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GEOMETRIC THEORY OF OPTIMUM DISORBIT PROBLEMS

by A. Busemann and N. X. Vinh

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A. Busemann and N. X. Vinh †

I. INTRODUCTION

This paper presents the general solutions of the problem of optimally disorbiting a vehicle initially in an elliptical orbit.

A vehicle is initially moving along an elliptical orbit (E_1) about a spherical planet with center at 0 (Fig. 1). The problem is to bring the vehicle along an optimal trajectory (E_2) (in the sense of minimum fuel consumption) which finally intersects the top of the sensible atmosphere of the central planet. The top of the sensible atmosphere is assumed to form a sphere of radius R and center 0, enclosing the planet, and the motion is assumed to be planar.

Several sub-classes of the problem will be considered, namely when the entry angle is given, or the entry speed is given or both the entry angle and the entry speed are pre-assigned, etc. These constraints are dictated by the safe recovery of the vehicle since the heating and deceleration during the re-entry portion of the trajectory depends on the entry angle and the entry speed.

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In Part II of the paper we assume an impulsive disorbit by a single change of velocity $\vec{\Delta V}$ at a certain point A of the initial orbit. The problem is to find this particular entry orbit which yields the minimum of ΔV. Thus the problem is a problem of ordinary maxima and minima. Special geometrical considerations will be introduced in order to get the solution in closed form or to reduce the resulting equation to a simple quartic or quintic algebraic equation, depending on the problem.

In Part III of the paper the number of orbital changes is not restrained. Thus the overall optimal trajectory may be composed of a series of sub-arcs, each of them must satisfy the conditions of optimality. The problem can be formulated as a Mayer's problem of the calculus of variations. Using the equivalent maximum principle the overall absolute optimal trajectory is found in closed form.

The notation used in this paper is defined along with the text.

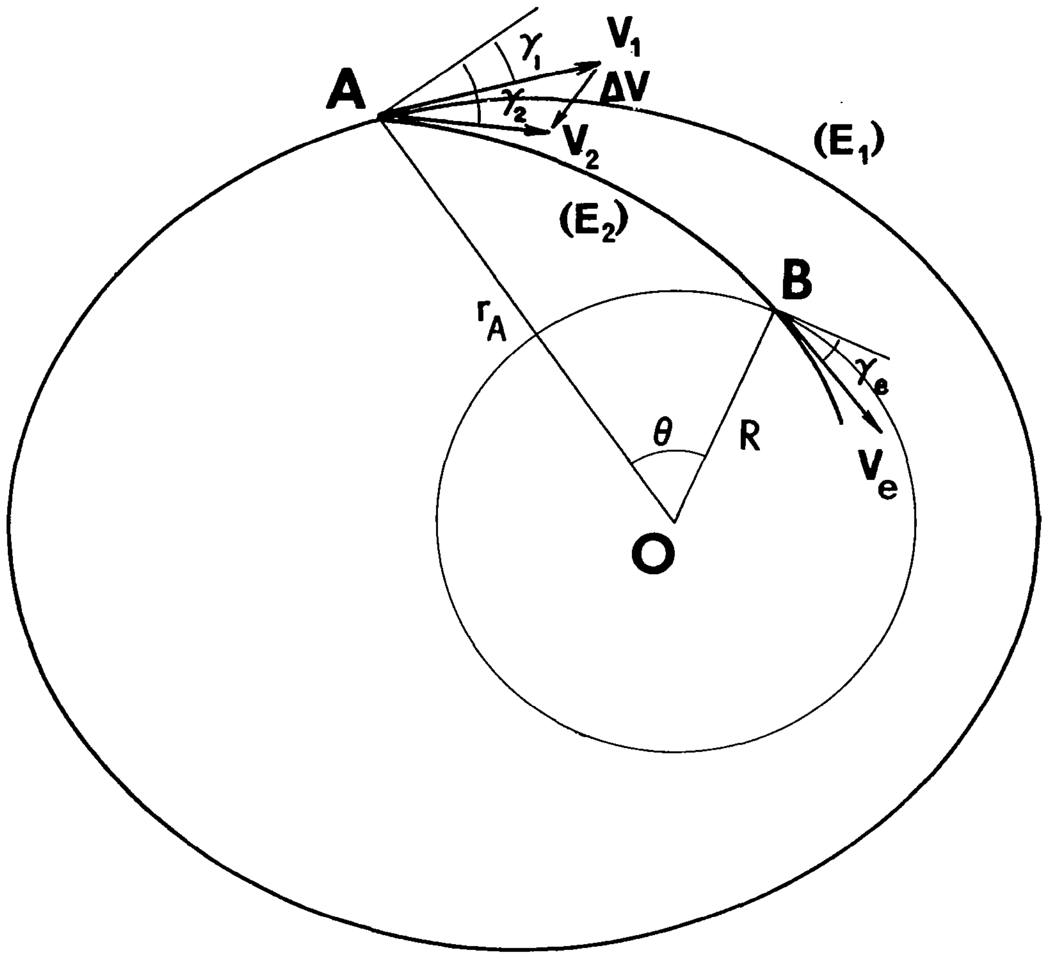


FIG. I. GEOMETRY OF THE DISORBIT PROBLEM

II. DISORBIT BY ONE IMPULSE

In this part, we first assume the position of disorbit A is fixed in advance. The disorbiting maneuver is assumed to be achieved by a single impulse $\vec{\Delta V}$ applied to the vehicle at the point A in its orbit. Thus the entry orbit is composed of a single Keplerian arc initiated at the point A. For the disorbit maneuver to be physically possible, this descending orbit must intersect the sensible atmosphere.

Let B be the entry point, and let F_2 be the second focus of the entry orbit, the first focus being the point 0 by Kepler's first law. Then, since A and B are two points on the entry orbit

$$AF_2 + A0 = BF_2 + B0$$

or

$$F_2B - F_2A = r_A - R.$$

where r_A is the radial distance from the center of the planet to the point A.

The locus of F_2 is a branch of the hyperbola (H_{AB}) with foci at the point A and B and transverse axis $(r_A - R)$. It is not difficult to show that when the point B moves, the envelope of this hyperbola is another hyperbola (H_A) with foci at the points A and 0 and transverse axis $|r_A - 2R|$ (Fig. 2)[1]. Hence if the point A is given, the region of admissible points F_2 is the portion of the plane of motion containing the point A and delimited by a branch of the hyperbola (H_A). The branch of hyperbola considered has

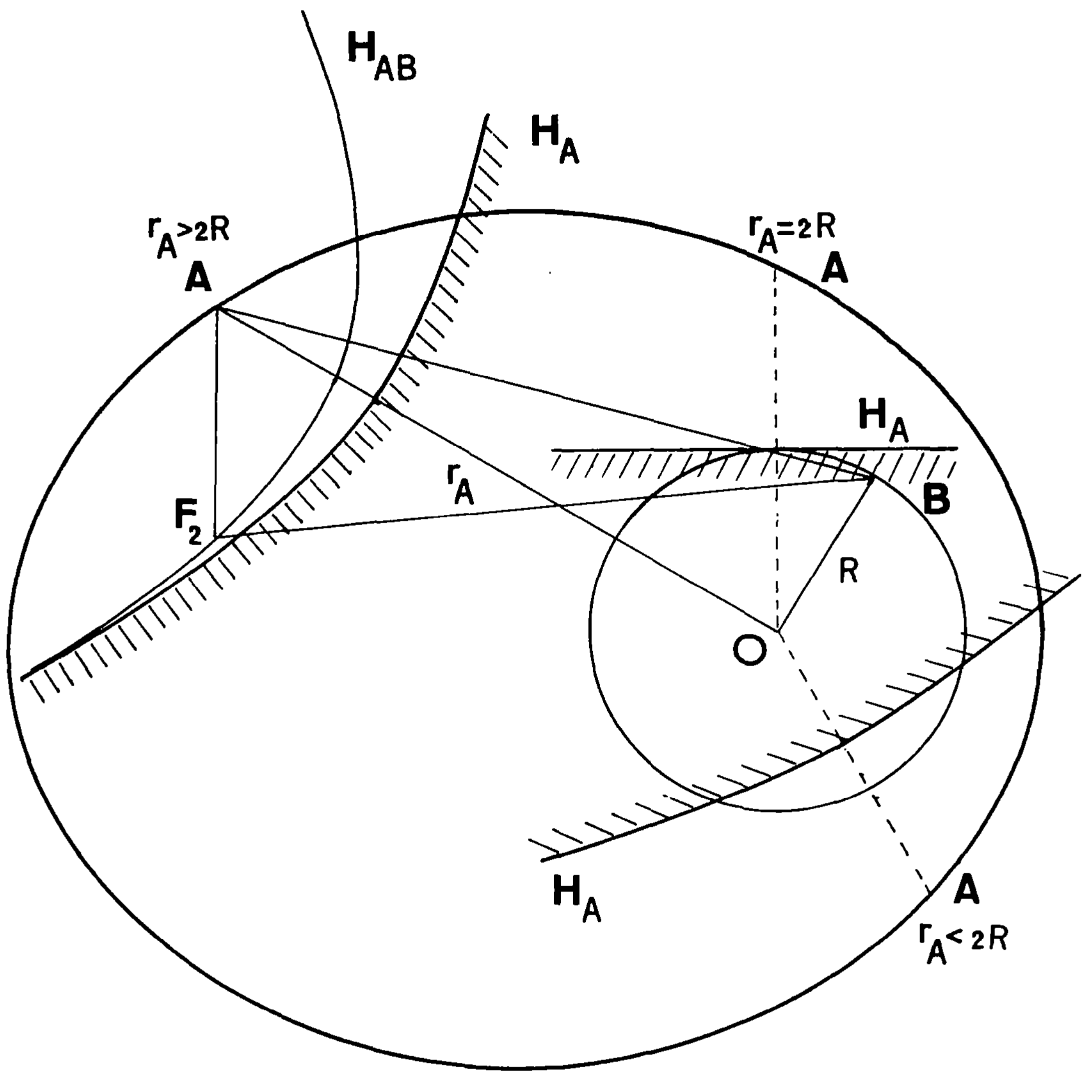


FIG. 2. REGION OF ADMISSIBLE F_2

its concavity turning toward the point A or the point 0 depending on $r_A > 2R$ or $r_A < 2R$.

When the point F_2 is on this hyperbola the entry angle γ_e is zero (limiting case for intersecting the atmosphere).

With these geometrical preliminaries, we now can successively consider the following different problems.

A. THE ENTRY SPEED IS GIVEN

In this case, we assume the entry speed V_e at the point B is given, but not the point B itself. Then by conservation of energy

$$V_2^2 - \frac{2\mu}{r_A} = V_e^2 - \frac{2\mu}{R} \qquad \text{(II-1)}$$

where μ is the gravitational constant and V_2 is the speed at A along the descending orbit. To non-dimensionalize the terms we take $\mu = 1$ and $R = 1$. Hence, all the distances will be dimensionalized with respect to R and all the velocities will be dimensionalized with respect to the circular speed at distance R. Then

$$V_2 = \sqrt{V_e^2 + \frac{2}{r_A} - 2} \qquad \text{(II-2)}$$

The locus of the terminus of the velocity vector $\vec{V}_2$ is a circle centered at A (Fig. 3). Then if $\vec{V}_1$ is the velocity at A before the impulse, it is clear that the optimum departure is tangential and the minimum impulse velocity is directed in the opposite direction of the motion. Its magnitude is given by

$$\Delta V^* = \sqrt{\frac{2}{r_A} - \frac{1}{a_1}} - \sqrt{V_e^2 + \frac{2}{r_A} - 2} \qquad \text{(II - 3)}$$

The entry angle is given by conservation of angular momentum

$$\cos \gamma_e^* = \frac{a_1}{V_e}\sqrt{\frac{r_A(1-e_1^2)}{2a_1 - r_A}\left(V_e^2 + \frac{2}{r_A} - 2\right)} \qquad \text{(II - 4)}$$

In the preceding equations a_1 and e_1 are respectively the semi-major axis and the eccentricity of the initial orbit, assumed known.

To have the entry position B we first calculate the angle $\omega = \measuredangle AOF_2$. We have

$$\tan \omega = \frac{a_1[2 - r_A(2-V_e^2)]\sqrt{(1-e_1^2)\,[r_A(2a_1 - r_A) - a_1^2\,(1-e_1^2)]}}{r_A\,(2a_1 - r_A) - a_1^2\,(1-e_1^2)\,[2 - r_A\,(2-V_e^2)]} \qquad \text{(II - 5)}$$

Then the longitude η of the perigee of the entry orbit is

$$\eta = \theta_A - (\pi + \omega) \qquad \text{(II - 6)}$$

The angle θ_B which defines the position of entry B is obtained by setting $r = 1$ in the polar equation of the entry orbit

$$r = \frac{p_2}{1 + e_2 \cos(\theta - \eta)} \qquad \text{(II - 7)}$$

where

$$p_2 = \frac{a_1^2\,(1-e_1^2)\,[2 - r_A(2 - V_e^2)]}{2a_1 - r_A} \qquad \text{(II - 8)}$$

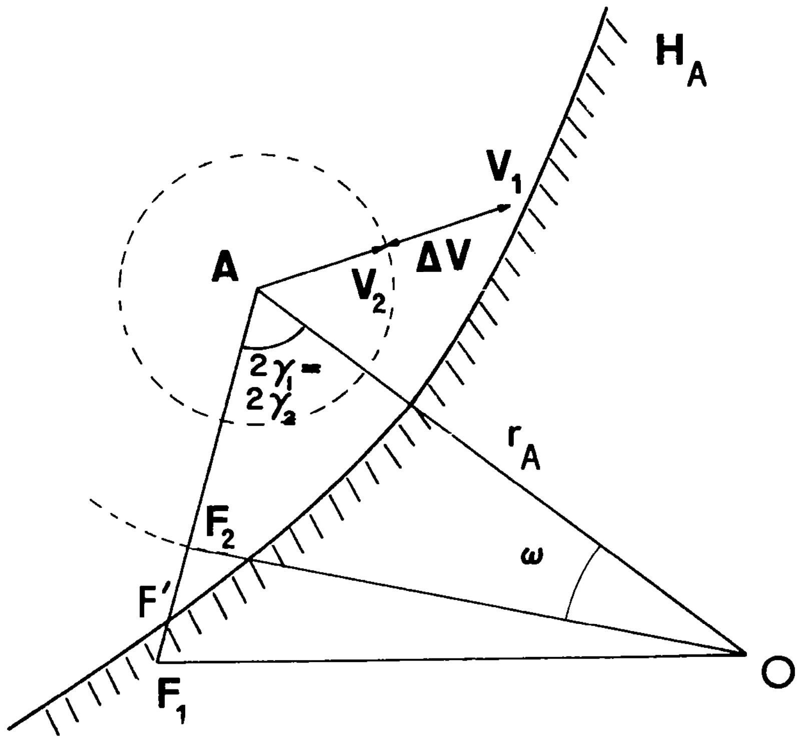


FIG. 3. OPTIMUM DISORBIT FOR SMALL V_e

$$e_2 = \sqrt{1 - \frac{a_1^2 (1 - e_1^2)(2 - V_e^2)[2 - r_A (2 - V_e^2)]}{2a_1 - r_A}} \qquad \text{(II - 8)}$$

Hence, the solution in this case is obtained in closed form. But, if we note that the locus of F_2 is also a circle, centered at A and of radius

$$AF_2 = 2a_2 - r_A = \frac{2 - r_A(2 - V_e^2)}{2 - V_e^2}$$

the condition for optimum tangential departure is (Fig. 3)

$$AF_2 \leq AF_2'$$

where F_2' is the intersection of the line AF_1, joining A to the second focus of the initial orbit, and the hyperbola (H_A). We have then the condition

$$V_e^2 \leq \frac{2a_1^2 (1 - e_1^2)(r_A - 1)}{r_A a_1^2 (1 - e_1^2) - (2a_1 - r_A)} \qquad \text{(II - 9)}$$

If the equality holds, the optimum departure is tangential and the optimum entry is grazing $(\cos \gamma_e^* = 1)$.

In the case of large entry speed i.e.

$$V_e > \left[\frac{2a_1^2 (1 - e_1^2)(r_A - 1)}{r_A a_1^2 (1 - e_1^2) - (2a_1 - r_A)}\right]^{1/2} \qquad \text{(II - 10)}$$

tangential disorbit is not possible since the descending trajectory will not intersect the atmosphere. The point F_2 will be outside of the permissible region for F_2 discussed above. For this case the optimum position of F_2 is on the hyperbola (H_A) and the optimum entry is grazing $(\gamma_e^* = 0)$. The magnitude of the minimum impulse velocity is (Fig. 4)

$$\Delta V^* = \left[V_1^2 + V_2^2 - 2V_1V_2 \cos(\gamma_1 - \gamma_2) \right]^{1/2} \qquad \text{(II - 11)}$$

where

$$V_1^2 = \frac{2}{r_A} - \frac{1}{a_1}, \quad V_2^2 = V_e^2 + \frac{2}{r_A} - 2 \qquad \text{(II - 12)}$$

and

$$\gamma_1 - \gamma_2 = \frac{1}{2} \measuredangle F_1AF_2$$

Explicitely, by considering the triangle OAF_1 and OAF_2 we have

$$\cos\gamma_1 = a_1 \sqrt{\frac{(1-e_1^2)}{r_A(2a_1 - r_A)}}$$

$$\cos\gamma_2 = \frac{V_e}{\sqrt{r_A[2 - r_A(2 - V_e^2)]}} \qquad \text{(II -13)}$$

The angle ω is given by the law of cosines applied to the triangle OAF_2. We have

$$\cos\omega = \frac{r_A - V_e^2}{r_A(V_e^2 - 1)} \qquad \text{(II - 14)}$$

The longitude η of the perigee of the entry orbit is given by (II -6) and the angle θ_B which defines the position of entry B is obtained by setting $r = 1$ in equation (II - 7) where we now have

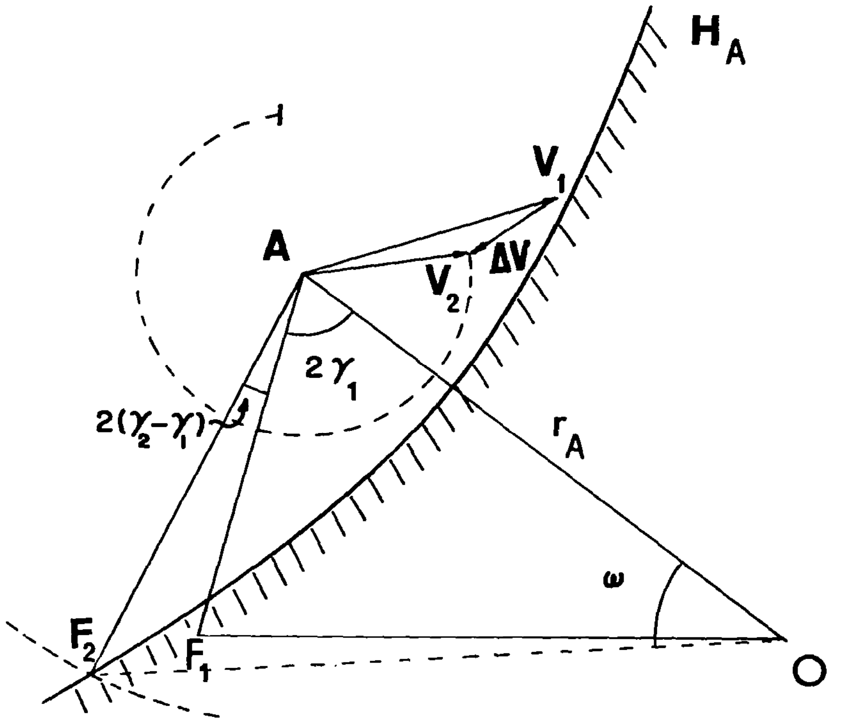


FIG. 4. OPTIMUM DISORBIT FOR LARGE V_e

$$p_2 = V_e^2$$

and (II - 15)

$$e_2 = V_e^2 - 1$$

B. THE ENTRY ANGLE IS GIVEN

In this case, we assume the entry angle γ_e at the point B is given but not the point B itself.

Let $\vec{V}_1$ be the velocity of the vehicle along the initial orbit at A. Let $\vec{V}_2$ be the velocity of the vehicle immediately after the impulse. $\vec{V}_2$ must be such that the entry angle at B is equal to the assigned value γ_e. Let $\vec{V}_e$ be the entry velocity at B

By conservation of angular momentum

$$r_A V_2 \cos \gamma_2 = R V_e \cos \gamma_e \qquad \text{(II - 16)}$$

where γ_2 is the angle between the local horizontal and the velocity $\vec{V}_2$ and by conservation of energy

$$\frac{2\mu}{r_A} - V_2^2 = \frac{2\mu}{R} - V_e^2 \qquad \text{(II - 17)}$$

Take a coordinate system AXY such that AX is along the local horizontal at A and AY is along the position vector $\vec{0A}$. Let X and Y be the components of the velocity vector $\vec{V}_2$ in this coordinate system. Then

$$V_2^2 = X^2 + Y^2 \qquad \text{(II - 18)}$$

and

$$V_2 \cos \gamma_2 = X \qquad \text{(II - 19)}$$

Then we can write Equations (II -16) and (II - 17)

$$r_A X = R V_e \cos \gamma_e \qquad \text{(II - 20)}$$

$$\frac{2\mu}{r_A} - (X^2 + Y^2) = \frac{2\mu}{R} - V_e^2 \qquad \text{(II - 21)}$$

By eliminating V_e between the two equations, we have

$$\frac{X^2}{\dfrac{2\mu (r_A - R) R \cos^2 \gamma_e}{r_A (r_A^2 - R^2 \cos^2 \gamma_e)}} - \frac{Y^2}{\dfrac{2\mu (r_A - R)}{r_A R}} = 1 \qquad \text{(II - 22)}$$

Therefore, the locus of the terminus of the velocity vector $\vec{V}_2$ is a branch of hyperbola $(H_{A\gamma_e})$ (Fig 5).

If we take $\mu = 1$, $R = 1$, the equation becomes

$$\frac{X^2}{\dfrac{2 (r_A - 1) \cos^2 \gamma_e}{r_A (r_A^2 - \cos^2 \gamma_e)}} - \frac{Y^2}{\dfrac{2 (r_A - 1)}{r_A}} = 1 \qquad \text{(II - 23)}$$

All the lengths have been dimensionalized with respect to the distance R and all the velocities have been dimensionalized with respect to the circular speed at distance R.

Thus, we have shown that the terminus of the velocity vector $\vec{V}_2$ must be on a branch of hyperbola given by equation (II -23) in order that the entry angle at B has a predetermined value γ_e. If we limit the entry trajectory to be elliptic, the terminus of $\vec{V}_2$ describes only a portion of the hyperbola delimited by the circle of center A and radius $\sqrt{2/r_A}$. This portion of hyperbola can be restricted further if we impose a lesser upper bound to the entry speed.

Now, at the point A, on its initial orbit, the vehicle possesses an initial velocity $\vec{V}_1$ with components along the AXY axes

$$X_1 = V_1 \cos \gamma_1$$

$$Y_1 = V_1 \sin \gamma_1$$

or

$$X_1 = \frac{1 + e_1 \cos \theta_1}{\sqrt{a_1 (1 - e_1^2)}}$$

$$Y_1 = \frac{e_1 \sin \theta_1}{\sqrt{a_1 (1 - e_1^2)}} \qquad \text{(II - 24)}$$

where a_1 and e_1 are respectively the semi-major axis and the eccentricity of the initial orbit, assumed known, and θ_1 is the true anomaly which defines the position A.

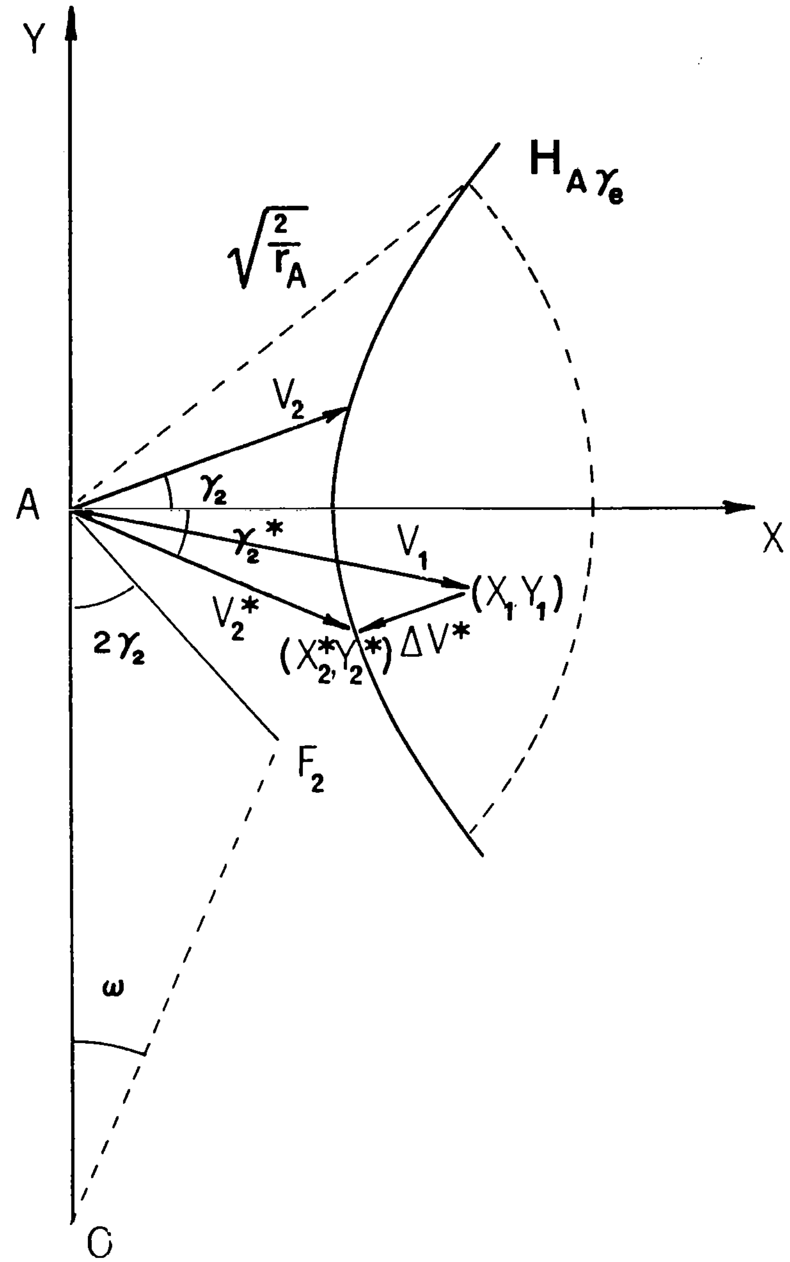


FIG. 5. HYPERBOLIC LOCUS OF V_2

The minimum impulse velocity ΔV^* is therefore equal in magnitude to the shortest distance from the point (X_1, Y_1) to the hyperbola. The solution is obtained by solving the quartic equation

$$B_o Z^4 + B_1 Z^3 + B_2 Z^2 + B_3 Z + B_4 = 0 \qquad \text{(II - 25)}$$

where

$$Z = \tan \gamma_2^* = \frac{Y^*}{X^*} \qquad \text{(II - 26)}$$

X^* and Y^* are the components of the optimum velocity $\vec{V}_2^*$, and γ_2^* is the optimum initial flight path angle. The coefficients B_i of the resulting equation (II -25) are given by

$$\begin{aligned} B_o &= \beta^3 X_1^2 \\ B_1 &= 2\alpha\beta^2 X_1 Y_1 \\ B_2 &= \alpha^2 \beta Y_1^2 - \alpha\beta^2 X_1^2 + (\alpha+\beta)^2 \\ B_3 &= -2\alpha^2\beta X_1 Y_1 \\ B_4 &= -\alpha^3 Y_1^2 \end{aligned} \qquad \text{(II - 27)}$$

where

$$\alpha = \frac{r_A (r_A^2 - \cos^2\gamma_e)}{2(r_A - 1)\cos^2\gamma_e}$$

$$\beta = \frac{r_A}{2(r_A - 1)} \qquad \text{(II - 28)}$$

The magnitude of the minimum impulse velocity is simply

$$\Delta V^* = [(X_1 - X^*)^2 + (Y_1 - Y^*)^2]^{1/2} \qquad \text{(II -29)}$$

where

$$X^* = \frac{\alpha Y_1 + \beta X_1 \tan\gamma_2^*}{(\alpha+\beta)\tan\gamma_2^*}$$

$$\text{(II - 30)}$$

$$Y^* = \frac{\alpha Y_1 + \beta X_1 \tan\gamma_2^*}{(\alpha+\beta)}$$

The direction of $\vec{\Delta V}^*$ is the direction of the normal to the hyperbola which has components $(\alpha X^*, -\beta Y^*)$; therefore, if δ^* is the optimum angle of $\vec{\Delta V}^*$ with the X-axis, we have

$$\tan\delta^* = -\frac{\beta}{\alpha}\tan\gamma_2^* \qquad \text{(II - 31)}$$

To have the other elements of the entry orbits, we first calculate the angle ω. It can be seen that

$$\tan\omega = \frac{r_A X^* Y^*}{1 - r_A X^{*2}} \qquad \text{(II - 32)}$$

The longitude η of the perigee of the entry orbit is given by (II -6) and the angle θ_B which defines the position of entry B is obtained by setting $r = 1$ in equation (II - 7) where we now have

$$p_2 = r_A^2 X^{*2}$$

and $$\text{(II - 33)}$$

$$e_2 = [(r_A X^{*2} - 1)^2 + r_A^2 X^{*2} Y^{*2}]^{1/2}$$

The entry speed is given by the energy equation

$$V_e = [X^{*2} + Y^{*2} + 2 - \frac{2}{r_A}]^{1/2} \qquad \text{(II - 34)}$$

The problem is then completely solved.

C. BOTH THE ENTRY SPEED AND ENTRY ANGLE ARE GIVEN

In this case, both the entry speed and entry angle are given. If we also fix the disorbit position A, then the trajectory is unique and there is nothing to optimize. Hence in this case we allow the position A to vary. If the entry speed and entry angle at a distance R are given, the entry orbit is fixed in shape and size. Its relative orientation with respect to the initial orbit is to be found such that the impulse required for disorbiting the vehicle is a minimum.

First we notice that the parameters V_e and γ_e at entry cannot be given arbitrarily because the entry orbit must be such that it intersects the initial orbit. Taking this into consideration we have the constraints

$$\cos^2 \gamma_e \leq \frac{a_1(1 - e_1)[2 - a_1(1 - e_1)(2 - V_e^2)]}{V_e^2} \qquad \text{(II - 35)}$$

and

$$V_e^2 \geq \frac{2a_1(1-e_1)[a_1(1+e_1) - 1]}{a_1^2(1 - e_1)^2 - \cos^2\gamma_e} \qquad \text{(II - 36)}$$

For a given V_e there exists a lower bound for the entry angle γ_e, and also, if the entry angle is fixed, lower bound exists for the entry speed.

The magnitude of the impulse velocity required to transfer is given by

$$(\Delta V)^2 = V_1^2 + V_2^2 - 2V_1V_2 \cos(\gamma_1 - \gamma_2)$$

The minimum solution can be obtained in a simple form if we take as variable the distance r from the center of attraction to the disorbit position A. Then we have

$$(\Delta V)^2 = f(r) \pm g(r) \qquad \text{(II - 37)}$$

where

$$f(r) = -\frac{2h_1h_2}{r^2} + \frac{4}{r} - \left(\frac{1}{a_1} + \frac{1}{a_2}\right) \qquad \text{(II - 38)}$$

$$g(r) = 2\left[\frac{h_1^2h_2^2}{r^4} - \frac{2(h_1^2 + h_2^2)}{r^3} + \left(4 + \frac{h_1^2}{a_2} + \frac{h_2^2}{a_1}\right)\frac{1}{r^2} - 2\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{1}{r} + \frac{1}{a_1a_2}\right]^{1/2}$$

(II - 39)

and a_1, a_2 and h_1, h_2 are, respectively, the semi-major axes and angular momenta of the initial orbit and the entry orbit.

$$h_1 = \sqrt{a_1(1 - e_1)}$$

$$a_2 = \frac{1}{2 - V_e^2} \; ; \; h_2 = V_e \cos \gamma_e \qquad \text{(II - 40)}$$

The ($\pm$) sign in Equation (II -37) arises from the fact that in the orientated plane of motion the product $\sin \gamma_1 \sin \gamma_2$ may be positive or negative. In the orientated plane of motion (Fig. 6) the heavy lines represent the first halves of the orbits from perigee to apogee and the dotted lines are the second halves. It is clear that the product $\sin \gamma_1 \sin \gamma_2$ is positive if the point of intersection is on the halves of the same kind and negative in the alternate case. Let

$$(\Delta V)^2 = F(r) = f(r) - g(r) \qquad \text{(II - 41)}$$

$$(\Delta V)^2 = G(r) = f(r) + g(r) \qquad \text{(II - 42)}$$

be the square of the impulse velocity in these two cases. Since $g(r) \geq 0$, we have $F(r) \leq G(r)$. The minimum ΔV occurs at an intersection of the first kind and the maximum ΔV occurs at an intersection of the second kind.

The selection of r as variable, instead of say the angle η between the axes of the conics, yields a simple expression for ΔV^2 and make an analytic study possible. When we rotate the entry orbit (E_2) in the plane of motion, two different cases may materialize: or the two ellipses are always intersecting, or they may become tangential at a certain configuration.

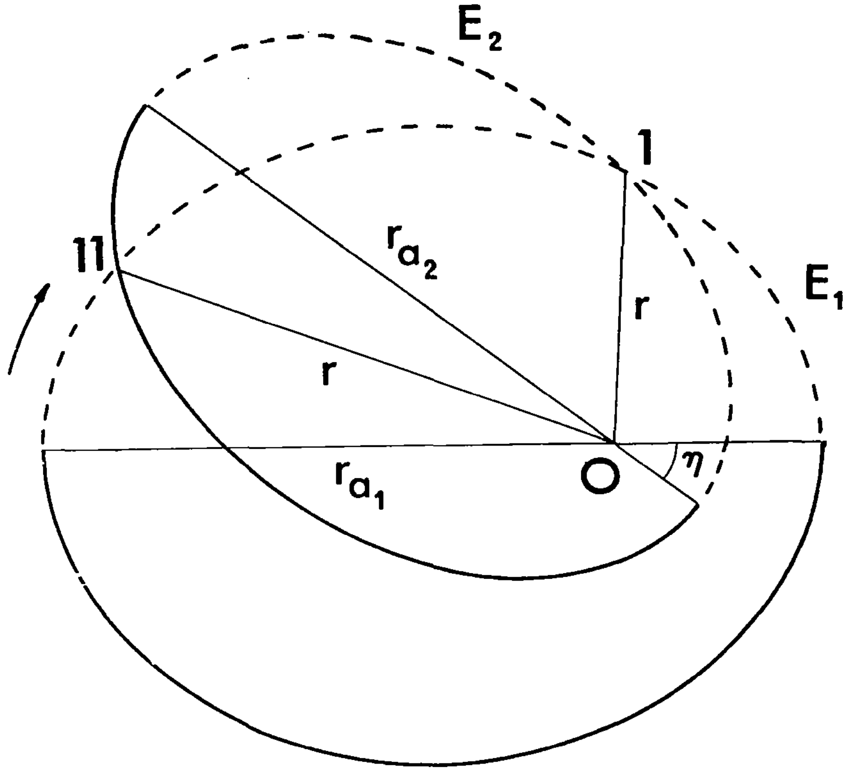


FIG. 6. I : INTERSECTION OF THE FIRST KIND, $\Delta V^2 = F(r)$

II : INTERSECTION OF THE SECOND KIND, $\Delta V^2 = G(r)$

If

$$r_{a_2} \geq r_{a_1}$$

or

$$\frac{1}{2 - V_e^2}\left[1 + \sqrt{V_e^4 \cos^2 \gamma_e - 2V_e^2 \cos^2 \gamma_e + 1}\right] \geq a_1(1 + e_1) \qquad \text{(II - 43)}$$

the orbits are always intersecting. If the inequality reverses, the orbits become tangential at a certain point. In the first case, using subscripts a and p to denote the conditions at apogee and perigee respectively, we have

$$r_{p_1} \leq r \leq r_{a_1} \qquad \text{(II - 44)}$$

In the second case

$$r_{p_1} \leq r \leq r_{a_2} \qquad \text{(II - 45)}$$

The derivative of $F(r)$ with respect to r is

$$\frac{dF}{dr} = f'(r) - g'(r)$$

It is easy to verify that $f'(r)$ remains finite at the end points. But $g(r)$ has the form

$$g(r) = 2P_1^{1/2} P_2^{1/2}$$

where

$$P_1 = (V_1 \sin \gamma_1)^2 = -\frac{h_1^2}{r^2} + \frac{2}{r} - \frac{1}{a_1}$$

$$P_2 = (V_2 \sin \gamma_2)^2 = -\frac{h_2^2}{r^2} + \frac{2}{r} - \frac{1}{a_2}$$

therefore

$$g'(r) = \frac{1}{P_1^{1/2} P_2^{1/2}} [P_1 P_2' + P_1' P_2]$$

In this form, it can be seen that $g'(r)$ becomes infinite at the end points. Furthermore if we consider the sign of $g'(r)$ we can see that $F(r)$ has infinite negative slope at the lower bound and infinite positive slope at the upper bound of the independent variable r. The inverse is true for the function $G(r)$. Hence, for one impulse disorbit, in general ΔV is not an optimum at the apses of the conics. Typical variations of $F(r)$ and $G(r)$ are shown in Fig. 7.

There is a special case when $r_{a_2} = r_{a_1}$. In this case at the upper limit of r we have

$$\lim F'(r) = \frac{4}{a_1^2 (1 + e_1)^2} \left[\sqrt{1 + e_1 e_2 - (e_1 + e_2)} - (1 - \sqrt{e_1 e_2})\right] < 0$$

and

$$\lim G'(r) = \frac{4}{a_1^2 (1 + e_1)^2} \left[\sqrt{1 + e_1 e_2 - (e_1 + e_2)} - (1 + \sqrt{e_1 e_2})\right] < 0$$

Typical variations of $F(r)$ and $G(r)$ in this special case are shown in Fig. 8 and the upper limit of r may give a minimum. With this preliminary analysis, it is readily seen that the optimum values of r are obtained by solving the equation

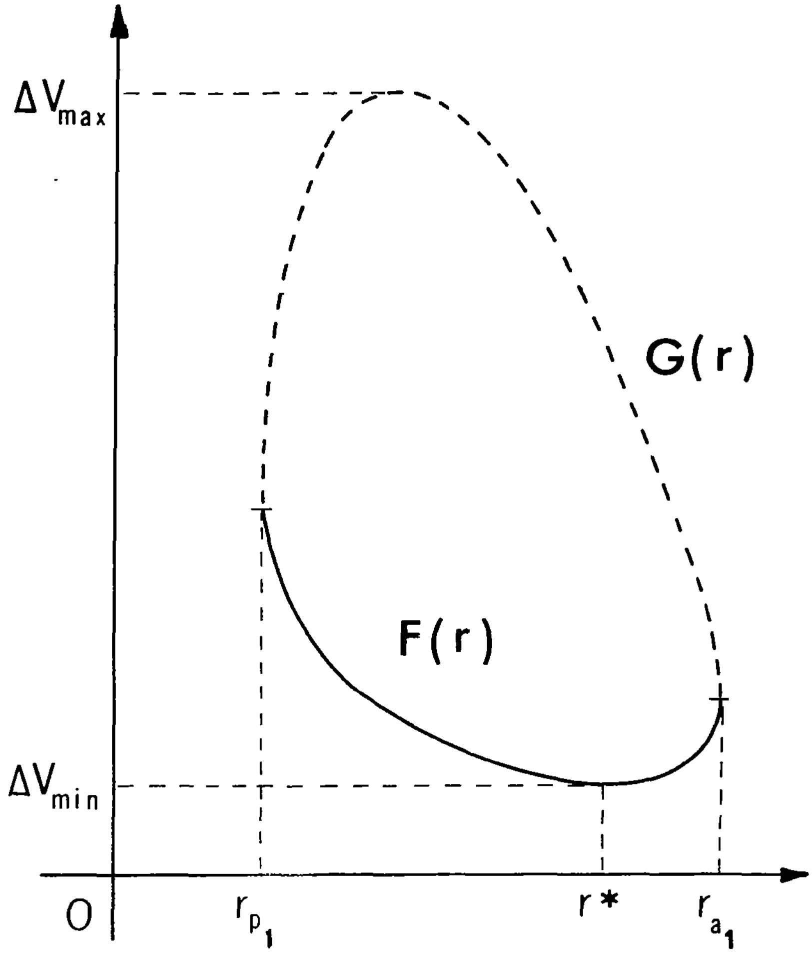


FIG. 7. TYPICAL VARIATIONS OF ΔV^2

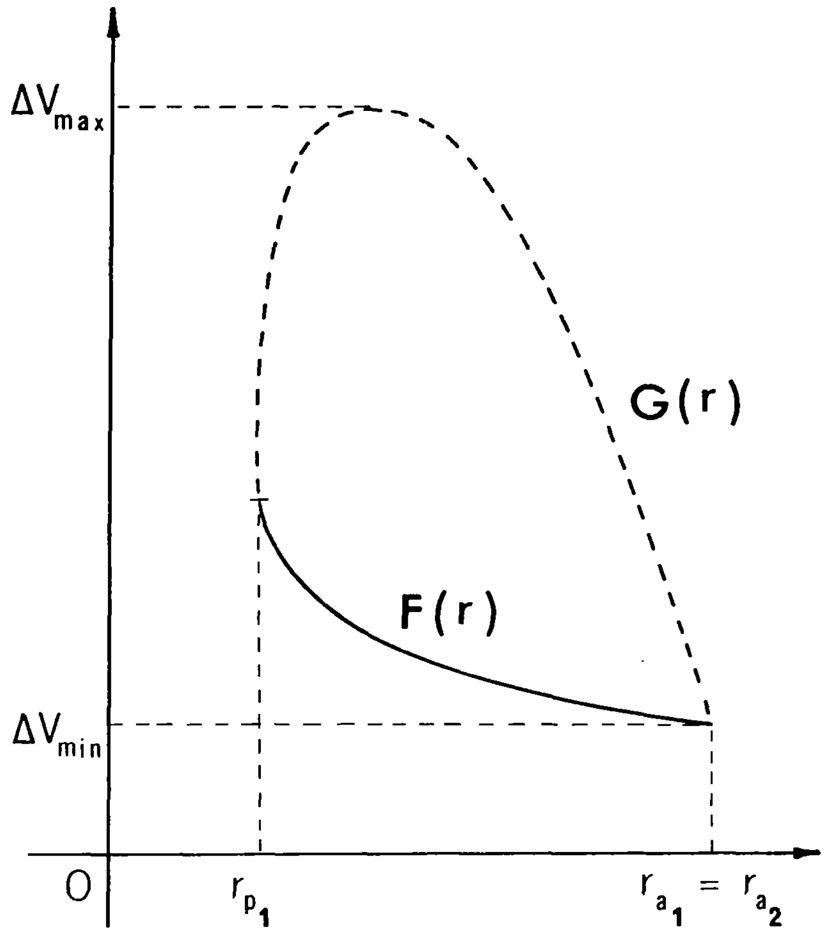


FIG. 8. VARIATIONS OF ΔV^2 WHEN $r_{a_1} = r_{a_2}$

$$\frac{d\,(\Delta V)^2}{dr} = 0$$

This results in a quintic equation in r

$$A_o r^5 + A_1 r^4 + A_2 r^3 + A_3 r^2 + A_4 r + A_5 = 0 \qquad \text{(II -46)}$$

where

$$A_o = -\left(\frac{a_1 - a_2}{a_1 a_2}\right)^2$$

$$A_1 = \frac{2}{a_1\, a_2}\,(h_1 - h_2)^2 + 2\left(\frac{h_1}{a_2} - \frac{h_2}{a_1}\right)^2$$

$$A_2 = \frac{2}{a_1}\,(h_1 - h_2)(5h_2 - 3h_1) - \frac{2}{a_2}\,(h_1 - h_2)(5h_1 - 3h_2) - \left(\frac{h_1^2}{a_2} - \frac{h_2^2}{a_1}\right)^2$$

$$A_3 = 2\,\frac{h_1^2}{a_2}\,(h_1 - h_2)(3h_1 - h_2) - 2\frac{h_2^2}{a_1}\,(h_1 - h_2)(3h_2 - h_1) + 16\,(h_1 - h_2)^2$$

$$A_4 = -\,(h_1 - h_2)^2\,(9h_1^2 + 2h_1h_2 + 9h_2^2)$$

$$A_5 = 4h_1^2h_2^2\,(h_1 - h_2)^2 \qquad \text{(II - 47)}$$

By the form of the coefficients A_i, it can be seen that an exact solution of this equation can be obtained in the special case where $h_1 = h_2$, i.e. when the re-entry parameters satisfy the relation

$$V_e \cos \gamma_e = \sqrt{a_1(1-e_1^2)} \qquad \text{(II -48)}$$

Equation (II - 46) reduces to a quadratic form and has a double root

$$r = h^2 = \text{semi-latus rectum} = a_1(1 - e_1^2)$$

In the optimum configuration, the two orbits are coaxial, and the position of disorbit is at 90^o angle from the apses.

In the general case, the minimum ΔV occurs when the two orbits are nearly coaxial in the first case, or when the two orbits are nearly tangential in the second case. Thus, the discrepancies between the $r_{optimum}$ and $r_{coaxial}$ (or $r_{tangential}$) and the respective ΔV are small. We have

$$r_{coaxial} = \frac{e_1 h_2^2 - e_2 h_1^2}{e_1 - e_2} \qquad \text{(II - 49)}$$

$$r_{tangential} = \frac{2a_1 a_2 \, (h_1^2 - h_2^2)}{a_1 h_1^2 - a_2 h_2^2} \qquad \text{(II - 50)}$$

and $r_{optimum}$ is obtained by solving Equation (II - 46)

As an example, if we take $a_1 = 2$, $h_1 = 1.28$ and $a_2 = 1.6$, $h_2 = 1.2$, then

$$r_{tan} = 1.3053, \qquad r_{opt} = 1.2810$$

and

$$\Delta V_{tan} = 0.0635, \qquad \Delta V^* = 0.0630$$

The optimum values of r are obtained by solving the quintic equation (II -46). It remains to show that other elements of the entry orbit can be expressed as functions of r.

First, if r is known, the position of disorbit A can be obtained from the polar equation of the initial orbit

$$r_A = \frac{p_1}{1 + e_1 \cos \theta_A} \qquad \text{(II - 51)}$$

The longitude η of the perigee of the entry orbit is obtained by setting $r = r_A$, $\theta = \theta_A$ in equation (II -7) where we now have

$$p_2 = V_e^2 \cos^2 \gamma_e$$

and (II - 52)

$$e_2 = \sqrt{V_e^4 \cos^2 \gamma_e - 2V_e^2 \cos^2 \gamma_e + 1}$$

Once η is known, the entry position B is obtained by setting $r = 1$ in the same equation. The range angle is simply

$$\theta = \theta_B - \theta_A \qquad \text{(II - 53)}$$

The flight path angles γ_1 and γ_2 along the two orbits at the disorbit position A and the optimum direction of the impulse velocity $\Delta\vec{V}$ are easily obtained from geometry of conic section and by the law of sines applied to the velocities triangle at point A.

Thus the optimum one impulse disorbit in this case of given entry speed V_e and entry angle γ_e is completely solved. The governing equation (II -46) is a quintic equation and it is simple enough to be solved using a desk calculator.

D. THE DISORBIT POSITION IS VARYING

In cases A and B of this part, we have initially assumed that the disorbit position A is fixed in advance. If we also vary A, then the characteristic velocity ΔV is a function of two independent variables and in each case the solution can be obtained by solving simultaneously a system of two equations. We can foresee that using this straight forward analytic derivation, it is difficult to get closed form solution. In this case, geometric considerations prove to be very useful.

First we consider the case where the entry speed V_e is given. If V_e is small, then we know that the optimum departure is tangential and the minimum characteristic velocity ΔV^* is given by Equation (II - 3) It is not difficult to show that ΔV^* is monotonically increasing with the distance r_A from the center of the planet to the disorbit position. Therefore if V_e is very small the optimum disorbit position is at the perigee of the initial orbit. Since inequality (II -9) and (II - 36) must be satisfied, we have for this case, by putting $r_A = a_1(1 - e_1)$

$$\frac{2[a_1(1 - e_1) - 1]}{a_1(1 - e_1)} \le V_e^2 \le \frac{2a_1(1 - e_1)}{a_1(1 - e_1) + 1} \qquad \text{(II - 54)}$$

If this condition is not satisfied i.e. the given entry speed V_e is large, then we know that the optimum trajectory is such that the entry is grazing. Therefore, we reach the conclusion:

"For one impulse disorbit for entry at given speed, the optimum trajectory is grazing."

We have the solution by solving the quintic equation (II -46) where in calculating the coefficients A_i we use the value $\gamma_e = 0$.

Next let us consider the case where only the entry angle γ_e is given. The search for the absolute optimum is obtained by varying either the entry speed or the disorbit position. The study of Fig. 7 and 8 shows that the optimum position is either near the apogee or at the apogee itself. This fact was confirmed analytically when we studied the derivative $F'(r)$ at apogee in the previous section. If we take the disorbit position at the apogee, then Equation (II - 25) becomes

$$\beta^3 V_1^3 Z^4 - [\alpha\beta^2 V_1^2 - (\alpha + \beta)^2] Z^2 = 0 \qquad \text{(II -55)}$$

where V_1 is the speed of the satellite at the apogee of the initial orbit. This equation gives two solutions

$$Z^* = \tan \gamma_2^* = 0 \qquad \text{(II - 56)}$$

and

$$Z^* = \tan \gamma_2^* = \sqrt{\frac{\alpha\beta^2 V_1^2 - (\alpha + \beta)^2}{\beta^3 V_1^2}} \qquad \text{(II - 57)}$$

The first solution always occurs and corresponds to a tangential disorbit at the apogee. The second solution is valid only when

$$V_1 > \frac{\alpha + \beta}{\beta\alpha^{1/2}} \qquad \text{(II - 58)}$$

This inequality expresses the condition that the point (X_1, Y_1) terminus of the vector velocity $\vec{V}_1$ is inside the evolute of the hyperbolic locus of the terminus of the vector velocity $\vec{V}_2^{(2)}$. If inequality (II - 58) holds, then the solution (II - 57) corresponds to a non tangential minimum disorbit from apogee, and the tangential disorbit given by solution (II - 56) is a maximum and thus it cannot be overall minimum.

Hence optimum disorbit at apogee occurs when

$$V_1 \leq \frac{\alpha + \beta}{\beta \alpha^{1/2}} \qquad \text{(II - 59)}$$

which can be written as

$$\cos^4 \gamma_e - r_{a_1}^2 \cos^2 \gamma_e + \frac{2 r_{a_1}^4 (r_{a_1} - 1)}{1 - e_1} \geq 0 \qquad \text{(II - 60)}$$

If $e_1 = 0$, this inequality reduces to the one already derived in Ref. 3 for the case of disorbit from circular orbit.

For most practical cases inequality (II -60) is usually satisfied, and the overall optimum disorbit for given entry angle occurs at the apogee of the initial orbit.

If it is not the case, then

$$\cos^4 \gamma_e - r_{a_1}^2 \cos^2 \gamma_e + \frac{2 r_{a_1}^4 (r_{a_1} - 1)}{1 - e_1} < 0 \qquad \text{(II - 61)}$$

The two roots of the equation are real positive or imaginary.

$$\cos^2 \gamma_e = \frac{r_{a_1}^2}{2} \left[1 \pm \sqrt{\frac{9 - 8 r_{a_1} - e_1}{1 - e_1}} \right] \qquad \text{(II - 62)}$$

First it is necessary that

$$r_{a_1} \leq \frac{9 - e_1}{8}$$

or

$$\frac{1}{1 - e_1} < a_1 \leq \frac{9 - e_1}{8(1 + e_1)} \qquad \text{(II - 63)}$$

The first inequality expresses that the initial orbit is completely outside sensible atmosphere.

Therefore , we get an upper limit for the eccentricity

$$e_1 \leq 9 - 4\sqrt{5} = 0.055728 \qquad \text{(II - 64)}$$

Also, the maximum value for the semi-major axis of the initial orbit is

$$a_1 \leq 1.125 \qquad \text{(II - 65)}$$

which is obtained when $e_1 = 0$. Beside these two conditions which occur only for nearly circular and low altitude orbit we must also have

$$\frac{r_{a_1}}{\sqrt{2}} \left[1 - \sqrt{\frac{9 - 8r_{a_1} - e_1}{1 - e_1}} \right]^{1/2} < \cos \gamma_e < \frac{r_{a_1}}{\sqrt{2}} \left[1 + \sqrt{\frac{9 - 8r_{a_1} - e_1}{1 - e_1}} \right]^{1/2} \qquad \text{(II - 66)}$$

for the optimum disorbit position to be at a point other than the apogee.

The results of the three conditions above are plotted in Fig. 9.

Now , we know that if the three above conditions are satisfied simultaneously, the optimum positions for disorbiting for fixed entry angle is at a point off the apogee. If this case occurs, then no closed

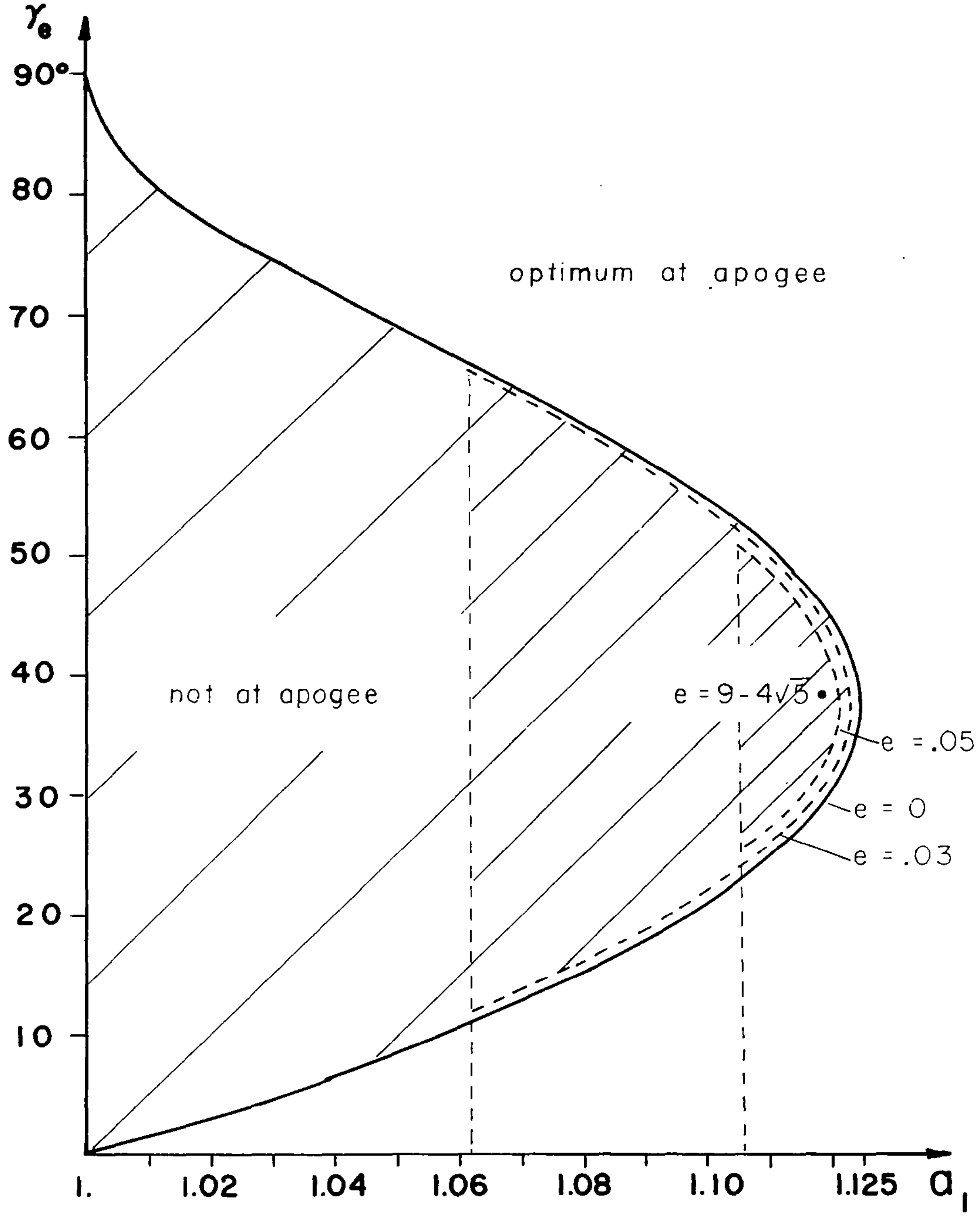


FIG. 9. CONDITIONS FOR OPTIMUM DISORBIT AT APOGEE

form solution is possible and we have to solve numerically a system of two equations for two unknowns.

One remark which facilitates the work is that this case invariably corresponds to an initial orbit with small eccentricity since the upper bound of e_1 is small by Equation (II - 64). Then we can first take the initial orbit as circular. Reference 3 gives exact solution for this case. Therefore we can calculate a first approximate value for V_e . The corresponding position for the disorbit position is given by Equation (II - 46). From this solution we can calculate a first approximate value for the characteristic velocity ΔV. The optimum value ΔV^* is obtained by varying V_e near its approximate value.

E. THE DISORBIT POSITION IS GIVEN

In this last section of one-impulse disorbit we will consider the case where only the disorbit position is fixed in advance. The situation may arise from an emergency landing or from a ground controlled maneuver where the position of disorbit is restrained to a portion of the trajectory.

The characteristic velocity is a function of two variables, say the entry speed V_e and the entry angle γ_e. We will see that this case can be reduced to the case of single variable as follows.

First we assume that the entry speed is given and find the optimum trajectory. Next we will vary V_e in order to find the absolute optimum.

We start from the smallest possible value of V_e. This value of V_e is such that the entry orbit is reduced to the straight line A0 (Fig. 3). To achieve this orbit the velocity V_1 of the satellite at A is cancelled completely and the vehicle will drop from zero velocity at A and follows the line A0 to finally intersect the atmospheric sphere at B. The speed V_e at B is the minimum possible entry speed and is given by

$$V_e = \sqrt{\frac{2(r_A - 1)}{r_A}} \qquad \text{(II - 67)}$$

Now as V_e increases while it is still small we have the case 1 of section A in this part. From Fig. 3 we can see that ΔV^* decreases while V_2 is increasing. But V_2 is increasing with V_e as it can be seen by Eq. (II - 2). Hence we must at least take V_e such that it is maximum in case 1, i.e. it is such that the optimum entry angle γ_e^* is zero. Next, as V_e continues to increase, we fall into case 2 of section A for large V_e. We know that in this case, the optimum trajectory is such that $\gamma_e^* = 0$.

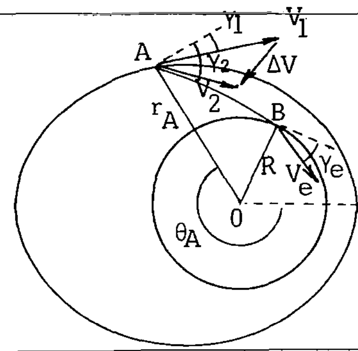
Therefore we reach the important conclusion that: "For one-impulse disorbit from a given position the optimum trajectory is grazing."

Hence, the solution is obtained by solving the quartic equation (II -25) where in calculating the coefficients B_i we take $\gamma_e = 0$.

A summary of the solutions of different cases considered in this part is presented in Table I in the next page.

TABLE I

OPTIMUM DISORBIT BY ONE IMPULSE

NOMENCLATURE		
The initial orbit is known as well as the distance R to entry position.		
KNOWN QUANTITIES	SOLUTIONS GIVEN BY	REMARKS
θ_A and V_e	Eq. II-3 if V_e is small Eq. II-11 if V_e is large (closed form solutions)	$\gamma_1 = \gamma_2$ $\gamma_e = 0$
θ_A and γ_e	Eq. II - 25 (Quartic equation)	
V_e and γ_e	Eq. II - 46 (Quintic equation)	
V_e only	Disorbit at perigee if V_e is small Eq. II-46 if V_e is large	$\gamma_1 = \gamma_2 = 0$ $\gamma_e = 0$
γ_e only	In general disorbit at apogee If $e_1 \leq 0.055...$, $a_1 \leq 1.125$ and γ_e relatively large, optimum disorbit is not at apogee	$\gamma_1 = \gamma_2 = 0$
θ_A only	Eq. II - 25	$\gamma_e = 0$

III. DISORBIT BY SEVERAL IMPULSES

The solutions derived in Part II are optimal in the sense that they give the best trajectory for the case considered, when only one impulse is allowed for disorbiting. Obviously this optimum one-impulse is not the absolute optimum solution since if we permit the optimal trajectory to have more than one ballistic arc, the solution may change considerably.

In this part we propose to investigate this problem of multiple orbital changes. For this purpose, to define a particular orbit, we introduce a set of three variables.

$$\ell = 2a = \text{major-axis of the orbit}$$
$$f = 2c = \text{focal distance} \qquad \text{(III - 1)}$$
$$\eta = \text{longitude of the perigee of the orbit}$$

A summary of characteristic properties of this space of coplanar orbits will be presented in the next section.

A. THE CONFIGURATION SPACE

Let E be a point defined by the cylindrical coordinates (III - 1) (Fig. 10). For E to represent a certain ellipse (E) with one focus at 0 in the physical plane $0 f_x f_y$, we must have

$$e = \frac{f}{\ell} < 1 \qquad \text{(III - 2)}$$

where e is the eccentricity of the ellipse. Therefore, the point E must be inside a cone of revolution (Σ) with axis 0ℓ, vertex at 0 and half angle at the vertex equal to $\frac{\pi}{4}$. At the point E we define an associated cone (S), vertex at E, axis vertical EF and half angle at the vertex of $\frac{\pi}{4}$. It can be shown that[1], every point inside the cone (S) (and inside (Σ)) represents an ellipse non intersecting to the ellipse (E). Every point outside the cone (S) (and inside (Σ)) represents an ellipse intersecting the ellipse (E). The line 0ℓ represents the set of concentric circular orbits. They are non intersecting as it can be seen by using the system of associated cones.

The associated cone (S) cut the cone (Σ) along an ellipse (Γ) passing through the lowest point M and the highest point N. The projection of (Γ) into the plane $0 f_x f_y$ is precisely the ellipse (E) considered. The point M is projected at the perigee and the point N is projected at the apogee of the ellipse. Hence if we follow the line EM we obtain a series of tangential elliptic orbits having the same perigee. If we follow the line EN we obtain a series of tangential elliptic orbits having the same apogee. We also note that the lines EM and EN are inclined at 45° angle with respect to 0ℓ.

On the axis 0ℓ if we take a point C such that $0C = 2R$, then the point C represents a circle (C) in the physical plane centered at 0 and radius R. If we take (C) as the circle representing the top of the sensible atmosphere, the point C must be inside the associated

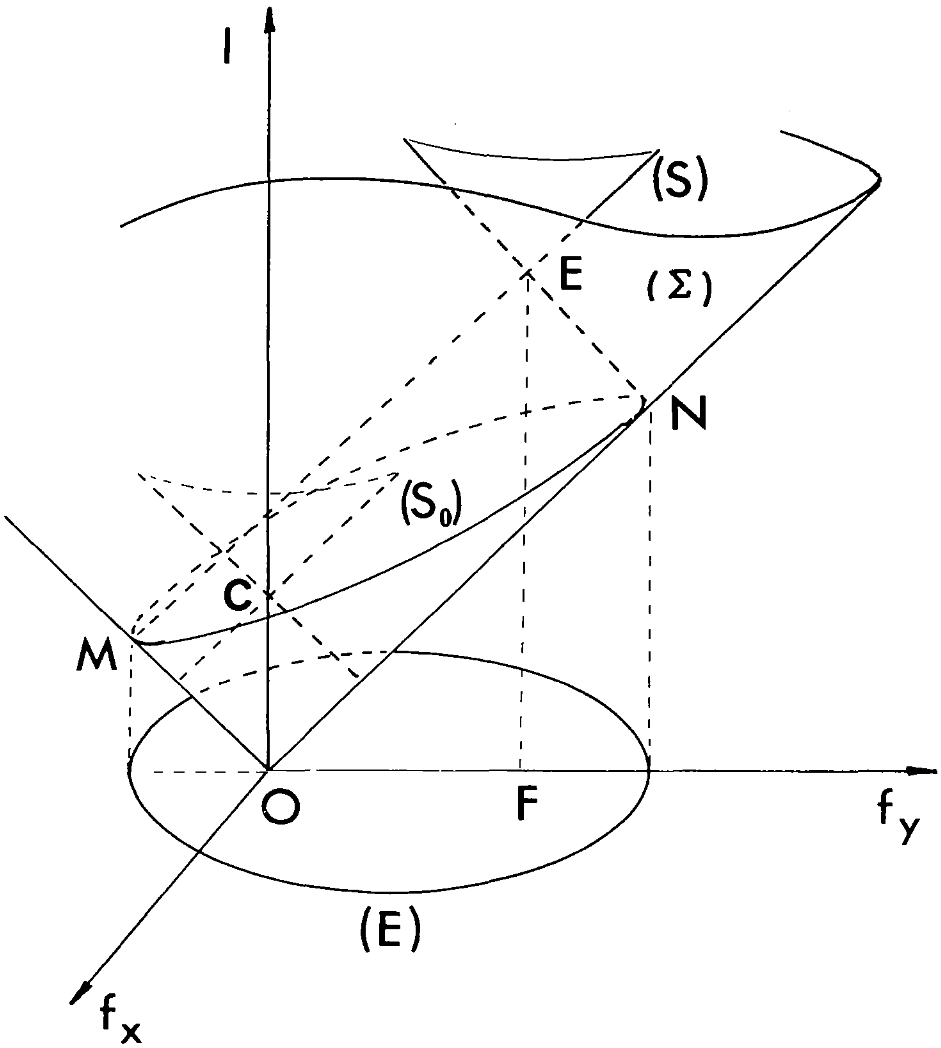


FIG. 10. SPACE OF COPLANAR ORBITS

cone (S) for the elliptic orbit (E) to be completely outside of the atmosphere. The cone associated to the point C is labelled (S_o).

The entry speed is given by

$$V_e^2 = \mu\left(\frac{2}{R} - \frac{1}{a_2}\right) \qquad \text{(III - 3)}$$

where a_2 is the major axis of the entry orbit. Hence the set of all entry orbits at given speed V_e is a plane, parallel to the physical plane, given by

$$\ell = 2a_2 = \frac{2\mu - RV_e^2}{2\mu R} \qquad \text{(III - 4)}$$

Only the portion of this plane, inside (Σ) and outside the cone (S_o) associated to C, can represent real, elliptic entry orbits.

Also, it has been shown that[1] the set of all entry orbits at given angle γ_e is a hyperboloid of revolution represented by the equation

$$f^2 = \ell^2 - 4\ell R \cos^2\gamma_e + 4R^2\cos^2\gamma_e \qquad \text{(III - 5)}$$

only the portion of this hyperboloid, inside (Σ) and outside the cone (S_o) can represent real, elliptic entry orbits.

A more detailed analysis of this configuration space is presented in Reference 1.

B. THE OPTIMAL TRAJECTORIES

To go from one point E to a neighboring point E' using a small variation in the velocity du, the variations in the coordinates are given by the following relations obtained from the classical equations of variations in celestial mechanics:

$$\frac{d\ell}{du} = \frac{1}{nb\ell}\left[2f\ell \sin v \sin\alpha + 2\ell(\ell + f\cos v)\cos\alpha\right]$$

$$\frac{df}{du} = \frac{1}{nb\ell}\left[(\ell^2+f^2)\sin v \sin\alpha + \frac{f(\ell^2+f^2)\cos^2 v + 2\ell(\ell^2+f^2)\cos v + f(3\ell^2-f^2)}{\ell + f\cos v}\cos\alpha\right]$$

$$f\frac{d\eta}{du} = \frac{1}{nb\ell}\left[-(\ell^2-f^2)\cos v \sin\alpha + (\ell^2-f^2)\frac{2\ell + f\cos v}{\ell + f\cos v}\sin v \cos\alpha\right]$$

(III - 6)

where, in addition to definition (III - 1)

b = semi-minor axis , $b = 1/2\,(\ell^2 - f^2)^{1/2}$

u = characteristic velocity

v = true anomaly of the point where the impulse is being applied

n = mean motion, $n^2 = \frac{8\mu}{\ell^3}$

α = direction of impulse, measured from the local horizontal

The first two equations of (III - 6) do not contain η. Hence if the entry orbit is to be achieved with conditions imposed on V_e and γ_e only, i.e. one looks for a particular final shape and size of the final orbit regardless of its orientation, we have only to consider the first two equations

Let $\ell = y$ and $f = x$ we can rewrite the equations of motion

$$\frac{dy}{du} = \frac{1}{nby}\left[2xy \sin v \sin\alpha + 2y(y+x\cos v)\cos\alpha\right] = f_1(x,y,v,\alpha) \qquad \text{(III - 7)}$$

$$\frac{dx}{du} = \frac{1}{nby}\left[(x^2+y^2)\sin v \sin\alpha + \frac{x(x^2+y^2)\cos^2 v + 2y(x^2+y^2)\cos v + x(3y^2-x^2)}{y + x\cos v}\cos\alpha\right] =$$

$$= f_2(x,y,v,\alpha)$$

The control parameters are v which defines the position where the thrust is being applied and α which gives the direction of the thrust. The quantity to minimize is the characteristic velocity u. Hence, the problem is a problem of time optimum control.

At each instant, the generalized Hamiltonian is

$$H = \psi_1 f_1 + \psi_2 f_2 \qquad \text{(III - 8)}$$

where ψ_1 and ψ_2 are the associated Lagrange multipliers.

It is a maximum with respect to the control variables when

$$v = 0 \text{ or } \pi$$

and

$$\alpha = 0 \text{ or } \pi$$

or

$$\cos v = \epsilon_1 = \pm 1$$

$$\cos\alpha = \epsilon_2 = \pm 1 \qquad \text{(III - 9)}$$

$$\sin v = \sin\alpha = 0$$

Therefore the optimum position for transfer is at the apses of the orbit, and the optimum dierction is horizontal. Equations (III - 7) now become

$$\frac{dx}{du} = \frac{2\epsilon_1\epsilon_2(y + \epsilon_1 x)}{nb}$$

$$\frac{dy}{du} = \frac{2\epsilon_2(y + \epsilon_1 x)}{nb} \qquad \text{(III - 10)}$$

and the Hamiltonian has the form

$$H = \frac{2\epsilon_2(y + \epsilon_1 x)}{nb} (\epsilon_1 \psi_1 + \psi_2) \qquad \text{(III - 11)}$$

Equations (III - 10) can be integrated immediately to give

$$y = \epsilon_1 x + \text{constant} \qquad \text{(III - 12)}$$

Hence, in the xy plane the extremals are straight lines inclined at a 45° angle with respect to the axes.

C. THE ABSOLUTE OPTIMUM DISORBIT

If the entry conditions are completely arbritrary, then the only physical constraint is that the transfer orbit must initiate from the initial orbit and ultimately intersect the atmospheric sphere. In our configuration space (Fig. 11) the problem is to go the quickest way from the point E_1 representing the initial orbit to the region outside the associate cone (S_o). Since E_1 is inside the cone, any admissible trajectory will cross the cone (S_o). Hence, the problem is the same as going from the point E_1 to the surface (S_o) which is degenerated into the two lines CI and CJ in this case. Since the final orbit is on (S_o), the optimum entry invariably is grazing.

We know that the extremals are lines inclined at a 45^o angle to the y-axis. Hence E_1I and E_1J are two possible solutions. E_1I corresponds to a disorbit at apogee and E_1J corresponds to a disorbit at perigee.

We can rule out the solution E_1J, for E_1J intersects the y-axis at C_o. From the circle C_o, it is trivial that the trajectories C_oJ and C_oJ' yield the same amount of fuel consumption. So we have symbolicly, in terms of fuel consumption

$$(E_1J) = (E_1C_o) + (C_oJ')$$

But it is also known that

$$(E_1C_o) + (C_oJ') > (E_1I) + (IJ') > (E_1I)$$

Therefore the overall optimum disorbit is at the apogee and the entry is grazing.

But the trajectory $(E_1 I)$ can be unfavorably compared with the trajectory $(E_1 \infty)$ as is shown in the Figure. This trajectory, also a solution, can be physically achieved by transferring the vehicle at the perigee of the initial orbit into a parabolic orbit. At a point at infinity in this parabolic orbit, by an infinitesimal impulse δu, we propel the vehicle into another parabolic orbit with its perigee inside the sensible atmosphere. Hence $(E_1 I)$ is the absolute optimum trajectory if and only if

$$\sqrt{\frac{(1 - e_1)}{a_1(1 + e_1)}} - \sqrt{\frac{2}{a_1(1 + e_1)[a_1(1 + e_1) + 1]}} \leq \sqrt{\frac{2}{a_1(1 - e_1)}} - \sqrt{\frac{(1 + e_1)}{a_1(1 - e_1)}} \qquad \text{(III - 13)}$$

where in calculating the characteristic velocities for the two optimum modes of transfer, we have taken $R = 1$, $\mu = 1$. After some manipulations

$$a_1 \leq \frac{2\,[\sqrt{2(1 + e_1)} + (1 + e_1)]}{1 - e_1^2} \qquad \text{(III - 14)}$$

Let x_1, y_1 be the coordinates of the point E_1, we must have

$$(x_1 - y_1)(x_1^2 - y_1^2 + 8x_1 + 8y_1 + 16) \leq 0 \qquad \text{(III - 15)}$$

The hyperbola delimiting the point E_1 for absolute optimum disorbit at apogee is plotted in heavy line in Figure 11.

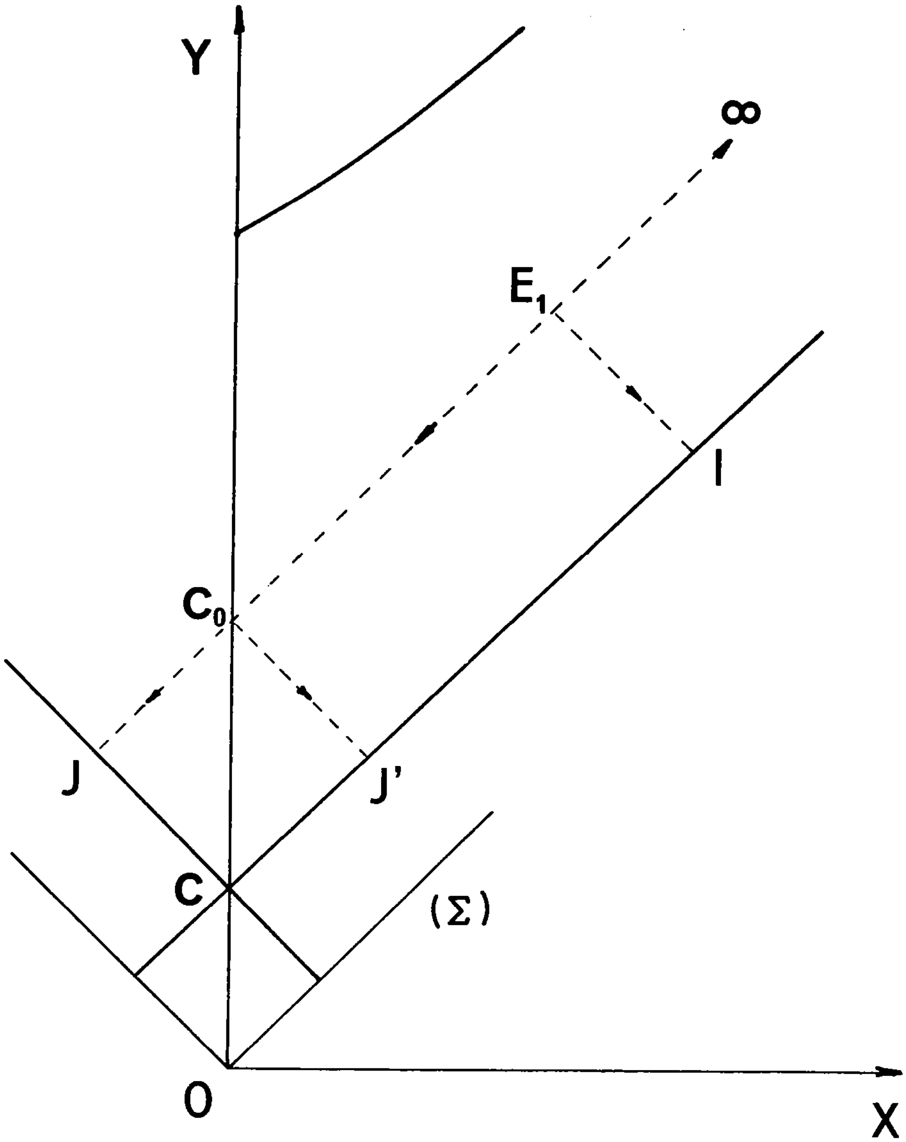


FIG. 11. ABSOLUTE OPTIMUM DISORBIT

If the point E_1 is outside the region, the absolute optimum disorbit is via parabolic orbits. This will require infinite time for transfer. Practically it can be done by injecting the vehicle into an elongated elliptic orbit at the perigee of the initial orbit, by applying an impulse ΔV_1 in the direction of motion. Then when the vehicle arrives at the apogee of the new orbit we apply another impulse ΔV_2 in the opposite direction of motion to change the orbit into a final entry orbit such that its perigee is at the top of the sensible atmosphere. By taking the eccentricity of the intermediary orbit large enough, we can make the sum of the increment velocities $(\Delta V_1 + \Delta V_2)$ less than the characteristic velocity required to disorbit at apogee.

D. CONCLUSION

The optimal disorbit found in the preceding section is absolute optimum in the sense that it is the best trajectory connecting the initial orbit and the sensible atmosphere. It is obvious that any solution for different particular cases in Part II of this paper requires a larger amount of fuel consumption than for this absolute optimum transfer.

If we required that the entry must be made at a given entry speed V_e or at a given entry angle γ_e, without limiting the number of the orbital changes, then the problem is to go the quickest way from the point E_1 to the plane defined by Equation (III - 4) or to the hyperboloid of revolution defined by Equation (III - 5). Unfortunately, for these cases there will be another physical constraint involved, namely the last impulse cannot be applied at a distance less than R. The full discussions are very much involved, and they will be the subject of a separate paper by the same authors.

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