scalar $\zeta(i)$ has been defined earlier in (14). At the i^{th} stage, error in the estimator = $\{x-\hat{x}(i)\}$.

From (21) we can write the recursive relation for $(x-\hat{x}(i))$

$$\begin{aligned} x - \hat{x}(i+1) = \{1-\alpha\,\zeta(i)\}\,\{x-\hat{x}(i)\} \\ -\zeta(i)\,(\alpha\tilde{x}(i) + \eta(i)) \end{aligned} \tag{22}$$

From (22) and (20) we can obtain the recursive equation (16) for the scalar q(i).

In the appendix it is shown that asymptotically

$$q(i) \approx \frac{\sigma_\eta^2}{\alpha^2 i}\{1+2\sqrt{\frac{N_b}{N_o}} + 2\,\frac{N_b}{N_o}\} \qquad , \quad \frac{N_b}{N_o} < 1 \text{ and sufficiently small}$$

2. <u>Optimal Estimator of $\hat{x}$ (i+1) at the Transmitter</u>

Rewriting (21)

$$\hat{x}\,(i+1) = \hat{x}\,(i) + \alpha\,\zeta(i)\,(x-\hat{x}\,(i)) + \zeta(i)\,\eta(i) \tag{23}$$

Equation for the measurement z(i+1) is

$$z(i+1) = \alpha\hat{x}\,(i+1) + \xi(i+1) \tag{24}$$

We want to evaluate the ML estimator of the random vector $\hat{x}$ (i+1) given the measurements z(1) , ... , z(i+1). Rewriting (23)

$$\hat{x}\,(i+1) = \overline{x}\,(i) + \alpha\zeta(i)\,(x-\overline{x}\,(i)) + \widetilde{x}\,(i) + \zeta(i)\eta(i) \tag{25}$$

Hence, by definition, given $\hat{x}$ (i) and z(i+1), $\hat{x}$ (i+1) is obtained by minimizing J w. r. t vector u

$$J = \| u-\overline{x}\,(i) + \alpha\zeta(i)\,(x-\overline{x}\,(i))\,\|^2_{\left(\frac{I}{p(i)+\zeta^2(i)\,\sigma_\eta^2}\right)} + \|z(i+1)-\alpha u\|^2_{\frac{I}{\sigma_\xi^2}} \tag{26}$$

Performing the minimization, we get

$$\overline{x}\,(i+1) = \overline{x}(i) + \alpha\zeta(i)\,(x-\overline{x}(i)) + \theta(i+1)\,[z(i+1) - \alpha\overline{x}(i) - \alpha\zeta(i)\,(x-\overline{x}(i))] \tag{27}$$

where

$$\theta(i+1) = \alpha p(i+1)/\sigma_\xi^2$$

and the recursive equation for p(i) is given earlier.

It should be noted that

$$\overline{x}(i+1) = E(\hat{x}\,(i+1)\,/z(1)\,,\ldots,z(i+1)\;;x)$$

since the ML estimator is identical with the Kalman-Bucy estimator.

Let

$$\tilde{x}(i) = \hat{x}(i) - \overline{x}(i) = \text{error in the optimal estimate.}$$

Subtracting (25) from (27) we get the difference equation for the error $\tilde{x}(i)$

$$\tilde{x}(i+1) = \tilde{x}(i)\{1-\alpha\theta(i+1)\} + \zeta(i)\,\eta(i)\,(1-\alpha\,\theta(i+1)) - \theta(i+1)\xi(i+1) \tag{28}$$

we can derive the recursive relation for $p(i)$ from (28) and the various definitions. Alternatively, we can invoke the Kalman-Bucy theory to get the relation (17). Similarly using (28) and (22), we can get the recursive relation (18) for $r(i)$

Asymptotically

$$p(i) = \frac{\sigma_\eta^2}{\alpha^2 i}\sqrt{\frac{N_b}{N_o}}\left(1+\frac{3}{2}\sqrt{\frac{N_b}{N_o}}\right), \qquad (N_b/N_o) < 1$$

$$r(i) = -\frac{\sigma_\eta^2}{\alpha^2 i}\sqrt{\frac{N_b}{N_o}}\left(1+\frac{3}{2}\sqrt{\frac{N_b}{N_o}}\right), \qquad (N_b/N_o) < 1$$

These relations have been proved in the appendix 1.

From these we obtain (in appendix)

$$E[(x-\overline{x}(i))\,(x-\overline{x}(i))^T] = \frac{\sigma_\eta^2}{\alpha^2 i}\left\{1+\sqrt{\frac{N_b}{N_o}}+\frac{1}{2}\,\frac{N_b}{N_o}\right\}I \tag{29}$$

3. Determination of the Criticle Transmission Rate R_c

THEOREM:

Let P_{AV} be the constraint on the average transmitted power. Let $(N_o/2)$ and $(N_b/2)$ be respectively the two-sided spectral densities of the additive white Gaussian noises in the forward and backward channel respectively. Suppose one of $M \triangleq \exp(RT)$ messages has to be transmitted over a time T seconds (where R is known as the transmission rate). Suppose the coding scheme mentioned in Section II is used and the maximum likelihood decision rule is used to obtain the decision. Let $P_e(M,T)$ be the probability of error. Then there exists a constant R_c such that

$$\lim_{T\to\infty} P_e(M = \exp(RT), T) = \begin{cases} 0 & \text{if } R < R_c \\ 1 & \quad\;\; > R_c \end{cases}$$

An approximate expression for R_c is

$$R_c = \left(\frac{1}{1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\,\frac{N_b}{N_o}} \right) \cdot \left(\frac{P_{AV}}{N_o} \right)$$

Proof: Suppose that during the time T, N measurements $y(1), \ldots, y(N)$ have been taken at the receiver, the latest measurement being $y(N)$. Since all the M messages are equally probable, the decision rule $d(\cdot)$ at the receiver is:

$$d(\hat{x}(N+1)) = m^{(j)} \quad \text{if } \left| (x^{(j)} - \hat{x}(N+1))_i \right| \leq \left| (x^{(k)} - \hat{x}(N+1))_i \right|$$

$$\forall\; i=1,\ldots,D \text{ and } \forall\; k=1,\ldots,M.$$

i.e.,

$$d(\hat{x}(N+1)) = m^{(j)} \text{ if } -1/2M_1 < (x^j - \hat{x}(N+1))_i < 1/2M_1$$

$$\forall\; i=1,\ldots,D$$

where $M_1 = M^{1/D}$ (an integer)

Recall that $\mathrm{Cov}\,(x-\hat{x}(N+1)) \approx \frac{\sigma_\eta^2}{a^2}\,\frac{1}{N}\,(1+k_q)$, where $k_q = 2\sqrt{\frac{N_b}{N_o}}$

Let the probability of error $\triangleq P_e(M,T)$

$$P_c = 1-P_e\,(M, T)$$

$$= \left[\int_{\hat{x}(N+1) - (1/2M_1)}^{\hat{x}(N+1) + (1/2M_1)} \frac{\sqrt{N}\,a}{\sqrt{(2\pi)\sigma_\eta^2\,(1+k_q)}} \exp\{ - \frac{N\sigma^2}{2\sigma^2(1+k_q)} (u_j - \hat{x}_j(N+1))^2\} du_j \right]^D$$

$$= \{\mathrm{erf}\,(\frac{a}{2\sqrt{2\sigma_\eta}\sqrt{1+k_q}} \frac{\sqrt{N}}{M_1})\}^D$$

Let $M(T) \triangleq \exp\,(RT) = N^{D(1-\epsilon)/2}$

$$P_c = \{\mathrm{erf}\,(\frac{a}{2\sqrt{2\sigma_\eta}\sqrt{1+k_q}} N^{\epsilon/2})\}^D$$

Hence

$$\lim_{T\to\infty} P_e(M, T) = \begin{cases} 0 & \text{if } \epsilon > 0 \\ 1 & \phantom{\text{if }} < 0 \end{cases}$$

Therefore, the optimal signalling rate R_c is obtained by setting

$$M(T) = N^{D/2}$$

$$R_c \triangleq \frac{\ln M(T)}{T} = \frac{D \ln N}{2T} \tag{30}$$

P_{AV} = the average transmitted power

$$= E\,[\frac{1}{T} \int_0^T s^2(t)dt]$$

$$= \frac{1}{T} E\,[\sum_{i=1}^N a^2 \,\|\, x - \overline{x}\,(i)\,\|^2]$$

Let us assume that x_j is uniformly distributed in the interval $(-1/2, 1/2)$ for every $j=1, \ldots, D$. Since $\overline{x}\,(1)=0$, $E(x_j^2) = 1/12$ for all $j=1, \ldots, D$.

we know from (29) that

$$E \parallel x-\bar{x}(i) \parallel^2 \approx \frac{D\sigma_\eta^2}{\alpha^2 i}\left(1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\frac{N_b}{N_o}\right) \triangleq \sigma^2/(\alpha^2 i)$$

where

$$\sigma^2 \triangleq \sigma_\eta^2 \left\{ 1+\sqrt{N_b/N_o} + \frac{1}{2}\frac{N_b}{N_o} \right\}$$

$$P_{AV} = \frac{\alpha^2 D}{T}\left[\frac{1}{12} + \frac{\sigma^2}{\alpha^2}\left(\sum_{i=1}^{N} \frac{1}{i} + A_1 \right)\right]$$

where A_1 is the error due to the use of approximation formula.

But $\sum_{i=1}^{N} \frac{1}{i} \approx \ln N + A_2$, A_2 = Euler-Maschorini constant

Substituting for T from (30) we get

$$P_{AV} = \frac{2R_c}{\ln N}\left\{ \sigma^2 \ln N + \frac{\alpha^2}{12} + \sigma^2 (A_1 + A_2) \right\} \tag{31}$$

Hence $\lim_{N\to\infty} P_{AV} = 2R_c\,\sigma^2$

From the definition of σ^2 , we obtain

$$R_c = \left(\frac{1}{1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\frac{N_b}{N_o}} \right) \left(\frac{P_{AV}}{N_o} \right)$$

IV. PROPERTIES OF THE CODING SCHEME CS-1

1. Probability of error P_e for finite T with an optimal choice for the gain α .

Let

$$C = P_{AV}$$

and

$$C_2 \triangleq \frac{R_c}{C} = \frac{1}{1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\frac{N_b}{N_o}}$$

For given α and large T the probability of error is given by

$$P_e = 1 - \{\mathrm{erf}\,(\frac{\alpha}{2\sqrt{N_o(1+k_q)}} \quad \frac{\sqrt{N}}{M^{1/D}})\} \tag{32}$$

Let $M = e^{RT}$

From (31) we obtain

$$C = \frac{D}{T} [\frac{\alpha^2}{12N_o} + \frac{1}{2C_2} (\ln N + A_1 + A_2)]$$

Rewriting the above equation by neglecting A_1 and A_2, we have

$$N \approx \exp [2C_2 (\frac{CT}{D} - \frac{\alpha^2}{12N_o})] \tag{33}$$

Substituting for N from (33) in (32) and minimizing the overall expression for P_e with respect to α we get the optimal value of α as

$$\alpha^2_{opt} = \frac{6N_o}{C_2} \tag{34}$$

We can substitute the value of α_{opt} in (32) and simplify it by noting that in the expression for N given by (33), $(\alpha^2/12N_o)$ can be neglected w.r.t (CT/D)

Let

$$v_{opt} \triangleq \frac{\alpha_{opt}}{2\sqrt{N_o(1+k_q)}} \quad \frac{\sqrt{N}}{M^{1/D}}$$

$$= \{ \frac{3}{2} \quad \frac{1}{C_2(1+k_q)} \}^{1/2} \quad \exp \{ \frac{R_c - R)T}{D} \}$$

$$= (\frac{3C_3}{2})^{1/2} \quad \exp \{ (\frac{R_c}{R} - 1) \frac{RT}{D} \}$$

where

$$C_3 \triangleq \frac{1}{C_2(1+k_q)}$$

$$= \frac{1+\sqrt{\frac{N_b}{N_o}}+\frac{1}{2}\frac{N_b}{N_o}}{1+2\sqrt{\frac{N_b}{N_o}}+2\frac{N_b}{N_o}}$$

Then P_e (M=exp(RT), T)

$$\triangleq 1-\{\operatorname{erf}(v_{opt})\}^D$$

$$\approx [D\exp(-v_{opt}^2)]/\sqrt{\pi}\ v_{opt}$$

$$= \frac{D\exp\left[-\frac{3}{2}C_3\exp\left\{\left(\frac{R_c}{R}-1\right)\frac{2RT}{D}\right\}\right]}{\frac{3\pi C_3}{2}\exp\left\{\left(\frac{R_c}{R}-1\right)\frac{RT}{D}\right\}} \tag{35}$$

(35) is the basic expression for the probability of error for finite T. Note that for given T , N can be determined from (33) and hence (T/N), the time per iteration is also determined.

2. Noiseless Feedback Channel

Here $\xi(i) = 0$ and hence

$$\operatorname{Cov}[\xi(i)] \triangleq \sigma_\xi^2 I = 0$$

$$\operatorname{Cov}[\tilde{x}(i)] \triangleq p(i)I = 0$$

$$E[\{x-\hat{x}(i)\}\{\tilde{x}(i)\}^T] \triangleq r(i)\, I = 0$$

In this case, the recursive formula (13) becomes

$$\bar{x}(i+1) = z(i+1) \qquad \text{Since } \theta(i+1) = 1/a \qquad \forall\ i < \infty$$

$$= \hat{x}(i+1)$$

Correspondingly, the weighting factor $\zeta(i)$ in (12) becomes

$$\zeta(i) = 1/\alpha i$$

In other words, (12) becomes

$$\hat{x}(i+1) = \hat{x}(i) + \frac{1}{\alpha i} y(i) \tag{36}$$

This coding scheme will be referred to as CS-2

This simplified coding scheme is given in figure (5). Note that in this scheme $R_c = C$ and that the expression (34) for probability of error (when time T is finite) can be simplified by noting that $C_3 = 1$.

This coding scheme is nothing but an extension of the coding scheme in [6] to D-dimensional signals.

V. ALTERNATE CODING SCHEME FOR NOISY CHANNEL

This coding scheme is very similar to the one considered all along except that the recursive formulae for $\hat{x}(i)$ and $\overline{x}(i)$ are simpler. This coding scheme will be referred to as CS-3 .

$$\hat{x}(i+1) = \hat{x}(i) + \frac{1}{\alpha i} y(i) \tag{37}$$

$$\overline{x}(i+1) = \overline{x}(i) + \frac{x - \overline{x}(i)}{i} + \frac{p(i+1)\alpha}{\sigma_\xi^2}\left[z(i+1) - \alpha\overline{x}(i) - \frac{x - \overline{x}(i)}{i}\right] \tag{38}$$

where

$$p(i+1) = \frac{\{p(i) + (\sigma_\eta^2/\alpha^2 i^2)\}\ \sigma_\xi^2}{\sigma_\xi^2 + \alpha^2 p(i) + \frac{\sigma_\eta^2}{i^2}} \tag{39}$$

In this case the critical rate R_c is given by the same formula as before. The error covariance Cov $[x-\hat{x}(i)]$ which determines the error probability is given by the relation

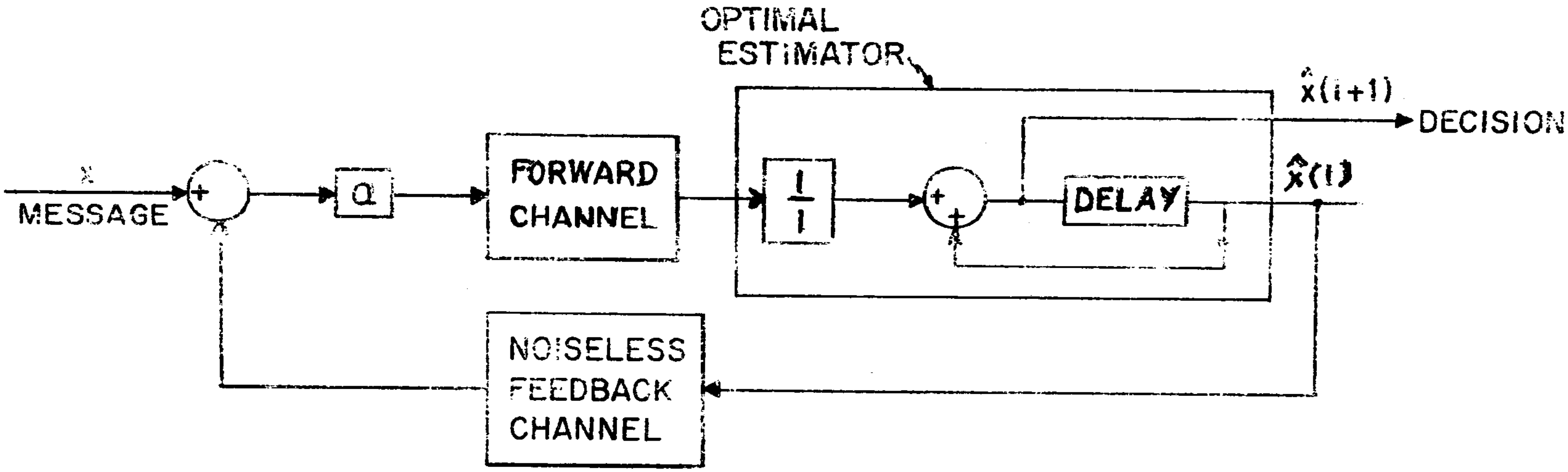

FIG. 5 FEEDBACK CODING SCHEME CS-2 WITH NOISELESS FEEDBACK CHANNEL.

$$\mathrm{Cov}\,[x-\hat{x}\,(i)] = \frac{\sigma_\eta^2}{\alpha^2 i}\,[1+2\sqrt{\frac{N_b}{N_o}} + \frac{N_b}{N_o}]\,I \tag{40}$$

The other relevant covariance matrices are given below:

$$\mathrm{Cov}[\tilde{x}\,(i)] = \frac{\sigma_\eta^2}{\alpha^2 i}\,[\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\,\frac{N_b}{N_o}]\,I \tag{41}$$

$$E[(x-\hat{x}\,(i))\,(\tilde{x}\,(i))^T] = -\,\frac{\sigma_\eta^2}{\alpha^2 i}\,[\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\,\frac{N_b}{N_o}]\,I \tag{42}$$

$$\mathrm{Cov}\,[x-\hat{x}\,(i)] = \frac{\sigma_\eta^2}{\alpha^2 i}\,[\,1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\,\frac{N_b}{N_o}]\,I \tag{43}$$

The derivations of these formulae can be found in Appendix 2. As before, an expression can be found for the probability of error when time T is finite and we can choose the gain α to minimize this probability of error. The expression for the optimal value of the gain, α_{opt}, is given below

$$\alpha_{opt}^2 = 6\,N_o/C_2$$

The probability of error is given by the formula:

$$P_e\,(M=e^{RT}\,,\;T) = \frac{D\exp\,[\,-\frac{3}{2}C_3\exp\,\{(\frac{R_c}{R}-1)\,\frac{2RT}{D}\}]}{\sqrt{\frac{3\pi\,C_3}{2}}\,\exp\,\{(\frac{R_c}{R}-1)\,\frac{RT}{D}\}} \tag{44}$$

where

$$C_3 = \frac{1+\sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\,\frac{N_b}{N_o}}{1+2\sqrt{\frac{N_b}{N_o}} + 2\,\frac{N_b}{N_o}} \tag{45}$$

The details are omitted since they are very similar to the ones used earlier.

The most intriguing feature of this coding scheme is that the probability of error is less than that for CS-1 for Same T and other parameters. This is due to the fact that Cov $[x-\hat{x}(i)]$ for CS-3 is slightly smaller than the corresponding quantity in CS-1. Even though this is intriguing, it can be easily explained by noting that the maximum likelihood estimate $\hat{x}(i)$ in CS-1 is computed from a set of dependent measurements $y(1), \ldots, y(i-1)$ and in this case the MLE estimate is not a minimum variance estimate. It is interesting to note that the estimate $\hat{x}(i)$ of x in CS-3 is obtained by minimizing J with respect to u.

$$J = E\left[\sum_{j=1}^{i-1} \| y(i)-\alpha(u-\hat{x}(j)) \|^2 / y(1), \ldots, y(i-1)\right]$$

Hence the estimate $\hat{x}(i)$ in CS-3 can be looked upon as the minimum variance estimate and hence its variance must be less than that of $\hat{x}(i)$ in CS-1 which is a MLE.

VI. COMPARISON

The best codes which use only the forward channel are the simplex codes which behave like the orthogonal codes for large M. For these codes, the probability of error is bounded by (46) [7].

$$\log_{10} P_e \leqslant \log_{10} 2 - E'(R) \left(\frac{RT}{\log_e 10} \right)$$

where

$$E'(R) = \begin{cases} \left(\frac{1}{2} \frac{C}{R} - 1\right) & \text{if } 0 < R/C \leqslant 1/4 \\ \left(\sqrt{\frac{C}{R}} - 1\right)^2 & \text{if } 1/4 \leqslant R/C < 1 \end{cases}$$

and $RT \triangleq \ln M$ = amount of information to be conveyed over a period of T seconds.

In the coding scheme CS-1

$$\log_{10} P_e = \log_{10} \left(\sqrt{\frac{2}{3\pi C_3}}\, D\right) - (\log_{10} e)\, \frac{3}{2}\, C_3 \exp\left\{\left(\frac{R_c}{R} - 1\right) \frac{2RT}{D}\right\} \tag{47}$$

$$-(\log_{10} e)\left(\frac{R_c}{R} - 1\right) \frac{RT}{D}$$

where

$$C_3 = \frac{1 + \sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\, \frac{N_b}{N_o}}{1 + 2\sqrt{\frac{N_b}{N_o}} + 2\, \frac{N_b}{N_o}} \tag{48}$$

In the coding scheme CS-3

$$\log_{10} P_e = \log_{10} \left(\sqrt{\frac{2}{3\pi C_3}}\, D\right) - (\log_{10} e)\left(\frac{R_c}{R} - 1\right) \frac{RT}{D}$$

$$-(\log_{10} e)\, \frac{3}{2}\, C_3 \exp\left\{\left(\frac{R_c}{R} - 1\right) \frac{2RT}{D}\right\} \tag{49}$$

where

$$C_3 = \frac{1 + \sqrt{\frac{N_b}{N_o}} + \frac{1}{2}\, \frac{N_b}{N_o}}{1 + 2\sqrt{\frac{N_b}{N_o}} + \frac{N_b}{N_o}} \tag{50}$$

and

$$R_c = \frac{C}{1 + \frac{1}{2}\, \frac{N_b}{N_o} + \sqrt{\frac{N_b}{N_o}}} \quad ; \quad C \triangleq P_{AV} / N_o$$

Inspection of expressions (46)-(50) establishes unambiguously the superiority of the coding schemes CS-1 and CS-3 over the orthogonal codes. In Figure 6, for a fixed R/C and D, the probability of error is plotted on a logarithmic scale versus (RT) in nats for various values of (N_b/N_o), the ratio of feedback noise power to forward noise power.

Another way of comparison is to compare the time delay T required to acheive the same probability of error for the different coding schemes.

Let $P_e = 1 \times 10^{-7}$, $R/C = 0.6$, $C = 1$ nat / sec , $D=1$

Let $N_o = 10$ joules. We can obtain the corresponding value of RT in nats from the formulae given earlier and hence compute the time delay T for the same probability of error for the different coding schemes.

Simplex codes: RT = 183.5 nats ; T = 306 seconds

Coding Schemes CS-1 and CS-3

$\frac{N_b}{N_o}$	$\frac{R_c}{C}$	Scheme CS-1		Scheme CS-3	
		RT	T secs	RT	T secs
0.1	0.732	5.825	9.7	5.65	9.42
0.02	0.87	2.7	4.5	2.64	4.4
0.0	1.0	1.68	2.79	1.68	2.79

We will also make a few remarks on the advantages and disadvantages of having a large D, the dimensions of the elementary signals in the feedback coding scheme. It is evident from (47) that for same RT, larger D implies smaller P_e. In order to offset this fact, consider the number of iterations N which occur in time T.

$$N = \exp \left\{ \frac{2R_c}{R} \quad \frac{RT}{D} \right\}$$

For fixed RT, larger D implies smaller N. In other words, the ratio of T/N, the time per iteration or the "pulse width" will increase with D. This clearly is an advantage since this reduces the "cost" of the system in some sense.

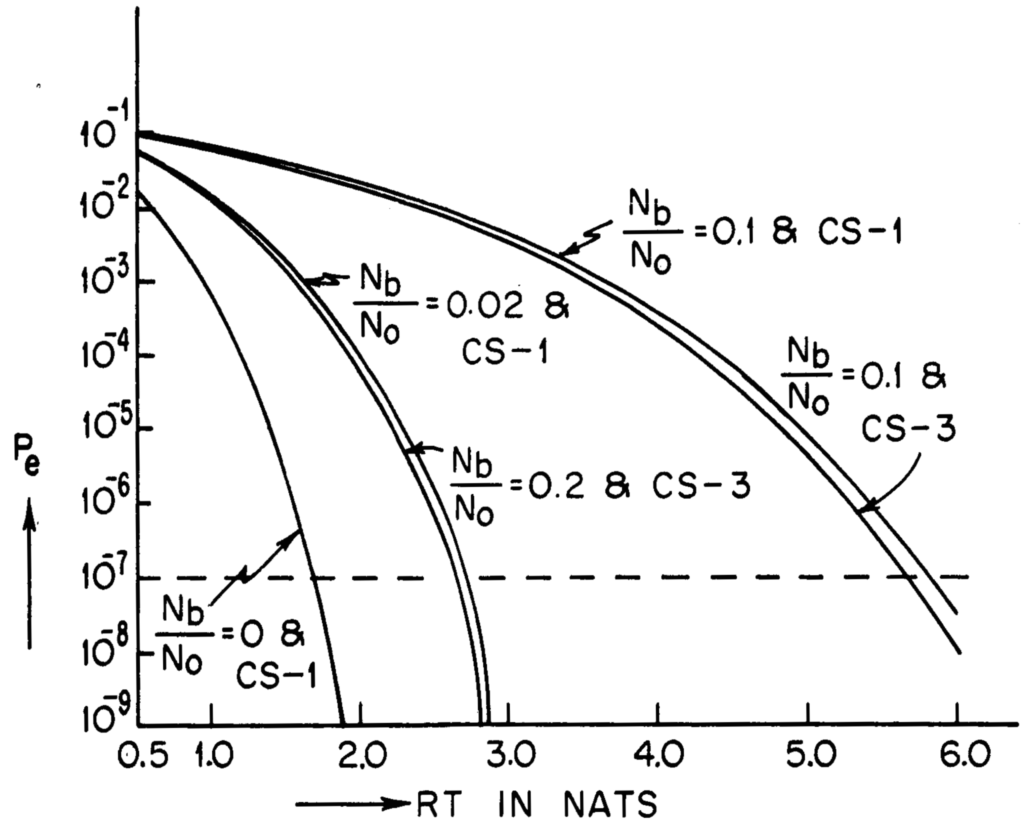

FIG. 6: PROBABILITY OF ERROR P_e AS A FUNCTION OF THE AMOUNT OF INFORMATION (RT) IN NATS FOR VARIOUS VALUES OF (N_b/N_o) USING THE CODING SCHEMES CS-1 AND CS-3.

VII. CONCLUSIONS

Feedback coding schemes using a noisy feedback channel and D-dimensional elementary signals have been developed in which to achieve a zero error transmission rate R_c nearly equal to the channel capacity C. We have also analyzed their performance when the transmission time T is finite and showed their superiority over traditional methods.

The only drawbacks of the coding schemes are the requirements of infinite bandwidth for the signal and infinite power at the receiver if time T is infinite. But for finite time T both these objections are not applicable. Moreover, in many space communication problems there is no limit on the available power or the receiver. Hence, the only important drawback is the requirement of infinite bandwidth for the signal. Methods of overcoming these disadvantages are presently under consideration.

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APPENDIX I

The asymptotic expression for the solution of the difference equations (16)-(18) will be given here.

It is not too hard to show that the homogeneous difference equations (16)-(18) have zero as the equilibrium state. Hence, p(i), q(i) and r(i) will be expanded in power series in terms of $\frac{1}{i}$

Let

$$\begin{cases} q(i) = \dfrac{\sigma_\eta^2}{a^2} \dfrac{1}{i} (1+k_q) + 0(\dfrac{1}{i}) \\ p(i) = \dfrac{\sigma_\eta^2}{a^2} \dfrac{1}{i} k_p + 0(\dfrac{1}{i}) \\ r(i) = \dfrac{\sigma_\eta^2}{a^2} \dfrac{1}{i} k_r + 0(\dfrac{1}{i}) \end{cases} \quad \dagger \tag{A-1}$$

Let $\sigma_\xi^2 / \sigma_\eta^2 \triangleq N_b/N_o \triangleq K_{bo}$

From (A-1), one can easily show that

$$\begin{cases} a\zeta(i) = (1+k_q + k_r) \dfrac{1}{i} + 0(\dfrac{1}{i}) \\ \theta(i)a = \dfrac{k_p}{K_{bo}} \dfrac{1}{i} + 0(\dfrac{1}{i}) \end{cases} \tag{A-2}$$

Moreover, we will use the following expansion throughout the Appendix.

$$\frac{1}{i+1} = \frac{1}{i} - \frac{1}{i^2} + 0(\frac{1}{i^2})$$

$\dagger$ By definition, $a_i \triangleq 0(b_i)$ if $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$

From (16), using (A-1) and (A-2), we get

$$(1+k_q)\left(\frac{1}{i}-\frac{1}{i^2}\right) = \left[\frac{1}{i}(1+k_q) + \frac{1}{i^2}\left\{(1+k_q+k_r)^2 - 2(1+k_q+k_r)(1+k_q)\right.\right.$$

$$\left.\left.-2(1+k_q+k_r)(1+k_q) - 2k_r(1+k_q+kr)\right\} + 0\left(\frac{1}{i^2}\right)\right]$$

Equating the coefficients of $\frac{1}{i^2}$ on either side of the above equation, we get

$$(k_q+k_r)^2 + (k_q+2kr) = 0 \qquad \text{(A-3)}$$

Let us consider equation (17).

$$k_p\left(\frac{1}{i}-\frac{1}{i^2}\right) = \frac{K_{bo}\left\{\frac{1}{i}k_p + \frac{1}{i^2}(1+k_q+k_r)^2\right\}}{\left\{K_{bo} + \frac{1}{i}k_p + \frac{1}{i^2}(1+k_q+k_r)^2\right\}} + 0\left(\frac{1}{i^2}\right)$$

$$= \frac{1}{i}k_p + \frac{1}{i^2}\left\{(1+k_q+k_r)^2 - \frac{1}{K_{bo}}k_p^2\right\} + 0\left(\frac{1}{i^2}\right)$$

Equating the coefficients of $\frac{1}{i^2}$ on both sides of the above equation, we get

$$k_p^2 - K_{bo}\,k_p - (1+k_q+k_r)^2\,K_{bo} = 0 \qquad \text{(A-4)}$$

Let us simplify (18)

$$k_r\left(\frac{1}{i}-\frac{1}{i^2}\right) = \left[\frac{1}{i}k_r - \frac{1}{i^2}\left\{(1+k_q+k_r)\,k_r + \frac{k_pk_r}{K_{bo}} + (1+k_q+k_r)\,k_p + (1+k_q+k_r)^2\right\}\right.$$

$$\left.+\,0\left(\frac{1}{i^2}\right)\right]$$

Equating the coefficients of $\left(\frac{1}{i^2}\right)$ on both sides of the above equation, we get

$$\left[1+k_p + \frac{k_pk_r}{K_{bo}} + 2(k_q+k_r) + (k_q+k_r)(k_q+2k_r+k_p) = 0\right] \qquad \text{(A-5)}$$

We have to solve (A-3)-(A-5) simultaneously for k_p, k_q and k_r. The simplest way is to assume power series solutions for all of them in terms of $\sqrt{K_{bo}}$.

Let

$$\sqrt{K_{bo}} = d$$

Let

$$\begin{cases} k_p = g_1 d + g_2 d^2 + 0(d^2) \\ k_r = g_3 d + g_1 d^2 + 0(d^2) \\ k_q = g_5 d + g_6 d^2 + 0(d^2) \end{cases} \qquad \text{(A-6)}$$

Parameters g_1 through g_6 have to be determined. Substitute (A-6) in (A-3) and equate the coefficients of the two most significant powers of d on either side of equation

$$\begin{cases} g_5 + 2g_3 = 0 \\ g_6 + 2g_4 + (g_5 + g_3)^2 = 0 \end{cases} \qquad \text{(A-7)}$$

Repeating the same process with (A-4) and (A-6) we get

$$\begin{cases} g_1^2 - 1 = 0 \\ 2g_1 g_2 - g_1 - 2\,(g_3 + g_5) = 0 \end{cases} \qquad \text{(A-8)}$$

Repeating the same process with (A-5) and (A-6), we get

$$\begin{cases} g_1 g_3 = -1 \\ g_2 g_3 + g_1 g_4 + g_1 + 2(g_3 + g_5) = 0 \end{cases} \qquad \text{(A-9)}$$

Solving the equations (A-7) - (A-9) for $g_1, \ldots, g_6$ we get

$$g_1 = 1 \qquad g_3 = -1 \qquad g_5 = 2$$

$$g_2 = 3/2 \qquad g_4 = -3/2 \qquad g_6 = 2$$

Hence

$$\mathrm{Cov}\,[\tilde{x}(i)] \approx \frac{\sigma_\eta^2}{a_i^2}\, k_p = \frac{\sigma_\eta^2}{a_i^2}\left(\sqrt{K_{bo}} + \frac{3}{2} K_{bo}\right) I$$

$$\mathrm{Cov}\,[x-\hat{x}(i)] \approx \frac{\sigma_\eta^2}{a_i^2}\,(1+k_q)\, I$$

$$= \frac{\sigma_\eta^2}{a_i^2}\left(1+2\sqrt{K_{bo}} + 2K_{bo}\right) I$$

$$E[\{x-\hat{x}(i)\}\{\tilde{x}(i)\}^T] \approx \frac{\sigma_\eta^2}{a_i^2}\, k_r\, I$$

$$= -\frac{\sigma_\eta^2}{a_i^2}\left(\sqrt{K_{bo}} + \frac{3}{2} K_{bo}\right) I$$

Finally

$$\mathrm{Cov}\,[x-\bar{x}(i)] \triangleq \mathrm{Cov}\,[x-\hat{x}(i) + \tilde{x}(i)]$$

$$= \frac{\sigma_\eta^2}{a_i^2}\,(1+k_q+k_p+2k_r) I$$

$$= \frac{\sigma_\eta^2}{a_i^2}\left(1+\sqrt{K_{bo}} + \frac{1}{2} K_{bo}\right) I$$

APPENDIX 2

We will establish the covariance formula (40-43) for coding scheme CS-3 by following the methods similar to those in Appendix 1.

We will use the expansions (A-1) in Appendix 1.

We know that

$$\zeta(i) = 1/\alpha i$$

We can easily show that

$$\theta(i)\alpha = \frac{1}{i}\frac{k_p}{K_{bo}} + 0\left(\frac{1}{i}\right)$$

From (16) , using (A-1) , we get

$$(1+k_q)\left(\frac{1}{i} - \frac{1}{i^2}\right) = \left[\frac{1}{i}(1+k_q) + \frac{1}{i^2}\{1-2(1+k_q) - 2k_r\} + 0\left(\frac{1}{i^2}\right)\right]$$

Equating the coefficients of $\frac{1}{i^2}$ on either side we get

$$k_q + 2k_r = 0 \tag{A-10}$$

Let us simplify (17) using (A-1)

$$k_p\left(\frac{1}{i} - \frac{1}{i^2}\right) = \frac{K_{bo}\{\frac{1}{i}k_p + \frac{1}{i^2}\}}{K_{bo} + \frac{1}{i}k_p + \frac{1}{i^2}} + \left(\frac{1}{i^2}\right)$$

$$\approx \frac{1}{i}k_p + \frac{1}{i^2}\left\{1 - \frac{k_p^2}{K_{bo}}\right\} + 0\left(\frac{1}{i^2}\right)$$

Equating the coefficients of $\frac{1}{i^2}$ on either side, we get

$$\{k_p^2 - K_{bo}\, k_p - K_{bo}\} = 0 \tag{A-11}$$

Let us simplify (18) using (A-1)

$$k_r\left(\frac{1}{i} - \frac{1}{i^2}\right) = \left[\frac{1}{i}k_r - \frac{1}{i^2}\left\{k_r + \frac{k_p k_r}{K_{bo}} + k_p + 1\right\} + 0\left(\frac{1}{i^2}\right)\right]$$

Equating the coefficients of $\left(\frac{1}{i^2}\right)$ on both sides of the above equation, we get

$$\left\{1+k_p+\frac{k_p k_r}{K_{bo}}\right\} = 0 \tag{A-12}$$

Comparing (A-11) and (A-12), we get

$$k_r = -k_p \tag{A-13}$$

Solving for k_p from (A-11) and retaining terms $\sqrt{K_{bo}}$ and K_{bo}, we get

$$k_p = \sqrt{K_{bo}} + \frac{1}{2}K_{bo}$$

$$k_r = -\sqrt{K_{bo}} - \frac{1}{2}K_{bo}$$

$$k_q = 2\sqrt{K_{bo}} + K_{bo}$$

These relations in conjunction with (A-1) give the expressions (40)-(42).

In order to prove (43) note that

$$\begin{aligned}
\text{Cov}\,[x-\bar{x}\ (i)] &\triangleq \text{Cov}\,[x-\hat{x}\ (i) + \tilde{x}(i)] \\
&= \{q(i) + p(i) + 2r(i)\}\, I \\
&= \frac{\sigma_\eta^2}{\alpha^2 i}\{1+k_q+k_p+2k_r\}\, I \\
&= \frac{\sigma_\eta^2}{\alpha^2 i}\{1+\sqrt{K_{bo}} + \frac{1}{2} K_{bo}\}\, I
\end{aligned}$$

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A coding scheme for additive Gaussian channel is developed using a noisy feedback link and D-dimensional elementary signals with no bandwidth constraint. This allows error free transmission at a rate $R < R_c$ where R_c is slightly less than the channel capacity C. When there is no noise in the Feedback channel, the coding scheme reduces to a D-dimensional generalization of the coding scheme of Schalkwijk and Kailath. In addition, the expression for the probability of error is determined when T, the time of Transmission rate is finite. Our scheme is also compared with the best codes which use only the forward channel.

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