THE FIRST PASSAGE PROBABILITY AND CONTROL

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NOTATION AND SYMBOLS

$\vec{x}$	state vector
$\vec{y}$	control vector
$\vec{r}$	stochastic input of white noise type
$\vec{c}$	initial value of state vector
R	the "allowed" or "safe" region of state space
$\underset{\vec{r}_i}{Exp}$	the expected value over $\vec{r}_i$
$\underset{\{\vec{r}_i\}}{Exp}$	the expected value over the set $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$
ε	"is contained in"
$\notin$	"is not contained in"
b	damping coefficient
$\langle\ \rangle$	ensemble average
τ	$t_1 - t_2$
D	strength of noise (chosen to be 1 in the example)
$\vec{g}$	vector function which governs system
F_n	cost function
h_i	contribution to total cost from ith to i + 1st stage
$f_n(\vec{c})$	optimal return over n stages starting with initial state $\vec{c}$

ABSTRACT

This report shows how one may use the first-passage probability as a measure of performance in a stochastic control problem. Some general background is presented along with a mathematical formulation which lends itself to numerical computation. To demonstrate some of the details of computation an actual example is included.

1. Introduction

There are many problems in science and engineering which involve the excitation of a system by some random disturbance. The output of such a system is then a random variable and must be treated from a statistical rather than a deterministic standpoint. Averages, mean-squares, and higher statistical moments may be used as yardsticks to evaluate the response. For wide classes of problems these measures may provide as much information as one really needs; for other problems one would like to know much more about the response.

The specific type of problem to be considered here is one for which there are sufficiently large excursions of the response to make failure an imminent possibility. Failure will be defined simply as the first crossing of some predefined level, or more precisely, as the entrance of the state vector into some forbidden region of state space for the first time in a given interval of operation.

For this type of problem the notions of average level of response or mean-square of response do not tell very much about how long one should expect a system to operate without failing. It would therefore be most desirable to actually compute the probability density of the time to first passage. Unfortunately this turns out to be a difficult problem and only the simplest of examples have been solved analytically. In an attempt to find approximate solutions investi-

gations have been and are being made into various computational schemes based upon Monte Carlo methods and numerical diffusion of probability mass.[1]

It is possible to envision even more complicated problems in which some sort of feedback control is involved, either for the explicit purpose of minimizing failure probability or for some other unrelated purpose. The question of synthesizing a control function to provide minimum failure probability or of finding the maximum failure probability with a given control function is then more complex because an optimization is involved in addition to the first passage problem. An example of the latter case may arise when a flexible launch vehicle is subjected to a random wind field. The resulting bending vibration may be so severe that the vehicle will actually break up. This paper shows how dynamic programming may be used as a computational means to attack such problems.

2. Dynamic Programming

Dynamic Programming is now a widely used tool in the study of optimal control problems and multistage decision processes. The method is based upon the Principle of Optimality which states: An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.[2]

In order to clarify the meaning of this statement consider the following example (see Fig. 1). Suppose it be required to find the route from each starting point (A, B, C, D) that gets to the finish line at minimum total cost. The cost for each leg is indicated by the number adjacent to the leg. At each intermediate junction one must choose to go either straight ahead or to follow a diagonal path.

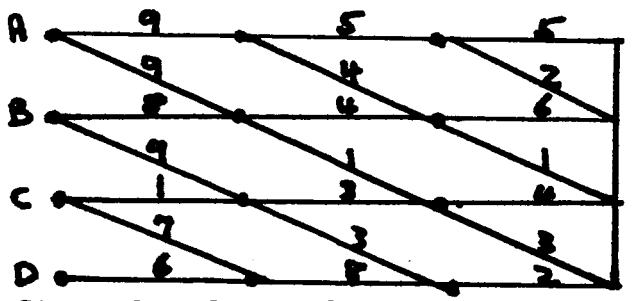


Fig. 1

One approach would be to enumerate all possible paths, calculate the total cost for each one, and find the one with least total cost. The obvious disadvantage in doing this is that the number of paths increases geometrically with the number of stages, and for large problems the situation gets

rapidly out of hand.

Consider on the other hand the following approach justified by the principle of optimality. Starting with stage 3 draw in the paths from each point that have minimum cost. The diagram now looks as follows.

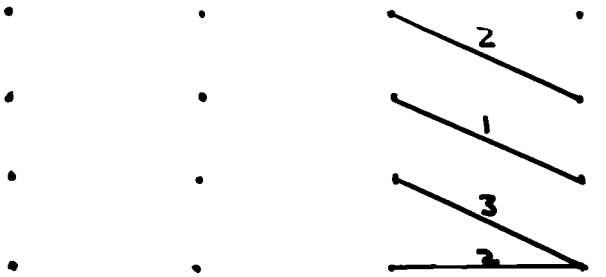


Fig. 2

The other paths at the last stage are eliminated once and for all because once we get to stage 3, no matter at which point, the remainder of the path must be optimal. Using the same reasoning, start at each point from stage 2 and compute the total cost for each possibility, and draw in only the routes with lowest cost. Our decision will be to choose

Min (5+2, 4+1) = 5

Min (4+1, 1+3) = 4

Min (3+3, 3+2) = 5

Min (8+2) = 10

The resulting new figure will be

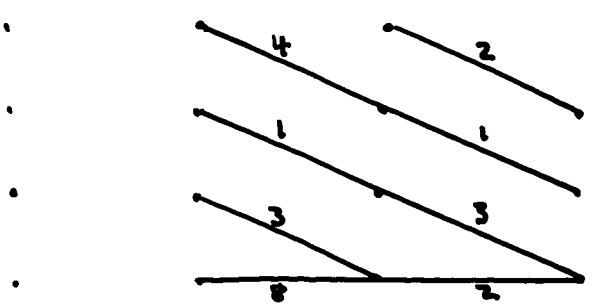


Fig. 3

Note that at each stage the amount of computation is the same, since the optimal routes along with their costs for the remainder of the path have already been computed. This is a simple application of the Principle of Optimality.

Starting finally at stage 1 the optimal paths are obtained.

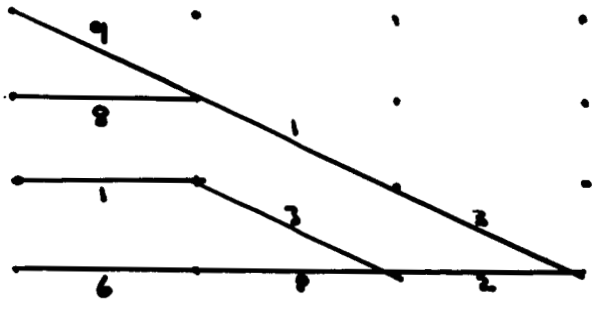


Fig 4

All legs which are not segments of optimal routes have been discarded.

The preceding example is a heuristic explanation of the Principle of Optimality, and does not touch at all on many of the far-reaching consequences of this seemingly simple concept. Rigorous mathematical derivations and many applications may be found in reference [2].

3. Application of Dynamic Programming in the Discrete Determinate Case

The same ideas which were used above may be applied to a more general example involving a difference equation, with decisions to be made at each stage on the selection of one of several vectors. The total cost will be computed on the basis of the route followed and of the values chosen for the control vectors. The difference equation is given by

$$\vec{x}_{i+1} = \vec{g}(\vec{x}_i, \vec{y}_i) \qquad \vec{x}_o = \vec{c} \tag{1}$$

where $\vec{x}_i$ is the state vector at stage i, $\vec{g}$ a vector function, and $\vec{y}_i$ a control vector chosen at stage i. Expanding upon the earlier idea of a total cost attached to a certain route, a cost function F_n will now be defined which depends upon the various states and the decisions made at each stage, that is

$$F_n = F_n(\vec{c}, \vec{x}_1, \ldots, \vec{x}_{n-1}; \vec{y}_o, \vec{y}_1, \ldots, \vec{y}_{n-1}) \tag{2}$$

Lest one become confused by the seeming complexity of the above function, it should be emphasized that F_n is simply a means to evaluate the performance of the system in some quantitative way.

Suppose now that F_n may be separated into a contribution from each stage:

$$F_n = h_o(\vec{c}, \vec{y}_o) + h_1(\vec{x}_1, \vec{y}_1) + \ldots + h_{n-1}(\vec{x}_{n-1}, \vec{y}_{n-1}) \tag{3}$$

To minimize F_n apply the same reasoning that was used in the earlier example. A systematic approach is to first define

$$f_n(\vec{c}) = \underset{\{\vec{y}_i\}}{\text{Min}} \left[h_o(\vec{c}, \vec{y}_o) + h_1(\vec{x}_1, \vec{y}_1) + \ldots + h_{n-1}(\vec{x}_{n-1}, \vec{y}_{n-1})\right] \tag{4}$$

where the notation $\underset{\{\vec{y}_i\}}{\text{Min}}$ means that the minimization is to be taken over all values of the set $\vec{y}_o, \vec{y}_1, \ldots, \vec{y}_{n-1}$. Observe that

$$f_1(\vec{c}) = \underset{\vec{y}_o}{\text{Min}}\, h_o(\vec{c}, \vec{y}_o) \tag{5}$$

and may easily be found by a search over the allowed values of $\vec{y}_o$. This is done for all values of the initial state $\vec{c}$.

To find $f_2(\vec{c})$, use the relationship established through the Principle of Optimality, that is

$$f_n(\vec{c}) = \underset{\vec{y}_o}{\text{Min}} \left[h_o(\vec{c}, \vec{y}_o) + f_{n-1}(\vec{g}(\vec{c}, \vec{y}_o))\right] \tag{6}$$

Once $f_2(\vec{c})$ has been found for all values of $\vec{c}$, the above relationship may then be used to find $f_3(\vec{c})$. Continuing in this manner one finally arrives at the solution to the problem $f_n(\vec{c})$.

4. Application of Dynamic Programming in the Discrete Stochastic Case

The formulation of the preceding section applies just as well in the stochastic case where

$$\vec{x}_{i+1} = \vec{g}(\vec{x}_i, \vec{y}_i, \vec{r}_i) \qquad \vec{x}_o = \vec{c} \tag{7}$$

and $\vec{r}_i$ is a random vector of the white noise type. The only difference is that now the cost function F_n will depend on the random vectors $\vec{r}_i$, where $i = o, 1, \ldots, n-1$. Since F_n is therefore a stochastic quantity it only makes sense to minimize its expected value. Define

$$f_n(\vec{c}) = \underset{\{\vec{y}_i\}}{\text{Min}} \; \underset{\{\vec{r}_i\}}{\text{Exp}} \, [F_n] \tag{8}$$

By the Principle of Optimality,

$$f_n(\vec{c}) = \underset{\vec{y}_o}{\text{Min}} \; \underset{\vec{r}_o}{\text{Exp}} \, [h_i(\vec{c}, \vec{y}_o, \vec{r}_o) + f_{n-1}(\vec{g}(\vec{c}, \vec{y}_o, \vec{r}_o))] \tag{9}$$

The computational procedure is exactly the same except that the expectation over $\vec{r}_o$ must be taken at each stage. The method of this section will now be applied to the first passage problem.

5. The First-Passage Problem

Let R be the "safe" or allowed region of state space. Define the function $h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i)$ by

$$h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i) = 1 \text{ if } \vec{x}_i \varepsilon R \text{ and } \vec{g}(\vec{x}_i, \vec{y}_i, \vec{r}_i) \notin R$$
$$= 0 \text{ otherwise} \tag{10}$$

Then

$$\underset{\{\vec{r}_i\}}{\text{Exp}}\, h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i) = \int h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i) p(\vec{r}_i) d\vec{r}_i \tag{11}$$

is the probability that failure occurs in the transition from the ith to the (i + 1)st stage given an initial state $\vec{x}_i$ and control vector $\vec{y}_i$. It is simply the sum of the probabilities $p(\vec{r}_i)d\vec{r}_i$ over those values of $\vec{r}_i$ which cause $\vec{x}_{i+1}$ to leave R. This requirement together with the absorption condition

$$\vec{x}_{i+1} = \vec{g}(\vec{x}_i, \vec{y}_i, \vec{r}_i) = \vec{x}_i \qquad \text{if } \vec{x}_i \notin R \tag{12}$$

allows failure to occur at most once in any evolution of the system. The total probability of failure over n stages is

$$\underset{\{\vec{r}_i\}}{\text{Exp}} \sum_{i=o}^{n-1} h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i) \tag{13}$$

The control problem now takes on the form of finding a sequence of $\vec{y}$'s, subject to various constraints, which minimizes (13).

As before, the n-stage cost function, dependent on the initial condition $\vec{c}$, is given by

$$f_n(\vec{c}) = \underset{\{\vec{y}_i\}\{\vec{r}_i\}}{\text{Min Exp}} \sum_{i=o}^{n-1} h_i(\vec{x}_i, \vec{y}_i, \vec{r}_i) \qquad (14)$$

It is the minimum probability of system failure over n stages. For n = 1,

$$f_1(\vec{c}) = \underset{\vec{y}_o}{\text{Min}} \underset{\vec{r}_o}{\text{Exp}} h_o(\vec{c}, \vec{y}_o, \vec{r}_o) \qquad (15)$$

The Principle of Optimality yields the recurrence relationship

$$f_n(\vec{c}) = \underset{\vec{y}_o}{\text{Min}} \underset{\vec{r}_o}{\text{Exp}} [h_o(\vec{c}, \vec{y}_o, \vec{r}_o) + f_{n-1}(\vec{g}(\vec{c}, \vec{y}_o, \vec{r}_o))] \qquad (16)$$

6. Case of No Control

To set the ground for an actual example of the method, consider the case of no control. Assume without loss of generality that $\vec{y}$ is identically zero. Then the cost function

$$f_n(\vec{c}) = \underset{\{\vec{r}_i\}}{\text{Exp}} \sum_{i=o}^{n-1} h_i(\vec{x}_i, o, \vec{r}_i) \tag{17}$$

gives the failure probability over n stages, and

$$f_1(\vec{c}) = \underset{\vec{r}_o}{\text{Exp}}\, h_o(\vec{c}, o, \vec{r}_o) \tag{18}$$

The recurrence relationship is

$$f_n(\vec{c}) = \underset{\vec{r}_o}{\text{Exp}} [h_o(\vec{c}, o, \vec{r}_o) + f_{n-1}(\vec{g}(\vec{c}, o, \vec{r}_o))] \tag{19}$$

Equations (17), (18), and (19) are identical with (14), (15), and (16) except for the choice of $\vec{y}_i = \vec{o}$ and the omission of the minimization. So in the special case of no control, the analysis simply yields the first passage probabilities. The meaning of Equation (19) may be made clear by considering separately each of the two terms appearing on the right hand side:

$$\begin{aligned} f_n(\vec{c}) &= \underset{\vec{r}_o}{\text{Exp}} [h_o(\vec{c}, \vec{o}, \vec{r}_o)] + \underset{\vec{r}_o}{\text{Exp}} [f_{n-1}[\vec{g}(\vec{c}, \vec{o}, \vec{r}_o)]] \\ &= f_1(\vec{c}) + \int d\vec{r}_o p(\vec{r}_o)\, f_{n-1}[\vec{g}(\vec{c}, o, \vec{r}_o)] \end{aligned} \tag{20}$$

The first term represents the probability of failure during the first stage of the process. The second term amounts to finding the probability that $\vec{r}_o$ will assume a certain value, multiplying by the probability of failure over the last n-1 stages given that $\vec{r}_o$ assumes this value, and integrating over all the values which $\vec{r}_o$ may assume. The second term is therefore the probability of failure over the last n-1 stages of the process given that the system has not failed during the first stage.

AN EXAMPLE

THE SYSTEM

Since so few first-passage-time problems have been solved analytically, it is difficult to find reliable solutions to use for the sake of comparison. It was finally decided to consider the problem of the Brownian motion of a particle in a viscous medium. For those more concerned with mechanical systems, it should be pointed out that this is precisely the same problem as a mass-damper system subject to white-noise excitation as shown in Figure 5 below.

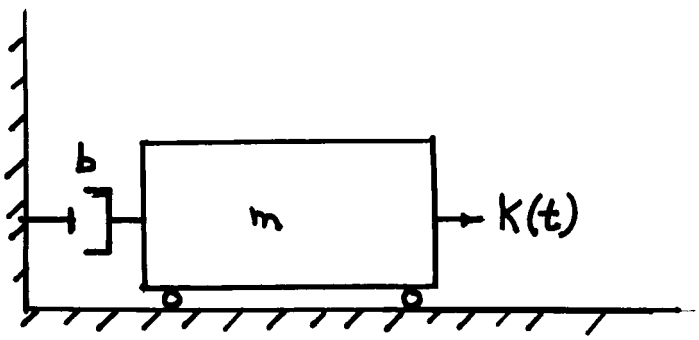


Fig. 5

Following the notation of Wang and Uhlenbeck[5], the equation of motion will be

$$m\left(\frac{dv}{dt}\right) + bv = K(t) \qquad (21)$$

where m = mass

v = velocity

b = coefficient of damping

and K(t) is the fluctuating force of which the average value

is zero and which has a practically white spectrum.

Then equation (21) may be written in the form

$$\left(\frac{dx}{dt}\right) + \beta x = Q(t) \tag{22}$$

The spectral density of Q(t) is taken to be 4D and Q(t) is assumed to be Gaussian with zero mean. Thus,

$$\langle Q(t) \rangle = 0 \tag{23}$$

$$R_{QQ}(t_1 - t_2) = \langle Q(t_1)Q(t_2) \rangle = 2D\delta(t_1 - t_2) \tag{24}$$

DISCRETE REPRESENTATION OF SYSTEM

The differential equation of motion may be considered in terms of the discrete approximation

$$\frac{x_{i+1} - x_i}{\Delta t} + \beta x_i = Q(t_i) \qquad (25)$$

This means that it is necessary to find a discrete approximation to the random function Q(t) as well.

DISCRETE APPROXIMATION TO WHITE NOISE

Following Cook[6], consider the random function composed of statistically independent square pulses with a Gaussian amplitude distribution

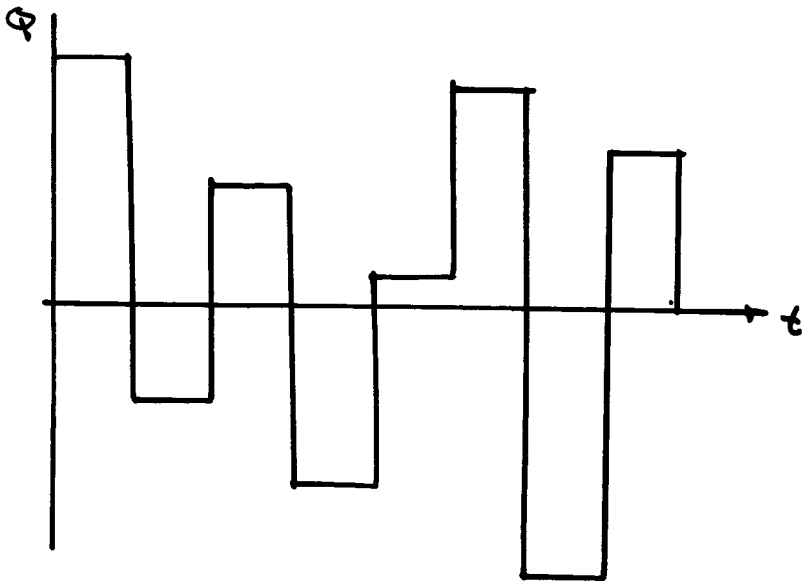


Fig 6

The auto correlation function of this time series is

$$R_{QQ}(\tau) = \sigma_Q^2[1 - |\frac{\tau}{t_o}|] \; ; \; |\tau| < t_o$$
$$= 0 \qquad\qquad ; \; |\tau| > t_o \tag{26}$$

Recalling our original assumption on Q(t)

$$R_{QQ}(\tau) = 2D\delta(\tau) \qquad \tau = t_1 - t_2 \tag{27}$$

and noting that

$$\lim_{t_o \to o} \frac{\left[1 - \left|\frac{\tau}{t_o}\right| \right]}{t_o} = \delta(\tau) \tag{28}$$

$$\text{2D} \lim_{t_o \to o} \left[\frac{1 - \left|\frac{\tau}{t_o}\right|}{t_o} \right] = \sigma_Q^2 \left[1 - \left|\frac{\tau}{t_o}\right| \right] \tag{29}$$

We conclude that for small t_o

$$\frac{2D}{t_o} \approx \sigma_Q^2 \tag{30}$$

For an evaluation of this assumption, see Figure (3) in the Appendix, reprinted from Cook's dissertation[6].

NUMERICAL EVALUATION OF $f_n(c)$

Start with the equation

$$f_n(c) = \int_{-\infty}^{\infty} p(r)h(c, o, r)dr + \int_{-\infty}^{\infty} f_{n-1}(g(c, o, r)\, p(r)dr \quad (31)$$

Now form a set of grid points to store the value of the state variable. Note that if $c \notin R$, the first integral above is zero by the definition of h. The second integral is also zero since, by the definition of g and h, if $c \notin R$ then

$$\begin{aligned} f_1(c) &= \int_{-\infty}^{\infty} p(r)h(c, o, r)dr = 0 \\ f_2(c) &= 0 + \int_{-\infty}^{\infty} f_1(c)p(r)dr = 0 \qquad (32) \\ &\vdots \\ f_{n-1}(c) &= 0 \end{aligned}$$

Hence, it is not necessary to extend the grid to those values for which $c \notin R$.

Since $f_{n-1}(c)$ has only been computed for values of c lying on a grid of points and one is now required to compute $f_{n-1}(g(c, o, r))$, it would be convenient if g would fall exactly on the same grid points as c so that no interpolation would be necessary. The assumption is therefore made that $f_{n-1}(c)$ is constant for $x - \frac{\Delta x}{2} \leq c \leq x + \frac{\Delta x}{2}$
The second integral may then be approximated by

$$\sum_{i=1}^{m} f_{n-1}(x_i) \int_{r_i - \Delta r/2}^{r_i + \Delta r/2} p(r')dr' \qquad (33)$$

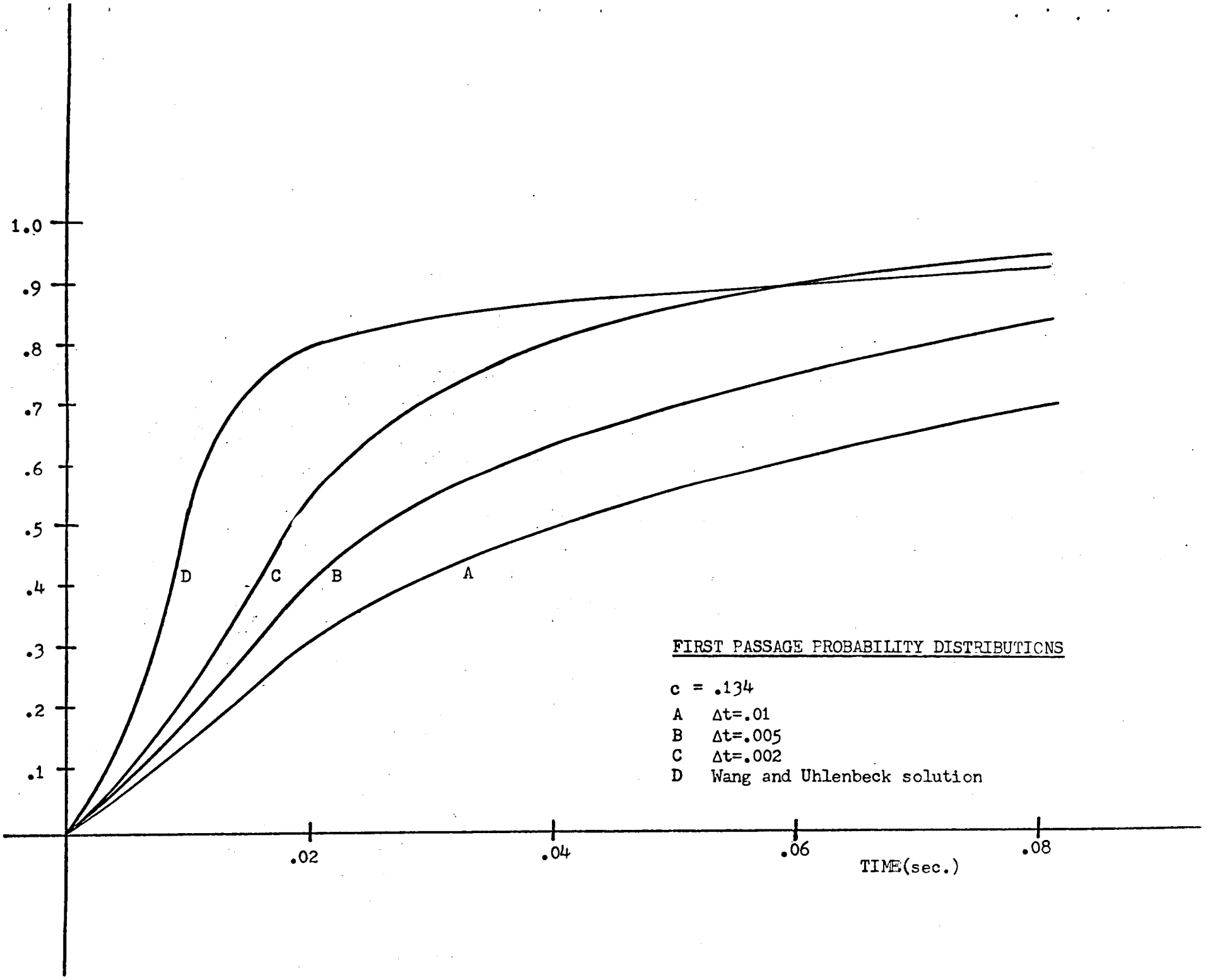
where x_i denotes the value of x at the i th grid location and r_i is found by inverting $x_i = g(c, o, r_i)$
Since r is assumed to have a Gaussian distribution, integrals of the type $\int_a^b p(r)dr$ may be numerically evaluated from a polynomial approximation of the error function.

In the usual theory of the Brownian motion based upon the ordinary diffusion equation $\frac{\partial P}{\partial t} = \frac{D\partial^2 P}{\partial x^2}$ a few first-passage-time problems have been considered and solved by Smoluchowski and others. Wang and Uhlenbeck[5] state that the method of Smoluchowski can also be used for a one-dimensional Gaussian Markov process x(t) and that one can show that the probability density w(c, t)dt of the first-passage time to reach x = 0 starting from c is given by

$$w(c, t)dt = c\left(\frac{2\beta}{\pi D}\right)^{1/2} \exp\left(\frac{-\beta c^2}{2D} z^2\right)dz \qquad (34)$$

where $z = e^{-\beta t}\,(1-e^{-2\beta t})^{-1/2}$

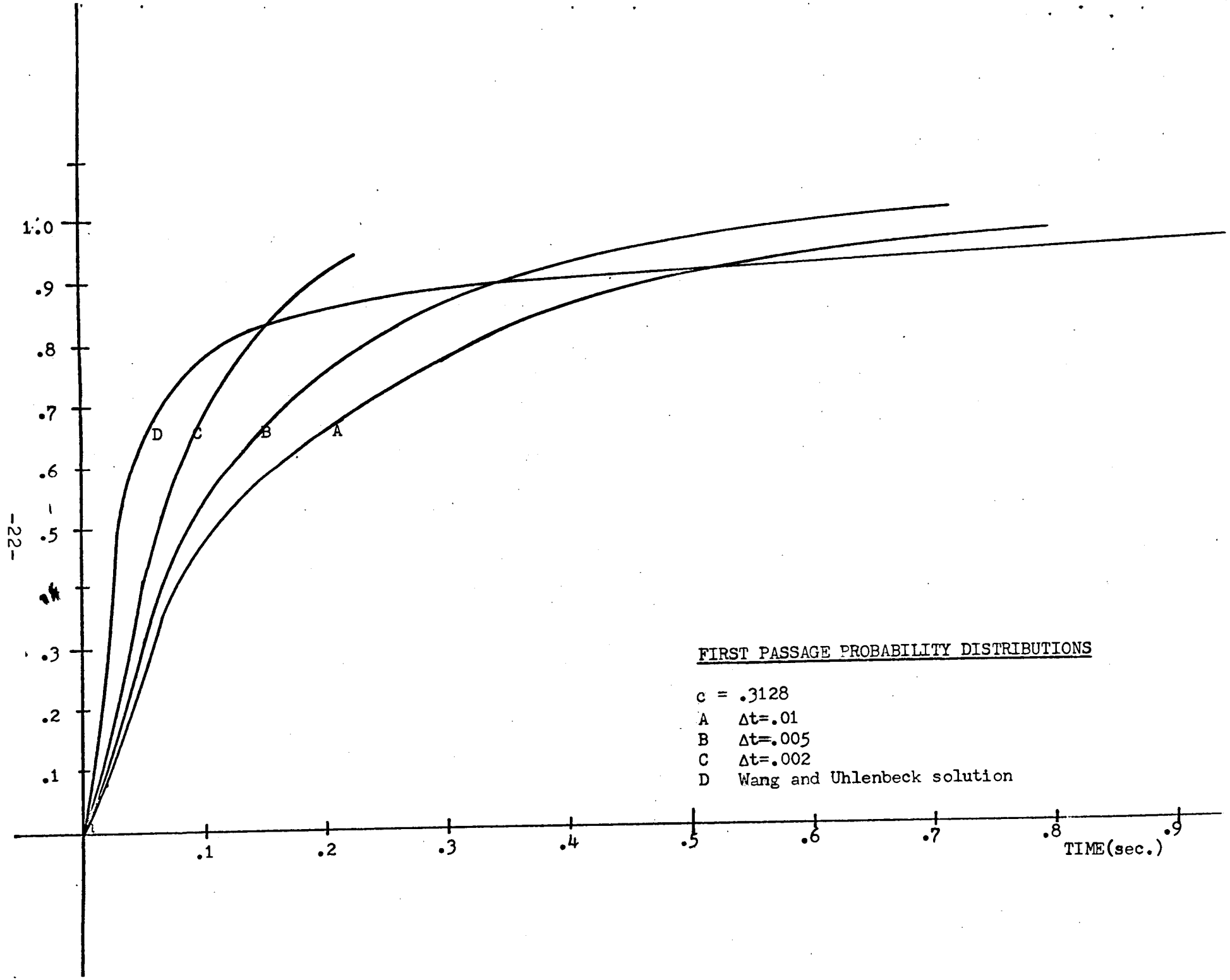
In the graphs which follow, the distribution function labeled (D) was obtained from the above equation by numerically integrating w(c, t)dt for various values of c and for $\beta = .5$. It should be mentioned that the normalizing condition on the probability distribution requires a normalizing constant of -1 if $c > 0$.

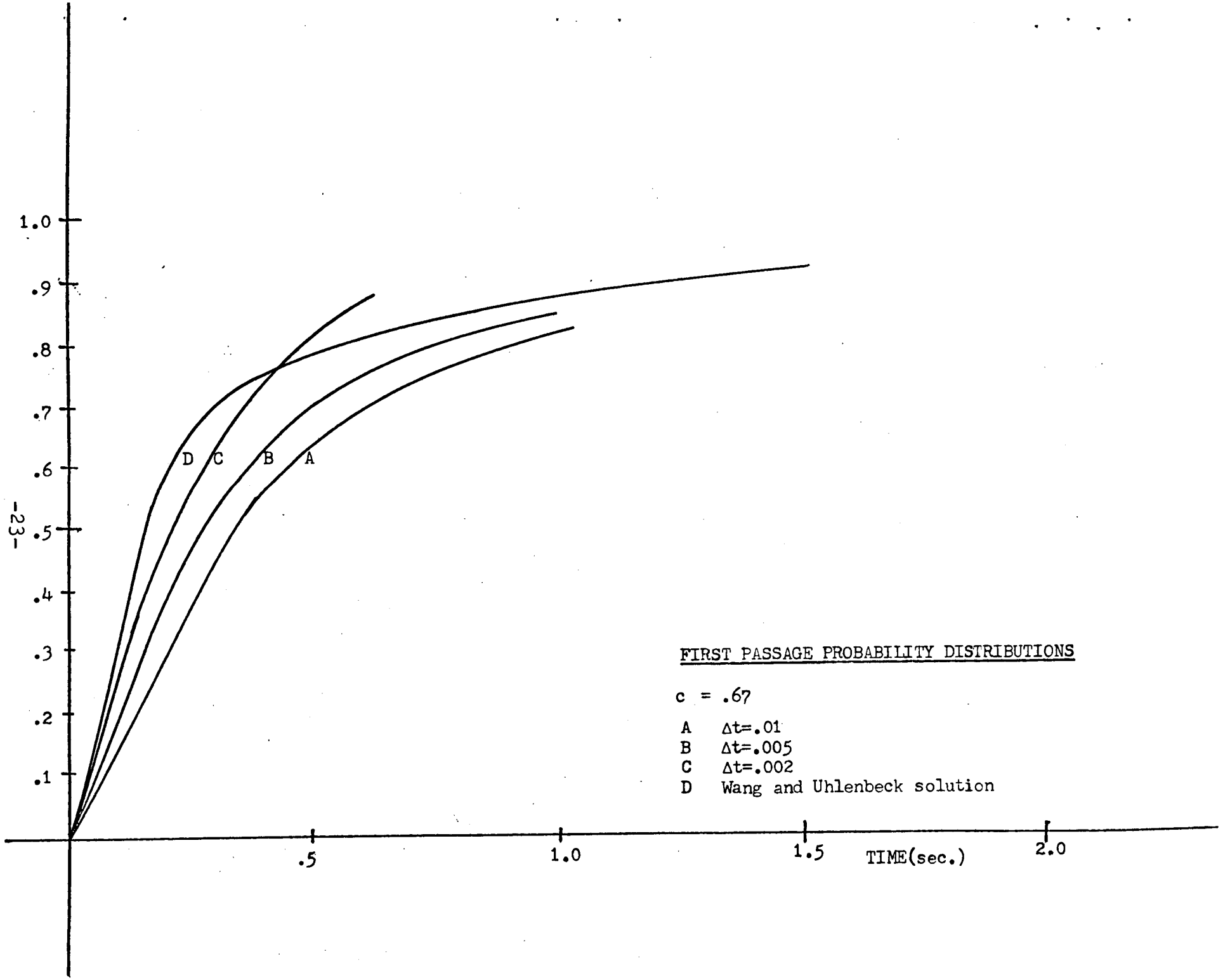


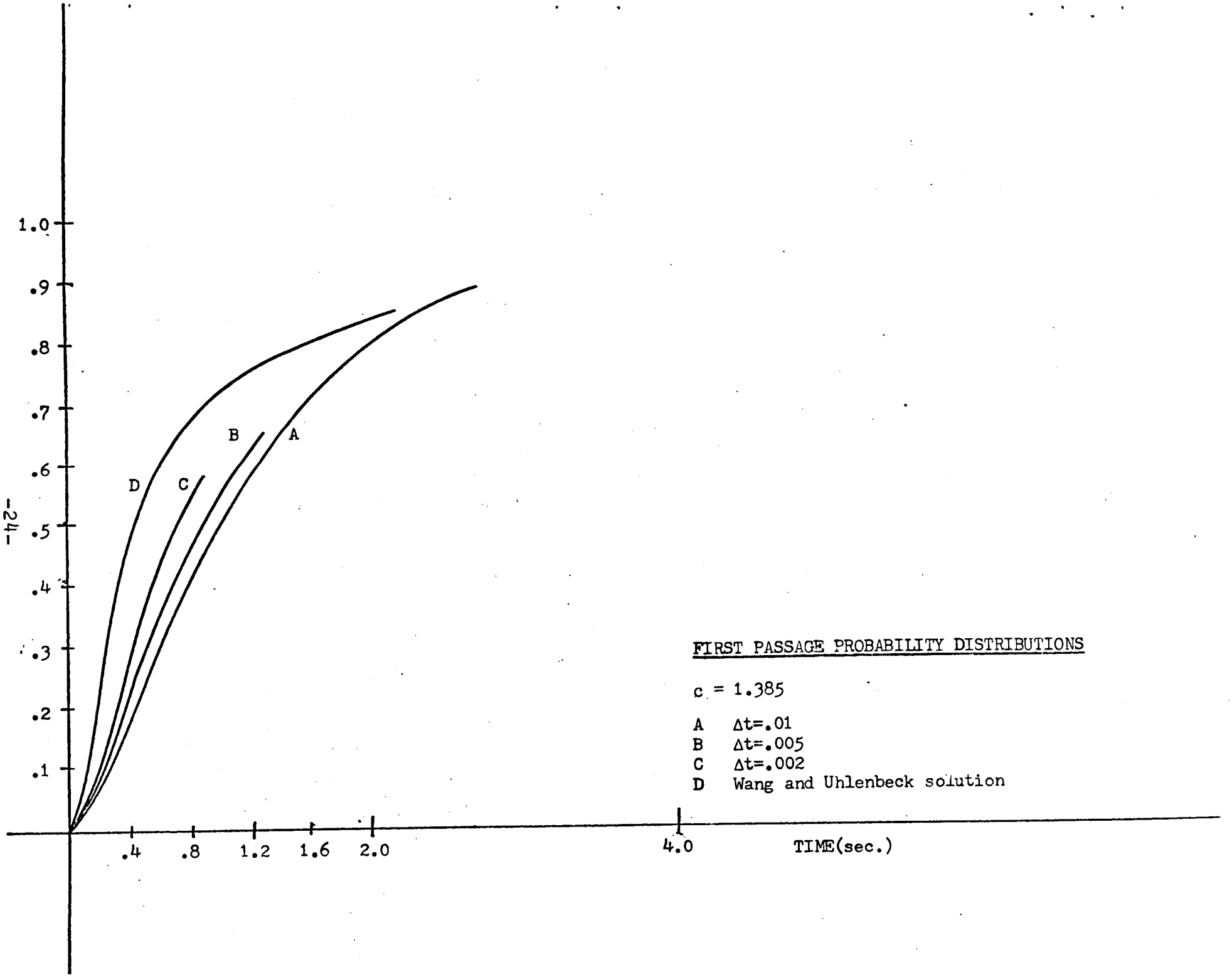
FIRST PASSAGE PROBABILITY DISTRIBUTIONS

c = .134

A $\Delta t = .01$
B $\Delta t = .005$
C $\Delta t = .002$
D Wang and Uhlenbeck solution


1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
D
C
B
A
.1
.2
.3
.4
.5
.6
.7
.8
.9
TIME(sec.)
FIRST PASSAGE PROBABILITY DISTRIBUTIONS
c = .3128
A Δt=.01
B Δt=.005
C Δt=.002
D Wang and Uhlenbeck solution

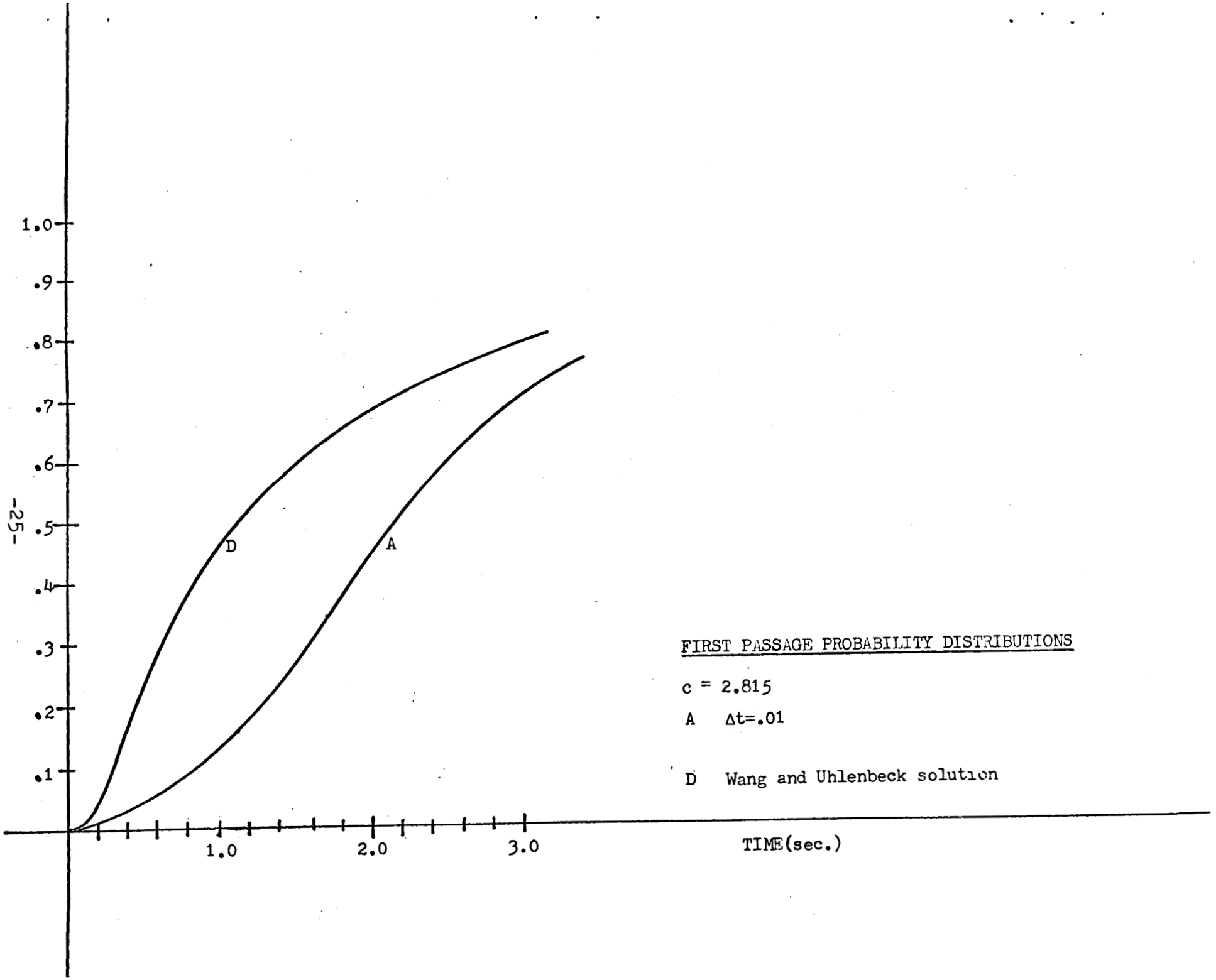

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
D
C
B
A
.5
1.0
1.5
2.0
TIME(sec.)
FIRST PASSAGE PROBABILITY DISTRIBUTIONS
c = .67
A Δt=.01
B Δt=.005
C Δt=.002
D Wang and Uhlenbeck solution



FIRST PASSAGE PROBABILITY DISTRIBUTIONS

c = 1.385

A $\Delta t = .01$
B $\Delta t = .005$
C $\Delta t = .002$
D Wang and Uhlenbeck solution



FIRST PASSAGE PROBABILITY DISTRIBUTIONS

c = 2.815

A $\Delta t = .01$

D Wang and Uhlenbeck solution

EXTENSION OF RESULTS

The method outlined in this paper lends itself quite easily to some important generalizations. Note that in the example treated, the region R was chosen so that results could be compared with the solution given by Wang and Uhlenbeck. Yet any region in state space might just as well have been chosen without significant complications.

Another possibility would be to introduce time-varying coefficients into the system equation and again no serious difficulties would arise.

One might also consider the case of non-white noise. Since most noise can be thought of as the result of filtering white noise (if necessary through a time-varying filter), one has only to increase the dimensionality of the system. This offers no real conceptual difficulty although certain computational problems may arise. For example, as the system dimensionality increases, it finally becomes impossible to store a sufficiently accurate set of values of the return function on a grid of points in the core memory. One must then resort to one of a variety of special techniques.

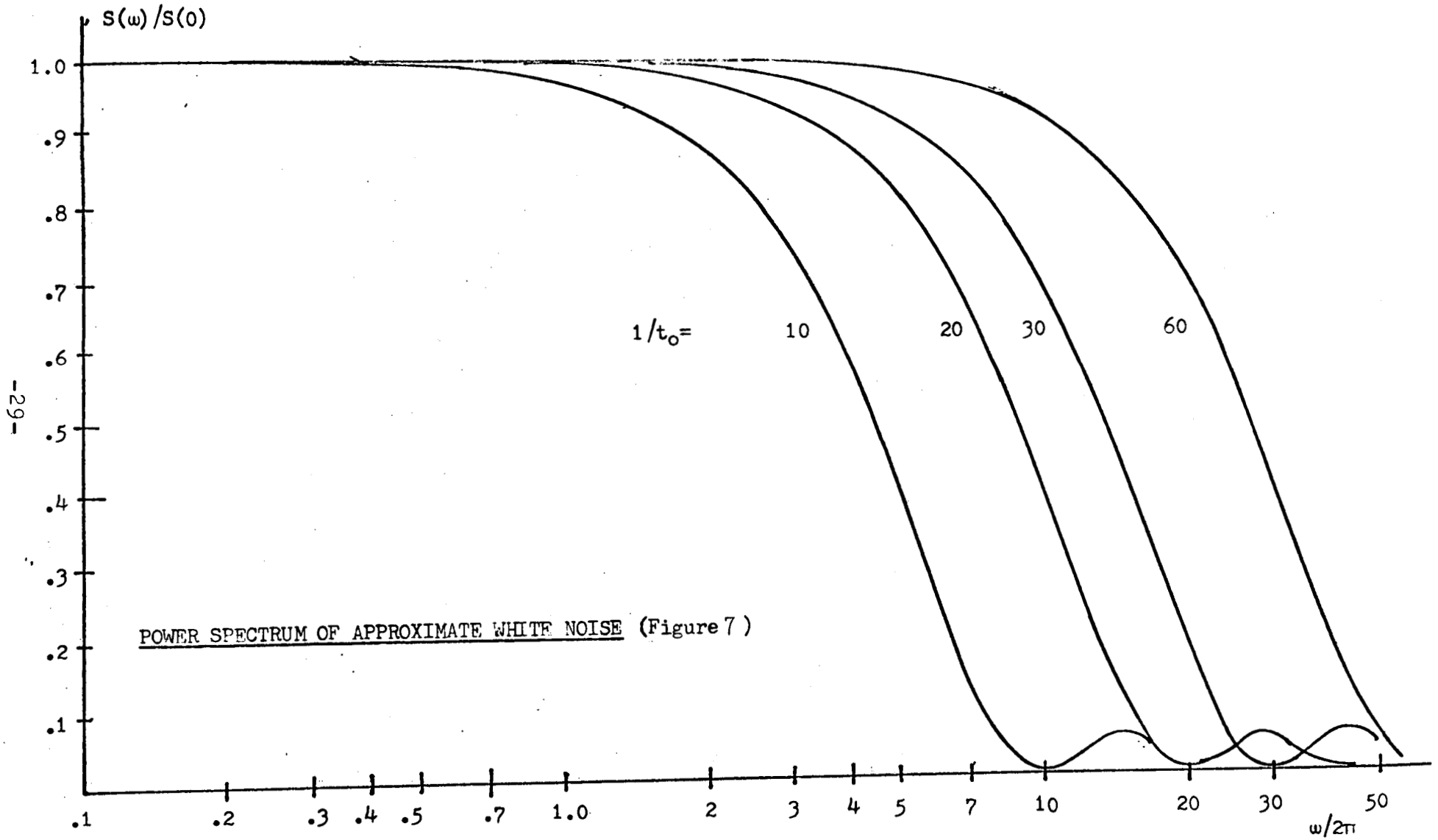
Of these special techniques the most useful is probably a polynomial approximation scheme in which the cost function is represented by a polynomial in the components of the state vector. Not only is core storage tremendously reduced

but in addition the actual number of computations is cut down, and it may even be possible to perform some of the necessary integrations analytically. The details of this method are discussed by Bellman in reference 2.

APPENDIX

The graph which follows represents the normalized power-spectrum of approximate white noise obtained by Cook[6]. It is based upon the auto-correlation function given in the text in the section on white noise.

Calculations to obtain the first passage probabilities were done on the CDC-6600 computer at New York University.



POWER SPECTRUM OF APPROXIMATE WHITE NOISE (Figure 7)

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