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Volume I

SYNTHESIS OF STRUCTURAL DAMPING

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by

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APPROVED BY: [signature]

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FOREWORD

A major portion of the work described in this report was carried out by Lockheed Missiles & Space Company for the Marshall Space Flight Center under Contract NAS8-20387.

This report is Volume I of three volumes which comprise the Final Report under Contract NAS8-20387, as follows:

- Volume I - "Synthesis of Structural Damping," by C. S. Chang and R. E. Bieber (LMSC/HREC A783975)
- Volume II - "Nonlinear Dynamic Analyses," by R. O. Hultgren (LMSC/HREC A783963)
- Volume III - "A Study of Hereditary Springs in Relation to Hysteretic Damping," by G. A. Ramerez (LMSC/HREC A783201)

SUMMARY

For a structure exhibiting hysteresis behavior, the force-deflection relationship under arbitrary loading can be derived in terms of four physical properties. This relationship relies upon geometrical similarities between the loading and the unloading branches of the hysteresis loop and a genetic curve obtained by considering the rate of energy dissipation during steady-state oscillations. A damping law is introduced to obtain results which show agreement with experimental data for a 1/5-scale structural model of the Saturn I launch vehicle.

CONTENTS

Section 1
INTRODUCTION

A common cause of difficulty in obtaining consistant structural damping data is their dependency on the amplitude of vibration. An acknowledged source of nonlinearity is the hysteretic relationship between force and deformation.

Caughey (Reference 1), studied a linear hysteresis damping model which can be made to satisfy the requirement that the energy dissipation rate be independent of the frequency in steady-state vibrations. In so doing, however, one finds that his model displays a force-displacement relationship that varies with the frequency of oscillation. The current study extends the bilinear hysteresis model of Caughey (Reference 2) to the curvelinear case in the manner of Whiteman (Reference 3) who used it for material hysteresis.

This investigation is restricted to applications in which the structure can be characterized by individual vibration modes, and where the representation of each mode by a one-degree-of-freedom system is essentially correct. The hysteresis relationship itself, however, is quite general and may be used in other applications.

Section 2
HYSTERETIC FORCE-DEFLECTION RELATIONSHIP

A model consisting of a large number of elastoplastic elements (i.e., springs in series with Coulomb friction units) can be used to construct the essential properties of a hysteretic force-deflection relationship for studying structural damping. This model, outlined by Timoshenko (Reference 4, 1940), was used by Whiteman (References 3 and 5, 1957), in connection with his study of metal fatigue, and later by Rosenbluth and Herrera (Reference 6, 1964), who cited Tanabashi and Keneta (Reference 7, 1962).

The mathematical model of hysteresis has the following properties:

- Variation of the force, F, or the displacement, X, between fixed limits $(\overline{F}, \overline{X})$ and $(-\overline{F}, -\overline{X})$ follows two distinct paths which intersect at the limiting points, see Figure 1.
- Using the sign convention of Figure 1, the hysteresis loop is always traversed in the clockwise direction, and the area enclosed by the loading and the unloading branches of the hysteresis loop represents D_o, the amount of energy dissipated per cycle.
- The locus of $(\overline{F}, \overline{X})$ for hysteresis loops of various amplitudes coincides with the initial loading curve in the positive direction, the locus of $(-\overline{F}, -\overline{X})$ coincides with the initial loading curve in the negative direction.
- The loading branch of the hysteresis loop between $(\overline{F}, \overline{X})$ and $(-\overline{F}, -\overline{X})$ is geometrically similar to the initial positive loading curve but is scaled in both directions, and is displaced from the origin so that it begins at $(-\overline{F}, -\overline{X})$ and ends at $(\overline{F}, \overline{X})$ (see Equation 6). The construction of the unloading branch is similar (see Equation 8).
- The relationship between force and displacement amplitudes, i.e., the shape of the $\overline{F}$ vs $\overline{X}$ curve, can be determined by the manner in which D_o is dependent upon $\overline{X}$. From geometrical considerations it may be shown that the area between the $\overline{F}$ vs $\overline{X}$ curve and the straight line from the origin to the point $(\overline{F}, \overline{X})$ is $1/8\ D_o$. Referring to Figure 2, it is seen that

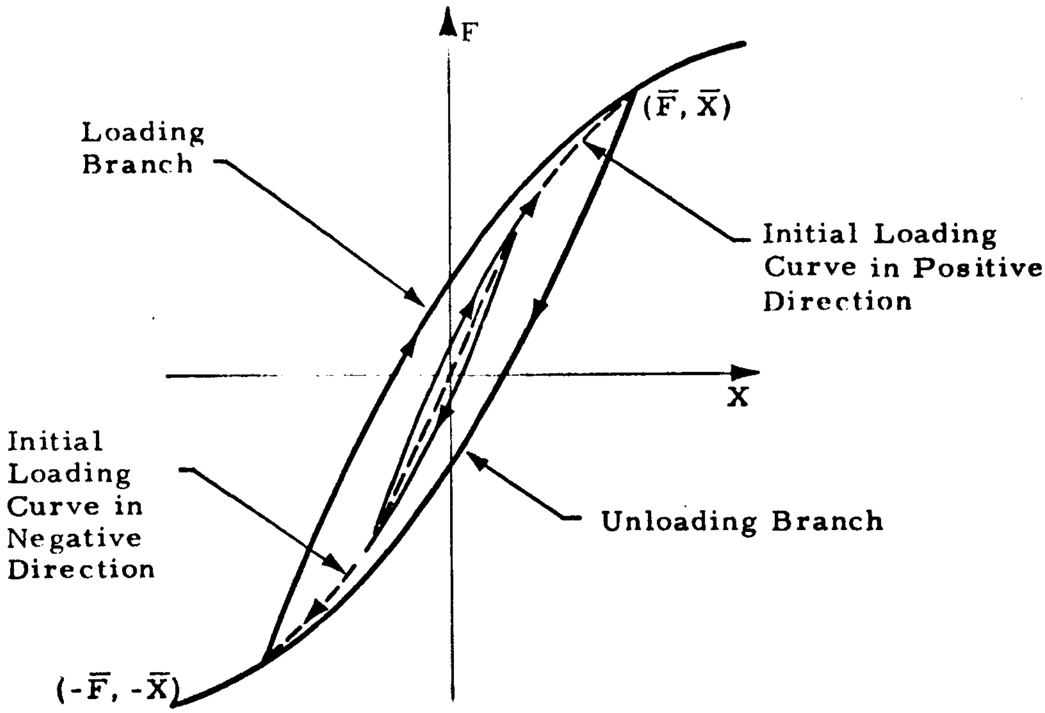

Figure 1 - Typical Hysteresis Loops

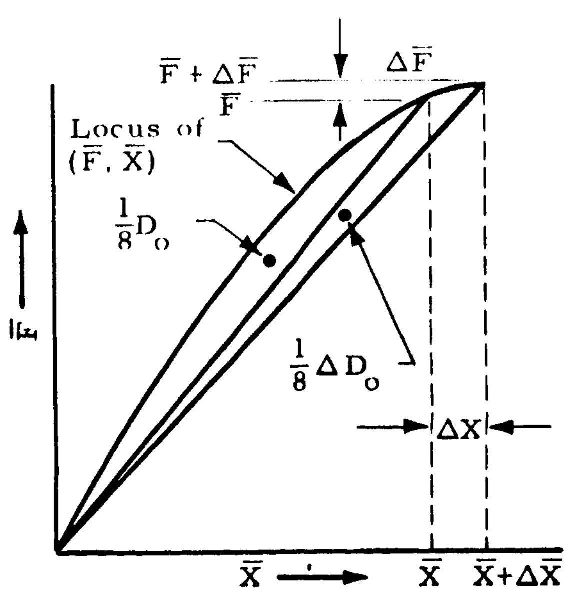

Figure 2 - Derivation of the Equation of $(\bar{F}, \bar{X})$

$$\frac{1}{8} D_o = \int_o^{\overline{X}} \overline{F}\, d\overline{X} - \frac{1}{2} \overline{F}\, \overline{X},$$

and, corresponding to a small change $\Delta\overline{X}$,

$$\frac{1}{8} D_o + \frac{1}{8} \Delta D_o = \int_o^{\overline{X}+\Delta\overline{X}} \overline{F}(\overline{X})\, d\overline{X} - \frac{1}{2} (\overline{F} + \Delta\overline{F})(\overline{X} + \Delta\overline{X}).$$

or,

$$\frac{1}{8} \Delta D_o = \overline{F} - \frac{1}{2} (\overline{F}\Delta\overline{X} + \overline{X}\Delta\overline{F} + \Delta\overline{F}\,\Delta\overline{X}).$$

and, in the limit as $\Delta\overline{X} \rightarrow 0$,

$$\overline{F} - X \frac{d\overline{F}}{d\overline{X}} = \frac{1}{4} \frac{d D_o}{d\overline{X}} . \tag{1}$$

The initial condition is $\overline{F}(0) = 0$.

- The general solution of Equation (1) can be written in the form

$$\overline{F}(\overline{X}) = K_o \overline{X} + f(\overline{X}); \tag{2}$$

where the term $K_o\overline{X}$ is the complimentary solution, and is independent of D_o, while $f(\overline{X})$ is the particular solution with

$$f(0) = 0 . \tag{3}$$

- Let X_p be the point where

$$\frac{d\overline{F}}{d\overline{X}} = 0 . \tag{4}$$

For displacements greater than $\overline{X}_p$,

$$F = \overline{F}_p . \tag{5}$$

- The loading path from $(-\overline{F}, -\overline{X})$ to $\overline{F}, \overline{X})$ on the hysteresis loop is described by the equation

$$\frac{1}{2}(F+\overline{F}) = K_o\left(\frac{\overline{X}+\overline{X}}{2}\right) + f\left(\frac{\overline{X}+\overline{X}}{2}\right), \qquad (6)$$

or equivalently,

$$F = K_o X + 2 f\left(\frac{X+\overline{X}}{2}\right) - f(\overline{X}); \qquad (7)$$

similarly, the unloading branch of the loop is described by the equation

$$F = K_o X - 2 f \frac{\overline{X} - X}{2} + f(\overline{X}). \qquad (8)$$

- The constant K_o in the solution of F(X) may be identified with the small-amplitude natural frequency of the corresponding linear, undamped system.

Experimental data have been collected on the bending variations of a structural model of the Saturn I launch vehicle*. The energy dissipated per cycle, D_o, for an extended range of response amplitudes is shown in Figure 3 for the second free-free mode. These experimental results lead to the following general form for D_o:

$$\begin{aligned} D_o &= 0, && \overline{X} \le X_o; \\ D_o &= J_n(\overline{X} - X_o)^n, && X_o \le \overline{X} \le \overline{X}_p; \\ D_o &= J_n(\overline{X}_p - X_o)^n + 4\overline{F}_p(\overline{X} - \overline{X}_p), && \overline{X}_p \le \overline{X}, \end{aligned} \qquad (9)$$

where all quantities are non-negative, and X_o is that displacement amplitude below which hysteresis damping is zero; n is an exponent and is not necessarily an integer; J_n is a constant of proportionality. Two special cases

*Obtained under Contract NAS8-20088, "Experimental Damping Studies," by Lockheed Missiles & Space Company, Huntsville Research & Engineering Center.

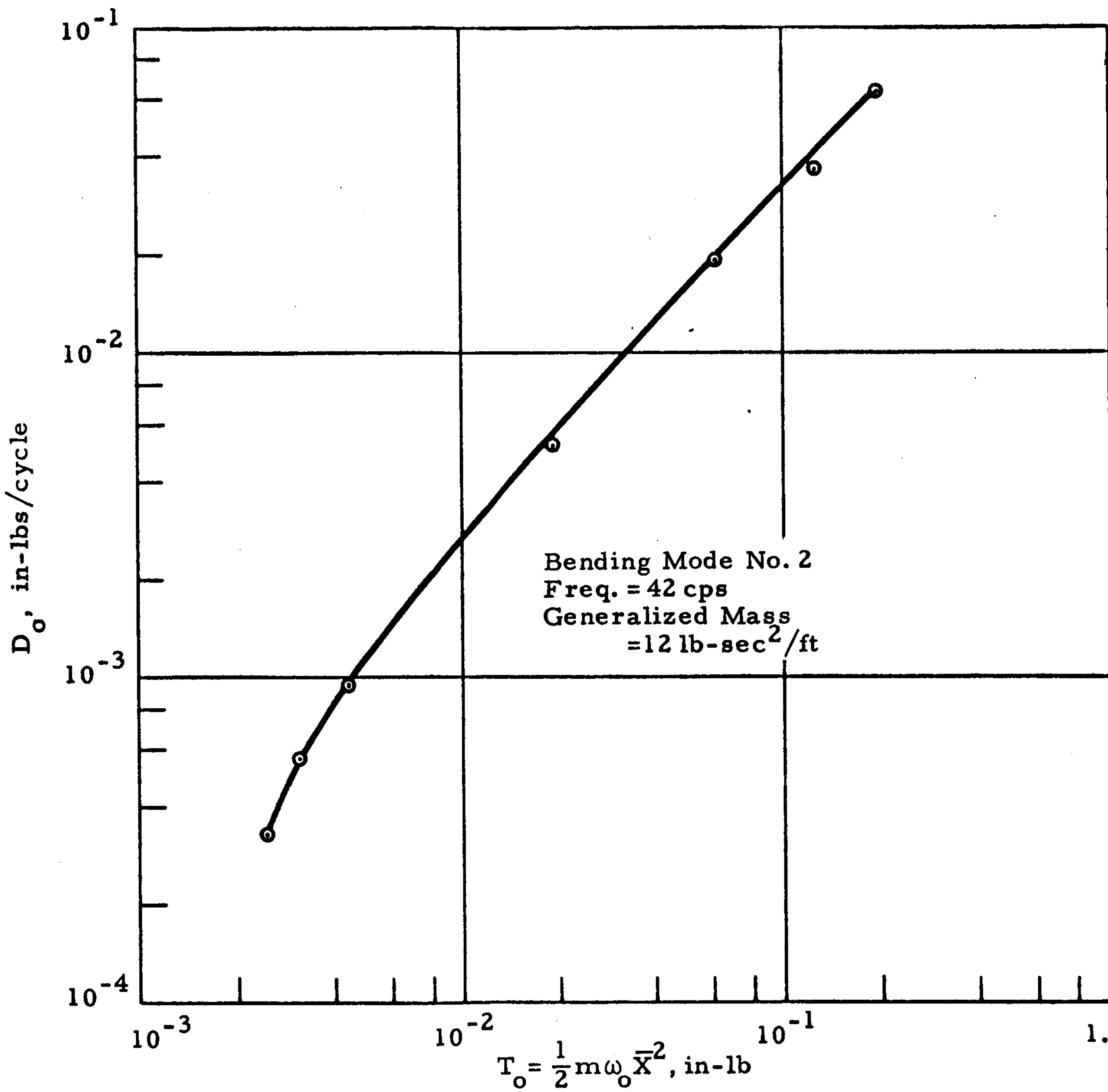

Figure 3 - Measured Damping Law for 1/5-Scale Model of Saturn I Vehicle

of interest are: (a) $n = 2.0$ and $X_o > 0^*$, so that

$$D_o = 0, \qquad 0 \le \bar{X} \le X_o;$$
$$D_o = J_2 (\bar{X} - X_o)^2, \qquad X_o \le \bar{X} \le \bar{X}_p;$$
$$D_o = J_2 (\bar{X}_p - X_o)^2 + 4 F_p (\bar{X} - \bar{X}_p), \qquad \bar{X}_p \le \bar{X}, \tag{10}$$

and (b) $n > 2.0$, $X_o = 0$, so that

$$D_o = J_n \bar{X}^n, \qquad 0 \le \bar{X} \le \bar{X}_p;$$
$$D_o = J_n \bar{X}_p{}^n + 4 F_p (\bar{X} - \bar{X}_p), \qquad \bar{X}_p \le \bar{X}. \tag{11}$$

For case (a), when $X \gg X_o$, the rate of dissipation, D_o, is nearly proportional to the peak stored energy and independent of frequency, approaching the case of "linear structural damping" that one often encounters in the structural vibration literature,; while case (b) bears a close similarity with material damping laws (Reference 8).

The solution of Equation (1) for case (a) above is

$$\bar{F} = K_o \bar{X} \qquad 0 \le \bar{X} \le X_o;$$
$$\bar{F} = K_o \bar{X} - J_2 \bar{X} \ln \frac{\bar{X}}{\bar{X}_o} + J_2 (\bar{X} - X_o), \qquad X_o \le \bar{X} \le \bar{X}_p;$$
$$\bar{F} = F(\bar{X}_p), \qquad \bar{X}_p \le \bar{X}. \tag{12}$$

The point $\bar{X}_p$ may be determined by differentiating Equation (12) and setting the result to zero, and is given by the expression

$$\bar{X}_p = e^{K_o/J_2} X_o \tag{13}$$

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*The case $X_o = 0$ must be ruled physically impossible since it leads to a dynamic system with an infinite small-amplitude natural frequency.

For typical structures, the exponent K_o/J_2 is usually very large, so that the assumption

$$\bar{X}_p \gg \bar{X}$$

can be made without loss of generality.

Let

$$H(X, \dot{X}) = F - K_o X \tag{14}$$

Substituting Equations (7), (8) and (12) into Equation (14), the following are obtained

$$\text{for } \bar{X} \le X_o, \; H(X, \dot{X}) = 0; \tag{15}$$

and for $X_o \le \bar{X} \le \bar{X}_p$,

$$H(X, \dot{X} > 0) = J_2 \bar{X} \ln \frac{\bar{X}}{X_o} - J_2(\bar{X} - X_o), \quad \frac{\bar{X} + X}{2} \le X_o \; ; \tag{15a}$$

$$H(X, \dot{X} > 0) = 2 J_2 \left(\frac{\bar{X} + X}{2}\right) \ln \frac{\bar{X} + X}{2X_o} - 2 J_2 \left(\frac{\bar{X} + X}{2} - X_o\right)$$

$$+ J_2 \bar{X} \ln \frac{\bar{X}}{X_o} - J_2(\bar{X} - X_o), \quad \frac{\bar{X} + X}{2} \ge X_o \; ; \tag{15b}$$

$$H(X, \dot{X} < 0) = - J_2 \bar{X} \ln \frac{\bar{X}}{X_o} + J_2(\bar{X} - X_o), \quad \frac{\bar{X} - X}{2} \le X_o \; ; \tag{15c}$$

$$H(X, \dot{X} < 0) = -2 J_2 \left(\frac{\bar{X} - X}{2}\right) \ln \frac{\bar{X} - X}{2X_o} + 2 J_2 \left(\frac{\bar{X} - X_o}{2}\right) - X_o$$

$$- J_2 \bar{X} \ln \frac{\bar{X}}{X_o} + J_2(\bar{X} - X_o), \quad \frac{\bar{X} - X}{2} \ge X_o \, . \tag{15d}$$

Similarly, for case (b), $D_o = J_n\bar{X}^n$,

$$\bar{F} = K_o\bar{X} - \frac{1}{4}\frac{n}{(n-2)} J_n\bar{X}^{n-1}, \quad n > 2, \quad \bar{X} < \bar{X}_p , \tag{16}$$

where:

$$X_p = \left[\frac{4 K_o(n-2)}{J_n n(n-1)}\right]^{\frac{1}{n-2}} \tag{17}$$

$$H(X, \dot{X} > 0) = -\frac{1}{2}\frac{n}{(n-2)} J_n\left(\frac{\bar{X}+X}{2}\right)^{n-1} + \frac{1}{4}\frac{n}{(n-2)} J_n(\bar{X})^{n-1} , \tag{18}$$

and

$$H(X, \bar{X} < 0) = \frac{1}{2}\frac{n}{n-2} J_n\left(\frac{\bar{X}-X}{2}\right)^{n-1} - \frac{1}{4}\frac{n}{(n-2)} J_n(\bar{X})^{n-1} . \tag{18a}$$

Equations (15) through (18a) will be needed for the solution of vibration problems in the following sections.

Figure 4 shows a typical relationship between $\bar{F}$ and $\bar{X}$ for both cases (a) and (b) of the above discussion.

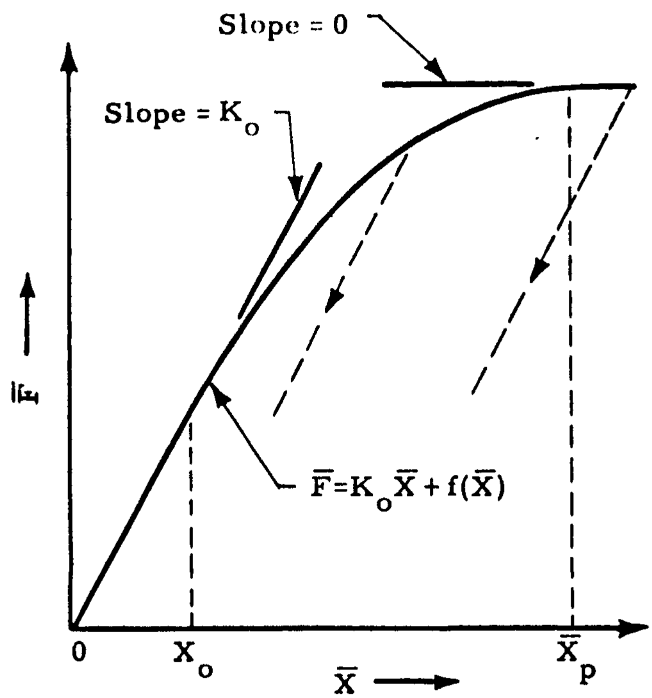

Figure 4 - Typical $\overline{F}$ vs $\overline{X}$ Curve

Section 3
SINUSOIDAL FORCED VIBRATION

For periodic excitations and for free vibration decay, as well as for a class of random excitations, where the solutions are expected to be similar to those for a system with small nonlinearities in viscous damping and in the restoring force, approximate solutions can be derived.

The steady-state equation of motion of a single-degree-of-freedom system can now be written:

$$m\ddot{X} + c\dot{X} + K_o X + H(X, \dot{X}) = P_o \cos\omega t \qquad (19)$$

where m is the mass, c the viscosity coefficient, P_o the forcing amplitude, and ω the forcing frequency. Let

$$\frac{K_o}{m} = \omega_o^2 , \qquad (20)$$

and

$$\frac{c}{m} = 2\zeta\omega_o . \qquad (21)$$

An approximate solution of the form

$$X = \bar{X} \cos(\omega t - \Phi) \qquad (22)$$

may be obtained in the following way:

Expand $H(X, \dot{X})$ into a Fourier series and retain only the fundamental frequency components,

$$H(X, \dot{X}) = I_1 \cos(\omega t - \Phi) + I_2 \sin(\omega t - \Phi) \qquad (23)$$

with

$$I_1 = \frac{1}{\pi} \int_0^{2\pi/\omega} H\left[\overline{X} \cos(\omega t - \Phi), \omega\overline{X} \sin(\omega t - \Phi)\right] \cos(\omega t - \Phi)\, \omega\, dt \tag{24}$$

and

$$I_2 = \frac{1}{\pi} \int_0^{2\pi/\omega} H\left[\overline{X} \cos(\omega t - \Phi), \omega\overline{X} \sin(\omega t - \Phi)\right] \sin(\omega t - \Phi)\, \omega\, dt$$

$$= - \frac{1}{\overline{X}\pi} \int_0^{2\pi/\omega} H(X, \dot{X})\, \dot{X}\, \omega\, dt \tag{25}$$

The last integral is obviously the energy dissipated over a cyle of vibration, so that

$$I_2 = - \frac{D_o}{\overline{X}\pi} \tag{26}$$

Substituting Equations (20) to (26) into Equation (19) and requiring the coefficients of $\cos(\omega t - \Phi)$ and $\sin(\omega t - \Phi)$ to vanish individually, the following equations are obtained for amplitude and phase responses of the steady-state problem:

$$\frac{K_o\overline{X}}{P_o} \left(1 - \frac{\omega^2}{\omega_o^2} + \frac{I_1}{K_o\overline{X}}\right) = \cos\Phi, \tag{27}$$

and

$$\frac{K_o\overline{X}}{P_o} \left(2\zeta\frac{\omega}{\omega_o} + \frac{D_o}{K_o\overline{X}^2\pi}\right) = \sin\Phi\ . \tag{28}$$

The effects of hesteresis damping are reflected in the presence of I_1 in Equation (21) and the presence of the term $\frac{D_o}{P_o\overline{X}\pi}$ in Equation (28). The quantity I_1 causes the frequency of phase resonance (the ω at which $\Phi = \pi/2$) to shift away from $\omega/\omega_o = 1.0$. Since I_1 is in general a function of the response amplitude, the shift in peak response natural frequency is a function of the excitation amplitude also. In the limiting case of zero forcing amplitude, ω/ω_o achieves the value of unity. To show the effect of D_o more directly, let D_c be the dissipation per cycle due to viscous damping, then

$$D_c = c\omega \bar{X}^2 \pi ,$$

and

$$\zeta \frac{\omega_o}{\omega} = \frac{D_c}{4\pi T_o} ,$$

where

$$T_o = \frac{1}{2} m\omega^2 \bar{X}^2$$

is the peak kinetic energy of the system. Equation (28) may now be written in the form

$$\frac{K_o \bar{X}}{P_o} \left(\frac{\omega^2}{\omega_o^2} \frac{D_c}{2\pi T_o} + \frac{D_o}{K_o \bar{X}^2 \pi} \right) = \sin\Phi \qquad (29)$$

showing the respective roles of D_c, which is an explicit function of ω, and D_o, which, on the contrary, is independent of ω.

Squaring Equations (27) and (28) and adding, one obtains

$$\left(\frac{K_o \bar{X}}{P_o} \right)^2 \left[\left(1 - \frac{\omega^2}{\omega_o^2} + \frac{I_1}{K_o \bar{X}} \right)^2 + \left(2\zeta \frac{\omega}{\omega_o} + \frac{D_o}{K_o \bar{X}^2 \pi} \right)^2 \right] = 1 ;$$

or,

$$\frac{\omega^2}{\omega_o^2} = \left(1 + \frac{P_o}{K_o \bar{X}} \right)^2 \pm \left[\left(\frac{P_o}{K_o \bar{X}} \right)^2 - \left(2\zeta \frac{\omega}{\omega_o} + \frac{D_o}{K_o \bar{X}^2 \pi} \right)^2 \right]^{1/2} . \qquad (30)$$

Since D_o is independent of frequency, the right-hand side of Equation (30) will be independent of frequency if viscous damping is absent. Therefore, for the case $\zeta = 0$,

$$\frac{\omega^2}{\omega_o^2} = \left(1 + \frac{I_1}{K_o \bar{X}}\right) \pm \frac{1}{K_o \bar{X}} \sqrt{P_o^2 - \frac{D_o^2}{X^2 \pi^2}} \; . \tag{31}$$

Peak response, $\bar{X} = \bar{X}_m$, is found where the radical in Equation (31) vanishes, and

$$\left. \frac{\omega^2}{\omega_o^2} \right|_{\bar{X}=\bar{X}_m} = 1 + \frac{I_1(\bar{X}_m)}{K_o \bar{X}_m} \tag{32}$$

Therefore, peak response occurs at the same frequency at which the response lags the force by a phase angle of $\pi/2$, if damping is entirely due to hysteresis loss. Obviously, the addition of viscous damping causes the separation of these frequencies.

At the peak response frequency,

$$P_o = \frac{D_o(\bar{X}_m)}{\pi \bar{X}_m} \tag{33}$$

which may be regarded as a relationship to determine either $\bar{X}_m$ from D_o, or vice versa, for given P_o. The nonlinear nature is quite obvious since in general D_o is not proportional to the square of the response amplitude.

Substituting Equation (33) into Equation (32), the peak response frequency is found in terms of $I_1(\bar{X}_m)$, $D_o(\bar{X}_m)$, and P_o.

$$\left. \frac{\omega^2}{\omega_o^2} \right|_{\bar{X}=\bar{X}_m} = 1 + \frac{\pi P_o I_1(\bar{X}_m)}{D_o(\bar{X}_m)} \tag{34}$$

As an example of the sinusoidal forced vibration solution, the force-deflection relationship of Equations (15) through (15d) (case (a)) are used to obtain the frequency and phase responses of a dynamic system described by Equation (19), with c=0.

It is convenient to introduce the parameter $X_s \equiv P_o/K_o$ to describe the forcing amplitude, and $\alpha \equiv 2\,J_2/K_o$ for a measure of the damping capacity.

For $\alpha = 0.1$, amplitude response curves are shown in Figure 5 for several values of X_s. In the lower region of the figure, where $\bar{X} \leq X_o$, the system is undamped.

Typical phase response curves are shown in Figure 6.

For each value of α, the peak response amplitude is a nonlinear function of the forcing amplitude. This is evidenced in the set of curves shown in Figure 7. The corresponding shift in frequencies is shown in Figure 8.

Experimental data on peak response and on frequency shift corresponding to the energy dissipation curve of Figure 3 for the Saturn I model are also plotted in Figures 7 and 8.

Similarities between experimental data and theoretical results indicate that the inherent nonlinearity of the hysteretic force-deflection relationship can be responsible for the observed nonlinear response.

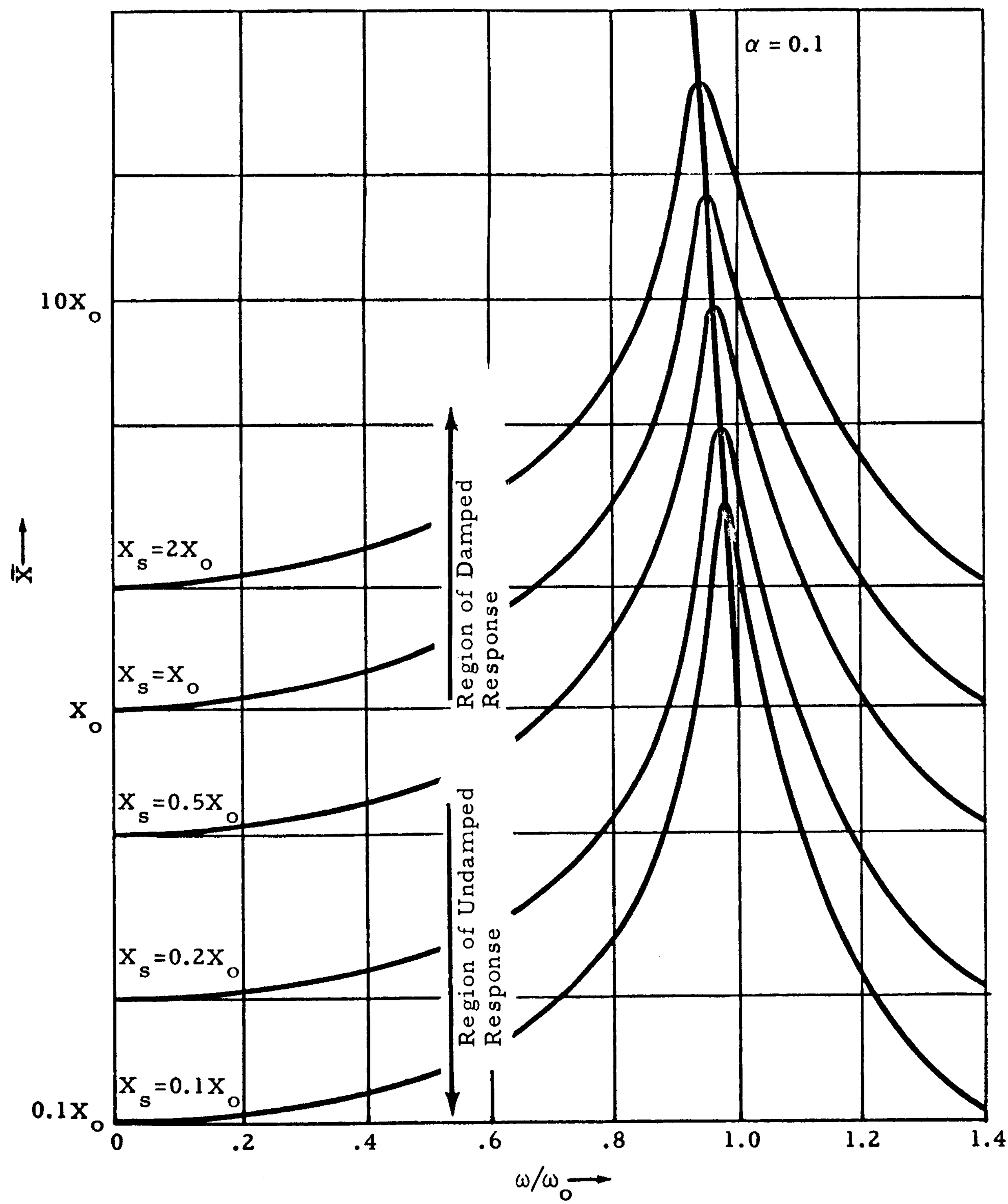

Figure 5 - Forced Vibration Amplitude Response Curves

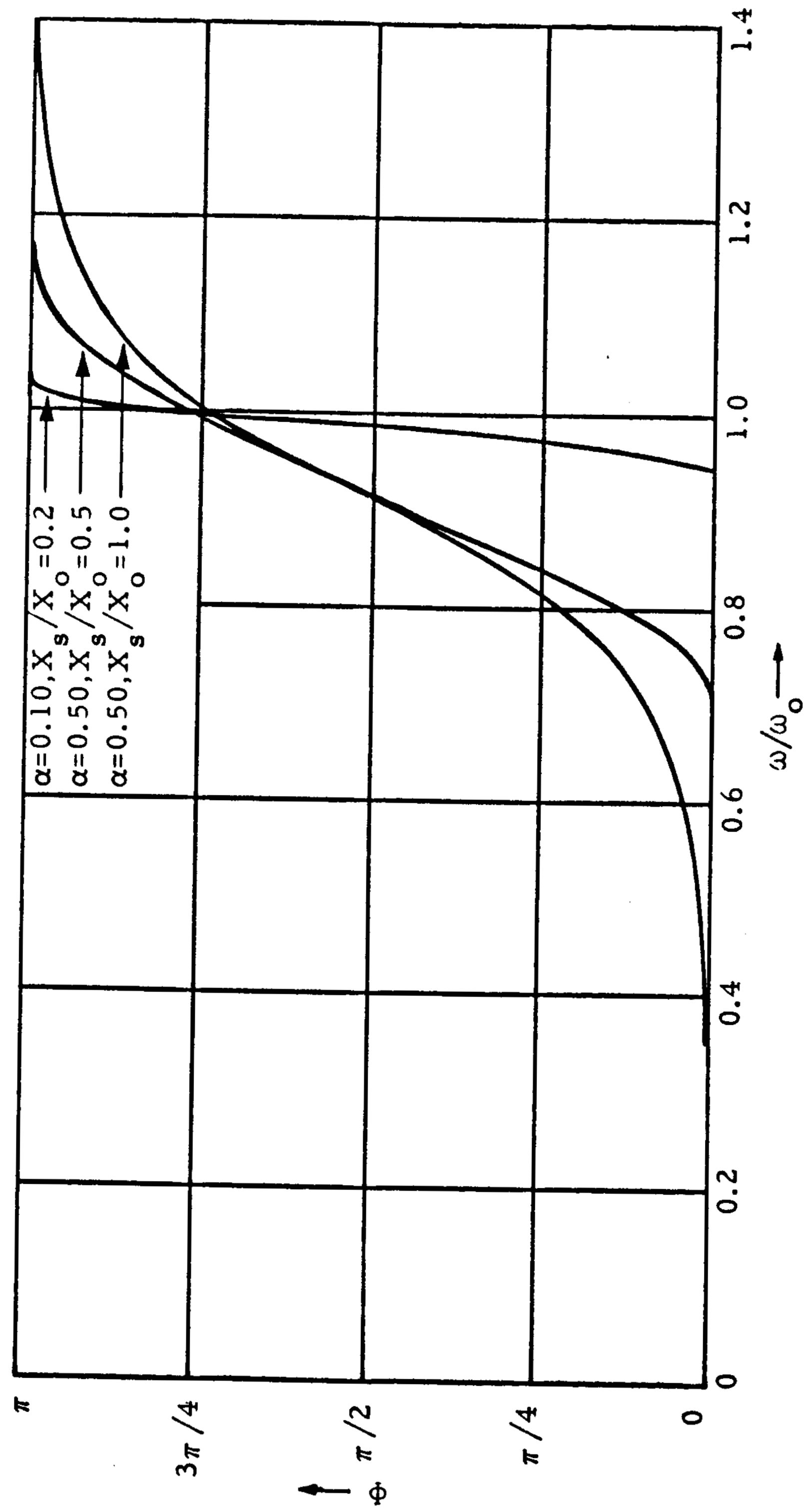

Figure 6 - Phase Response Curves

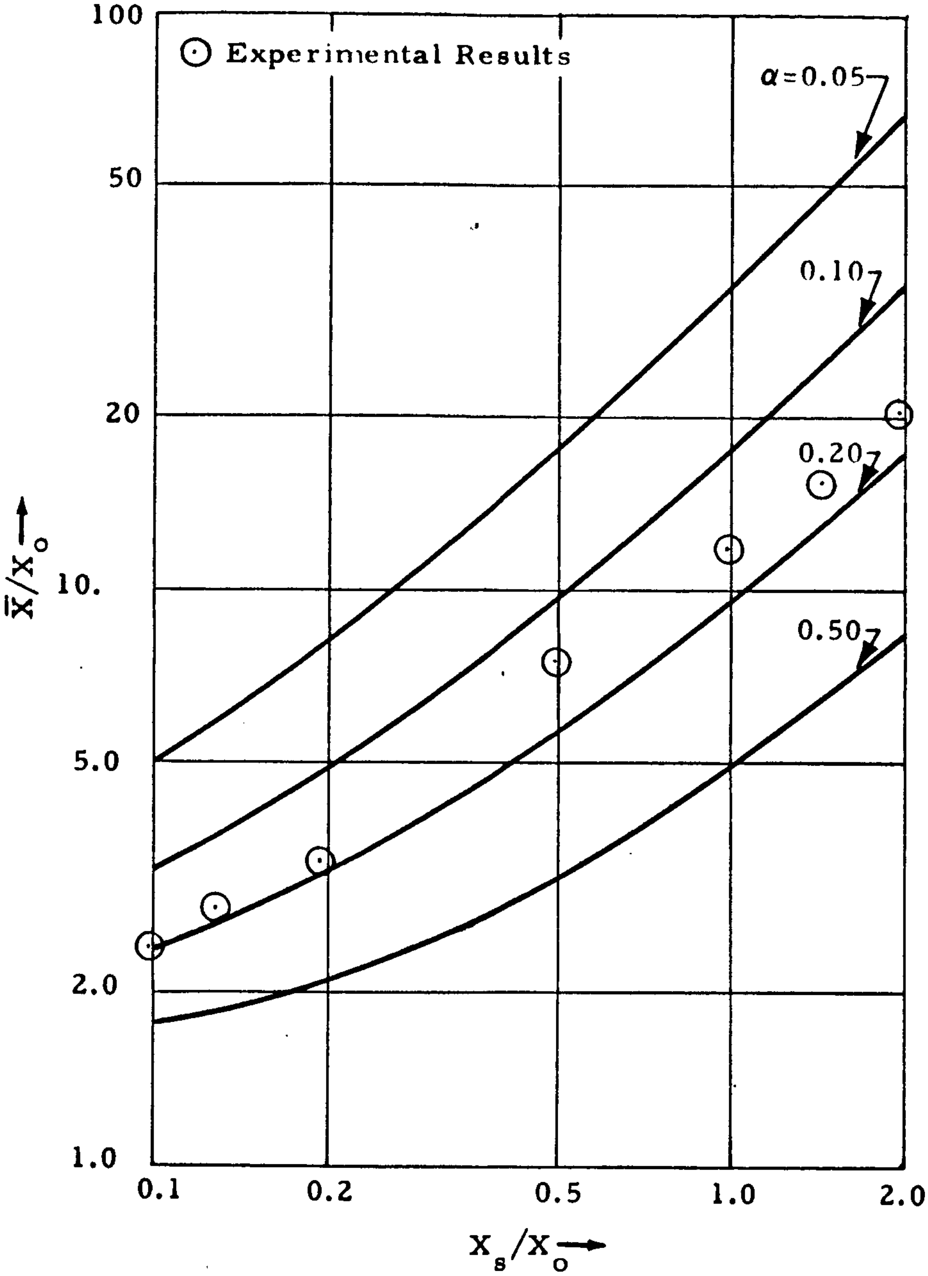

Figure 7 - Peak Response Amplitude vs Force Amplitude

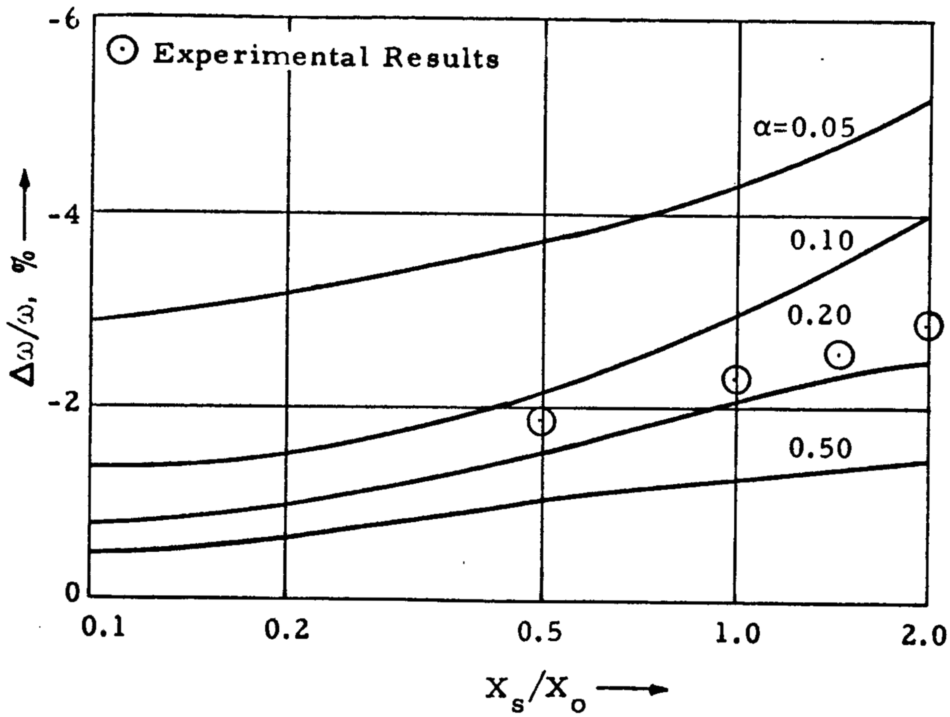

Figure 8 - Frequency Shift with Amplitude Due to Hysteresis Damping

Section 4
FREE VIBRATION

The equation of motion for the free vibration with both viscous and hysteretic types of damping is formally obtained by setting the excitation to zero in the equation for the steady-state

$$\ddot{X} + 2\zeta\omega_o\dot{X} + \omega_o^2 X + \frac{1}{m} H(X, \dot{X}) = 0 . \tag{35}$$

A solution of the form

$$X(t) = \bar{X}(t) \cos\left[\omega_o t - \theta(t)\right] \tag{36}$$

may be obtained by the method of Kryloff and Bogoliuboff (Reference 9).

The time-varying functions $\bar{X}(t)$ and $\theta(t)$ are defined such that when $\cos\left[\omega_o t - \theta(t)\right] = \pm 1$, the expression (36) will yield the correct values of X(t), and such that between successive peaks of X(t), both X(t) and $\theta(t)$ vary slowly and monotonically.

Imposing upon the two introduced functions $\bar{X}$ and θ, the relationship

$$\dot{\bar{X}} \cos\Psi + \dot{\theta}\bar{X} \sin\Psi = 0 , \tag{37}$$

where

$$\Psi = \omega_o t - \theta ,$$

and substituting the results into the original equation of motion, two first order equations are obtained.

$$\dot{\bar{X}} + 2\zeta\omega_o\bar{X} \sin^2\Psi = \frac{1}{m\omega_o} H \sin\Psi , \tag{38}$$

and

$$\dot{\theta} = \zeta\omega_o \sin 2\Psi + \frac{H}{m\omega_o \bar{X}} \cos\Psi . \tag{39}$$

A first order approximate solution of Equation (39) is

$$\theta = \dot{\theta} t . \tag{40}$$

Let

$$\omega_1 = \omega_o - \dot{\theta} ,$$

so that

$$\Psi = \omega_1 t ; \tag{41}$$

and expand the function H sinΨ into a Fourier series

$$H \sin\Psi = A_o + \sum_{i=1}^{\infty} A_i \cos i\Psi + B_i \sin i\Psi . \tag{42}$$

The coefficient A_o is given by the expression

$$A_o = \frac{1}{2\pi} \frac{1}{m\omega_o} \int_o^{2\pi} H(\bar{X} \cos\Psi, \omega_o \bar{X} \sin\Psi) \sin\Psi \, d\Psi = \frac{1}{2m\omega_o} I_2 = - \frac{D_o}{2\pi \, m\omega_o \bar{X}} \tag{43}$$

The solution $\bar{X}(t)$ may be obtained by integration of both sides of Equation (38), and can be put into the following form:

$$\frac{\bar{X}(t + 2\pi/\omega_1)}{\bar{X}(t)} = 1 - \frac{1}{2 T_o} (D_c + D_o) \tag{44}$$

The logarithmic decrement is given by

$$\delta \equiv \ln \frac{\bar{X} (t + 2\pi/\omega_1)}{\bar{X}(t)} \cong \frac{1}{2 T_o} (D_c + D_o) \tag{45}$$

In general, D_o is not proportional to T_o, so that δ is not a constant for the decay of free vibrations of structures with hysteresis damping.

The frequency, ω_1, is still to be determined. Referring to Equation (41), this amounts to finding $\dot{\theta}$, which is the average value of the quantity $(\zeta\omega_o \sin 2\Psi - H \cos\Psi / m\omega_o \bar{X})$ and is just the constant of the Fourier series expansion of the second term

$$\dot{\theta} = \frac{1}{2\pi} \frac{1}{m\omega_o \bar{X}} \int_0^{2\pi} H \cos\Psi \, d\Psi$$

According to Equations (24) and (41), this is equivalent to

$$\omega_o - \omega_1 = 1 \frac{I_1(\bar{X})}{2 m\omega_o \bar{X}} , \qquad (46)$$

or,

$$\frac{\omega_1^2}{\omega_o^2} = 1 + \frac{I_1(\bar{X})}{m\omega_o^2 \bar{X}} , \qquad (46a)$$

if a small term, θ^2, is omitted.

Equation (46a) is derived to show the similarity with Equation (33) which is an expression for the forced vibration natural frequency. For computational purposes, however, Equation (46) should be used.

Section 5
RANDOM VIBRATION RESPONSE

Because of the current interest in random vibrations and their effects on aerospace structures, the mean-square response of a hysteretically damped system to "white noise" excitation will be investigated. Unfortunately exact solutions for the mean-square are not available for the hysteresis loops of interest. An approximate solution will be obtained following the method of equivalent linearization employed by Caughey (Reference 10).

The equation of motion is of the same form as Equation (19) i.e.,

$$\ddot{X} + \beta\dot{X} + \omega_o^2\left[X + \frac{1}{K_o} H(X, \dot{X})\right] = N(t) , \tag{47}$$

where N(t) is "white noise" with spectral intensity Φ. It is convenient, to rewrite Equation (47) as,

$$\ddot{X} + \beta_{eq}\dot{X} + \omega_{eq}^2 X + \Delta(X, \dot{X}) = N(t) . \tag{48}$$

The method of solution is to minimize the mean-square error, $E\Delta^2$ with respect to ω_{eq} and β_{eq}. The known mean-square response associated with Equation (48), ignoring the Δ term, then becomes the approximate solution of Equation (47), i.e.,

$$EX^2 = \sigma_X^2 \cong \frac{\pi\Phi}{2\beta_{eq}\omega_{eq}^2} . \tag{49}$$

Solving for the error $\Delta(X, \dot{X})$,

$$\Delta = (\beta - \beta_{eq})\dot{X} + (\omega_o^2 - \omega_{eq}^2) X + \frac{\omega_o^2}{K_o} H ; \tag{50}$$

$$E\Delta^2 = E\left[(\beta - \beta_{eq})\,\dot{X} + (\omega_o^2 - \omega_{eq}^2)\,X + \frac{\omega_o^2}{K_o}\,H\right]^2 . \tag{51}$$

Minimizing with respect to ω_{eq}^2 and β_{eq} yields,

$$\omega_{eq}^2 = \omega_o^2\left(1 + \frac{E\,X\,H}{K_o\,E\,X^2}\right) , \tag{52}$$

$$\beta_{eq} = \beta + \frac{\omega_o^2\,E\,X\,H}{K_o\,E\,X^2} . \tag{53}$$

Solving Equation (49) for σ_X^2 and normalizing by σ_o^2,

$$\frac{\sigma_X^2}{\sigma_o^2} = \frac{1}{\left(\beta_{eq}/\beta\right)\left(\omega_{eq}^2/\omega_o^2\right)} \tag{54}$$

where σ_o^2 is the small vibration mean-square response,

$$\sigma_o^2 = \frac{\pi\Phi}{2\beta\omega_o^2} . \tag{55}$$

Representing X(t) by,

$$X(t) = \bar{X}(t)\cos(\omega_{eq}t - \varphi(t)) , \tag{56}$$

where $\bar{X}$ and φ are slowly varying random envelope and phase functions respectively; then the expectations $E\,X\,H$, $E\,\dot{X}\,H$, $E\,X^2$ and $E\,\dot{X}^2$ can be solved for in terms of conditional expectations $E[X\,H\,|\,\bar{X} < \bar{X}_p]$, $E[X\,H\,|\,\bar{X} > \bar{X}_p]$, ------, $E[\dot{X}^2\,|\,\bar{X} < \bar{X}_p]$, $[E\,\dot{X}^2\,|\,\bar{X} > \bar{X}_p]$.

Then,

$$\begin{aligned}
EXH &= E\left[E[XH|\overline{X} < \overline{X}_p]p_1 + E[XH|\overline{X} > \overline{X}_p]p_2\right] \\
E\dot{X}H &= E\left[E[\dot{X}H|\overline{X} < \overline{X}_p]p_1 + E[\dot{X}H|\overline{X} > \overline{X}_p]p_2\right] \\
EX^2 &= E\left[E[X^2|\overline{X} < \overline{X}_p]p_1 + E[X^2|\overline{X} > \overline{X}_p]p_2\right] \\
E\dot{X}^2 &= E\left[E[\dot{X}^2|\overline{X} < \overline{X}_p]p_1 + E[\dot{X}^2|\overline{X} > \overline{X}_p]p_2\right]
\end{aligned} \tag{57}$$

where:

$$\begin{aligned}
p_1 &= PR[\overline{X} < \overline{X}_p] \\
p_2 &= PR[\overline{X} > \overline{X}_p]
\end{aligned} \tag{58}$$

As assumed earlier, $\overline{X} << \overline{X}_p$ so that,

$$p_2 \cong 0. \tag{59}$$

Under this restriction the necessary conditional expectations are,

$$\begin{aligned}
E[XH|\overline{X}] &= \frac{\overline{X}\, I_1}{2}, \\
E[\dot{X}H|\overline{X}] &= \frac{-\omega_{eq}\overline{X}\, I_2}{2}, \\
E[X^2|\overline{X}] &= \frac{\overline{X}^2}{2}, \\
E[\dot{X}^2|\overline{X}] &= \frac{\omega_{eq}^2\, \overline{X}^2}{2},
\end{aligned} \tag{60}$$

where: I_1 and I_2 are given by Equations (24) and (25).

If the nonlinearities are small then it is expected that X(t) will be nearly Gaussian so that the probability density function of $\overline{X}$ can be approximated by,

$$p(\overline{X}) \cong \frac{\overline{X}}{\sigma_X^2}\, e^{-\overline{X}^2/2\sigma_X^2}, \tag{61}$$

and hence

$$E X H \cong \frac{1}{2\sigma_X^2}\int_o^{\overline{X}_p} \overline{X}^2 \, I_1(\overline{X})\, e^{-\overline{X}^2/\sigma_X{}^2}\, d\overline{X} \, ,$$

$$E \dot{X} H \cong -\frac{\omega_{eq}}{2\sigma_X^2}\int_o^{X_p} \overline{X}^2 \, I_2(\overline{X})\, e^{-\overline{X}^2/2\sigma_X^2}\, d\overline{X} \, , \tag{62}$$

$$EX^2 = \sigma_X^2 \, ,$$

$$E\dot{X}^2 \cong \omega_{eq}^2 \sigma_X^2 \, .$$

The solution of Equations (52), (53) and (54) has been obtained for case (b), i.e., for a structure which follows the hysteresis damping law of Equation (11).

$$\left.\begin{aligned} H(X, \dot{X} > 0) &= -\frac{J_n}{2}\frac{n}{n-2}\left(\frac{\overline{X}+X}{2}\right)^{n-1} + \frac{J_n}{4}\frac{n}{n-2}\,\overline{X}^{n-1}, \\ H(X, \dot{X} < 0) &= \frac{J_n}{2}\frac{n}{n-2}\left(\frac{\overline{X}-X}{2}\right)^{n-1} - \frac{J_n}{4}\frac{n}{n-2}\,\overline{X}^{n-1}, \end{aligned}\right\} \begin{aligned} &n > 2, \\ &X < \overline{X}_p \end{aligned}$$

As expected, both ω_{eq}^2 and β_{eq} are functions of the mean-square response, σ_X^2, and are given by,

$$\omega_{eq}^2/\omega_o^2 = 1 + \frac{(J_n\sigma_o^{n-2}/K_o)(\sigma_X/\sigma_o)^{n-2}\, n^2\, \rho_n \Gamma(n/2)}{2^{n/2}\,(n-2)\pi} \, , \tag{63}$$

$$\beta_{eq}/\beta = 1 + \frac{(J_n\sigma_o^{n-2}/K_o)(\sigma_X/\sigma_o)^{n-2}\, n\, 2^{n/2}\Gamma(n/2)}{4\pi\,(\beta/\omega_o)(\omega_{eq}/\omega_o)} \tag{64}$$

where:

$$\rho_n = \int_0^{\pi} (1 - \cos\Psi)^{n-1} \cos\Psi \, d\Psi$$

The simultaneous solution of Equations (63) and (64) together with Equation (54)yields the mean-square response σ_X^2/σ_o^2 versus the nonlinear parameter $J_n \sigma_o^{n-2}/K_o$. A numerical solution has been carried out and results are shown in Figure 9 for $\beta/\omega_o = .01$ and several values of n between 2.1 and 3.0. For small nonlinearities there is a beneficial effect due to hysteresis damping. For large values of the nonlinear parameter the mean-square response increases and eventually exceeds the linear case ($\sigma_X/\sigma_o > 1$); however, for this range of nonlinearities the approximate solutions would require additional verification.

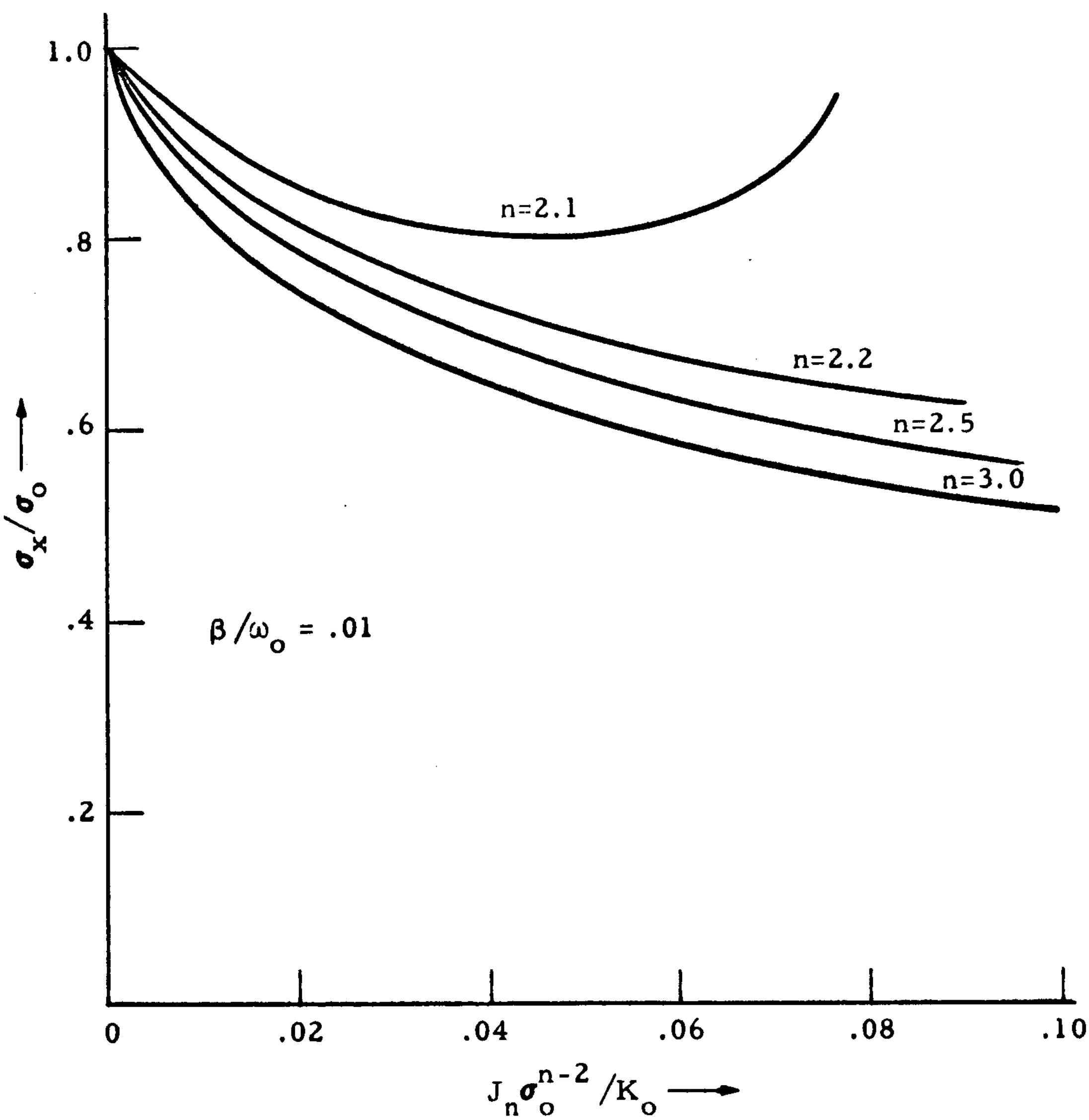

Figure 9 - Mean-Square Response vs Nonlinear Parameter

Section 6
CONCLUSIONS

It has been delineated that the presence of hysteresis damping implies a nonlinear force-deflection relationship. Analytical consistency of structural vibrations with hysteresis damping can be achieved if this nonlinearity is preserved in the dynamic equations. The actual derivation of equations and their approximate solutions presents no major difficulties.

The proposed hysteresis model thus affords additional means for analyzing the dynamic behavior of complex structures.

Section 7
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