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AN APPROACH TO THE PROBLEM OF OPTIMIZING ORBITAL MANEUVERS

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Abstract

The general problem of time open maneuvering of an orbital vehicle in a Newtonian gravitational field while conserving the characteristic velocity of the maneuver is considered. The problem is first formulated as a pseudodynamical system in the classical terms of a Hamiltonian. The canonical equations of Hamilton are presented and the maximum principle is applied.

The equivalent formulation in terms of the necessary conditions of the calculus of variations is made along with the interpretation of the problem as a search for geodesics, or minimum length paths, in a metric space. The approach which has been so successful for this problem and has produced the recent outpouring of results is presented. This approach considers separately the Weierstrass necessary condition and the Weierstrass-Erdmann corner condition. Since the metric of the state space is very much nonconvex, this eliminates most maneuvers as non-optimal, leaving a relatively small proportion of maneuvers as possible optimals.

This approach is applied in detail to the coplanar motion problem. However, the ideas discussed apply to the completely general noncoplanar case.

The further requirements of the maximum principle are briefly discussed. The equivalence of the maximum principle and Huygen's

wave principle is examined. The complete optimization approach is applied to the coplanar, coaxial case.

Nomenclature

c - exhaust speed of propellant

e - eccentricity, f/ℓ

F - second focus of elliptical orbit

f - linear eccentricity, distance between foci of orbit

f_i - functions governing the pseudodynamical system, eq. (4)

H - Hamiltonian, or pseudo Hamiltonian, eq. (5)

ℓ - length of major axis of orbit

m - mass of space vehicle

p_i - generalized momenta, costate variables, adjoint variables, Lagrange multipliers, eq. (6)

$\hat{p}$ - unit vector defining a direction in dx_i space

t - time

v - speed of space vehicle

x_i - state variables of the orbit treated as a dynamical system

α - a control variable, angle between thrust direction and plane of motion

β - orientation of major axis of orbit in its plane of motion, Fig. 1

β_i - the three Eulerian angles of the orbit in 3 space

γ - flight path angle, angle velocity vector makes with local horizontal, Fig. 1

θ - a control variable, true anomaly of space vehicle, polar angle in the plane measured from perigee in the direction of motion, Fig. 1

τ - acceleration due to thrust

Φ - value of φ at end of maneuver

φ - characteristic velocity, latent velocity, eq. (2)

ψ - a control variable, thrust angle in the plane of motion, Fig. 1

I

In the aerospace engineering sciences there is a basic problem of optimization which has not been solved and in which little progress had been made until recently. This problem is that of changing orbits with a space vehicle and performing the maneuver in some optimum sense. Classically, this problem has been treated under the constraint of vastly simplifying assumptions, but even so, up until now results have been sparse. The approach which has produced the recent outpouring of results for the simplified problem is the subject of this paper.

The first and obvious simplification for this problem is to assume that there is a single attracting gravitational body with an attractive force varying inversely with the square of the distance from it. This assumption implies that the vehicle is far enough from the planet so that atmospheric forces and oblateness effects can be neglected, and yet not so far from the planet that third body forces are important. The trajectories are conic sections: hyperbolas, parabolas, and ellipses.

The simplest criterion of optimization is equally obvious. From the very first and far into the foreseeable future the space mission has been and will be limited by the available payload. Since the payload is directly related to the final mass of the vehicle, the obvious goal is to maximize the final vehicle mass With this in mind, the

simplest criterion of optimization is to minimize the fuel expenditure during a maneuver. The quantity which furnishes the direct measure of the fuel expenditure is the "characteristic" velocity, or "latent" velocity. This is the theoretical maximum achievable velocity increment for a given expenditure of propellent. For a simple rocket traveling in a straight line in the absence of any external force, the change of velocity effected by a differential amount of propellent, dm, is

$$dv = -\frac{dm}{m} c \tag{1}$$

where m is the instantaneous mass of the rocket and unexpended fuel and c is the instantaneous exhaust velocity of the propellent. The characteristic velocity may thus be defined as

$$\varphi \equiv -\int_{m_o}^{m} \frac{c}{m} dm = \int_{t_o}^{t} \tau \, dt \tag{2}$$

where τ is the instaneous magnitude of acceleration due to thrust. If the magnitude of acceleration is unlimited then the velocity change due to thrust alone can be assumed to take place discontinuously and the characteristic velocity is simply the magnitude of the impulsive velocity change.

The most far-reaching simplification which is classically made for this problem is the assumption that the duration of the maneuver

is unlimited. This is referred to as the time open assumption. Many of the consequences of this assumption are completely unrealistic, but the assumption is necessary in order that the problem be tractable to solution in general. Some major consequences of this assumption are:

1) Only elliptical transfer between elliptical trajectories need be considered, along with the two limiting cases of the parabolic trajectories tangent at the periapses of the initial and the final ellipse. For the time open problem in its broadest sense all portions of an hyperbolic trajectory are made available by starting far enough back in time. In this case the transfer between two hyperbolic trajectories can be effected by a maneuver with an infinitesimal characteristic velocity. In general this requires six infinitesimal impulses:

i) Change the angular momentum to zero at infinity on the original hyperbolic trajectory.

ii) Change the energy to zero (parabolic) at the origin.

iii) Enter a large circular orbit at infinity on the degenerate parabola.

iv) Return on another degenerate parabola with the proper orientation.

v) Change the energy at the origin to the desired final energy.

vi) Change the angular momentum at infinity to the desired final value.

Similarly, transfer between an elliptical orbit and an hyperbolic trajectory can be accomplished through a maneuver with the same characteristic velocity as the minimum escape from the elliptical orbit, which is the parabolic escape tangent at the periapsis of the elliptical orbit. Thus, it is shown that for the time open problem hyperbolic trajectories need never be considered, and the only parabolic trajectories which enter are the minimum escape trajectories from the initial and final elliptical orbits.

2) The number of state variables is reduced by one. Since duration is of no concern and only closed (repeatable) orbits are involved, a vehicle may coast freely in an orbit at no cost. Thus, all points in an orbit are equivalent in the sense that they are all equally available. Position in orbit is eliminated as a discriminatory state variable.

3) Impulsive transfers are available to all space vehicles. A vehicle with a finite maximum thrust may approach impulsive maneuvers in the limit by applying its maximum thrust for an infinitesimal time, coasting completely around the orbit and again applying maximum thrust at the same place. This has the effect of multiplying the maximum thrust by any factor until, in the limit, it becomes infinite.

4) The rendezvous problem is the same as the transfer problem. If there is another vehicle in the final orbit which must be joined, the

solution to the simple transfer problem suffices for this rendezvous problem when one of the maneuvers is split in two. By interrupting one of the maneuvers and waiting in a chosen intermediate orbit with a slightly different period, the vehicle can simply bide its time until the other vehicle is in the proper position. This waiting period requires only a finite time as soon as one knows that all constant period maneuvers are nonoptimal and consequently there is always a finite interval of periods available during an optimal maneuver from which a suitable rational period may be chosen. The period of an orbit is a function only of the length of its major axis, and hence, only of the energy of the orbit. Thus, a constant period maneuver must have the velocity change always at right angles to the velocity of the vehicle. In the coplanar case such a velocity change is easily shown to be nonoptimal, and it is reasonable to suspect the same for the general case.

In the light of these four consequences it should be apparent that the assumption that the duration of the maneuver be unlimited becomes unrealistic. However, it is an unrealistic assumption only if the resulting optimal maneuver for a particular transfer requires an infinite amount of time. But many solutions turn out to require only a finite elapsed time which is compatible with the restrictions of typical space projects. The optimal maneuver under the time open assumption provides a lower bound for the characteristic velocity

requirements of an orbit transfer in which duration is considered. If this optimal maneuver involves a duration which is acceptable to the actual problem, then it provides a very good lower bound. Therefore, the greatly simplified problem as stated is of strong current interest and only now are general results beginning to appear.

The problem as posed can be considered as an optimal control problem, as well as a problem of the calculus of variations, or as a geodesic or minimum distance problem in a metric space. The discussion of this paper will attempt to point out this equivalence and use it to great advantage.

II

The problem in various guises has been studied by a number of researchers. Hohmann in 1925 displayed one famous solution which became known as the Hohmann transfer. The problem lay dormant until after World War II when Lawden began his work on the more general problem. Lawden worked virtually alone on this problem for ten years and produced practically all of the early results. Lawden used time as the independent variable necessitating special handling for impulsive velocity changes and less-than-maximum-thrust maneuvers.

The intrinsic independent parameter for this time open problem is the quantity to be conserved, the characteristic velocity. Then the problem is what is referred to in control theory as a time optimal problem, since it is the independent variable which is to be conserved. With this choice of independent variable, one can handle impulses, continuous thrust arcs of all types, and coasting arcs equally well. This is the choice made by Busemann in 1958 working at the Langley Research Center of NASA. Busemann's work culminated in his Prandtl Lecture in Vienna in 1965. A similar approach was used by Contensou (1961). Breakwell, (1963), followed Contensou and applied the maximum principle. The basic approach is applicable to the general problem outlined at the first of this paper, but to achieve concrete results all of these investigators have thus far restricted the problem to coplanar motion.

The first step is to formulate the problem in classical Hamiltonian terms and apply the maximum principle. Every orbit is treated as a possible state of a dynamical system described by a state vector $\vec{x}$ and a set of differential equations describing the change in the state vector under the influence of permissible controls. For this problem there are five components of the state vector. A convenient choice of state variables is:

$$\left.\begin{array}{l} x_1 = \beta_1 \\ x_2 = \beta_2 \\ x_3 = \beta_3 \end{array}\right\} \quad \text{orientation in space} \qquad (3)$$

$$x_4 = \ell \qquad \text{energy or period}$$

$$x_5 = f \text{ (or e)} \qquad \text{shape parameter}$$

where the β_i are the Eulerian angles of the orbit treated as a solid body in three space, ℓ is the length of the major axis of the orbit and is a measure of the energy of the orbit and the period of the orbit, f is the distance between foci of the orbit (e is f/ℓ, the eccentricity of the orbit).

The controls available are those governing the use of the rocket to achieve a state change. The first of these is the position along the orbit of the rocket engine during its use. This will be measured by the true anomaly, θ, of the vehicle. Two other controls measure the direction in space of the velocity increment (aligned with the rocket thrust). The first of these will be the angle ψ in the plane of motion measured in the same manner as the flight path angle. The second is the angle α which

the thrust direction makes with the plane of motion (Figure 1).

The differential equations governing this dynamical system are of the form

$$\frac{dx_i}{d\varphi} = f_i\,(\vec{x};\ \theta,\ \psi,\ \alpha) \qquad i = 1,\ \ldots,\ 5 \tag{4}$$

The Hamiltonian, or pseudo Hamiltonian as it is sometimes called, is formed as:

$$H(\vec{p},\ \vec{x};\ \theta,\ \psi,\ \alpha) \equiv \sum_i p_i \frac{dx_i}{d\varphi} = \sum_i p_i f_i \tag{5}$$

where $\vec{p}$ is the generalized momentum vector, also known variously as the adjoint vector in mathematics, the costate vector in control theory, or the Lagrange multiplier vector in the calculus of variations. The components of $\vec{p}$ satisfy the equations

$$\frac{dp_i}{d\varphi} = -\frac{\partial H}{\partial x_i} \tag{6}$$

and the differential equations for the state variables can be rewritten as

$$\frac{dx_i}{d\varphi} = \frac{\partial H}{\partial p_i} \tag{7}$$

forming the canonical equations of Hamilton for this dynamical system. The maximum principle (or minimum principle as it is called with a sign change) applies and states that along an optimal trajectory H is a maximum and is constant.

$$H(\vec{p}(\varphi),\ \vec{x}(\varphi);\ \theta(\varphi),\ \psi(\varphi),\ \alpha(\varphi)) = \max_{\theta,\psi,\alpha} H \tag{8}$$

$$H \equiv H_o = \text{constant} \tag{9}$$

For this problem the domain of the control variables is open. The control variables are all angles with no restrictions on their values. Then equation (8) yields the additional equations which must be satisfied at any local extremum of H:

$$\frac{\partial H}{\partial \theta} = 0 \quad \frac{\partial H}{\partial \psi} = 0 \quad \frac{\partial H}{\partial \alpha} = 0 \tag{10}$$

The problem can be treated as either an initial value problem with initial conditions

$$\vec{x}(\varphi_o) = \vec{x}_o \qquad \vec{p}(\varphi_o) = \vec{p}_o \tag{11}$$

or a boundary value problem with boundary conditions

$$\vec{x}(\varphi_o) = \vec{x}_o \qquad \vec{x}(\Phi) = \vec{x}_f \tag{12}$$

where Φ is the value of φ (variable) at the terminal state. The boundary value problem is the natural formulation for specific problems, and $\vec{x}_f$ can be generalized to a target set of state vectors, in which case the additional end condition, or transversality condition, must be added that $\vec{p}(\Phi)$ must be normal to the target set. There are numerous iterative techniques for solving such a problem, but it is very difficult to draw general results or conclusions from the solutions of these specific problems.

For the general problem as is being considered in this paper it is desired to know all the optimal paths leading from every point of the state space. Thus, the natural formulation is as an initial value problem. However, unless the problem can be solved analytically this also is a

laborious method of attaining general results. Breakwell and Moyer have used this technique and while they do produce some general results, their papers are illustrative of how this method clouds the problem.

Before presenting the approach which appears to yield the greatest insight to the general problem, it should be pointed out that the Hamiltonian formulation does produce some valuable general analytical results rather quickly. It turns out that the Hamiltonian does not depend explicitly on the three state variables, the Eulerian angles. This is apparent from the fact that the spatial axes can be rotated arbitrarily without altering the physics of the problem. Thus, the Eulerian angles as state variables are what are called cyclic or ignorable variables. It follows that the generalized momenta conjugate to these three variables are constants, as can be seen from equations (6). This has several important implications.

1. If the initial and final orbits have one or more of their Eulerian angles identical, then throughout the optimal transfer these angles remain unchanged. In terms of the initial value problem, if a maneuver starts out with no change in one or more Eulerian angles, then those angles remain constant from then on. In particular:

    a) Coplanar, coaxial transfer (the three initial Eulerian angles identical to the three final ones) is accomplished optimally only by coplanar, coaxial

maneuvers. As will be seen, this restricts the control to that of tangential impulses at the apses.

b) Coplanar transfer (the first two Eulerian angles constant) is accomplished optimally by coplanar maneuvers only.

c) Coaxial transfer (the third Eulerian angle constant) is accomplished by coaxial maneuvers only.

2. The variation of each of the Eulerian angles is strictly monotone throughout an optimal maneuver. At no intermediate point of the maneuver do any of the Eulerian angles equal that of the final orbit unless it does at every point of the maneuver. An Eulerian angle cannot take on the same value at two separate points of a maneuver unless it is constant throughout.

3. The characteristic velocity of a maneuver is a strictly increasing function of $|d\beta_i|$, the magnitude of the changes in each of the Eulerian angles. The one exception is the minimum escape and return maneuver (biparabolic maneuver) which does not depend on the relative orientation of the initial and final ellipses.

This is a considerable contribution which comes directly from the Hamiltonian formulation. Using other techniques, researchers have

spent much effort to prove that optimal transfer between two coplanar orbits is always coplanar, and between two coaxial orbits is always coaxial. It falls out immediately, as shown above, when the Hamiltonian approach is used. However, beyond this point general results are difficult to obtain directly. In the next section the approach which proves so fruitful is presented.

III

The conditions of the maximum principle of the Hamiltonian system set forth in the preceding section are local necessary conditions for optimality. This point brings one back to the basic idea behind the search for optimals. The basic idea is one of elimination. Tests are devised which an optimal maneuver must satisfy, and then possible maneuvers are checked by these tests. All maneuvers which fail one or more of the necessary conditions are eliminated. Ideally, it is known that an optimal path exists, and all but one path are eliminated as nonoptimal. The remaining maneuver must be the optimal one. However, it seldom works out this nicely. A major step will be achieved if some simple tests can be devised which eliminate most possible maneuvers as nonoptimal. This is what will be done.

The formulation of the previous section can be expressed equivalently in terms of the local necessary conditions of the calculus of variations. Equations (6) for the generalized momenta are equivalent to the first necessary condition of the calculus of variations. Equation (8), the requirement that the Hamiltonian be maximum under an optimal control, is equivalent to the second necessary condition, or the Weierstrass necessary condition. Equation (9), the requirement that the Hamiltonian be constant throughout an optimal maneuver, contains the Weierstrass-Erdmann corner condition of the calculus of variations. A maneuver

which fails to satisfy any one of these conditions is nonoptimal. Thus, it may simplify the task if these conditions are considered separately as is encouraged by the calculus of variations.

The Weierstrass necessary condition may be formulated as follows. The displacements

$$dx_i = f_i(\vec{x};\ \theta,\ \psi,\ \alpha) \qquad i = 1,\ \ldots,\ 5 \tag{13}$$

form a three dimensional hypersurface, a function of $\vec{x}$, in the five dimensional space of dx_i, which has the curvilinear coordinates θ, ψ, α. This hypersurface represents the displacements dx_i which can be had from the initial point $\vec{x}$ at a cost of characteristic velocity of $d\varphi$. On this hypersurface the curvilinear coordinates φ, ψ, α of a point specify the control required to reach that point. A control θ, ψ, α from an initial point $\vec{x}$ is said to satisfy the necessary condition of Weierstrass in case the displacement corresponding to this control, equations (13), is a maximum displacement over all possible controls for some direction in dx_i space. If $\hat{p}$ is the unit vector specifying a direction in dx_i space, then the displacements which satisfy the Weierstrass condition for this direction are the solutions (one or more) of the equation

$$\hat{p} \cdot d\vec{x} = \max_{\theta,\psi,\alpha} \left[\hat{p} \cdot d\vec{x}\ (\vec{x};\ \theta,\ \psi,\ \alpha) \right] \tag{14}$$

That is, displacements satisfying the necessary condition of Weierstrass are all those displacements which satisfy equation (14) for some direction $\hat{p}$ in dx_i space.

The Weierstrass-Erdmann corner condition applies across discontinuities in the control functions $\theta(\varphi)$, $\psi(\varphi)$, $\alpha(\varphi)$. Where the control functions are continuous this corner condition is satisfied trivially. At a corner, or discontinuity in the control program, let θ_1, ψ_1, α_1 be the limiting values of the controls as the discontinuity is approached with φ increasing, and θ_2, ψ_2, α_2 be the limiting values as the discontinuity is approached from the other side. The Weierstrass-Erdmann corner condition is that across such a discontinuity the relation

$$\hat{p} \cdot d\vec{x}(\vec{x};\ \theta_1,\ \psi_1,\ \alpha_1) = \hat{p} \cdot d\vec{x}(\vec{x};\ \theta_2,\ \psi_2,\ \alpha_2) \tag{15}$$

holds, where $\hat{p}$ is the direction vector of the Weierstrass necessary condition and must be the same at both sides of the discontinuity. That is, the corner condition applies when there is more than one solution of equation (14) for a given $\hat{p}$, and across a corner the controls (or displacements) must correspond to two solutions of equation (14) for the same $\hat{p}$.

It should be noted that the unit vector $\hat{p}$ is colinear with the generalized momentum (adjoint, costate, or Lagrange multiplier) vector $\vec{p}$. Where the displacement hypersurface has a well-defined normal direction the Weierstrass necessary condition requires among other things that $\hat{p}$ be the outer unit normal vector at $d\vec{x}$. Figure 2 illustrates these statements with a one dimensional hypersurface (a line) in a two dimensional space.

This problem may also be thought of as a problem of finding the

geodesics or minimum distance paths in a five dimensional $\vec{x}$ space over which a metric is defined by equations (4) or (13). One of the requirements, mathematically, for a metric is that locally the length of the path from one point to a second point and then to a third point must be greater than or equal to the length of the direct path between the first and third point. This is the triangle inequality required of a metric, and is equivalent to the requirement that the metric be convex.

In this case the metric tensor is represented by the displacement hypersurface. The Weierstrass condition is simply a check of the convexity of the metric. Those portions of the metric which fail this test, that is, which are nonconvex, are replaced by internal linear combinations of other displacements. The new, or improved, metric consists of the original metric and all points on the straight lines joining any two points of the original metric. Only points on the surface of this new metric are considered. That is, the original metric is completed by the smallest possible convex hull. This new metric satisfies the triangle inequality (which is simply the convexity requirement) and may be used to determine the geodesics of the space in the usual manner.

Figure 2 illustrates the principle of this technique. The nonconvex portions of the metric (or displacement) tensor are covered by a developable surface (a surface generated by straight lines) formed by rolling the metric on a hyperplane of the same dimension as the

hypersurface forming the displacement body. The straight line generators of this developable surface have endpoints corresponding to the control discontinuity across a corner which satisfies the Weierstrass-Erdmann corner condition.

With this groundwork laid, the approach which has produced the recent results for this problem and which is the subject of this paper becomes clear. The object of using local necessary conditions is to eliminate as many maneuvers as possible by showing them to be nonoptimal. It turns out that a separate application of the Weierstrass necessary condition and the Weierstrass-Erdmann corner condition eliminates practically all possible maneuvers. This is the approach used by Busemann and Contensou.

IV

This approach will be demonstrated by applying it to the coplanar case of the problem. For the coplanar case α is identically zero, and two of the Eulerian angles are constants of the motion and may be neglected. The remaining Eulerian angle will be referred to simply as θ, and is the angle the major axis of an orbit makes with some reference line in the plane of motion. The state space is now three dimensional

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \ell \\ x_3 &= f \text{ (or e)} \end{aligned} \tag{16}$$

with the three differential equations of this pseudodynamical system

$$\frac{dx_i}{d\varphi} = f_i(\vec{x}; \theta, \psi) \qquad i = 1,2,3 \tag{17}$$

where there are now only the two control variables, θ, the position in the orbit, and ψ the direction of the velocity change in the plane.

The displacement hypersurface is now just an ordinary two dimensional surface in the three dimensional displacement space. This displacement surface has curvilinear coordinates θ and ψ. The nature of these coordinates makes the construction of this displacement tensor, or metric, very simple. For a given position in the orbit (fixed θ) the

curve of displacements as the direction of velocity change varies through 360 degrees (as ψ varies through 360 degrees) is an ellipse centered at the initial point $\vec{x}$ and tangent along its major axis to the 45° cone of revolution with axis parallel to the x_3 axis in state space (Figure 3). Actually the displacements should be drawn in displacement space, $\vec{dx}$ but superimposing this on the state space as in Figure 3 helps to indicate how the displacement body (or metric) is generated.

It is helpful to note that the 45° cone in state space (Figure 3) divides the state space into intersecting and nonintersecting orbits. Points in the region outside the cone represent orbits which physically intersect the original orbit which is represented by $\vec{x}$, which serves as the vertex of the cone. Points on the cone represent orbits which are tangent to the original orbit. Points inside the upper half of the cone represent orbits which are entirely outside the original orbit. Points inside the lower half of the cone represent orbits which are wholly inside the original orbit.

Returning now to the displacement tensor, one finds that the ellipses representing the displacements available from a given position in the orbit move around the central cone as the position θ moves about the initial orbit. As the ellipses move, their major and minor axes deform continuously. The result is the displacement body of Figure 4. This is the original metric which must be rendered convex

by the smallest possible convex hull. That is, the Weierstrass condition must be checked. Obviously, this metric is highly nonconvex, so the Weierstrass condition will eliminate a great many possible maneuvers as nonoptimal.

The improved metric which has been rendered convex by the method described in the previous section is shown in Figure 5. The body has a natural plane of symmetry formed by the $dx_2 - dx_3$ axes. It also has polar symmetry through the initial point $\vec{x}$ (the origin of $d\vec{x}$ coordinates). This polar symmetry is equivalent to the existence of the inverse of every optimal path. If a path is optimal from $\vec{x}_o$ to $\vec{x}_f$, then the optimal path from $\vec{x}_f$ to $\vec{x}_o$ is found by using the polar symmetric image of the first path. That is, the maneuver is performed in reverse order with the thrust vector turned 180 degrees.

Only those portions of the metric which were originally convex may represent optimal maneuvers. That is, only those maneuvers represented by points which are on the surfaces of both the original metric and the improved, convex metric survive the Weierstrass necessary condition for optimal maneuvers. As can be seen in Figure 5, only a very small portion of maneuvers, the points along the ridge around the top and bottom of the body, remain as possible optimals.

The rest of the metric body is covered by three developable surfaces. One such surface wraps around the side of the body out-

distancing most of the directly available displacements. The other two surfaces cover the top and bottom of the body bridging the holes inside the 45° cone associated with the point $\vec{x}$. The straight line generators of these surfaces connect the pairs of points on the displacement body which satisfy the Weierstrass-Erdmann corner condition. These connections represent the only permissible discontinuities in the control program.

Figure 6 illustrates how few maneuvers remain after the Weierstrass test. For a given position on the orbit, the directions ψ which satisfy the Weierstrass condition form a narrow band between the local horizontal and the flight path. Since the metric does not contain x_1, (θ), explicitly and since x_2 (ℓ), appears only as a scale factor, the controls which yield maneuvers which satisfy the Weierstrass necessary condition may be presented as a region in e, θ, ψ space. Figure 6 presents a slice of this region along an e plane. The symmetries of the metric are implicit in the coordinate scales.

The controls which survive the Weierstrass test are given in detail in Figures 7 and 8. The surface in e, θ, ψ space in Figure 7 represents the maximum deviation from the horizontal of the velocity change for a given e and θ. The similar surface in Figure 8 represents the nearest to the horizontal the direction of a velocity change may come. To satisfy the Weierstrass condition the control must lie on or beneath the surface of Figure 7 and on or above the surface of Figure 8. In all of these figures upper signs go with upper signs, and lower signs with lower signs.

The Weierstrass-Erdmann corner condition, corresponding to

the straight line generators of the developable surfaces on the improved metric body (Figure 5), dictates the only way in which two distinctly different control programs can be joined so that the entire maneuver is optimal. Only impulses precisely on the two surfaces of Figures 7 and 8 may participate in such a juncture, since these surfaces represent the edges of the convex ridges in Figure 5 (the endpoints of the generators). The corner condition may be considered as connections between pairs of points, both on the same surface, of the surfaces of Figures 7 and 8. On each surface every point is connected uniquely to one other point on the same surface. The connection is always such that e remains constant, since the vehicle simply coasts around the orbit during the control discontinuity (when φ is constant; that is, when the rocket is not firing). Thus, the corner condition may be presented for every given e as a relation between the positions where the rocket is extinguished and where it is later refired. These relations are given in Figures 9 and 10.

These connections are one way only. That is, there is a definite order to the points. (The inverse of a maneuver is found by reversing the maneuver and turning the thrust vector 180 degrees, as discussed above.) One point is where the rocket is turned off, the other point is where the rocket is reignited. The direction of these connections is shown in Figure 5. This shows that there are only three possible

kinds of discontinuities: a forward thrust followed by a rearward thrust (the side developable surface of Figure 5), a forward thrust followed by a forward thrust (the top surface), and a rearward thrust followed by a rearward thrust (the bottom surface). In connecting two forward thrusts, one finds that the thrust nearer periapsis must come first. For two rearward thrusts, the one nearer apoapsis comes first.

A tangential thrust is optimal only where the velocity of the vehicle is horizontal, that is, at the apses. This, along with the corner condition, yields the well-known Hohmann type of transfers.

Near e equal to one the analysis becomes more complex and is not included here. For these highly eccentric orbits it turns out that there are arcs of the orbit between the apses from which no maneuver other than a coast can be optimal.

These, then, are the results of applying separately only the Weierstrass necessary conditons and the Weierstrass-Erdmann corner conditon. This simplified approach has resulted in the elimination of practically all maneuvers as nonoptimal. Of course those maneuvers which have survived this test must al pass all other necessary conditions, but these need be considerd only when it is desired to further narrow the selection of possible optimals. The most important remaining test is the global optimality test. Of all the extremals connecting two points, one is the best. This must be found by direct comparison.

One other test which is a local test which is usually applied is the first necessary condition. This will be discussed next.

V

The remaining parts of the Hamiltonian formulation of the maximum principle which must be checked are the constancy of the Hamiltonian throughout a maneuver and the first necessary condition as embodied in equations (6) for the generalized momenta. It must be emphasized that even when this is done the remaining extremals are not known to be optimal. The maximum principle is a local necessary condition for optimality and is not a sufficient condition. It is true that if a displacement is directionally convex in an open region of directionally convex displacements (such as the displacements in the interior of the convex ridges (Figure 5) in the example of the preceding section) then the extremal path given by the maximum principle is locally optimal for some finite distance. But even when the local optimals are found they must be checked globally to find the absolute optimals.

For the coplanar case just discussed in detail, Breakwell and Moyer have verified numerically that throughout an impulsive maneuver which satisfies the conditions of the previous section the Hamiltonian is constant and the generalized momenta are given by equations (6). This remains to be shown analytically. One particular continuous thrust maneuver was also checked by these researchers and found to satisfy all of the maximum principles. This is the Lawden spiral maneuver, so-

called because it represents a spiral trajectory about the center of attraction. This maneuver is found by following the $\theta_1 = \theta_2$ line in Figure 9. This is the line represented by the dashed line in Figure 8 which corresponds to a connection of one position in orbit with itself. The rocket is turned off and then on rapidly until a continuous intermediate thrust is simulated. Illustrative of the fact that the maximum principle does not provide sufficient conditions for optimality is the fact that this maneuver, which does satisfy the maximum principle, has been shown to be nonoptimal through the use of additional necessary conditions, usually referred to as the second variations.

A case to which the entire maximum principle can be readily applied is the coplanar, coaxial case. The extremals for this case are the Hohmann group of transfers. As noted earlier, the optimal transfer between two coplanar, coaxial orbits is a coplanar, coaxial maneuver. It can be determined from the displacement tensor, Figure 5, that the only maneuvers which satisfy the Weierstrass necessary condition are tangential thrust at the apses of the orbit. The corner condition requires that one velocity change at an apse can only be continued with the same sense or followed by a thrust at the opposing apse. Thus, the optimals must consist of segments of the 45° lines in Figure 11.

It is interesting to consider the propagation of the generalized momenta for this case. It enables one to pay tribute to the originator

of this entire optimization principle, Huygen. Huygen's principle was developed for the propagation of waves. Basically, Huygen's principle states that the wave propagation problem and the geodesic problem are equivalent. Huygen's wave principle is precisely equivalent to the maximum principle, and it preceded the maximum principle by three centuries. In Huygen's terms, the displacement body is a "wavelet", the generalized momentum vector is normal to the wave front, and the geodesic or optimal path is the path of the disturbance. The propagation of the generalized momenta is simply a consideration of the local changes of the wavelets and the consequent turning of the wave front.

The result of this for the coplanar, coaxial case is shown in Figure 11. A velocity increase at periapsis can be followed by either a velocity increase or a velocity decrease at apoapsis. A velocity increase at apoapsis can be followed only by a velocity decrease at periapsis. A velocity decrease at apoapsis can only be followed by a velocity decrease at periapsis. A velocity decrease at periapsis cannot be followed by any of the other maneuvers. Of course a sufficiently large velocity change can cause the state point to cross the central axis in Figure 11, thus interchanging the apses. The above rules may be used to form the extremals in this space. These maneuvers are the Hohmann class of transfers. Of course they must be compared globally. There is usually more than one extremal between two orbits. The best

known global competitor is the frequently optimal biparabolic transfer through infinity.

The purpose of this paper has been to show how the use of the Weierstrass necessary condition and the Weierstrass-Erdmann corner condition provides the desired result of eliminating most maneuvers for this problem as nonoptimals. This approach of using these separate conditions rather than the entire maximum principle has been responsible for the recent results in this problem of optimizing orbital maneuvers.

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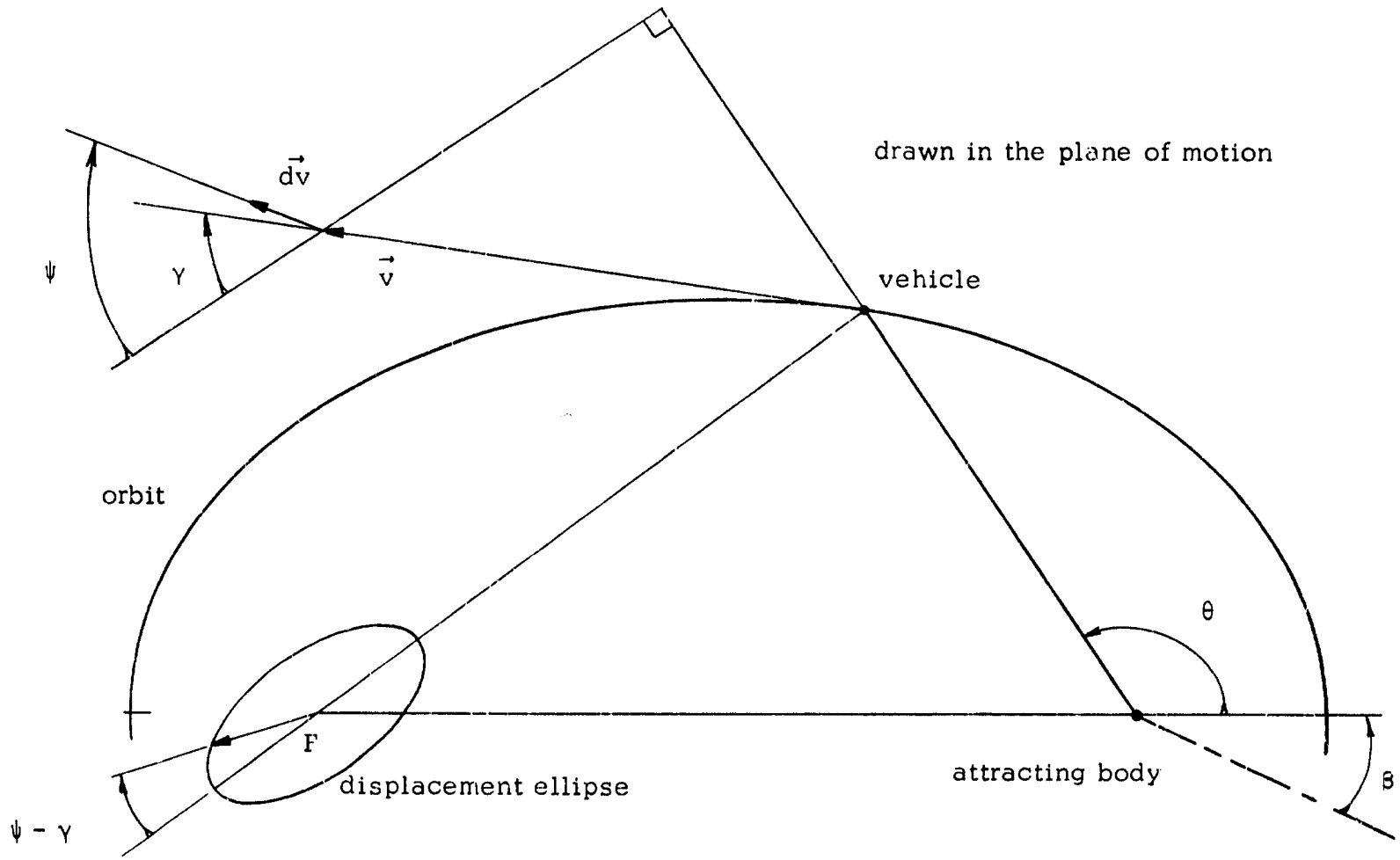


Figure 1. The notation of this analysis.

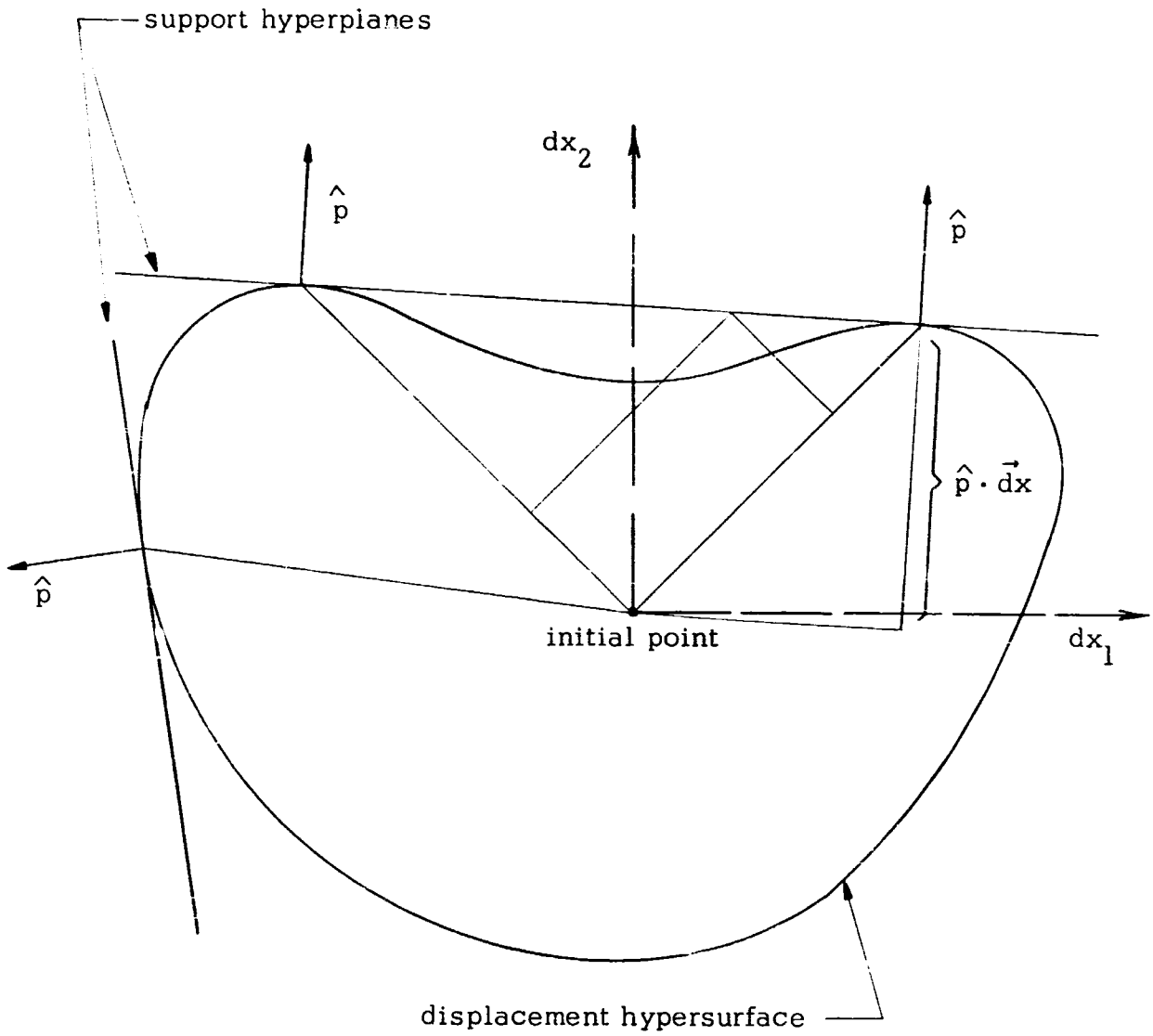


Figure 2. Two dimensional illustration of displacement space.

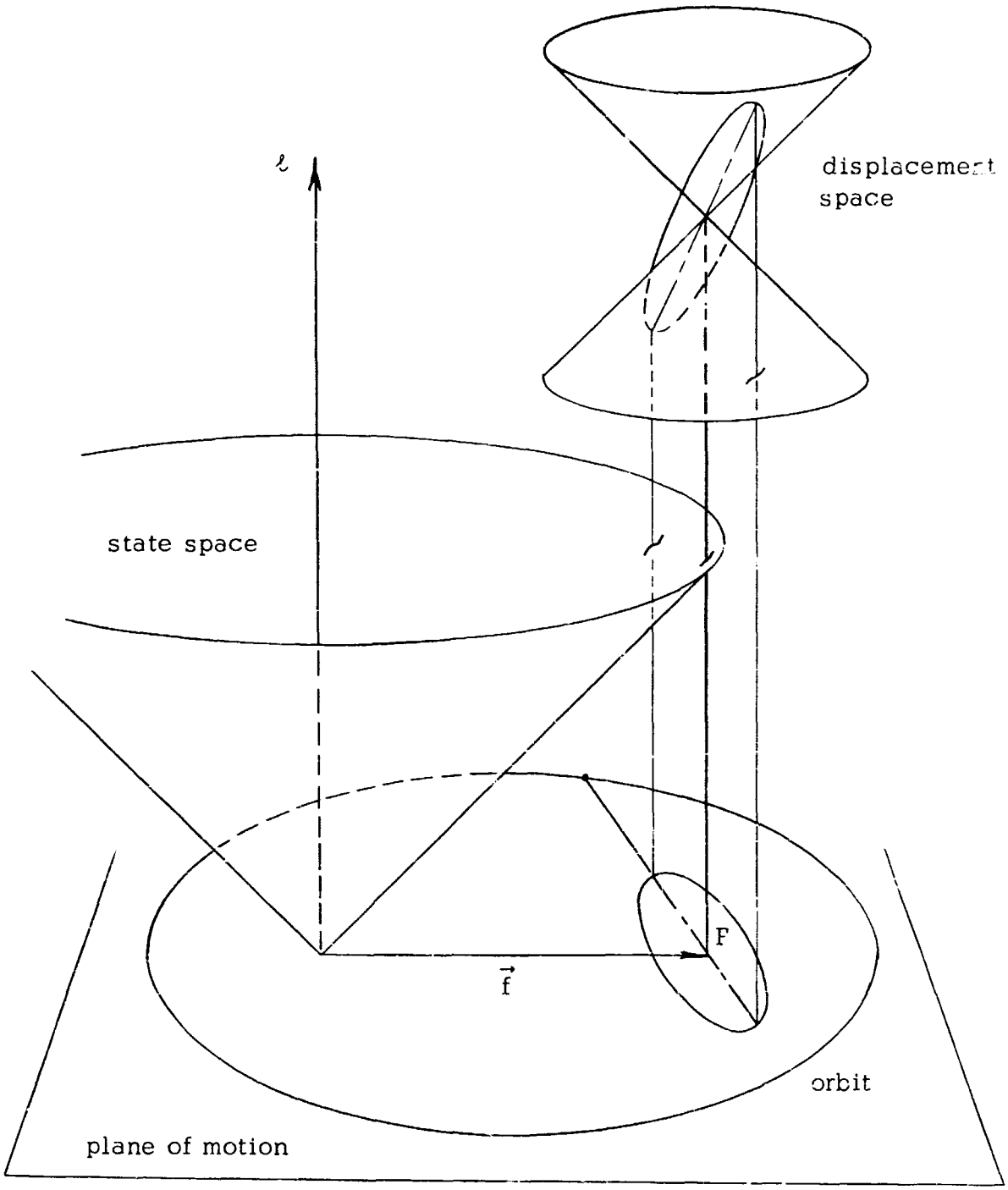


Figure 3. State space and displacement space for coplanar motion showing generation of displacement ellipse.

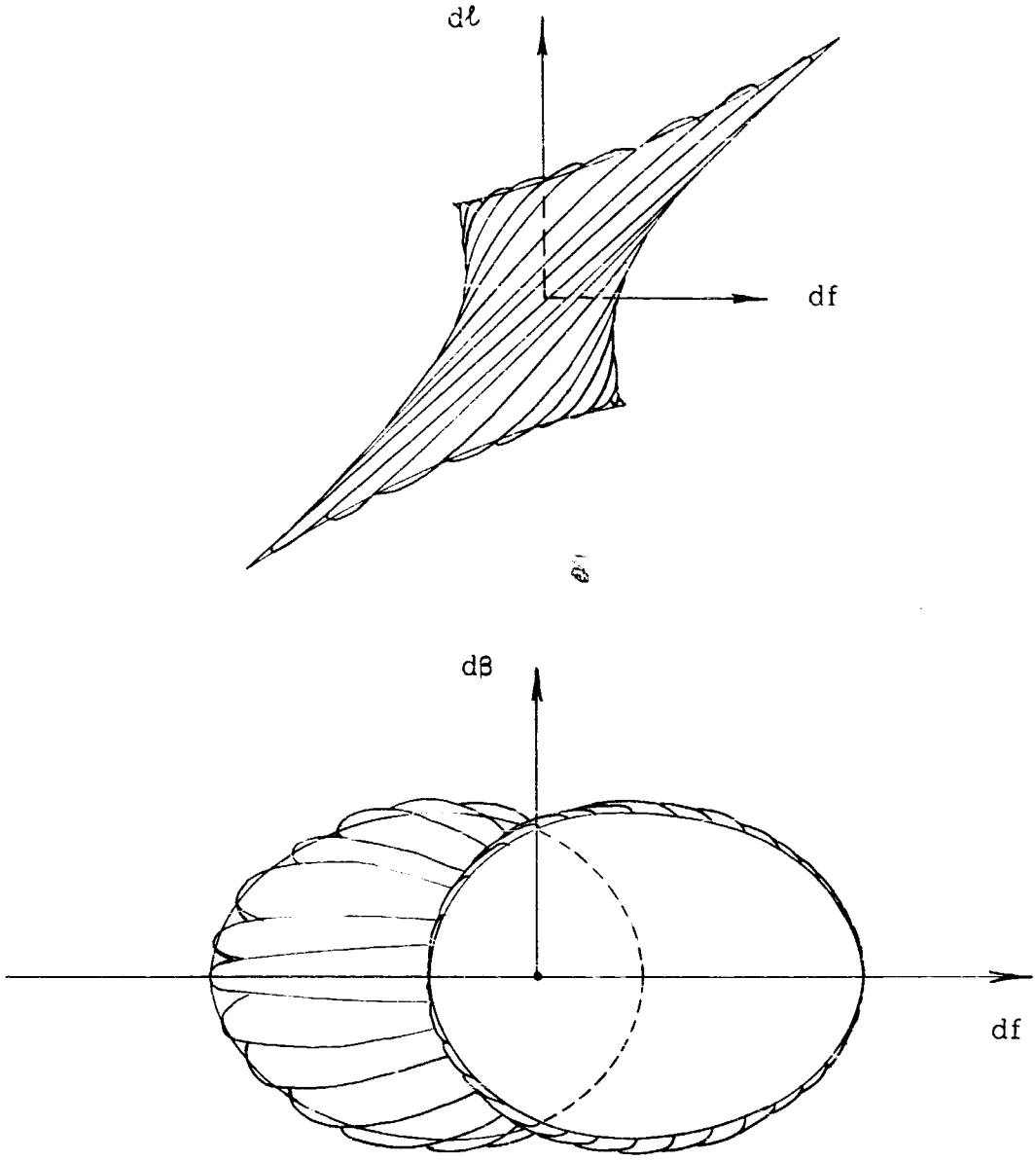


Figure 4. The displacement body for coplanar motion.

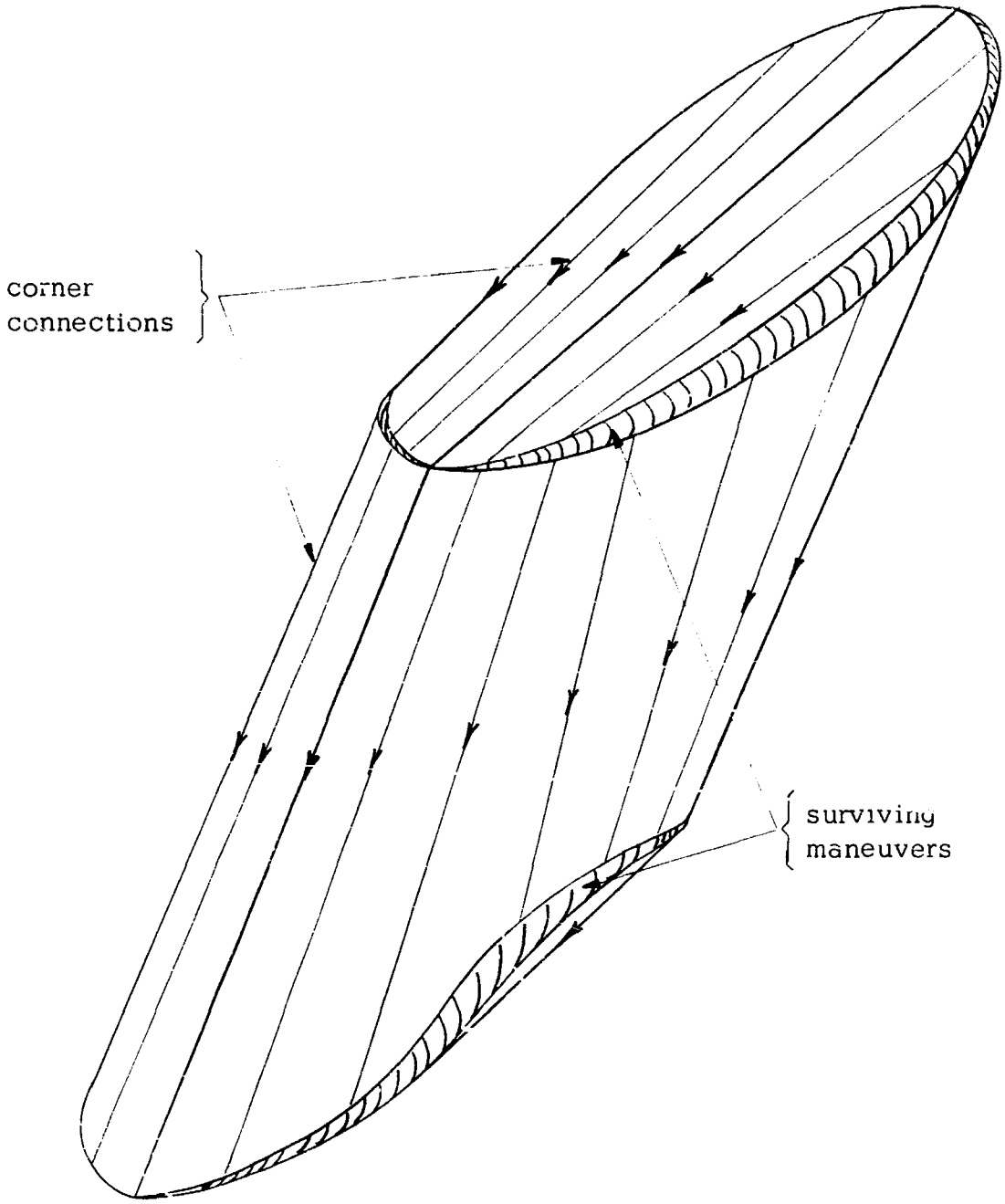


Figure 5. The displacement body after the Weierstrass tests.

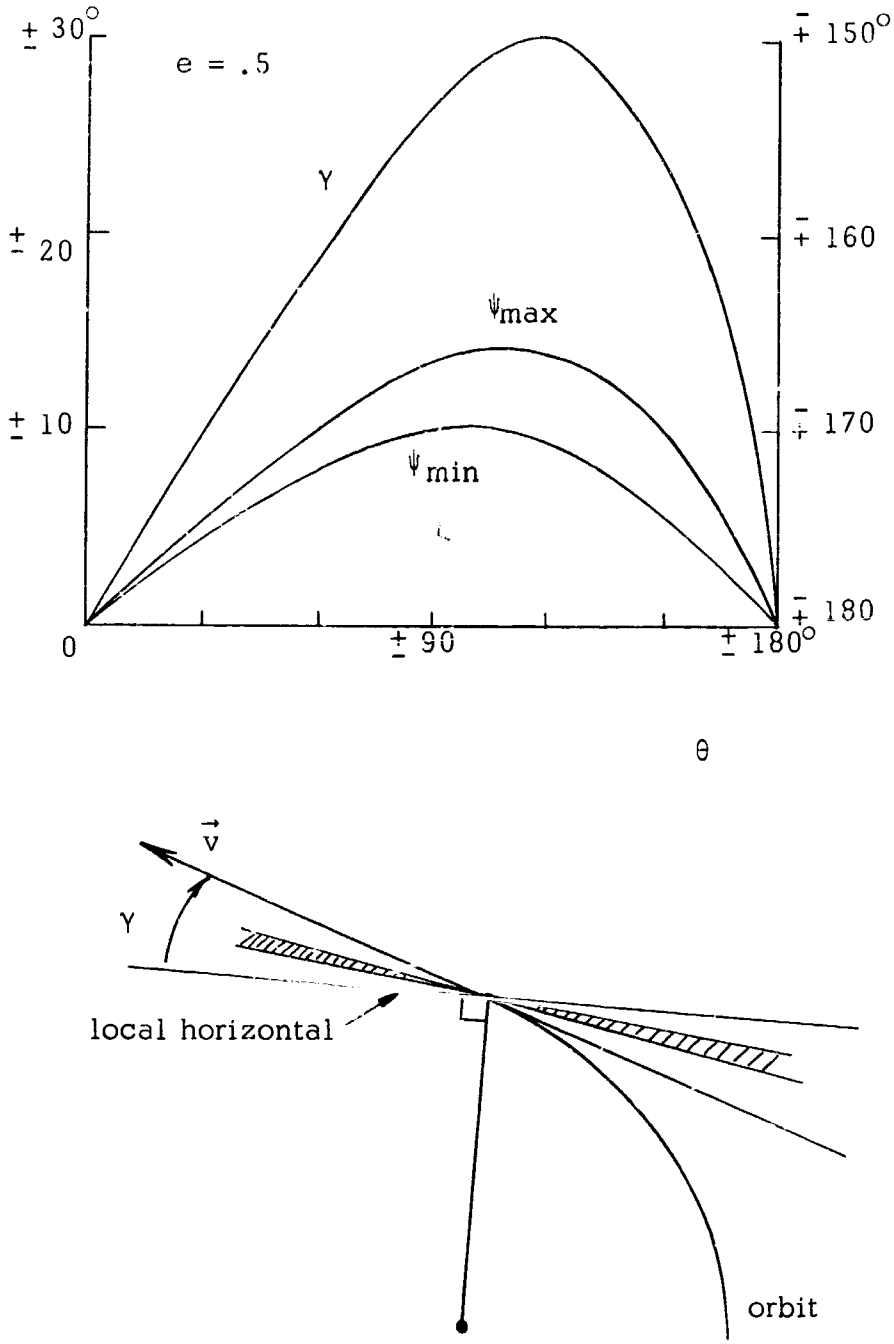


Figure 6. The controls which survive the Weierstrass test.

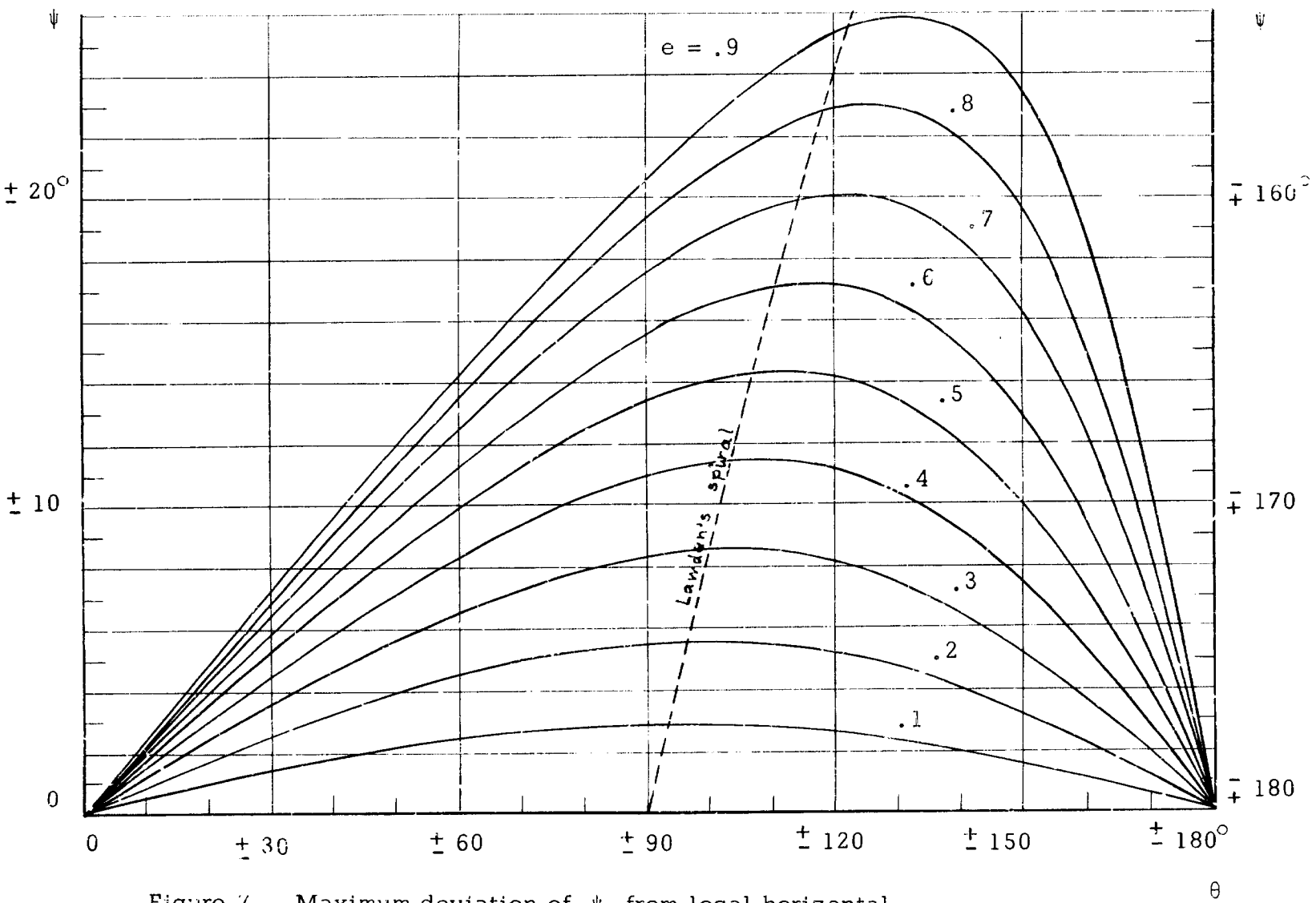


Figure 7. Maximum deviation of ψ from local horizontal.

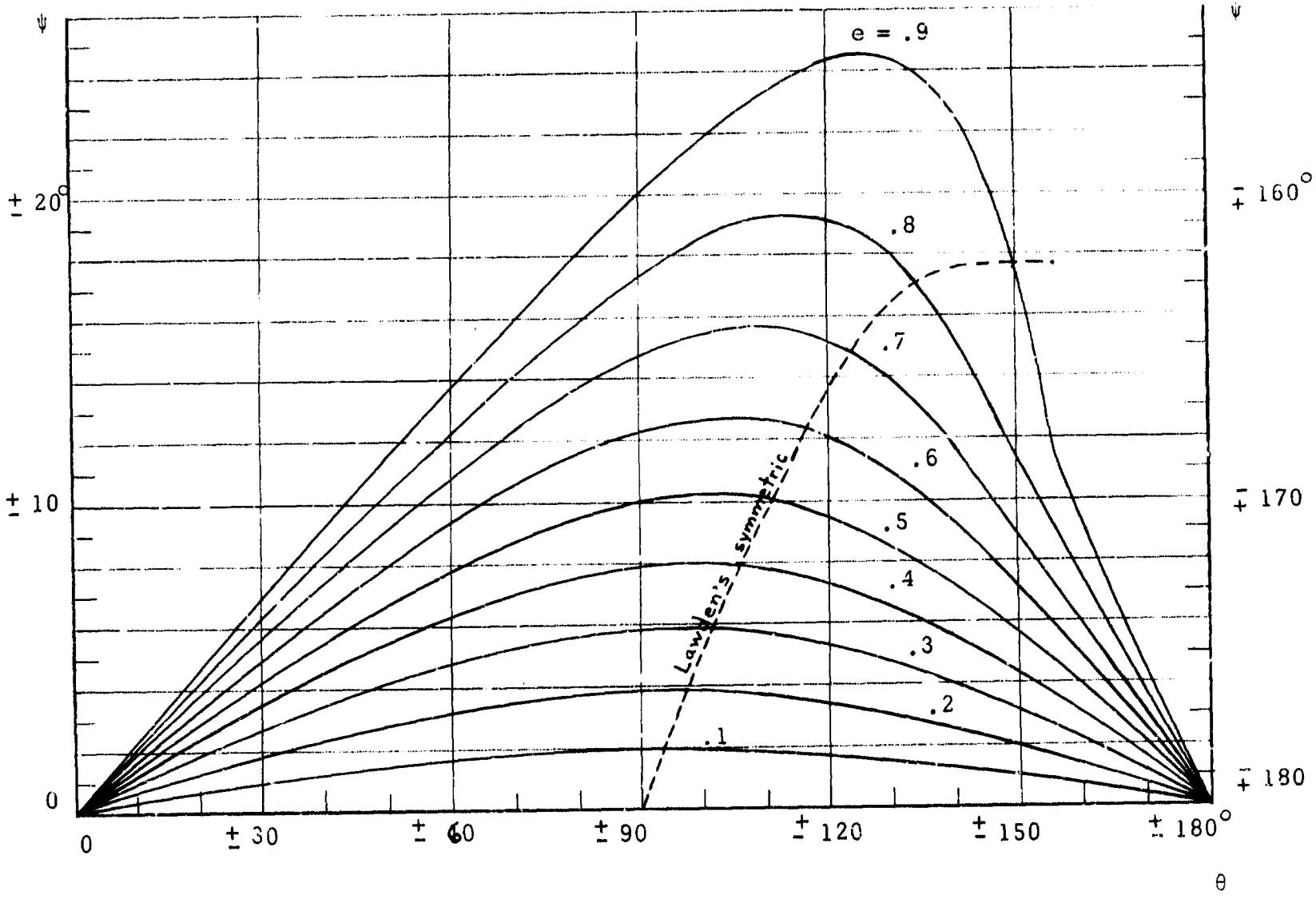


Figure 8. Minimum deviation of ψ from local horizontal.

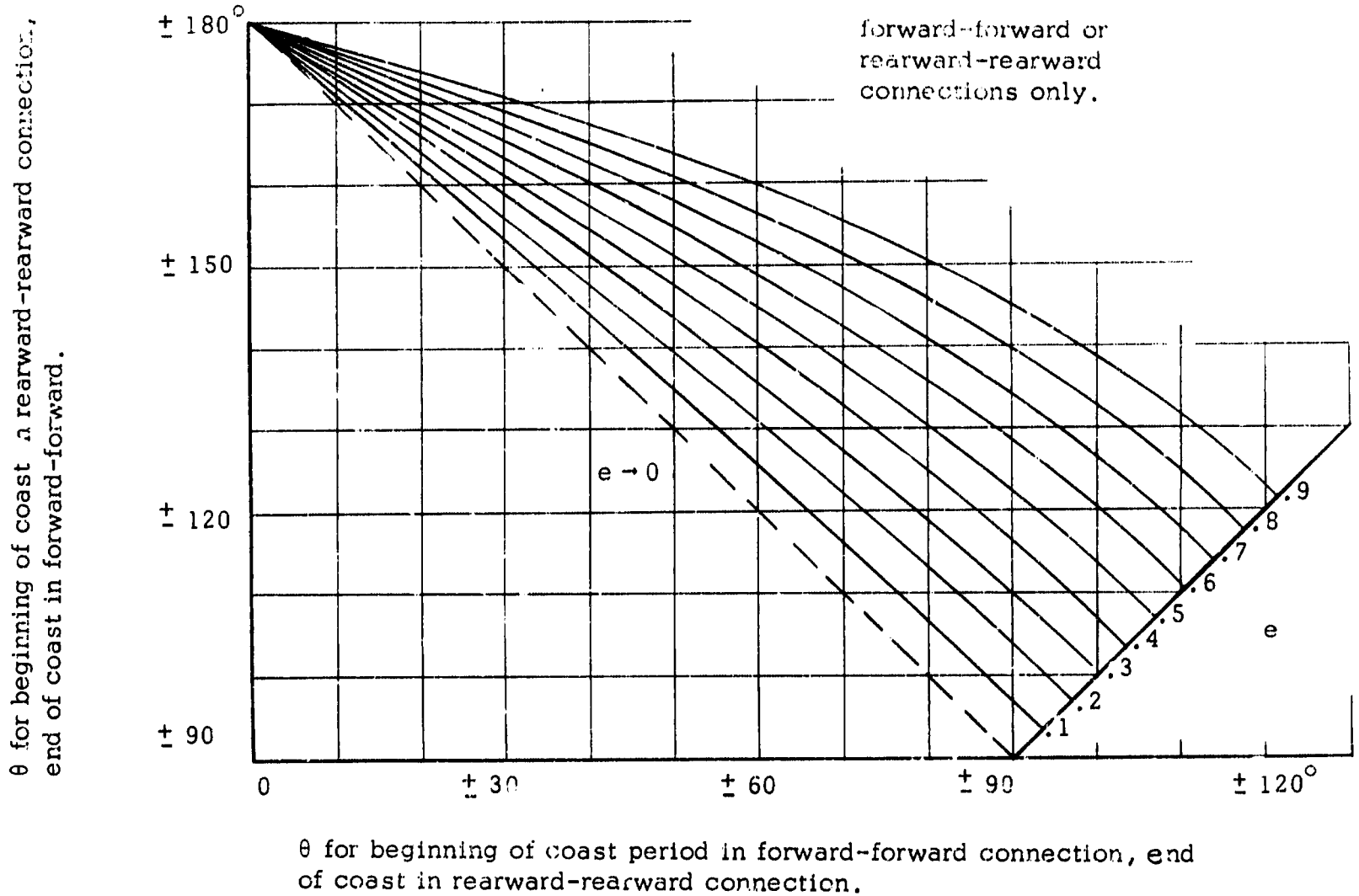


Figure 9. Weierstrass corner condition for top and bottom of displacement body.

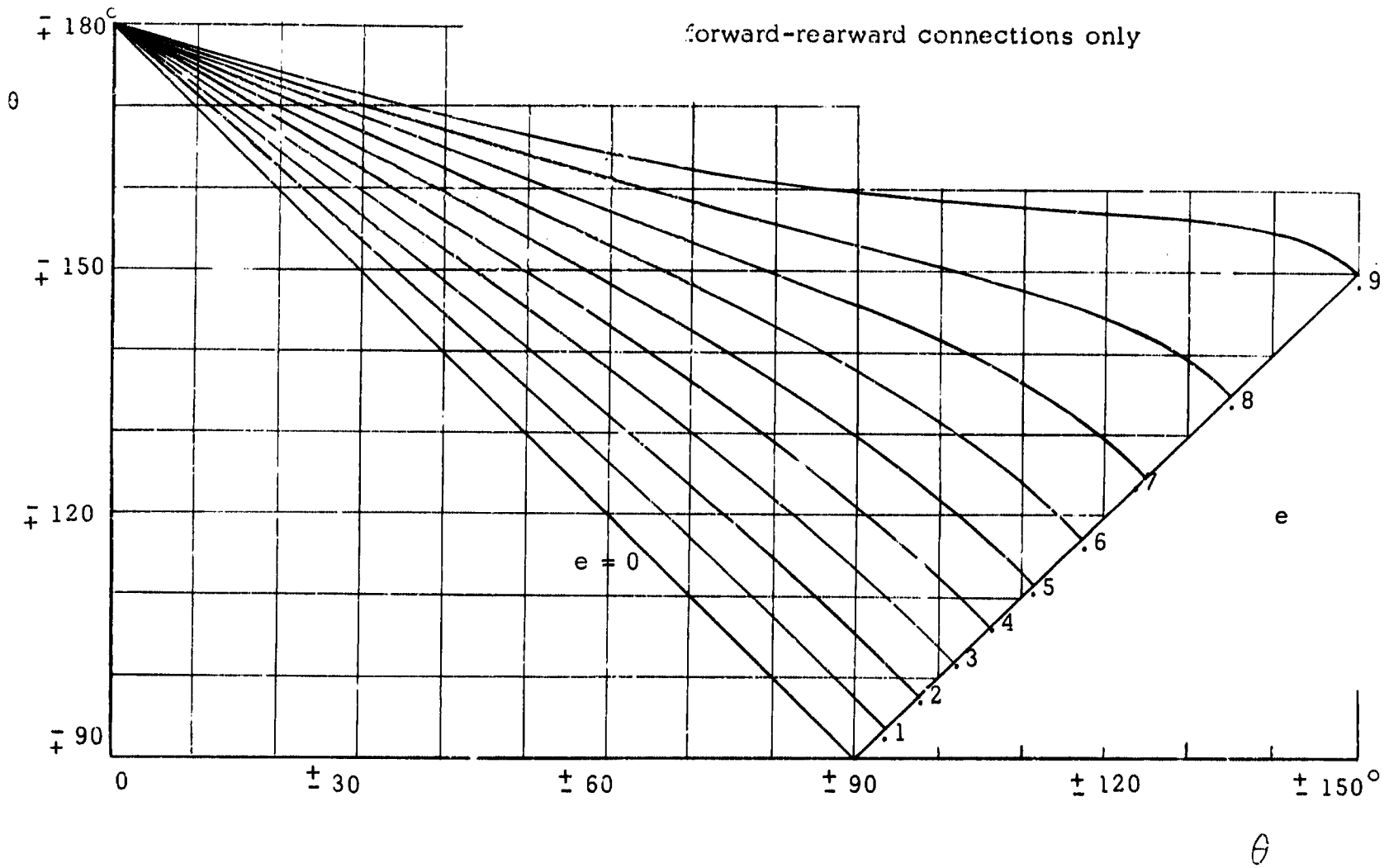


Figure 10. Weierstrass corner condition for sides of displacement body.

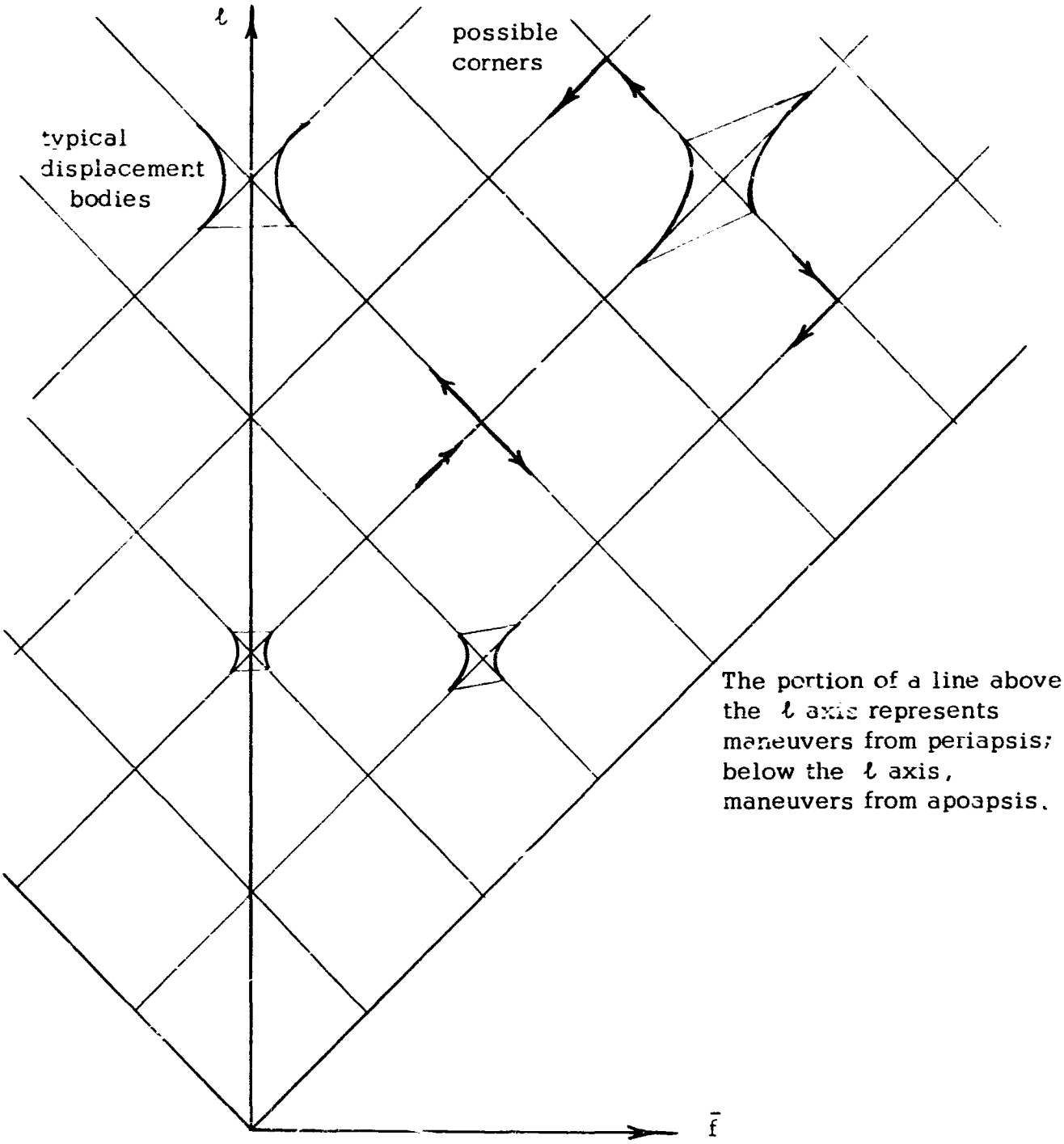
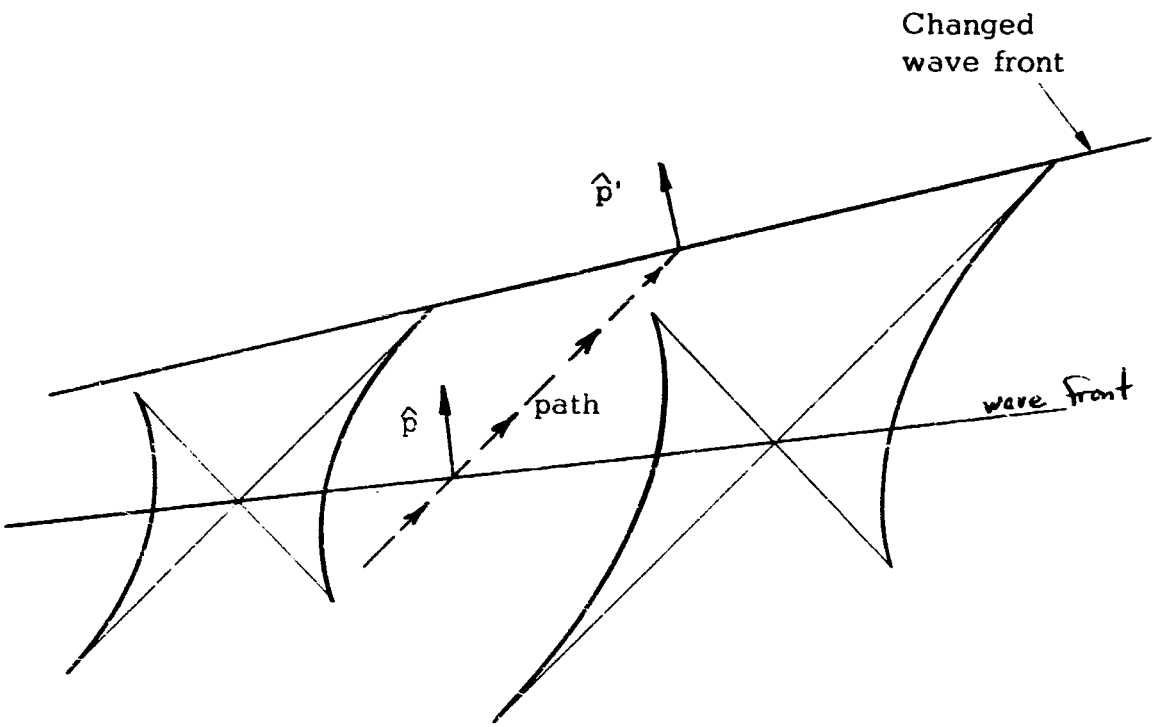


Figure 11. The case of coplanar, coaxial motion.

Corner, or switch, occurs when wave front changes its support position.



displacement bodies, or wavelets, along wave front

Figure 12. Huygen's principle applied to the coplanar, coaxial case.