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PERTURBATIONS OF AES ORBITS FROM TESSERAL HARMONICS OF GRAVITATIONAL POTENTIAL EXPANSION

by

A. M. Zhandarov

(USSR)

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SUMMARY

A method is given for the determination of AES orbit perturbations from tesseral harmonics of Earth's gravitational potential expansion, whereupon is utilized the intermediate orbit obtained by D. Brouwer [1].

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A method is described in the work [1] for the determination of perturbations in the motion of an artificial Earth's satellite (AES) caused by zonal harmonics of Earth's gravitational potential expansion. Subsequently a series of canonical transformations are performed, which allow the approximate integration of a system of equations of motion of an AES.

It is shown in the present paper that the problem of AES motion in the gravitational field devoid of axial symmetry is reduced to the problem with an axisymmetrical potential with the aid of a complementary canonical transformation.

The differential equations of perturbed motion of an AES may be represented for the osculating elements in the following form [2]:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}, \\ \frac{dK}{dt} &= \frac{\partial F}{\partial k}, & \frac{dk}{dt} &= -\frac{\partial F}{\partial K}, \end{aligned} \qquad (1)$$

(*) 5 VOZMUSHCHENIYA ORBIT ISZ OT TESSERAL'NYKH GARMONIK RAZLOZHENIYA GRAVITATSIONNOGO POTENTSIALA. 6

where
$$L = \sqrt{\mu a}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos i,$$
$$l = M, \quad g = \omega, \quad h = \Omega, \quad k = -s_0 - \omega^* t;$$

a, e, i, ω, Ω, M are the osculating elements of the orbit, ω^* is the angular velocity of the Earth, s_0 is the Greenwhich stellar time at the moment t_0. The function F constitutes a Hamiltonian system

$$F = \frac{\mu^2}{2L^2} + \omega^* K + R(L, G, H, l, g, h, k), \tag{2}$$

where R is the perturbing function.

The system of equations (1) differs from the system considered in [1] only by the pair of conjugate canonical variables k and K (the first of which being a linear function of time and the second having a formal sense) and the form of the perturbing function into which the tesseral harmonics of the Earth's gravitational potenital (longitudinal terms) are introduced in addition.

Evidently, if R = R(L, G, H, l, g), the integration of the system (1) is reduced to the case considered in [1].

The integration of the system (1) may be performed as follows. From the variables L. G. H, K, l, g, h, k we may pass to variables L', G', H', K', l' g', h', k', so that the Hamiltonian F* of the transformed system do not depend on the variables h' and k':

$$F(L, G, H, K, l, g, h, k) = F^*(L', G', H', K', l', g'), \tag{3}$$

Further, we may pass from variables L', G', H', K', l', g', h', k' to new variables L'', G'', H'', K'', l'', g'', h'', k'', so that the Hamiltonian F**, expressed in the new variables, be not dependent on the variable

$$F^*(L', G', H', K', l', g') = F^{**}(L'', G'', H'', K'', g''). \tag{4}$$

Finally, we may pass from the variables L'', G'', H'' ... to new variables L''', G''', H''',...., so that the Hamiltonian F*** include neither of the angular variables:

$$F^{**}(L'', G'', H'', K'', g'') = F^{***}(L''', G''', H''', K'''). \tag{5}$$

It is assumed here that the generating functions of the corresponding canonical transformations are not dependent on the independent variable. Obviously, the indicated transformations may be completed only with a specific degree of precision.

We shall be interested only in the first transformation of variables, for the second and the third are determined in [1]. The transformation of variables may be determined with the aid of the generating function

$$S^{(1)}(L', G', H', K', l, g, h, k). \tag{6}$$

The perturbing function depends on the small parameters c_{nk}, d_{nk} representing the coefficients of Earth's gravitational potential expansion in series by spherical functions. We shall postulate that the function $S^{(1)}$ is expandable in Taylor series by powers

$$S^{(1)} = S_0^{(1)} + S_1^{(1)} + S_2^{(1)} + \dots,$$

where $S_i^{(1)}$ constitutes a homogenous form of i-th order relative to c_{nk} and d_{nk}. We shall choose the function $S_0^{(1)}$ in the following manner:

$$S_0^{(1)} = L'l + G'g + H'h + K'k.$$

We shall rewrite the equality (3) by utilizing relations (6), (7)

$$F\left(L' + \frac{\partial S_1^{(1)}}{\partial l} + \frac{\partial S_2^{(1)}}{\partial l} + \dots,\quad G' + \frac{\partial S_1^{(1)}}{\partial g} + \frac{\partial S_2^{(1)}}{\partial g} + \dots,\right.$$

$$H' + \frac{\partial S_1^{(1)}}{\partial h} + \frac{\partial S_2^{(1)}}{\partial h} + \dots,\quad K' + \frac{\partial S_1^{(1)}}{\partial k} + \frac{\partial S_2^{(1)}}{\partial k} + \dots,$$

$$\left. l' - \frac{\partial S_1^{(1)}}{\partial L'} - \frac{\partial S_2^{(1)}}{\partial L'} - \dots,\quad \dots k' - \frac{\partial S_1^{(1)}}{\partial K'} - \frac{\partial S_2^{(1)}}{\partial K'} - \dots\right)$$

$$= F^*(L', G', H', K', l', g'). \qquad (7)$$

The approximate expression of Hamiltonian F contains two addends

$$F = F + F,$$

of which the first does not depend on c_{nk}, d_{nk} and the second includes them in first powers. Hamiltonian F* may, obviously, also be represented in an analogous form

$$F^* = F_0^* + F_1^*.$$

Taking this into account we may expand the left-hand part of equality (8) in series by powers

$$\sum_{i=1}^{\infty} \frac{\partial S_i^{(1)}}{\partial l},\quad \sum_{i=1}^{\infty} \frac{\partial S_i^{(1)}}{\partial g} \quad \dots,\quad \sum_{i=1}^{\infty} \frac{\partial S_i^{(1)}}{\partial K'}.$$

Assembling in both parts of the equality the terms containing identical powers of parameters c_{nk}, d_{nk}, we may obtain an infinite system of equations

$$F_0(L', K') = F_0^*(L', K'),$$

$$\frac{\partial F_0}{\partial L'} \frac{\partial S_1^{(1)}}{\partial l} + \frac{\partial F_0}{\partial K'} \frac{\partial S_1^{(1)}}{\partial k} + F_1(L', G', H', l', g', h', k') =$$

$$= F_1^*(L', G', H', l', g') \qquad (8)$$

. . . .

The first equation of this system is not differential, the second is integrated directly and the third may be integrated after the solution of the second, and so forth. We shall perform the calculations with a precision to the first power of c_{nk} and d_{nk}. This is why we shall determine the generating transformation function with the precision to $S_1^{(1)}$ with the aid of the second of Eqs.(9)

$$-\frac{\mu^2}{L'^3}\frac{\partial S_1^{(1)}}{\partial l}+\omega^*\frac{\partial S_1^{(1)}}{\partial k}+F_1=F_1^*. \tag{10}$$

F_1 may obviously be represented with the assumed degree of precision in the form of two addends, of which one contains only tesseral harmonics and is not dependent on c_{nk} and d_{nk}, $k\neq 0$, and the other -- only zonal harminics and the multipliers c_{n0}

$$F_1(L,G,H,l,g,h,k)=F_1'(L,G,H,l,g)+F_1''(L,G,H,l,g,h,k).$$

In view of the fact that we strive to determine

$$F_1^*=F_1^*(L',G',H',l',g',-,-),$$

Eq.(10) must be represented as two equations

$$F_1^*=F_1'(L',G',H',l',g',-,-)$$

$$F_1''-\frac{\mu^2}{L'^3}\frac{\partial S_1^{(1)}}{\partial l}+\omega^*\frac{\partial S_1^{(1)}}{\partial k}=0. \tag{11}$$

For the determination of $S_1^{(1)}$ a linear equation was obtained in partial derivative. It is equivalent to the system of ordinary differential equations in standard form:

$$\frac{dl}{dk}=-\frac{\mu^2}{\omega^* L'^3}=-\frac{n'}{\omega^*},$$

$$\frac{dS_1^{(1)}}{dk}=-\frac{1}{\omega^*}F_1'', \tag{12}$$

where n' is the average motion of the AES.

The system of Eqs.(12) is comparatively simply integrated. The first equation is integrated directly. The dependence $l=l(k)$ obtained from the first equation, may be substituted into function F_1'' in the right-hand part of the second equation. However this may be dome only with the aid of series by eccentricity powers, for the true anomaly f enters into function F_1''.

The operations, then required for the determination of function $S_1^{(1)}$, are in this case extremely cumbersome.

Because of cumbersomeness of the obtained expressions for $S_1^{(1)}$ we shall omit them for various tesseral harmonics. Note only that $S_1^{(1)}$ includes terms of the form

$$\varphi\left(\frac{\alpha}{N - Mn'(\omega^*)^{-1}}\right),$$

where N is a whole non-negative number, M = 1, 2, ..., N, α is a periodical function of time.

The appearance of such terms in a generating transformation function attests to the possibility of resonance emergence. Without investigating this question in detail we shall simply note that for most of the existing AES, $N - Mn'(\omega^*)^{-1}$ can not vanish at N = 12 ÷ 16, which conditions the possibility of appearance of notable orbit perturbations from tesseral harmonics of 12 ÷ 16-th orders.

Both equations of system (12) are integrated directly in the case when instead of function F_1'' we substitute in the right-hand part of the second equation its value(*)

$$F_1'' = \frac{1}{2\pi}\int_0^{2\pi} F_1''\,dl. \qquad (13)$$

averaged by the variable l.

In this case we may obtain long-period oscillations constituting fundamental interest. As an example, we shall find these perturbations from the harmonic with coefficients c_{22}, d_{22}.

It is not difficult to show that function F_1'', corresponding to harmonic P_{22}, may be expressed as follows:

$$F_1'' = \frac{3\mu^4 R_0^2}{L'^6} c_{22}[\cos(2h+2k)\sigma_3' - \sin(2h+2k)\sigma_4'] +$$
$$+ \frac{3\mu^4 R_0^2}{L'^6} d_{22}[\sin(2h+2k)\sigma_3' + \cos(2h+2k)\sigma_4'],$$
$$\sigma_3' = \frac{a'^3}{r'^3}\left[\cos^2(g+f) - \frac{H'^2}{G'^2}\sin^2(g+f)\right],$$
$$\sigma_4' = 2\frac{a'^3}{r'^3}\frac{H'}{G'}\cos(g+f)\sin(g+f)$$

where R_0 is the equatorial radius of the Earth,

$$\frac{a'}{r'} = \frac{1+e'\cos f}{1-e'^2}, \qquad e' = \sqrt{1-\frac{G'^2}{L'^2}}. \qquad (14)$$

Averaging F_1'' by $l = \int \frac{L'}{G'}\frac{r'^2}{a'^2}\,df$, we shall obtain

$$F_1'' = \frac{3}{2}\frac{\mu^4 R_0^2}{L'^3 G'^3}\left(1 - \frac{H'^2}{G'^2}\right)[c_{22}\cos(2h+2k) + d_{22}\sin(2h+2k)]. \qquad (15)$$

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* the averaging error is of the order $o(c_{20}, c_{22}, c_{20}, d_{22})$.

Then

$$S^{(1)} = S_0^{(1)} + S_1^{(1)} = L'l + G'g + H'h + K'k -$$
$$- \frac{3}{4\omega^*}\frac{\mu^4 R_0^2}{L'^3 G'^3}\left(1 - \frac{H'^2}{G'^2}\right)[c_{22}\sin(2h+2k) - d_{22}\cos(2h+2k)]. \qquad (16)$$

Knowing $S_1^{(1)}$, it is easy to determine the perturbations in the Delaunay elements

$$L = \frac{\partial S^{(1)}}{\partial l} = L', \qquad (17)$$

$$G = \frac{\partial S^{(1)}}{\partial g} = G', \qquad (18)$$

$$H = \frac{\partial S^{(1)}}{\partial h} = H' - \frac{3}{2\omega^*}\frac{\mu^4 R_0^2}{L'^3 G'^3}\left(1 - \frac{H'^2}{G'^2}\right)\times$$
$$\times [c_{22}\cos(2h+2k) + d_{22}\sin(2h+2k)], \qquad (19)$$

$$l' = \frac{\partial S^{(1)}}{\partial L'} = l + \frac{9}{4\omega^*}\frac{\mu^4 R_0^2}{L'^4 G'^3}\left(1 - \frac{H'^2}{G'^2}\right)\times$$
$$\times [c_{22}\sin(2h+2k) - d_{22}\cos(2h+2k)], \qquad (20)$$

$$g' = \frac{\partial S^{(1)}}{\partial G'} = g + \frac{3}{4\omega^*}\frac{\mu^4 R_0^2}{L'^3 G'^4}\left(3 - 5\frac{H'^2}{G'^2}\right)\times$$
$$\times [c_{22}\sin(2h+2k) - d_{22}\cos(2h+2k)], \qquad (21)$$

$$h' = h + \frac{3}{2\omega^*}\frac{\mu^4 R_0^2}{L'^3 G'^4}\frac{H'}{G'}[c_{22}\sin(2h+2k) - d_{22}\cos(2h+2k)], \qquad (22)$$

$$k' = k. \qquad (23)$$

The variable h may be substituted in the right-hand parts of equalities (19)-(22) by h'. When computing the perturbations an error of the order $o(c_{22}^2, d_{22}^2)$ arises on account of such a substitution.

From perturbations in the Delaunay elements it is easy to pass to perturbations in osculating elements by utilizing relation (1). The expressions for perturbation of satellite orbits from harmonics $P_{31}, \ldots, P_{44}$ and for perturbations from harmonic P_{22} without Hamiltonian averaging were obtained analogously. Because these expressoons are sufficiently cumbersome, they were obtained with the aid of a computer. The algorithms for deriving with their aid analytical expressions for orbit perturbations are described in the paper by A. M. Zhandarov and L. M. Kharchenko, published in this same issue.

T H E E N D

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