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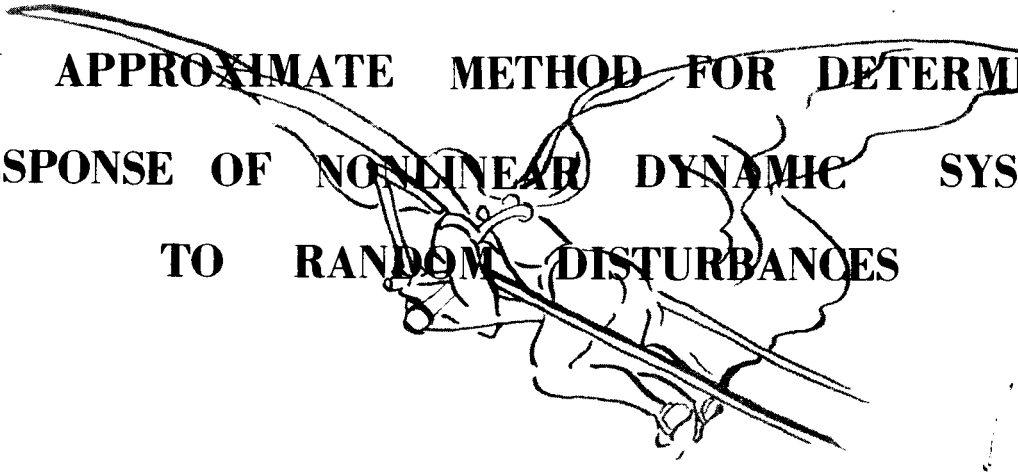

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February, 1967

# AN APPROXIMATE METHOD FOR DETERMINING RESPONSE OF NONLINEAR DYNAMIC SYSTEMS TO RANDOM DISTURBANCES

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## ABSTRACT

An approximate method is presented for determining the means and covariances of the state variables of nonlinear, nonstationary dynamic systems having random initial conditions and being excited by white noise or random disturbances which can be derived from white noise. Application of the method is illustrated on several simple examples and useful formulas for practical application as well as some helpful FORTRAN subprograms are given in the appendices. Comparison of results obtained by this approximate method with some known exact solutions and with solutions obtained by a Monte Carlo technique are given to aid in evaluating the accuracy of the method.

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# INTRODUCTION

There are many problems in engineering characterized by such needs as prediction of performance, estimation of structural loads, determination of trajectories, or optimization of parameter values for dynamic systems operating in environments and with tasks which might only be known in a probabilistic or statistical sense. Such problems have led to a search for practical methods of determining statistical properties of the state or outputs of a dynamic system from the mathematical model of the system and a statistical model of the disturbances and inputs.

Perhaps the oldest and most general approach to problems such as those indicated above is through the Fokker-Planck partial differential equation. This approach is based on techniques developed around the beginning of the century for the study of diffusion and Brownian motion, but a more recent presentation of the use of the Fokker-Planck (or Kolmogorov) equation for problems of current interest to engineers is given, for example, in Reference 1. The Fokker-Planck partial differential equation governs the evolution from a known initial condition of the joint probability density function of the state variables of a nonlinear, time-varying system described by ordinary differential equations and excited by white, gaussian noise.

Unfortunately, analytic or closed form solutions of the Fokker-Planck equation, even for stationary probability densities of the state variables, for systems of engineering interest are rare, and nonstationary solutions are almost nonexistant. Worse than this, however, is that numerical integration of the Fokker-Planck equation on a digital computer, at least by finite difference methods, requires a prohibitive amount of storage for systems of order greater than 3 or 4 since an n-dimensional density function must be stored for an $n^{th}$ order dynamic system.

Because of the difficulties of practical application of the Fokker-Planck approach, a number of other more practical approaches to the response of dynamic systems to random disturbances have evolved which usually sacrifice the generality or precision of the Fokker-Planck equation for computational ease. Some of the more well known methods are mentioned.

Several methods settle for finding the means and variances or covariances of the state variables (first and second joint moments of the probability density function) rather than the density function itself, thus greatly simplifying the statistical description of the system state variables. The power spectral density method (e.g., see Reference 2 or 5) can be used to find the power spectral density (Fourier transform of the autocorrelation function) of any state variable of a constant-coefficient, linear system subjected to stationary random inputs with known power spectral densities. Stationary means and variances can be found from the power spectral densities.

This method is fairly easy to use and lends itself well to machine computation but is restricted to linear, stationary problems.

An extension of the power spectral density method to the approximate treatment of a class of nonlinear, stationary problems is known as the equivalent linearization, gaussian input describing function, or Booton-Kazakov method and is treated extensively in Reference 3. This method produces the same type of results as the power spectral density method for linear systems, but the computations are much more difficult, especially for anything except very low order systems, and are not easily relegated to computer solution. The nature of the approximation is not well understood and the method is known to give grossly invalid results in some cases.

Nonlinear, stationary problems have also been treated by applying the power spectral density method to problems which can be reduced to a sequence of linear problems by the perturbation method of Poincare (see Reference 4). The type of nonlinear problem which can be treated by this approach is somewhat limited and the computational procedure is quite cumbersome for systems larger than first or second order.

Nonstationary problems can arise from considering statistical behavior of a system over a finite observation time, from dynamic systems described by nonautonomous or time-dependent differential equations, or from disturbances with time varying statistical properties. There are two basic

approaches to the treatment of nonstationary problems in linear systems. The first, called the adjoint equation method, allows the computation of the mean and variance of any linear combination of state variables at a single time by just integrating essentially the adjoint differential equations of the system once with a suitable set of final conditions. The adjoint equation method is particularly efficient for problems where the terminal value of a mean and variance is required (e.g., terminal guidance of a missile) and is well suited to both analog and digital computation. The second method is the mean and covariance equation approach which yields a time history of the complete set of means and covariances of the system states for an $n^{th}$ order linear system by integration of a set of $n(n+3)/2$ differential equations. This approach is also very suitable for analog and digital computation. It requires more work than the adjoint equation method but gives much more information. Both the adjoint equation method and the covariance equation method are presented in Reference 5.

A completely different approach to responses of dynamic systems of any type to random disturbances is by means of a "random experiment" or Monte Carlo method. In this method one would obtain a large number of solutions to the system equations with simulated "random" inputs and estimate from these runs the desired statistical measures by making estimates based on finite sample size. This technique is used for comparison purposes in the present study and is discussed in

Appendix D. Its chief advantages are its generality and simplicity of implementation. The main disadvantage is the long computation time required to obtain sufficiently accurate results.

The subject of this study is an approximate method for obtaining time histories of the complete set of means and covariances of the state variables of nonlinear, nonstationary systems subjected to nonstationary random disturbances. The method is closely related to both the equivalent linearization approach (Booton-Kazakov) and the mean and covariance equation approach for linear systems. The method is presented and illustrative applications given in Section II. Since the method is approximate, a discussion of its validity is given in Section III.

## II. MEAN AND COVARIANCE EQUATIONS FOR NONLINEAR SYSTEMS

### A. MEAN AND COVARIANCE EQUATIONS

Consider a dynamic process which can be described by the set of first order differential equations

$$\dot{x}_i = f_i(x_1,\ldots,x_n,t) + \sum_{i=1}^{m} e_{ik}(t)q_k(t) \qquad (2.1)$$

$$i=1,\ldots,n$$

where the $q_k$, $k=1,\ldots,m$, are independent white noise inputs with

$$\overline{q_k(t)q_i(t+\tau)} = \delta(\tau),\ k=i$$

$$= 0,\ k\neq i$$

If the means $m_i$ and covariances $\phi_{ij}$ of the state variables of the system (2.1) are defined as

$$m_i = \bar{x}_i$$

$$\phi_{ij} = \overline{(x_i-m_i)(x_j-m_j)}$$

the rates of change of these quantities can be expressed as

$$\dot{m}_i = \bar{f}_i \qquad (2.2)$$

$$\dot{\phi}_{ij} = \overline{(x_i - m_i) f_j} + \overline{(x_j - m_j) f_i} + \sum_{k=1}^{m} e_{ik} e_{jk} \qquad (2.3)$$

$$i,j = 1,\ldots,n$$

where the $f_i$, i=1,...,n, are those of the system (2.1). This result can be obtained either by averaging the Fokker-Planck equation (Reference 6) or, with slightly less restriction by averaging the system equations (2.1) directly (see Appendix A). Note that $\phi_{ij}=\phi_{ji}$ by definition so that there are really only n(n+1)/2 distinct covariances equations instead of the apparant $n^2$.

If the averages $\bar{f}_i$ and $\overline{(x_i - m_i) f_j}$ on the right-hand sides of (2.2) and (2.3) could be expressed completely as functions of $m_i$, $\phi_{ij}$, and t, (i,j = 1,...,n), equations (2.2) and (2.3) become a set of ordinary differential equations governing the evolution in time of the state variable means and covariances from their initial values. This is analogous to the evolution of the joint probability density function of the system state variables governed by the Fokker-Planck partial differential equation.

In the case where the system (2.1) is linear

$$f_i(x_1,\ldots,x_n,t) = \sum_{j=1}^{n} a_{ij}(t) x_j + b_i(t) \qquad (2.4)$$

and the averages $\bar{f}_i$ and $\overline{(x_j - m_j) f_i}$ are easily evaluated in terms of $m_i$, $\phi_{ij}$, and t (i,j=1,...,n) using the alternative forms of the definitions,

$$\phi_{ij} = \overline{(x_i - m_i)(x_j - m_j)} = \overline{(x_i - m_i)x_j} = \overline{x_i(x_j - m_j)} \qquad (2.5)$$

the resulting mean and covariance equations are the same as those discussed in Reference 5 for example.

In the nonlinear case the required averages on the right hand side of (2.2) and (2.3) cannot in general be expressed as functions of means and covariances alone, since either the complete joint probability density function or at least higher joint moments are necessary.

B. USE OF THE GAUSSIAN APPROXIMATION FOR NONLINEAR SYSTEMS

In order to allow equations (2.2) and (2.3) to be applied to the analysis of nonlinear systems, one must approximate the joint probability density function of the states with a density function which is completely determined by its first and second joint moments ($m_i$ and $\phi_{ij}$). In the present study the joint gaussian probability density function is used for the approximation.

Although there are various philoxophical rationalizations for using the joint gaussian density, the real reason is computational expediency. There are available many useful mathematical relations for gaussian distributed variables which are very helpful in forming the averages of nonlinear functions which are needed in equations (2.2) and (2.3). The validity of this approximation is discussed in Section III.

The general expression for an n-dimensional joint gaussian density function is

$$p_n(X_1,\ldots,X_n) = ((2\pi)^n|\Phi|)^{-1/2}\exp(-\tfrac{1}{2}Z'\Phi^{-1}Z) \qquad (2.6)$$

where the elements of the column matrix Z are

$$z_i = x_i - m_i$$

and the elements of the square matrix $\Phi$ are $\phi_{ij}$.

With the assumption that the system state variables have a gaussian joint density function, the required averages can be written formally as

$$\bar{f}_i = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i(X_1,\ldots,X_n)\, p_n(X_1,\ldots,X_n)\, dX_1 \cdots dX_n$$

$$\overline{(x_j - m_j) f_i} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_j - m_j)\, f_i(X_1,\ldots,X_n)\, p_n(X_1,\ldots,X_n)\, dX_1 \cdots dX_n \tag{2.7}$$

Since $p_n(X_1,\ldots,X_n)$ is dependent on means and covariances only, the averages expressed in (2.7) are functions of means and covariances only making (2.2) and (2.3) a set of differential equations for $m_i$ and $\phi_{ij}$ $(i,j=1,\ldots,n)$.

If the n-tuple integrals of (2.7) always had to be evaluated as n-tuple integrals, solutions of the mean and covariance equations would be an extremely cumbersome process even for a digital computer. Fortunately, the nature of many nonlinear problems of interest is such that the functions $f_i$ in equations (2.1) are composed of a linear part and one or more nonlinear functions of only one or two of the state variables at a time. Consideration of some simple example problems will serve to illustrate the practical aspects of application of the mean and covariance equations, the nature of the solutions,

and the types of problems which can be treated.

C. ILLUSTRATIVE EXAMPLES

Example 1. First consider the simple second order system shown in the block diagram below. The system could

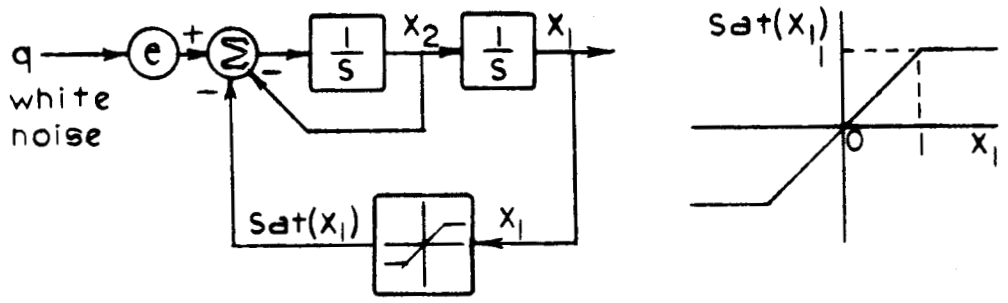



FIG. 2.1. Block diagram for example 1.

either represent a saturating regulator system or a nonlinear spring-mass system with a very broad-band, stationary random disturbing force.

In order to obtain the mean and covariance equations, the system equations must first be put into the first order form of (2.1).

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \text{sat}(x_1) + eq(t) \qquad (2.8)$$

where e is a constant which allows us to examine various magnitudes of random disturbance and q(t) is white noise with $\overline{q(t)q(t+\tau)} = \delta(\tau)$.

In writing the mean and covariance equations (2.2) and (2.3) for the system (2.8) the averages involving linear terms can be written directly in terms of means and covariances from the definition (2.5) and the following results are obtained:

$$\begin{aligned} \dot{m}_1 &= m_2 \\ \dot{m}_2 &= -m_2 - \overline{sat(x_1)} \end{aligned} \tag{2.9}$$

$$\begin{aligned} \dot{\phi}_{11} &= 2\phi_{12} \\ \dot{\phi}_{12} &= -\phi_{12} - \overline{(x_1-m_1)sat(x_1)} + \phi_{22} \\ \dot{\phi}_{22} &= 2[-\phi_{22} - \overline{(x_2-m_2)sat(x_1)}] + e^2 \end{aligned} \tag{2.10}$$

The relation $\phi_{ij}=\phi_{ji}$ has been used to eliminate one of the equations.

The next step is to use the gaussian approximation for the joint probability density of the state variables $x_1$ and $x_2$ to obtain $\overline{sat(x_1)}$, $\overline{(x_1-m_1)sat(x_1)}$, $\overline{(x_2-m_2)sat(x_1)}$ as functions of $m_1$, $m_2$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$. One immediate result of the gaussian approximation from Appendix B is

$$\overline{(x_j-m_j)g(x_i)} = \frac{\varphi_{ij}}{\phi_{ii}} \overline{(x_i-m_i)g(x_i)} \tag{2.11}$$

where $g(x_i)$ is any nonlinear function of a single variable. In the present example this means

$$\overline{(x_2-m_2)\,\mathrm{sat}(x_1)} = \frac{\phi_{12}}{\phi_{11}}\,\overline{(x_1-m_1)\,\mathrm{sat}(x_1)} \tag{2.12}$$

so that now only two troublesome averages $\overline{\mathrm{sat}(x_1)}$ and $\overline{(x_1-m_1)\,\mathrm{sat}(x_1)}$ remain to be evaluated. These must be obtained from

$$\overline{\mathrm{sat}(x_1)} = \frac{1}{\sqrt{2\pi\phi_{11}}}\int_{-\infty}^{\infty} \mathrm{sat}(X_1)\exp[-(X_1-m_1)^2/2\phi_{11}]\,dX_1$$

$$\overline{(x_1-m_1)\,\mathrm{sat}(x_1)} = \frac{1}{\sqrt{2\pi\phi_{11}}}\int_{-\infty}^{\infty} (X_1-m_1)\,\mathrm{sat}(X_1)\exp [-(X_1-m_1)^2/2\phi_{11}]\,dX_1 \tag{2.13}$$

In this example it happens that the integrals (2.13) can be obtained in a "closed" form as a rather involved expression containing error functions and exponentials, but usually no such analytic expression is available at all. For this reason several FORTRAN subprograms have been written to evaluate integrals of the type (2.13) for general nonlinear functions of a single variable by several different approximations. The description of these subprograms as well as one for nonlinear functions of two variables is given in Appendix C.

The integrals (2.13) when evaluated, whether analytically or numerically, are functions of $m_1$ and $\phi_{11}$. This fact

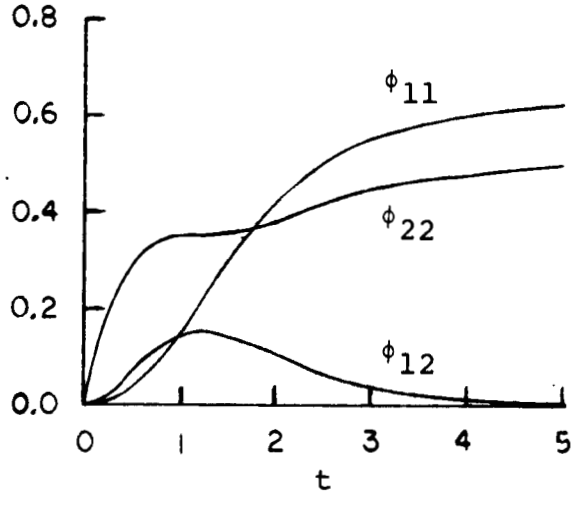
allows equations (2.9) and (2.10) to be considered as set of differential equations for $m_1$, $m_2$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$ which can be numerically integrated on a digital computer.

To obtain a solution to the differential equations a set of initial values for $m_1$, $m_2$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$ must be specified. If the system were at rest and its state completely known at time t=0, all initial values would be zero, i.e. the mean values of $x_1$ and $x_2$ would be zero and the variances about the mean values would be zero. The resulting solution for a specified value of e (the disturbance magnitude) would indicate the uncertainty of the system state at any subsequent time. Such a solution is shown in Figure (2.2). The solution for a non-zero initial mean value of $x_1$ is shown in Figure (2.3). Any initial uncertainty in $x_1$ or $x_2$ would be reflected in the initial values of $\phi_{11}$ and $\phi_{22}$. If both $x_1$ and $x_2$ were uncertain initially but correlated with each other $\phi_{12}$ would be non-zero initially.

Note that these example transient solutions are approaching the same steady state solution. Since this example problem is a stable, time-invariant system excited by a stationary random disturbance, a steady state or stationary solution to the mean and covariance equation can be expected. In principal the stationary solution could be obtained by setting the right-hand sides of equations (2.9) and (2.10) equal to zero and solving the resulting nonlinear equations. For this particular example the task is not too difficult to do graphically or numerically and the result corresponds exactly

to that obtained by the statistical linearization method described in Reference 3. The steady state results are discussed in Section III. For more realistic problems, however, the solution is tedious and often produces more than one solution which may or may not be a stable or even meaningful solution (e.g., a non-positive covariance matrix makes no sense physically). For this reason, solution of the differential equations for a reasonably long time period is probably the most attractive method of finding steady state solutions as it will converge only on stable, meaningful solutions and give additional useful information on transient behavior.

(a) e = 1.0




(b) e = 2.0

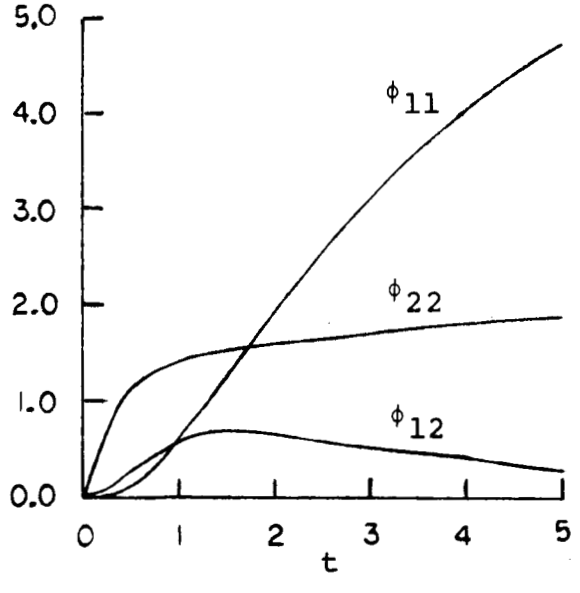
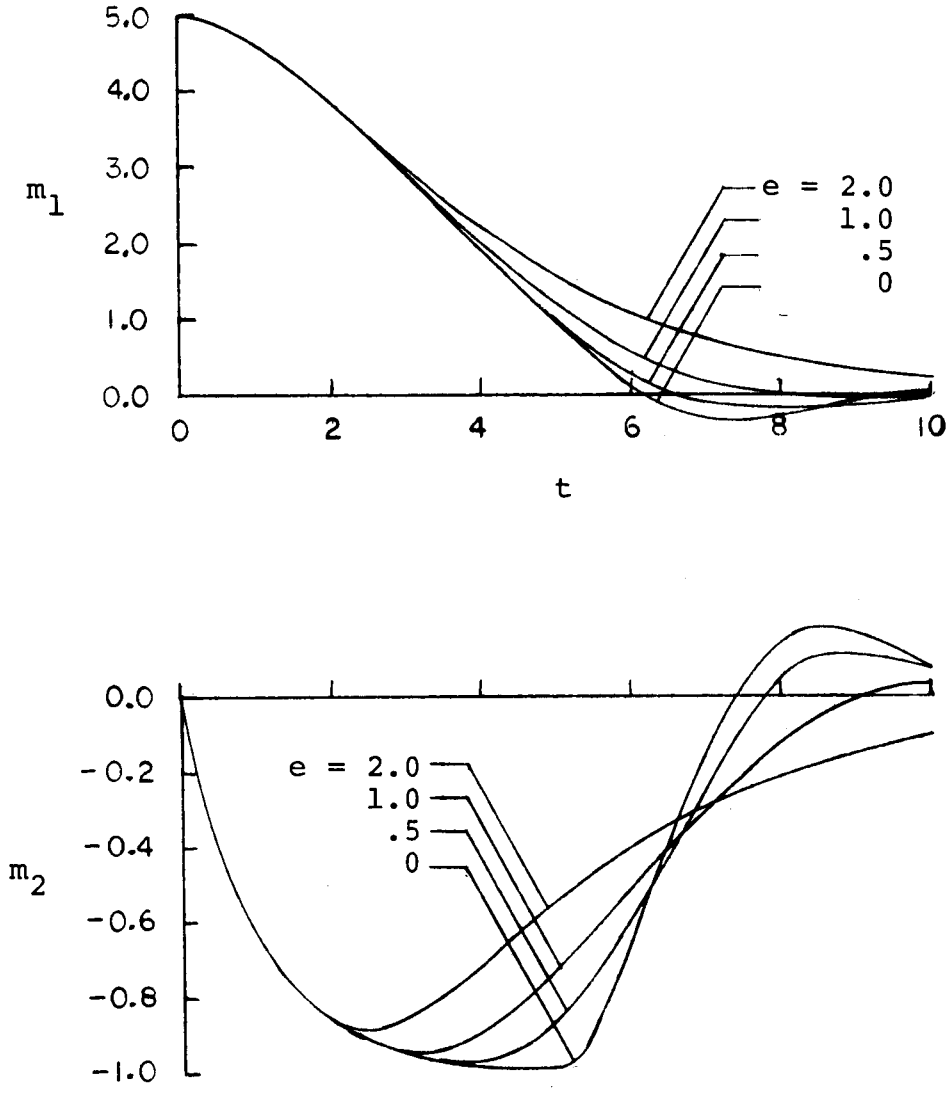
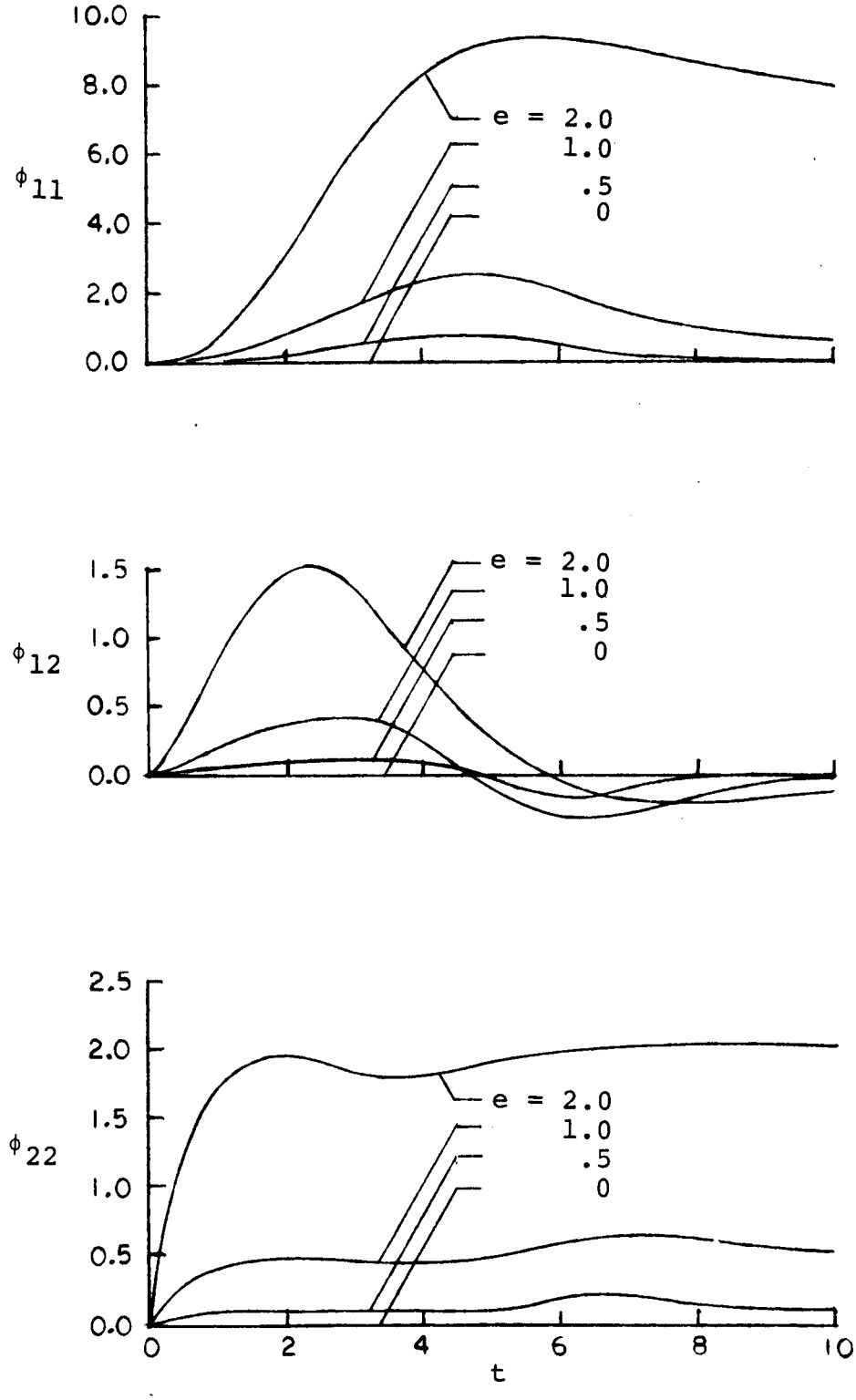



FIG. 2.2. Covariance solutions for example 1 with the initial values of all means and covariances zero.




(a) Means

FIG. 2.3. Solutions for example 1 with $m_1(0)=5.0$ and initial values of $m_2$ and covariances zero.




(b) Covariances

FIG. 2.3. Continued.

Example 2. Suppose we wish to examine the response of the system of example 1, Figure (2.1), to a random disturbance of much lower frequency content than could be reasonably approximated by white noise. For this example a disturbance is treated whose statistical model is generated by the response of a first order system to white noise. This is analogous to pre-filtering white noise in power spectral density analysis methods to produce non-white disturbance power spectra. The system block diagram is shown in Figure (2.4).

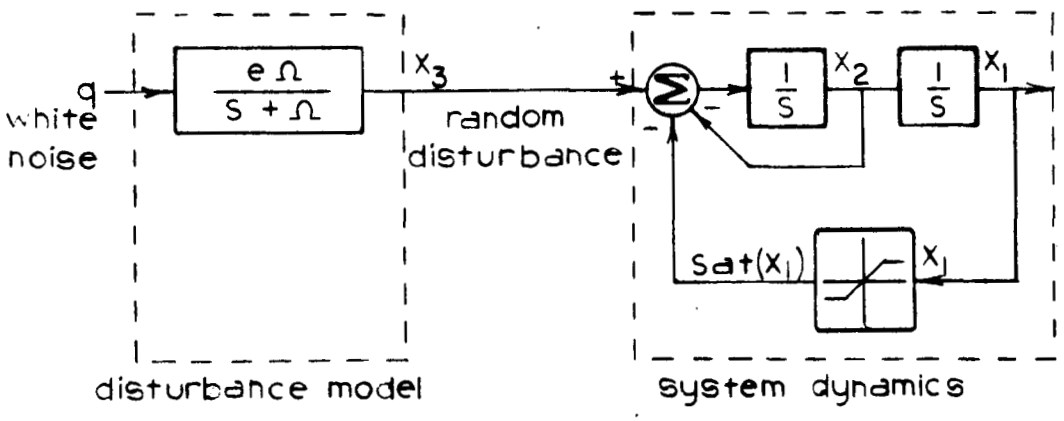



FIG. 2.4. Block diagram for example 2.

The system differential equations are written in first order form as

$$\text{system dynamics} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - \text{sat}(x_1) + x_3 \end{cases}$$

$$\text{disturbance model} \begin{cases} \dot{x}_3 = -\Omega x_3 + \Omega e q(t) \end{cases} \qquad (2.14)$$

where e again is a constant which determines the magnitude of the white noise input, $\Omega$ is the "band-pass" frequency of the disturbance model or filter, and $\overline{q(t)q(t+\tau)} = \delta(\tau)$ (white noise).

Now the differential equation which describes the dynamics of the disturbance model is treated as though it were just a part of a new system with a white noise disturbance in the form of equations (2.1). The mean and covariance equations, then, are written following the form of (2.2) and (2.3).

$$\begin{aligned} \dot{m}_1 &= m_2 \\ \dot{m}_2 &= -m_2 - \overline{\text{sat}(x_1)} + m_3 \\ \dot{m}_3 &= -\Omega m_3 \end{aligned} \qquad (2.15)$$

$$\dot{\phi}_{11} = 2\phi_{12}$$

$$\dot{\phi}_{12} = -\phi_{12} - \overline{(x_1-m_1)\,sat(x_1)} + \phi_{13} + \phi_{22}$$

$$\dot{\phi}_{22} = 2\left[\phi_{23}-\phi_{22} - \frac{\phi_{12}}{\phi_{11}}\,\overline{(x_1-m_1)\,sat(x_1)}\right]$$

$$\dot{\phi}_{13} = -\Omega\phi_{13} + \phi_{23}$$

$$\dot{\phi}_{23} = -(1+\Omega)\phi_{23} - \frac{\phi_{13}}{\phi_{11}}\,\overline{(x_1-m_1)\,sat(x_1)} + \phi_{33}$$

$$\dot{\phi}_{33} = -2\Omega\phi_{33} + e^2\Omega^2 \qquad (2.16)$$

The gaussian approximation result of equation (2.11) was used in the third and fifth covariance equations (2.16) to reduce the number of nonlinear terms to be averaged to $\overline{sat(x_1)}$ and $\overline{(x_1-m)\,sat(x_1)}$ as in example 1, so equations (2.16) are already approximate. Note again that only the diagonal elements and elements below the diagonal of the matrix of $\phi_{ij}$'s were used because of symmetry.

If we now consider that we have available the functional dependence of $\overline{sat(x_1)}$ and $\overline{(x_1-m_1)\,sat(x_1)}$ on $m_1$ and $\phi_{11}$ as implied by the integrals (2.13), equations (2.15) and (2.16) are ready to be integrated to obtain the time histories of $m_1$, $m_2$, $m_3$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$, $\phi_{13}$, $\phi_{23}$, $\phi_{33}$ or their steady state values. To obtain steady state results the only requirement on the initial conditions is that the initial matrix of $\phi_{ij}$'s be positive semi-definite, but some care is required in selecting proper initial conditions for the nonstationary or transient solution.

If we wish to obtain the transient response of the system means and covariances $m_1$, $m_2$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$ to the application

of a stationary, non-white noise disturbance $x_3$ at t=0, the values of the means $m_3$ and variance $\phi_{33}$ of the disturbance must initially be at their steady state or stationary values, otherwise $x_3$ would not be a stationary process. The stationary values of $m_3$ and $\phi_{33}$ can be found by setting the right-hand sides of their respective differential equations (number 3 of (2.15) and number 6 of (2.16)) equal to zero. The result is

$$m_3 = 0, \quad \phi_{33} = \frac{e^2\Omega}{2}$$

These two differential equations can be eliminated and $m_3$ and $\phi_{33}$ replaced in the remaining equations by their steady state values above. On the other hand, a class of intentionally nonstationary disturbances could be treated by using other initial values for $m_3$ and $\phi_{33}$.

Some solutions for various values of $\Omega$ are shown in Figure (2.5). The disturbance is stationary and the system is known exactly to be at rest at t=0.

Example 3. Another possible application of the mean and covariance equations might be for systems which have a constant physical parameter whose value is uncertain as well as disturbances which are random functions of time. As a simple illustration consider a system described by

$$\dot{x} = ax + eq(t) \tag{2.17}$$

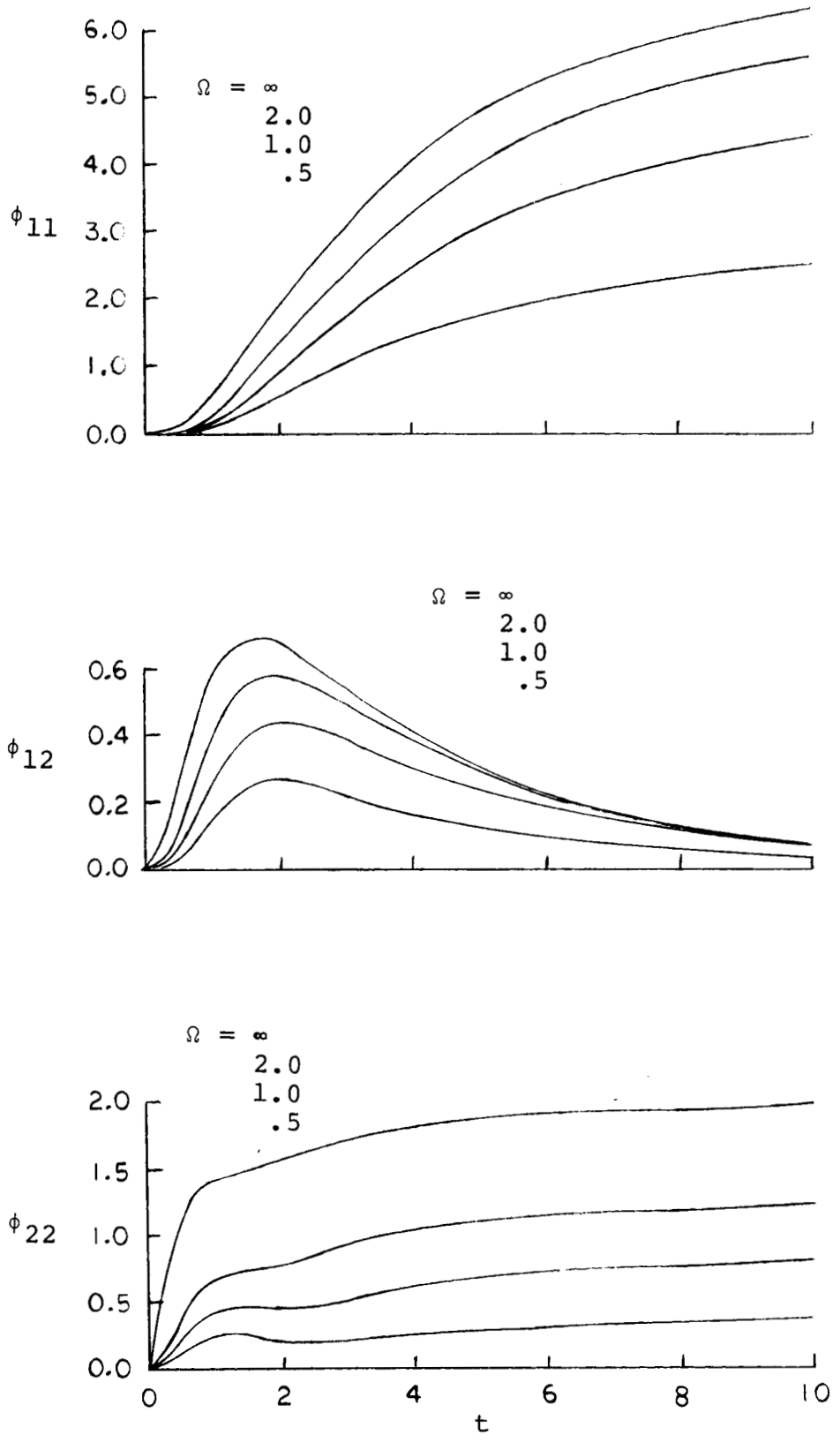



FIG. 2.5. Effect of disturbance cut-off frequency Ω on covariance solutions for example 2 with the initial values of all means and covariances zero and e=2.0.

where the constant a has an expected value $m_a$ and a variance $\phi_{aa}$.

To get the system into the form (2.1), a is treated as a second state variable satisfying the differential equation

$$\dot{a} = 0 \tag{2.18}$$

From the augmented system equations (2.17) and (2.18), the mean and covariance equations can be obtained using (2.2) and (2.3) as

$$\begin{aligned}
\dot{m}_x &= \overline{ax} \\
\dot{m}_a &= 0 \\
\dot{\phi}_{xx} &= +2\overline{(x-m_x)ax} + e^2 \\
\dot{\phi}_{ax} &= \overline{(a-m_a)ax} \\
\dot{\phi}_{aa} &= 0
\end{aligned} \tag{2.19}$$

The second and fifth equations of (2.19) are trivial since $m_a$ and $\phi_{aa}$ are constant statistical parameters describing our knowledge about a, and they can be eliminated. In the first equation $\overline{ax} = \phi_{ax} - m_a m_x$ from the definition of covariance. There are some third moments appearing in the third and fourth equations of (2.19) which with the gaussian approximation can be written in terms of $m_x$, $m_a$, $\phi_{xx}$, $\phi_{ax}$, $\phi_{aa}$ from the results in Appendix B. (It is known that x and a will not have a joint gaussian density distribution in general but the present method proceeds as though they will.)

The remaining equations in (2.19) after using the gaussian approximation become

$$\begin{aligned}\dot{m}_x &= \phi_{ax} + m_a m_x \\ \dot{\phi}_{xx} &= 2m_a\phi_{xx} + 2m_x\phi_{ax} + e^2 \\ \dot{\phi}_{ax} &= +m_x\phi_{aa} + m_a\phi_{ax}\end{aligned} \tag{2.20}$$

Assuming the initial value (at t=0) of the covariance $\phi_{ax}=0$ (i.e., initial value of x is not correlated with a) the solution for $m_x$ and $\phi_{xx}$ is found to be

$$m_x = \frac{m_x(0)}{2}\left[e^{(m_a+\sqrt{\phi_{aa}})t} + e^{(m_a-\sqrt{\phi_{aa}})t}\right] \tag{2.21}$$

$$\begin{aligned}\phi_{xx} = {} & \tfrac{1}{4}(m_x(0))^2\left[e^{2(m_a+\sqrt{\phi_{aa}})t} + e^{2(m_a-\sqrt{\phi_{aa}})t} - 2e^{2m_a t}\right] \\ & + \phi_{xx}(0)e^{2m_a t} + \frac{e^2}{2m_a}\left(1-e^{2m_a t}\right)\end{aligned} \tag{2.22}$$

The nature of this solution and its relation to two exact solutions are discussed in Section III.

Other Examples. Some additional examples are discussed in Section III where the accuracy of the gaussian approximation is examined.

## D. MEAN AND COVARIANCE EQUATIONS AS THE BASIS OF ESTIMATION AND OPTIMIZATION TECHNIQUES.

Although neither the estimation nor the optimization problem are treated here, a few words are in order to relate the mean and covariance equations as used in this study to

their application in estimation and optimization.

If the joint probability density function of the state variables of the dynamic system (2.1) is approximated by a gaussian probability density function, as suggested, equations (2.2) and (2.3) become a set of coupled first order differential equations which can be integrated for a given set of initial means and covariances to give an approximate set of means and covariances of the system state variables at some later time. The means and covariances at this later time can be considered as conditional means and covariances since they depend in general on the means and covariances at the initial time. Since the gaussian approximation is being used to make these equations integrable, another interpretation is that the integration of equations (2.2) and (2.3) provide a gaussian approximation to the conditional joint probability density function of the states--that is conditional on the knowledge or the joint probability density function of the states at the initial time.

There are many practical problems where this information alone is the desired result. For instance one might wish to determine the probability of a missile destroying a target from statistical knowledge of initial launch and target conditions, assumed statistical models of noise, target motion, disturbances, and a mathematical model of the missile and guidance dynamics. Or perhaps for the same missile a probabilistic description of structural loads is desired for design purposes. In estimation and optimization this is just

a necessary step in the overall problem.

Estimation. The nature of the so-called estimation problem is illustrated and the role of the mean and covariance equations in its treatment with a simple example. Suppose that a space vehicle is launched from earth at time $t=t_A$ on an interplanetary mission as illustrated in Figure 2.6. At time $t_A$ the state (i.e., velocity and position) of the vehicle is observed with a certain accuracy which is represented as a joint probability density or perhaps as means and covariances of the state variables at time $t=t_A$. At discrete times $t=t_B, t_C$, etc., during the flight it is intended to make radar observations of the vehicle's state to determine possible orbit corrections.

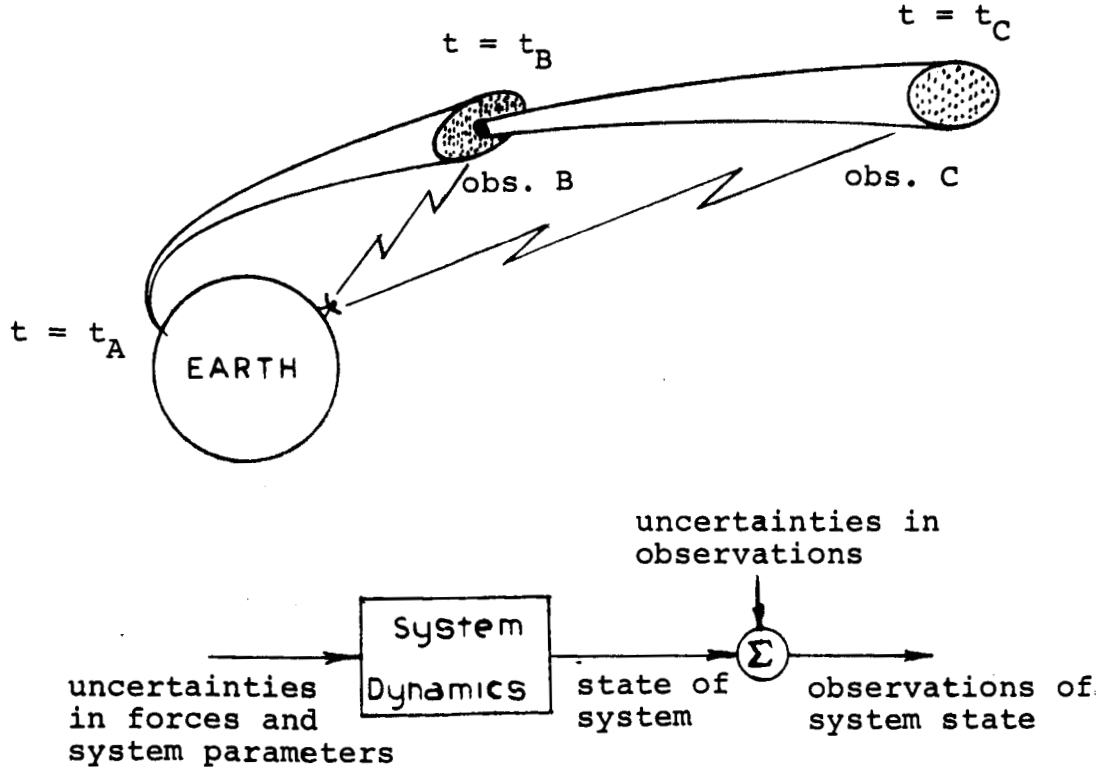



FIG. 2.6. An estimation problem.

The difficulty with this plan is that the uncertainty in the long range measurement of some of the state variables of the system is very great (e.g., radar directional information would be very poor although range and range rate would be very accurate) and it is desirable to make the best possible estimate of the state based on the available information. This is an estimation problem and we shall consider one possible approach to its solution.

First consider what information on the state of the vehicle at $t=t_B$ is available before observation B at $t=t_B$ is made. From observation A at $t=t_A$ initial means and covariances are known. If there is available a mathematical model of the vehicle dynamics and statistical models of the disturbances and unknown parameters, the mean and covariance equations for the system can be integrated to $t=t_B$ to obtain an approximation to the means and covariances or the implied gaussian joint probability density distribution at $t=t_B$ (represented by large shaded area at $t_B$). So, even though only an approximate solution is available, there is some knowledge of the system state just before the measurement at $t=t_B$.

If in addition the nature of the measurement error is known (e.g., its bias or mean and its covariances with itself and the system states) and the gaussian approximation is continued it is not difficult to obtain the conditional (on observation A) joint probability density of the state variables (the vector $\underline{x}_B$) and the observation variables (the vector $\underline{z}_B$) at $t=t_B$ given the observation $\underline{z}_A$ denoted as

$$p_{xz}(\underline{X}_B, \underline{Z}_B | \underline{Z}_A)$$

The conditional joint probability density

$$p_z(\underline{Z}_B | \underline{Z}_A)$$

may be obtained since it is just the marginal density obtained by integrating the preceding density over all the x's.

The reason for obtaining these particular conditional densities is that from them using Bayes' theorem (Reference 6 or 7) the conditional joint probability density of the system state $\underline{x}_B$ at $t=t_B$ given both observations $\underline{z}_A$ and $\underline{z}_B$ can be obtained as

$$p_x(\underline{X}_B | \underline{Z}_A, Z_B) = \frac{p_{xz}(\underline{X}_B, \underline{Z}_B | \underline{Z}_A)}{p_z(\underline{Z}_B | \underline{Z}_A)}$$

In other words, the joint density of the state variables (hence the means and covariances) can be up-dated based on the new observation at $t=t_B$ using this rule (represented by the small shaded area at $t=t_B$). If a "best" estimate of the system state at $t=t_B$ is desired one might take, for example, the expected values or means of the states from the new density function.

Now the mean and covariance equations are re-initialized based on the observation at $t=t_B$ and the procedure repeated at $t=t_C$, etc. This approach to trajectory estimation is treated in detail in Reference 6 using an approximation to

the mean and covariance equations almost identical to that of this study but reached by a much different approach. One can take the limit as observations are separated by less and less time until a method of treating continuous observations is obtained. This is also treated in Reference 6 for the nonlinear case as well as in Reference 7.

The mean and covariance equations, then, play the role of determining the "deterioration" of knowledge about the system state between observations and predicting something about what should be expected in the new observation based on old ones.

Optimization. The role of the mean and covariance equations in optimization of dynamic systems is much more direct and obvious. In the example of predicting hit probability of a missile, mentioned at the beginning of Part D, it is only natural to ask how missile parameters or gain changing programs might be altered to maximize the predicted hit probability. In the light of our approximate solution to the mean and covariance equations a similar but more tractable statement of the problem might be to minimize the expected value of the square of the terminal miss distance, which can be expressed in terms of the terminal means and covariances.

If a performance measure to be maximized or minimized can be expressed in terms of terminal means and covariances or as an integral over the run time of a function of these quantities, the optimization problem can be treated, in principle, like any other variational type of problem, the

only difference being that the mean and covariance equations replace the dynamic system equations as constraints. Practically, however, one of the most useful optimization methods, the gradient search (e.g., Reference 8), becomes very difficult to implement because the required partial derivatives of the nonlinear terms in $\bar{f}_i$ and $\overline{(x_i - m_i) f_j}$ are not easily evaluated when those averages must be obtained numerically. Therefore, other approaches, such as random search, which do not require the evaluation of a gradient must be used. Some preliminary studies on this subject were made in Reference 9 for optimal adjustment of system parameters.

# III. ACCURACY OF THE GAUSSIAN APPROXIMATION IN NONLINEAR SYSTEMS

## A. GENERAL COMMENTS

As indicated in Section II, it is certainly possible to integrate the mean and covariance equations for nonlinear systems if the joint probability density of the system state variables is approximated with the gaussian joint density function. The next problem to be treated, of course, is the relation of the solutions obtained by such a gross approximation to the true solution.

Ideally one would like to have a simple method to determine useful error bounds on the means and covariances for any system of interest. However, in the present study, at least, no such method has been found.

As a result of this inability to bound errors, an unsatisfactory but informative approach compares approximate solutions of several specific examples to exact solutions or Monte Carlo results. In Part B some examples are considered which produce reasonable results when treated by the approximate approach even though the systems are strongly nonlinear. In Part C two cases are treated in which great disagreement between approximate and exact solutions can be obtained even in the qualitative behavior of the solutions.

B. COMPARISON OF SOME APPROXIMATE MEAN AND COVARIANCE EQUATION SOLUTIONS WITH FOKKER-PLANCK AND MONTE CARLO SOLUTIONS.

Case 1. First consider Example 1 of Section II (Figure 2.1). This example has a known stationary solution for the joint probability density function of $x_1$ and $x_2$ by means of the Fokker-Planck equation (see Reference 1).

$$p_{12}(X_1,X_2) = C\exp[-\frac{\beta}{D}\{\frac{X_2^2}{2} + \int_0^{X_1} \text{sat}(\xi)d\xi\}] \qquad (3.1)$$

where for our case $\beta$=1.0 and $D=e^2/2$. The constant C is chosen to make the integral

$$\int_{-\infty}^{\infty}\int p_{12}(X_1,X_2)dX_1dX_2 = 1$$

From(3.1) it can be seen that $X_1$ and $X_2$ are statistically independent and that $X_2$ has a gaussian probability density. If second moments of the density function(3.1) are compared with stationary solutions of the covariance equations(2.10) in both cases $\phi_{12}$ is equal to zero and $\phi_{22}$ is equal to $\frac{e^2}{2}$. However, the values of $\phi_{11}$ do not agree and a plot of $\phi_{11}$ versus $e^2$ is shown in Figure 3.1. Remember that $e^2$ is the amplitude of the white noise power spectral density.

Note that the agreement between the standard deviations rather than variances would be even better. Also, remember that for $\phi_{11}>1$ the system is saturating heavily.

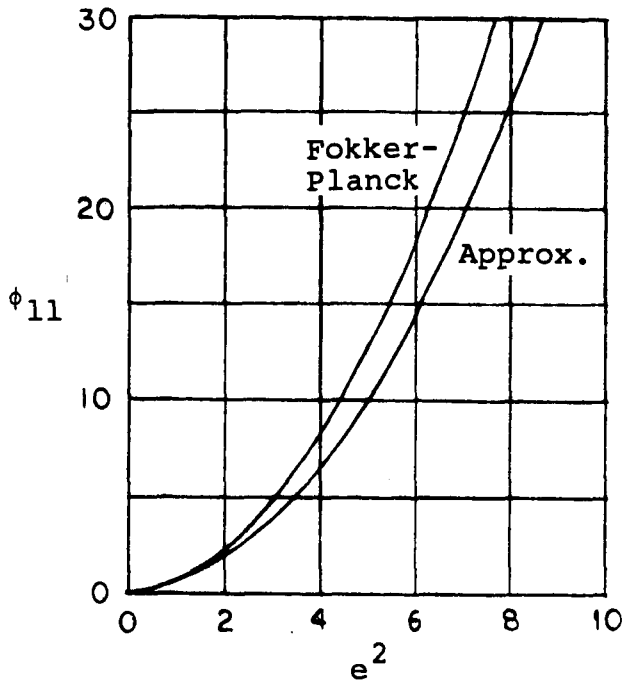



FIG. 3.1. Comparison of stationary solutions for $\phi_{11}$ for case 1 between Fokker-Planck and the gaussian approximation.

To get a feeling for the validity of the approximate mean and covariance approach for the non-stationary solution, the results for two different solutions of the mean and covariance equations for Example 1 (Equations (2.9) and (2.10)) are compared to Monte Carlo solutions using simulated gaussian white noise for 100 trials (see Appendix D). The results are shown in Figure 3.2 and Figure 3.3. The bands around the Monte Carlo solution points represent approximately plus and minus one standard deviation for the estimates of means and variances as described in Appendix D. The cases selected for Figures 3.2 and 3.3 are both well into the non-linear range of the system and still the agreement is reasonable.

Case 2. Next a Monte Carlo comparison is made with an approximate non-stationary solution of the mean and covariance

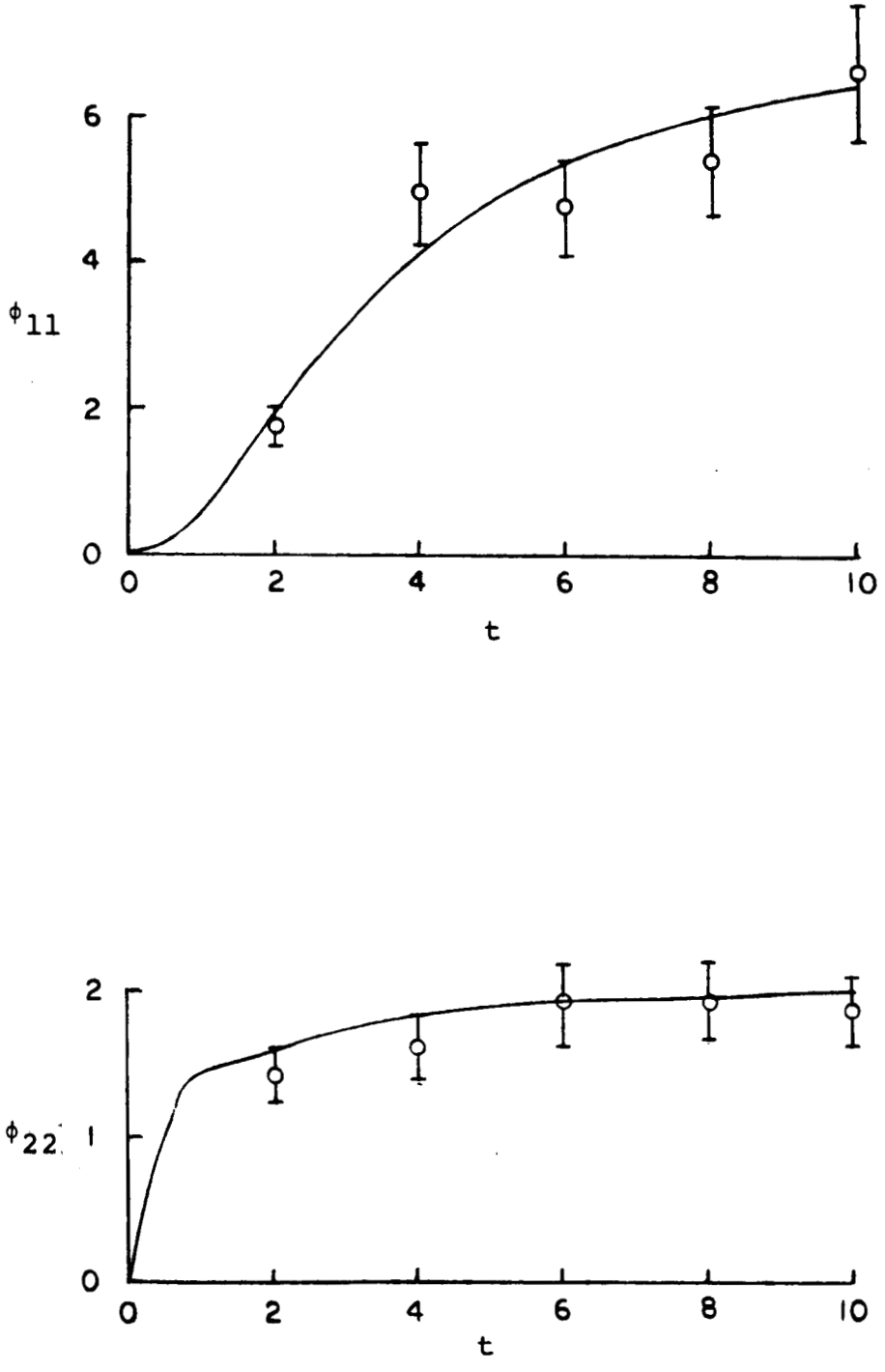



FIG. 3.2. Comparison with Monte Carlo results of the approximate solution for variances in case 1 with e=2.0 and zero initial values of means and covariances. (Monte Carlo results shown with one standard deviation range about estimate.)

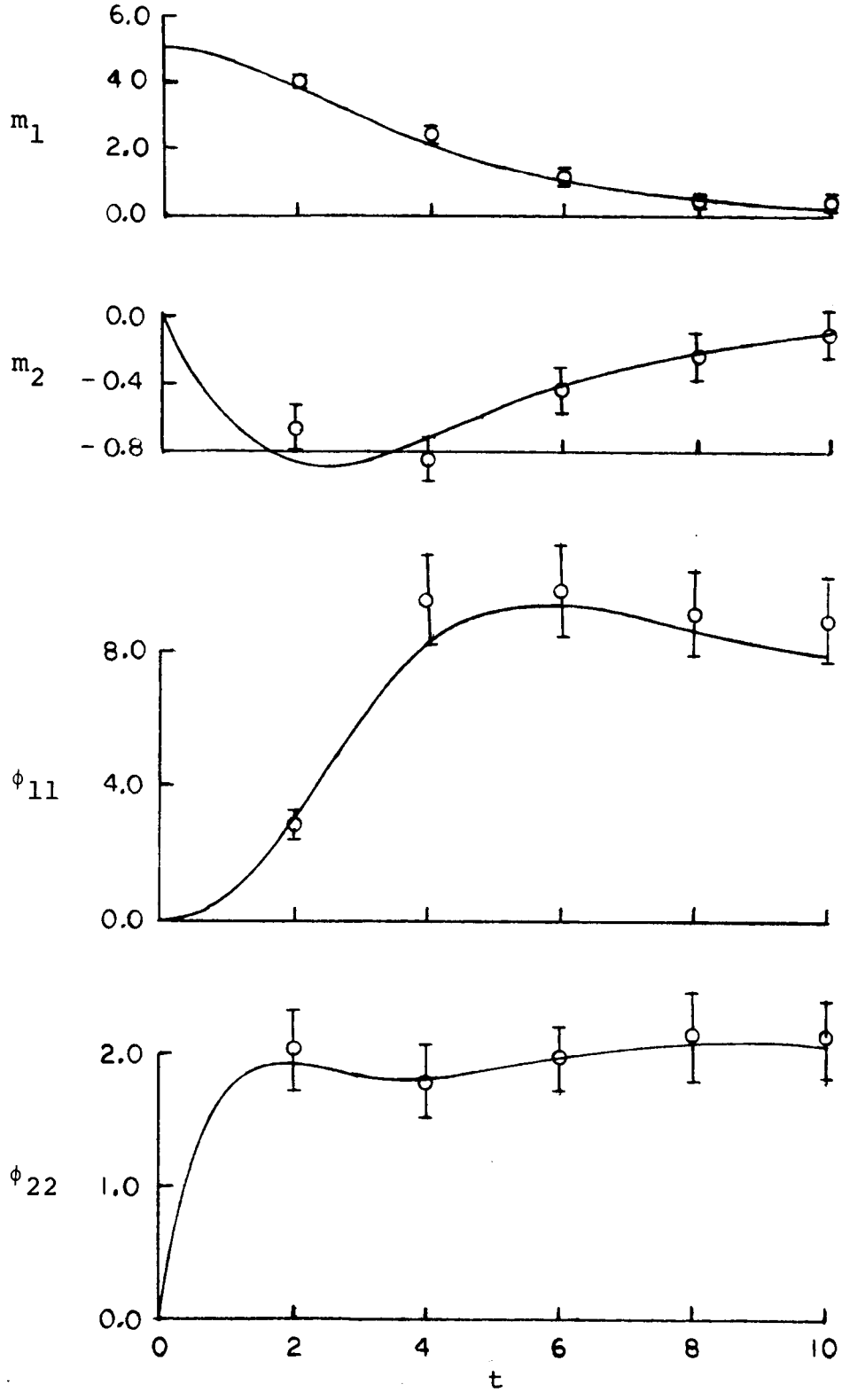
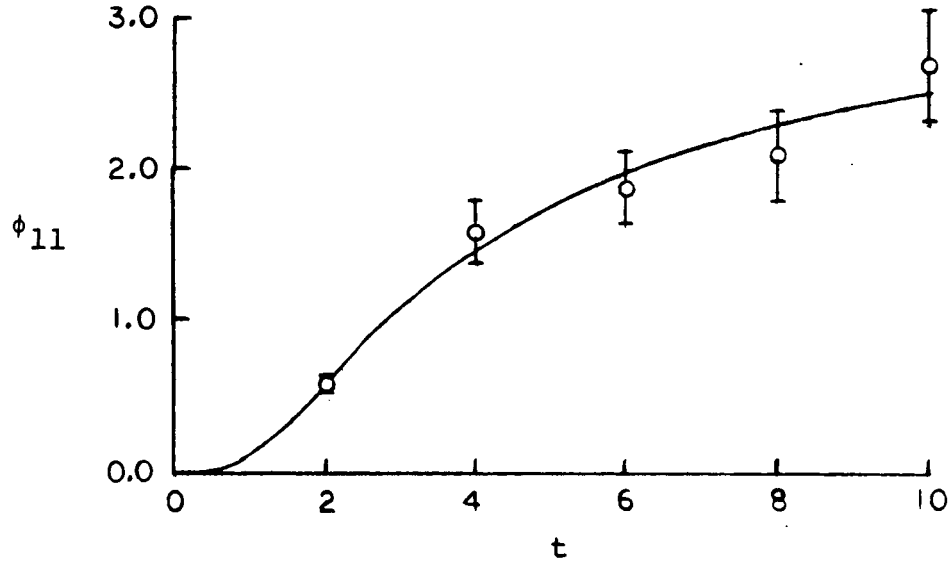



FIG. 3.3. Comparison with Monte Carlo results of the approximate mean and variance solutions in case 1 with e=2.0, $m_1(0)$=5.0, and all other initial conditions zero. (Monte Carlo results shown with one standard deviation range about estimate.)

equations (2.15) and (2.16) for Example 2 of Section II (Figure 2.4). The case chosen was for a stationary random disturbance $x_3$ produced by passing white noise through the low pass filter (Figure 2.4) with $\Omega$=.5, e=2.0. The results are shown in Figure 3.4. Again the comparison with Monte Carlo results is good even though the system was operating in a strongly nonlinear regime. The bands again represent an estimated plus or minus one standard deviation spread on the estimates of variances by the Monte Carlo approach.



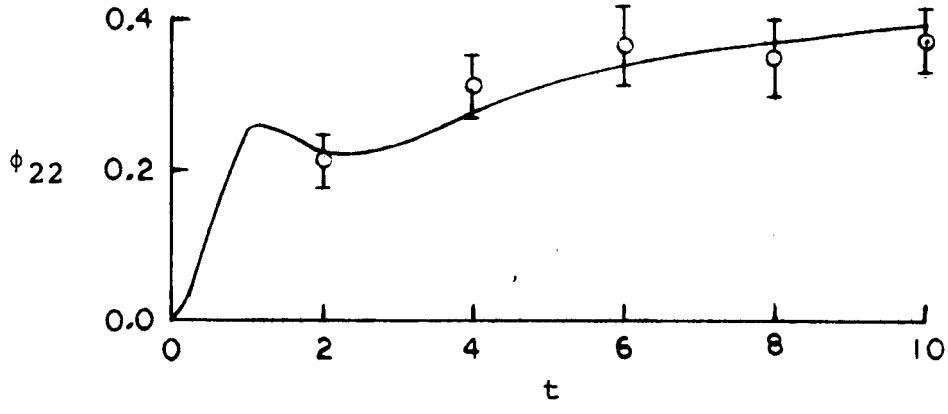




FIG. 3.4. Comparison with Monte Carlo results of the approximate mean and variance solutions in case 2 with e=2.0, $\Omega$=.5, initial conditions on $m_1$, $m_2$, $m_3$, $\phi_{11}$, $\phi_{12}$, $\phi_{22}$, $\phi_{13}$, $\phi_{23}$ zero, and $\phi_{33}(0) = e^2\Omega/2 = 1.0$ (stationary value.)

Case 3. The third case for comparison is a very old and very interesting example constructed in Reference 10 to allow an exact stationary solution of the Fokker-Planck equation to be obtained for a second order system having a stable limit cycle. The example is of interest here because the exact stationary joint probability density function is so obviously nongaussian.

The system is described by the two first order differential equations

$$\dot{x} = \{1-(x^2 + y^2)\}x - y + eq_1(t)$$
$$\dot{y} = \{1-(x^2 + y^2)\}y + x + eq_2(t) \qquad (3.2)$$

where e is constant and $q_1$ and $q_2$ are independent white noise disturbances with

$$\overline{q_1(t)q_1(t+\tau)} = \overline{q_2(t)q_2(t+\tau)} = \delta(\tau)$$

and

$$\overline{q_1(t)q_2(t+\tau)} \equiv 0.$$

The undisturbed solutions of the system 3.2 are characterized by an approach to a stable circular limit cycle of radius 1.0 and an unstable singular point at the origin on the phase plane. (Figure 3.5)

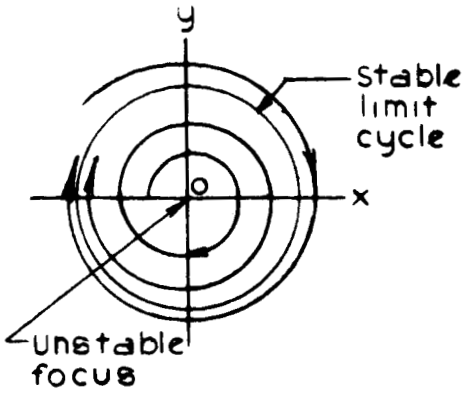



FIG. 3.5. Undisturbed solutions (Case 3) (from ref. 10)

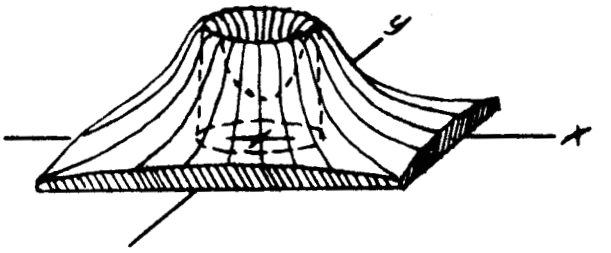


FIG. 3.6. Probability density (Case 3) (from ref. 10)

The steady state solution to the Fokker-Planck equation is found by transforming to polar coordinates. The joint probability density function for x and y (as found in Reference 10) is

$$p_{xy}(X,Y) = C\exp\{\frac{1}{e^2}(r^2 - \frac{1}{2}r^4)\} \qquad (3.3)$$

where $r^2=X^2+Y^2$ and C is a normalization constant. A sketch of the joint density function for a specified value of e is shown in Figure 3.6. The solution is a "crater-shaped" surface which has a minimum at the origin and whose maxima form a circle lying above the limit cycle of radius=1.0.

We shall now see how stationary solutions of the covariance equations using the gaussian approximation compare with the exact results. The covariance equations are written for the

zero mean case (since the stationary solution will obviously be zero mean) and the rules for expressing expected values of products of gaussian distributed variables discussed in Appendix B are used to approximate the required averages which occur for the system (3.2).

$$\dot{\phi}_{xx} = 2\phi_{xx} - 2\phi_{xy} - 6\phi_{xx}^2 - 4\phi_{xy}^2 - 2\phi_{xx}\phi_{yy} + e^2$$
$$\dot{\phi}_{xy} = 2\phi_{xy} + \phi_{xx} - \phi_{yy} - 6\phi_{xy}\phi_{xx} - 6\phi_{xy}\phi_{yy}$$
$$\dot{\phi}_{yy} = 2\phi_{yy} + 2\phi_{xy} - 6\phi_{yy}^2 - 4\phi_{xy}^2 - 2\phi_{xx}\phi_{yy} + e^2 \qquad (3.4)$$

The stationary solution of(3.4)shows that $\phi_{xy}=0$ and $\phi_{xx}=\phi_{yy}$ as expected. A plot of $\phi_{xx}$ versus $e^2$ obtained from the stationary solution of(3.4) is shown in Figure 3.7 along with a plot of $\phi_{xx}$ determined from the exact density function(3.3). The agreement is not bad, particularly since variances are being compared rather than standard deviations.

It is also interesting to compare cross sections (say, at the x axis) of the exact joint density function with a cross section of the approximating joint gaussian density function for various values of e shown in Figure 3.8. The disparity is particularly apparent at small excitation levels. It is remarkable that the variances agree as well as they do when the gaussian joint density is such a poor approximation.

C. SOME CASES OF EXTREME DISAGREEMENT WITH KNOWN EXACT SOLUTIONS.

Two cases have been found during this study where even the qualitative behavior of solutions to the approximate mean

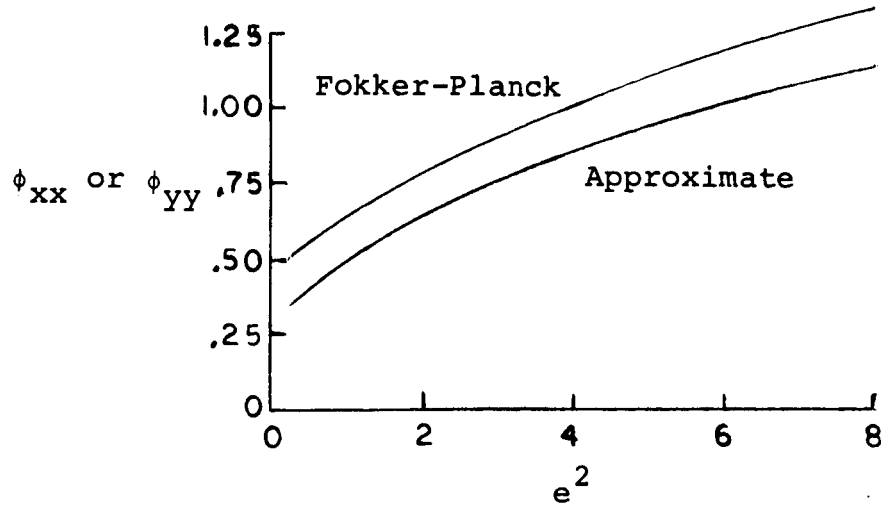



FIG. 3.7. Comparison of Fokker-Planck and approximate solutions for $\phi_{xx}$ or $\phi_{yy}$ in case 3.

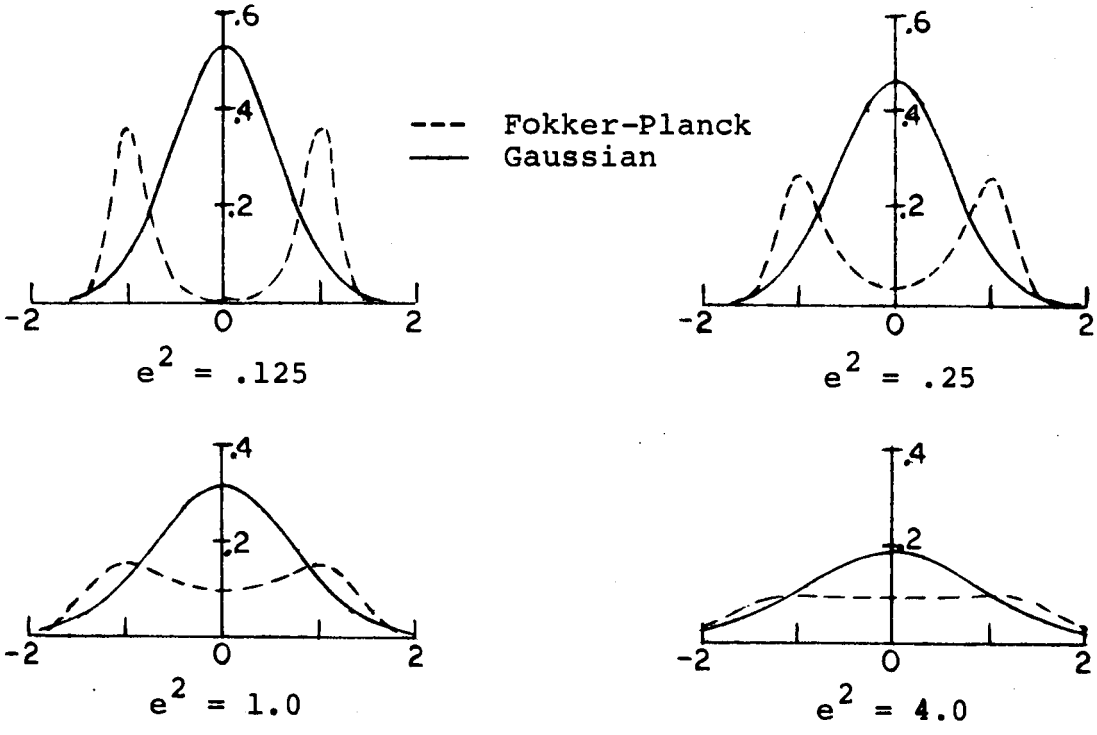



FIG. 3.8. Comparison of the central cross section of the joint probability density from Fokker-Planck with the gaussian approximation.

and covariance equations is different than the exact solutions. The first troublesome situation occurs in nominally stable systems with a random parameter or coefficient which has a finite probability of taking on a value which causes the system to become unstable. The second discrepency involves systems which have a limited region of stability (in state space) about a stable equilibrium point. When such a system is randomly excited in a manner which might drive it unstable, the covariance equation approach and Fokker-Planck approach to the analysis disagree on the nature of the solution. The examples chosen to illustrate these two cases were simple first order systems because of the ease of obtaining solutions by covariance and exact methods.

Case 1. First consider the simple random parameter example introduced in Section II (example 3) whose system equation is (2.17). In this case, however, no random excitation will be considered for simplicity (e=0). The system equation is simply

$$\dot{x} = ax \tag{3.5}$$

where a is a random, constant coefficient.

Assuming a and x have a joint gaussian probability density and using the covariance equation approach, as was done in Example 3, Section II, the mean and variance of x are found from (2.21) and (2.22) by setting the coefficient e=0.

$$m_x = \frac{m_x(0)}{2}\left[e^{(m_a + \sqrt{\phi_{aa}})t} + e^{(m_a - \sqrt{\phi_{aa}})t}\right] \tag{3.6}$$

$$\phi_{xx} = \frac{1}{4}(m_x(0))^2\left[e^{2(m_a + \sqrt{\phi_{aa}})t} + e^{2(m_a - \sqrt{\phi_{aa}})t} - 2e^{2m_a t}\right] + \phi_{xx}(0)e^{2m_a t} \tag{3.7}$$

Note that for negative a the system (3.5) is stable. For the problem of a random value of a, we must state our knowledge of a in terms of its expected value $m_a$ and its variance $\phi_{aa}$. From (3.6) and (3.7) it can be seen that if $m_a$ is negative and $\sqrt{\phi_{aa}} < |m_a|$, then the mean $m_x$ and variance $\phi_{xx}$ of the state variable x approach zero as $t \to \infty$ regardless of the initial values $m_x(0)$ and $\phi_{xx}(0)$. If, however, $\sqrt{\phi_{aa}} > |m_a|$ both $m_x$ and $\phi_{xx}$ grow without bound in magnitude.

Using a conditional probability density approach to the random parameter problem for the system (3.5), exact solutions can be obtained for any type of probability density of the parameter a. The solution for a gaussian distributed a (designated $m_{xg}$ and $\phi_{xxg}$) is

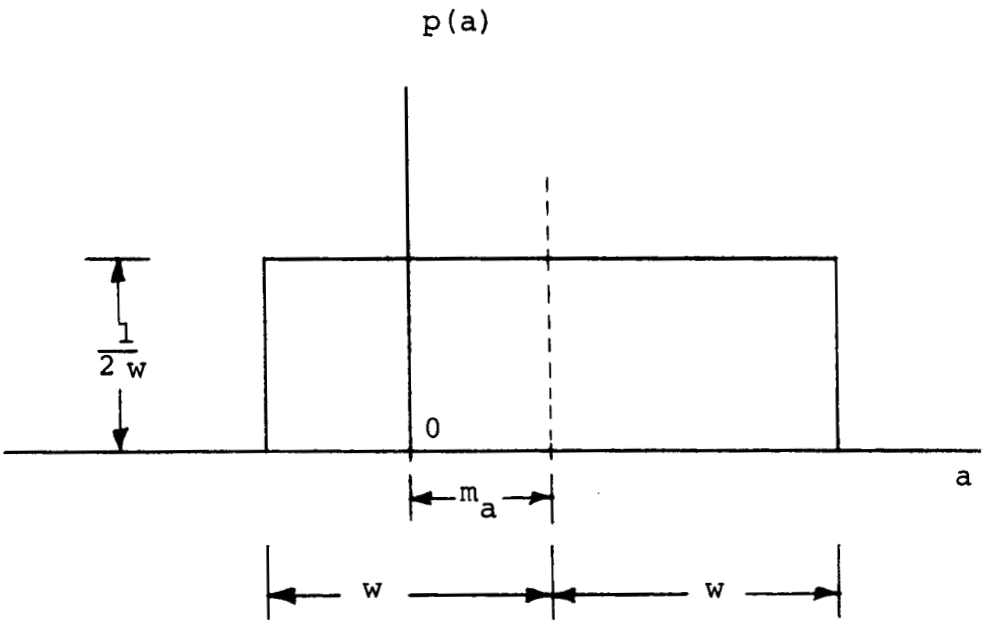
$$m_{xg} = m_x(0)e^{(m_a t + \frac{1}{2}\phi_{aa}t^2)} \tag{3.8}$$

$$\phi_{xxg} = (m_x(0))^2 e^{2(m_a t + \phi_{aa}t^2)}(1 - e^{-\phi_{aa}t^2}) + \phi_{xx}(0)e^{2(m_a t + \phi_{aa}t^2)} \tag{3.9}$$

The solutions (3.8) and (3.9) show that the mean and variance of x grow in magnitude without bound for any non-zero values of $m_a$ and $\phi_{aa}$ ($\phi_{aa}$, of course, must always be positive).

Although this differs markedly from the approximate covariance solution (3.6) and (3.9), it is quite reasonable since the unbounded nature of the gaussian distributed parameter a always gives a non-zero probability that the system will be unstable. It is the unbounded nature of the gaussian density which makes this result somewhat unrealistic.

To complete the picture an exact solution was found for a random parameter a having a bounded uniform or "flat top" probability density function (see sketch).




For this density function $\phi_{aa}=w^2/3$. The solutions for the mean and variance of x (designated $m_{xf}$ and $\phi_{xxf}$) are

$$m_{xf} = \frac{m_x(0)}{2wt}\left(e^{(m_a + w)t} - e^{(m_a - w)t}\right) \qquad (3.10)$$

$$\phi_{xxf} = \frac{(m_x(0))^2}{4wt}\{[e^{2(m_a+w)t} - e^{2(m_a-w)t}] + \frac{1}{wt}[e^{2(m_a+w)t} + e^{2(m_a-w)t} - 2e^{2m_a t}]\} + \frac{\phi_{xx}(0)}{4wt}[e^{2(m_a+w)t} - e^{2(m_a-w)t}] \qquad (3.11)$$

where $w=\sqrt{3\phi_{aa}}$. This solution is similar in nature to the covariance equation solution(3.6) and(3.7) in that for negative $m_a$ there is a threshold of $\phi_{aa}<|m_a/\sqrt{3}|$ for which $m_{xf}$ and $\phi_{xxf}$ approach zero as t approaches $\infty$.

The three different solutions discussed all have the form

$$m_x = m_x(0)A(t)$$
$$\phi_{xx} = \phi_{xx}(0)B(t) + (m_x(0))^2 C(t) \qquad (3.12)$$

The time functions A(t), B(t), and C(t) are plotted for $m_a=-1$ and various values $\phi_{aa}$ in Figure 3.9. For values of $\phi_{aa}\leq .4$ agreement between the approximate solution and the exact flat density solution is quite good. The agreement between the approximate and exact gaussian is even good for $\phi_{aa}\leq .4$ in the interval $0\leq t\leq 4$ which was plotted, but for larger t the exact gaussian solution begins to grow. For $\phi_{aa}=.8$ both exact solutions are growing significantly while the approximate solution is still decaying. At $\phi_{aa}=1.2$ the A and C parts of the approximate solution begin to grow (but not nearly fast enough) and the B part still decays.

It seems paradoxical, but the approximate method works

best in simulating situations where the probability density function of the parameter is bounded (even though it is based on a gaussian approximation) and when the standard deviation of the parameter is small compared to the expected value. Agreement with the unbounded, gaussian distributed parameter solution is good only for short times even with small standard deviation of the parameter.

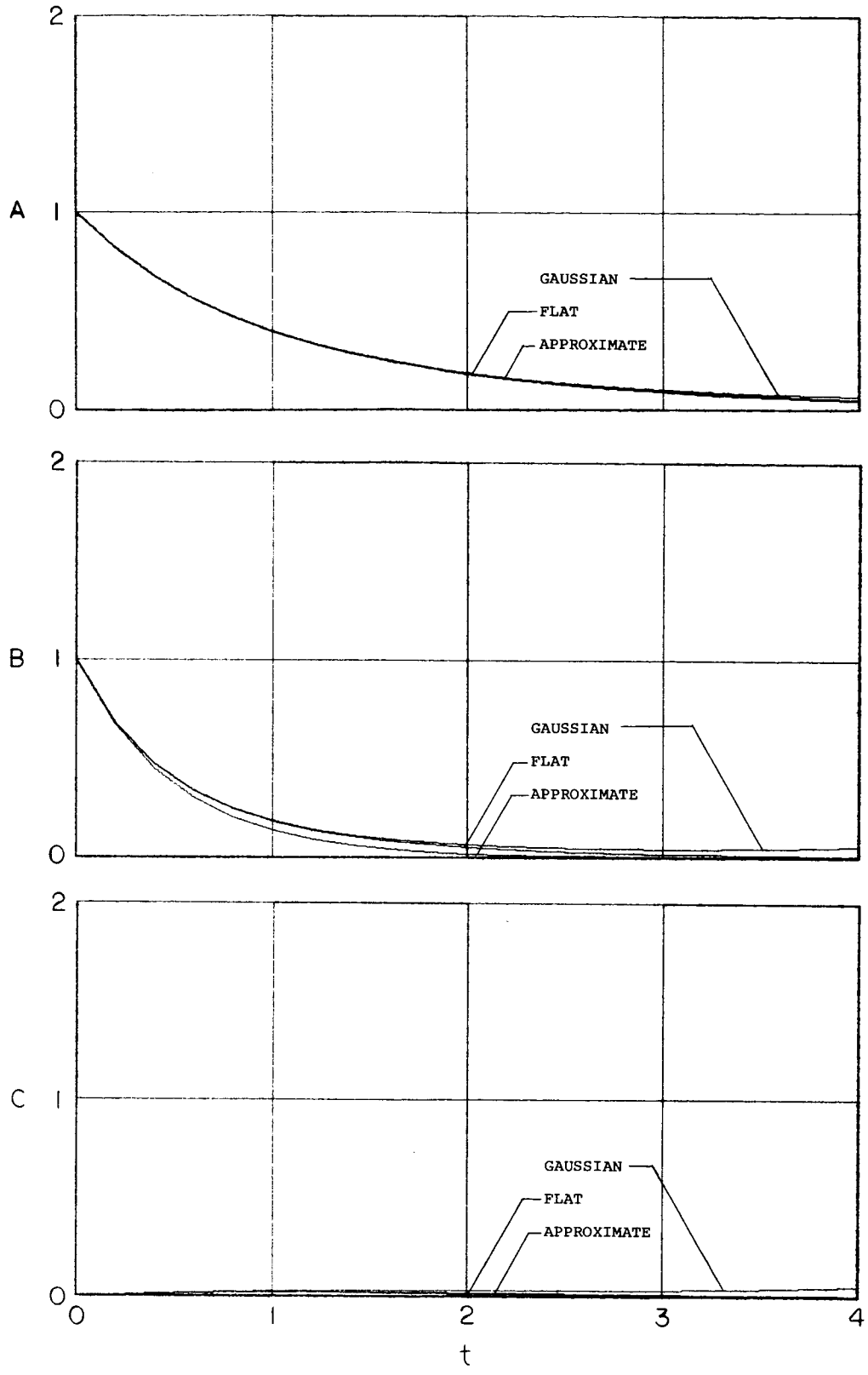
Case 2. Now consider another simple first order system.

$$\dot{x} = ax - \mathrm{sgn}(x) + eq(t) \tag{3.13}$$

where $\overline{q(t)q(t+\tau)} = \delta(\tau)$, $a>0$, and the function sgn(x) is defined by

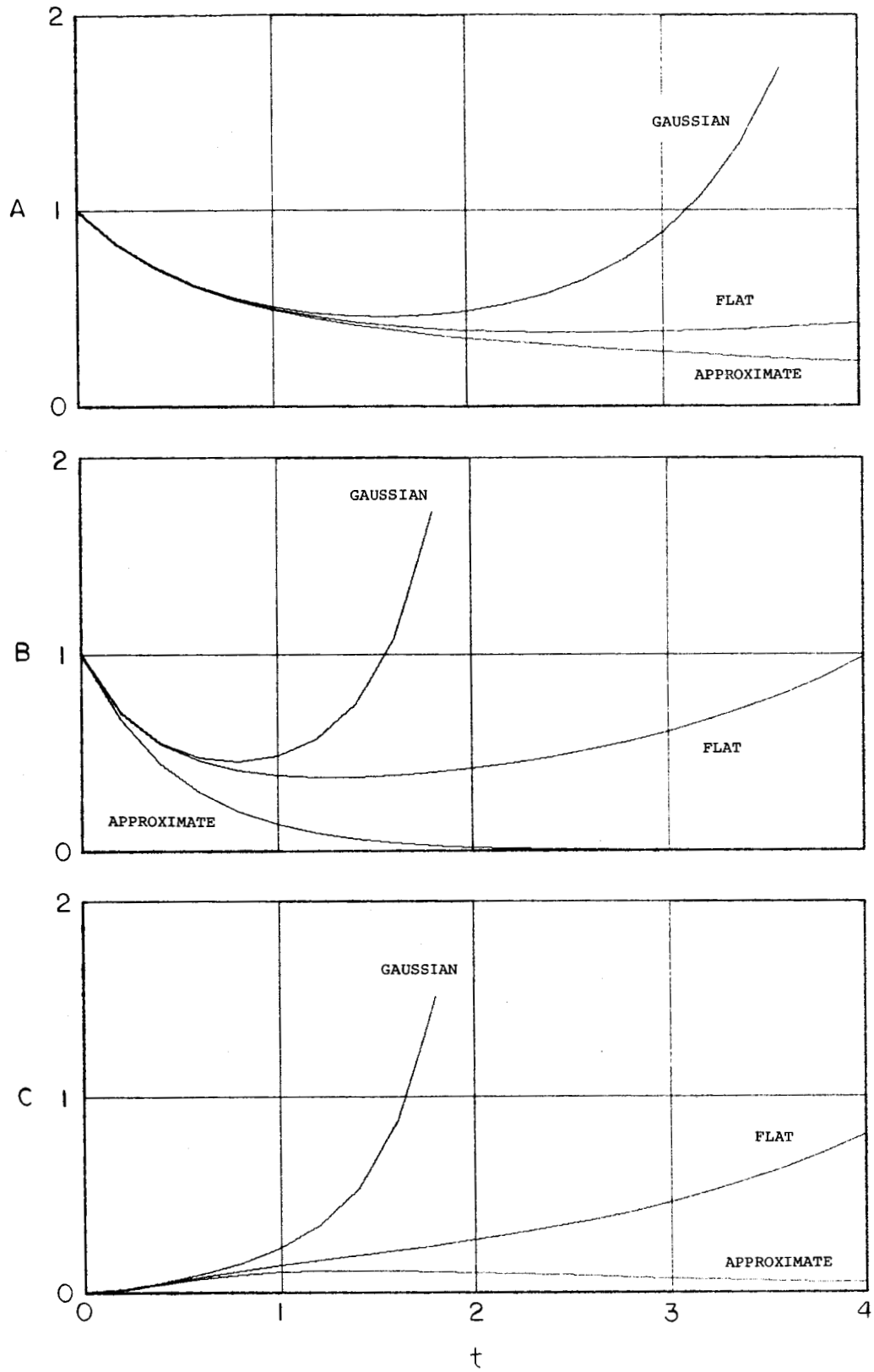
$$\mathrm{sgn}(x) = +1 \text{ if } x>0$$
$$= -1 \text{ if } x<0$$

The undisturbed system(3.13)has a stable equilibrium point at x=0 for initial conditions $|x_0|<\frac{1}{a}$. For initial conditions $|x_0|>\frac{1}{a}$, x grows indefinitely. This problem arises in automatic stabilization of unstable systems (Reference 11).



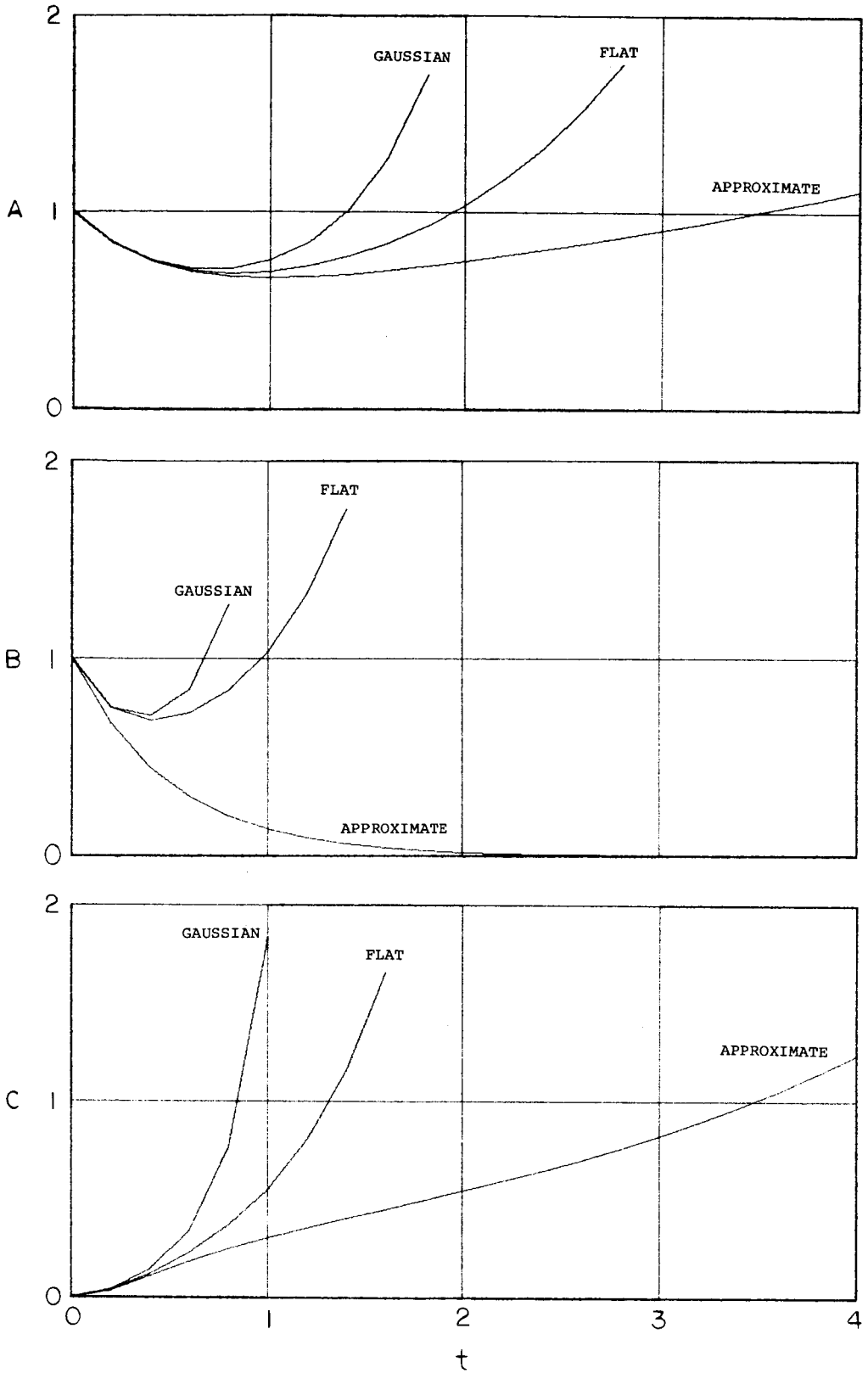

(a) $\phi_{aa} = .4$

FIG. 3.9. COMPARISON BETWEEN EXACT AND APPROXIMATE SOLUTIONS OF THE RANDOM PARAMETER EXAMPLE.




(b) $\phi_{aa} = .8$

FIG. 3.9. CONTINUED.




(c) $\phi_{aa} = 1.2$

FIG. 3.9. CONCLUDED.

If the zero mean case is treated, the variance equation based on a gaussian approximation becomes

$$\dot{\phi} = 2a - \sqrt{2/\pi}\sqrt{\phi} + e^2 \qquad (3.14)$$

Of particular interest are the stationary or equilibrium solutions of 3.14.

$$\phi = [1/4\pi a^2][1-2\pi a e^2 \pm \sqrt{1-4\ ae^2}], \quad e^2 < \frac{1}{4\pi a} \qquad (3.15)$$

The smaller solution (-sign) is a stable equilibrium and the larger (+sign) is unstable. If $\phi(0)$ is below the unstable equilibrium, $\phi$ will approach the stable equilibrium point as $t \to \infty$. If $\phi(0)$ is above the unstable equilibrium point $\phi$ will grow without bound. If $e^2 > \frac{1}{4\pi a}$ there are no equilibrium points and $\phi$ grows without bound regardless of the initial condition.

The nature of the solution just described seems to make sense physically, but the Fokker-Planck solution to this problem shows that no stationary solution exists under any condition, (see Reference 11). Here, as in Case 1 above, it again appears as if the covariance equation approach is simulating more accurately the case of a disturbance with a bounded probability density.

## D. REMARKS ON THE SUCCESSFUL APPLICATION OF THE APPROXIMATE MEAN AND COVARIANCE EQUATIONS TO NONLINEAR SYSTEMS.

Of course, the examination of a few simple examples can give no valid general conclusions about the accuracy of the gaussian approximation approach to solving the mean and covariance equations. From these and other examples, however, it appears that if the expected joint probability density function for the problem under consideration is at all similar to a gaussian density (i.e., it has a single maximum and diminishes for large arguments at least exponentially) reasonably good results can be expected.

Situations requiring extreme caution are those involving nominally stable systems which have a finite probability of becoming unstable either due to a random parameter or random excitation.

## IV. CONCLUSION

A method has been presented for computing approximate means and covariances of the state variables of nonlinear, nonstationary, dynamic systems subjected to white noise disturbances. The technique is based on a gaussian approximation of the joint probability density function of the system state variables allowing the formulation of a set of ordinary differential equations which govern the evolution in time of the means and covariances of the system state variables. This technique is applicable to a large class of nonlinear, time-varying problems and is well suited for digital computation.

The method is approximate, however, and the nature of the approximation is not well understood. The several examples for which comparisons with Fokker-Planck and Monte Carlo solutions were made, showed that the approximate method produced quite reasonable results except for certain problems involving nominally or conditionally stable systems which had finite probabilities of becoming unstable. The approach is not limited to stable systems, but systems which can be considered neither stable nor unstable appear to cause difficulties. On the basis of the few cases examined, no general conclusions on accuracy and applicability of the method could be made, of course.

The mean and covariance equation approach, while it has many advantages over power spectral density approaches, has one distinct disadvantage in that it gives no direct method of evaluating the frequency content or auto-correlation of a state variable. For stationary or quasi-stationary problems, however, the method used by Rice (Reference 12) for finding the expected frequency of crossing of a given threshhold level can easily be adapted for use with the mean and covariance equations.

In spite of its shortcomings, use of the approximate mean and covariance equation for treatment of randomly disturbed dynamic systems can, with judicious application, provide a practical means of computation for a much larger class of problems than other current methods.

## APPENDIX A

## DEVELOPMENT OF THE MEAN AND COVARIANCE EQUATIONS

For simplicity the mean and covariance equations will be developed for only a single white noise input, as the extention to several independent inputs is straight forward. Consider a dynamic system described by

$$\dot{x}_i = f_i(x_1,\ldots,x_n,t) + e_i q(t), \quad i=1,\ldots,n \tag{A-1}$$

where the coefficients $e_i$ can be functions of time and q(t) is zero mean, stationary "white" noise with an auto-correlation function

$$\overline{q(t)q(t+\tau)} = \delta(\tau) \tag{A-2}$$

Define the means

$$m_i = \bar{x}_i \tag{A-3}$$

and the covariances

$$\phi_{ij} = \overline{(x_i-m_i)(x_j-m_j)} = \overline{x_i x_j} - m_i m_j \tag{A-4}$$

The rate of change of the means can be written

$$\dot{m}_i = \lim_{\Delta t \to 0} \frac{m_i(t+\Delta t) - m_i(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{x_i(t+\Delta t) - x_i(t)}}{\Delta t} \qquad \text{(A-5)}$$

From the differential equations (1)

$$x_i(t+\Delta t) - x_i(t) = f_i(x_1, \ldots, x_n, t)\Delta t + \int_t^{t+\Delta t} e_i(\eta) q(\eta) d\eta + 0(\Delta t^2) \qquad \text{(A-6)}$$

Taking the expected value of both sides of (A-6) and substituting into (A-5)

$$\dot{m}_i = \lim_{\Delta t \to 0} \left[ f_i(x_1, \ldots, x_n, t) + \frac{1}{\Delta t} \int_t^{t+\Delta t} e_i(\eta) q(\eta) d\eta + \frac{1}{\Delta t} 0(\Delta t^2) \right] \qquad \text{(A-7)}$$

Since $\overline{q(\eta)} = 0$ and $\frac{1}{\Delta t} 0(\Delta t^2) = 0(\Delta t)$,

$$\dot{m}_i = \overline{f_i(x_1, \ldots, x_n, t)} \qquad \text{(A-8)}$$

To obtain an expression for the rate of change of the covariances, (A-4) can be written

$$\dot{\phi}_{ij} = \frac{d}{dt} (\overline{x_i x_j}) - \dot{m}_i m_j - \dot{m}_i m_j \qquad \text{(A-9)}$$

and from (A-8) and (A-9)

$$\dot{\phi}_{ij} = \frac{d}{dt} (\overline{x_i x_j}) - m_j \overline{f_i(1, \ldots, x_n, t)} - m_i \overline{f_j(x_1, \ldots, x_n, t)} \qquad \text{(A-10)}$$

By definition the first term on the right-hand side of(A-10) is

$$\frac{d}{dt}(\overline{x_i x_j}) = \lim_{\Delta t \to 0} \left[\frac{\overline{x_i(t+\Delta t)x_j(t+\Delta t)} - \overline{x_i(t)x_j(t)}}{\Delta t}\right] \quad \text{(A-11)}$$

Using the results of(A-6) in(A-11)

$$\frac{d}{dt}(\overline{x_i x_j}) = \lim_{\Delta t \to 0} [\underbrace{\overline{x_i f_j}}_{1} + \underbrace{\overline{x_j f_i}}_{2} + \underbrace{x_i \frac{1}{\Delta t}\int_t^{t+\Delta t} e_j(\eta)q(\eta)d\eta}_{3} + \underbrace{x_j \frac{1}{\Delta t}\int_t^{t+\Delta t} e_i(\xi)q(\xi)d\xi}_{4} + \underbrace{f_j \int_t^{t+\Delta t} e_i(\xi)q(\xi)d\xi}_{5} + \underbrace{f_i \int_t^{t+\Delta t} e_j(\eta)q(\eta)d\eta}_{6} + \underbrace{\frac{1}{\Delta t}\int_t^{t+\Delta t}d\eta\int_t^{t+\Delta t}d\xi e_j(\eta)e_i(\eta)q(\xi)q(\eta)}_{7}$$

$$+ 0(\Delta t)] \quad \text{(A-12)}$$

First consider term (3) in(A-12). From the theorem of the mean term (3) can be rewritten as

$$e_j(t+\varepsilon\Delta t)\overline{x_i(t)q(t+\varepsilon\Delta t)} \ , \ 0 \leq \varepsilon \leq 1 \quad \text{(A-13)}$$

In the limit as $\Delta t \to 0$,(A-13) becomes

$$e_j(t)\overline{x_i(t)q(t)} \quad \text{(A-14)}$$

where $\overline{x_i(t)q(t)}$ is just the covariance of $x_i$ and q. But $x_i$ can depend only on the past history of q ( not its present value) and since q is "white" noise (i.e., the present value is not correlated with any past value) we can expect $\overline{x_i q}$

and hence the contribution of term ③ to be zero. Similar arguments can be used on terms ④, ⑤ and ⑥ to show that they should also be zero in the limit as $\Delta t \to 0$.

Noting that

$$\overline{q(\xi)q(\eta)} = \delta(\xi-\eta) \tag{A-15}$$

term ⑦ in (A-12) can be evaluated.

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} d\eta \int_t^{t+\Delta t} d\xi e_j(\eta) e_i(\xi)\delta(\xi-\eta) = \frac{1}{\Delta t}\int_t^{t+\Delta t} e_j(\eta)e_i(\eta)\,d\eta$$

$$= e_j(t+\varepsilon\Delta t)e_i(t+\varepsilon\Delta t) \quad , \quad 0 \leq \varepsilon \leq 1 \tag{A-16}$$

In the limit as $\Delta t \to 0$ term ⑦ becomes $e_i(t)e_j(t)$ and (A-12) becomes

$$\frac{d}{dt}(\overline{x_i x_j}) = \overline{x_i f_j} + \overline{x_j f_i} + e_i e_j \tag{A-17}$$

The expression for rate of change of covariance can be obtained from (A-17) and (A-10)

$$\dot{\phi}_{ij} = \overline{(x_i - m_i) f_j} + \overline{(x_j - m_j) f_i} + e_i e_j \tag{A-18}$$

The results for more than one independent white noise input are given in equation (2.2) and (2.3) of Section II.

## APPENDIX B

## USEFUL RELATIONS RESULTING FROM THE GAUSSIAN APPROXIMATION

In this appendix several results are presented which are of great value in the practical application of the mean and covariance equations to nonlinear systems by means of the gaussian approximation. The results are not new but just presented in a form which is convenient for this application.

Nonlinear functions of a single variable.

If a dynamic system contains nonlinear functions of a single state variable $g(x_i)$, the mean and covariance equations for that system will contain terms of the type $\overline{g(x_i)}$ and $\overline{(x_j-m_j)g(x_i)}$.

The first simplifying result of the gaussian approximation is

$$\overline{(x_j-m_j)g(x_i)} = \frac{\phi_{ij}}{\phi_{ii}}\overline{(x_i-m_i)g(x_i)} \qquad \text{(B-1)}$$

This reduces the two-dimensional average to a one-dimensional average which is the same for any j. (Note: Care should be taken to avoid allowing $\phi_{ii}$ to become zero in machine computations. If an initial value $\phi_{ii}=0$ is required for a problem, $\phi_{ii}$ should be set initially to some very small positive value instead.)

Using the result(B-1), only two averages must be computed for each function $\overline{g(x_i)}$ occuring in the system:

$$\overline{g(x_i)} = (2\pi\phi_{ii})^{-1/2} \int_{-\infty}^{\infty} g(x_i) e^{-\frac{(x_i-m_i)^2}{2\phi_{ii}}} dx_i \qquad \text{(B-2)}$$

$$\overline{(x_i-m_i)g(x_i)} = (2\pi\phi_{ii})^{-1/2} \int_{-\infty}^{\infty} (x_i-m_i) g(x_i) e^{-\frac{(x_i-m_i)^2}{2\phi_{ii}}} dx_i \qquad \text{(B-3)}$$

The averages (B-2) and (B-3) are functions only of $m_i$ and $\phi_{ii}$. (B-2) and (B-3) can occasionally be evaluated analytically but often must be done numerically. FORTRAN subprograms for this purpose are presented in Appendix C.

Some special functions of more than one variable.

Sometimes because of the particular set of state variables chosen, especially in automatic control systems, the input z to a nonlinear function of a single variable g(z) is a linear combination of the state variables

$$z = \sum_{i=1}^{n} a_i x_i$$

thus making g(z) a degenerate function of several variables. The user is then faced with evaluating averages of the form $\overline{(x_j-m_j)g(z)}$. Applying (B-1) the following helpful result is obtained:

$$\overline{(x_j - m_j)g(z)} = \frac{\sum_{i=1}^{n} a_i \phi_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \phi_{ij}} \overline{(z-\bar{z})g(z)} \qquad \text{(B-4)}$$

The averages $\overline{g(z)}$ and $\overline{(z-\bar{z})g(z)}$ can be obtained from (B-2) and (B-3) noting that $\bar{z} = \sum_{i=1}^{n} a_i m_i$ and $\overline{(z-\bar{z})^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \phi_{ij}$.

Another type of special nonlinear function of more than one variable which often occurs is the product of two or more state variables. Results are given for the averages required with products of two variables, $x_i x_j$, and of three variables $x_i x_j x_k$. The results are obtained using the relations for higher moments of a joint gaussian density function. Two variable product, $x_i x_j$:

$$\overline{x_i x_j} = \phi_{ij} + m_i m_j \qquad \text{(B-5)}$$

$$\overline{(x_k - m_k) x_i x_j} = m_i \phi_{kj} + m_j \phi_{ki} \qquad \text{(B-6)}$$

Three variable product, $x_i x_j x_k$;

$$\overline{x_i x_j x_k} = m_i m_j m_k + m_i \phi_{jk} + m_j \phi_{ik} + m_k \phi_{ij} \qquad \text{(B-7)}$$

$$\overline{(x_\ell - m_\ell) x_i x_j x_k} = m_i m_j \phi_{\ell k} + m_i m_k \phi_{\ell j} + m_j m_k \phi_{\ell i} + \phi_{\ell k}\phi_{ij} + \phi_{\ell j}\phi_{ik} + \phi_{\ell i}\phi_{jk} \qquad \text{(B-8)}$$

General functions of more than one variable.

First consider a system containing general nonlinear

functions of two variables of the form $g(x_1,x_2)$. The mean and covariance equations require the evaluation of averages of the type $\overline{g(x_1,x_2)}$ and $\overline{(x_j-m_j)g(x_1,x_2)}$. A result analogous to (B-1) can be obtained with the gaussian assumption to reduce the second averages from several three-dimensional integrals to two two-dimensional integrals.

$$\overline{(x_j-m_j)g(x_1,x_2)} = k_1\overline{(x_1-m_1)g(x_1,x_2)} + k_2\overline{(x_2-m_2)g(x_1,x_2)} \qquad \text{(B-9)}$$

where $k_1$ and $k_2$ are solutions to the linear equations

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \end{bmatrix} \qquad \text{(B-10)}$$

The form of (B-9) and (B-10) can be directly extended to functions of three or more variables if required. The averages on the right-hand side as well as $\overline{g(x_1,x_2)}$ must be obtained by integrating with the two-dimensional joint gaussian density function. A FORTRAN subprogram for evaluating these averages with a two-dimensional numerical integration is given in Appendix C. (Note: Just as in the use of (B-1), care must be taken to assure

$$\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} \neq 0$$

in using (B-9) on a digital computer. Small values for $\phi_{11}$ and $\phi_{22}$ should be inserted instead of zeros when zero initial conditions are required.)

## APPENDIX C

## FORTRAN SUBROUTINES FOR COMPUTATION OF AVERAGES OF NONLINEAR FUNCTIONS

In this Appendix subprograms are presented which compute the averages required by the mean and covariance equations of systems having three different types of one-dimensional nonlinearities and one two-dimensional nonlinearity. All the subprograms are written in IBM 1620 FORTRAN II but are written so that they will run unaltered in any standard FORTRAN IV system. These programs have all been successfully run in a number of cases. Program listings are at the end of this Appendix.

### SUBROUTINE AVE1(PHI,VMEAN,INDEX,GAV,XGAV,SD,NPTS)

This subroutine computes by numerical integration the averages

$$\text{GAV} = \overline{g_1(x_i)} \quad \text{and} \quad \text{XGAV} = \overline{(x_i - m_i)\, g_1(x_i)}$$

based on a gaussian probability density function where $\bar{x}_i = m_i = \text{VMEAN}$ and $\overline{(x_i - m_i)^2} = \phi_{ii} = \text{PHI}$. The function $g_1(x_i)$ is entered through a function subprogram G1(ARG,INDEX) written by the user for his particular one-dimensional nonlinearities. G1(ARG,INDEX) defines the value of a function

for a given argument (ARG) and the INDEX allows more than one function to be defined by G1(ARG,INDEX) by branching on the value of INDEX supplied by the main program. This latter artifice (INDEX) can be avoided in FORTRAN IV if desired by allowing G1 to replace INDEX as an argument of AVE1.

SD is the number of standard deviations to either side of the mean of the gaussian density over which the integration takes place (SD = 3 or 4 is usually sufficient) and NPTS is the number of points used to define the functions in the interval specified by SD. The integration is effectively trapezoidal rule.

SUBROUTINE AVE2(F1,F2,F12,VM1,VM2,INDEX,GAV,X1G,X2G,SD,NPTS)

This subprogram is the same as AVE1 except that it computes the required averages for a two-dimensional non-linear function,

$$GAV = \overline{g_2(x_1,x_2)}$$

$$X1G = \overline{(x_1-m_1)g_2(x_1,x_2)}$$

$$X2G = \overline{(x_2-m_2)g_2(x_1,x_2)}$$

based on a two-dimensional joint gaussian probability density function where $m_1$ = VM1, $m_2$ = VM2, $\phi_{11}$ = F1, $\phi_{12}$ = F12, and $\phi_{22}$ = F2. Just as in AVE1 the function subprogram G2(ARG1,ARG2,INDEX) is written by the user and INDEX has the same use as in G1.

SD and NPTS have basically the same meaning as in AVE1.

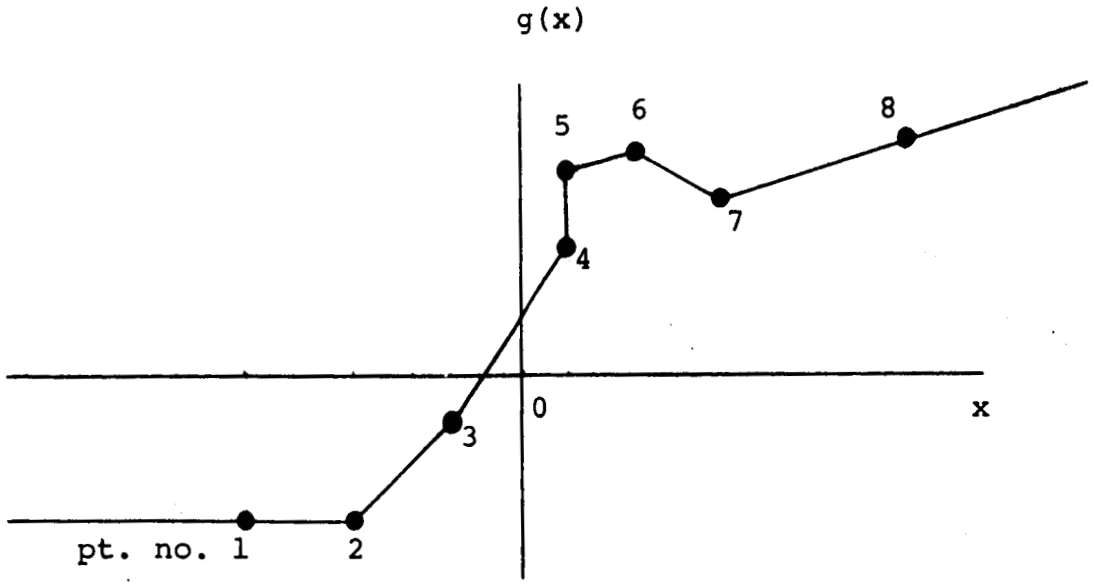
Before the two-dimensional integration is performed, a change of variables is made to the principal axes of the joint gaussian density so that the cut-off points defined by SD have more meaning.

SUBROUTINE APRO(NU,X,Y,VMEAN,PHI,XBAR,FXBAR)

This subroutine, like AVE1, computes the averages

$$XBAR = \overline{g(x_i)} \text{ and } \overline{FXBAR} = \overline{(x_i - m_i)g(x_i)}$$

based on a one-dimensional gaussian probability density function with mean $m_i$ = VMEAN and variance $\phi_{ii}$ = PHI. The function $g(x_i)$ must be represented by straight line segments connecting the points X(I), Y(I), I=1,NU.




The points must be numbered left to right. The function g(x) is assumed to extend to the left indefinitely using the extension of the line segment between the first two points and

to the right using an extension of the line segment through the last two points. Discontinuities (such as at points 4 and 5 in the sketch above) are permitted, but the function must be single valued, of course. The arrays X and Y are supplied by the calling program and there is no limit on their dimension.

The integrations are based on closed form evaluations for each line segment.

SUBROUTINE POLY(NU,A,VMEAN,PHI,XBAR,FXBAR)

This subroutine performs the same task as APRO except that the function is represented by a polynomial of degree NU (i.e., NU + 1 terms) of the form

$$g(x) = A(1) + A(2)x + \ldots + A(NU + 1)x^{NU}$$

The array A of polynomial coefficients is supplied by the calling program and the degree NU$\leq$14. VMEAN, PHI, XBAR, FXBAR all have the same meaning as in APRO.

The integration is achieved with closed form formulas based on the higher moments of gaussian density functions.

```
      SUBROUTINE AVE1(PHI,VMEAN,INDEX,GAV,XGAV,SD,NPTS)
      SIGMA=SQRT(PHI)
      GAV=0.0
      XGAV=0.0
      PTS=NPTS
      H=2.*SD/(PTS-1.0)
      DO 5 I=1,NPTS
      VI=I-1
      XI=VI*H-SD
      ARG=XI*SIGMA+VMEAN
      DUM=H*G1(ARG,INDEX)*EXP(-.5*XI*XI)
      GAV=GAV+DUM
    5 XGAV=XGAV+XI*SIGMA*DUM
      GAV=GAV*.3989
      XGAV=XGAV*.3989
      RETURN
      END
```

```
      SUBROUTINE AVE2(F1,F2,F12,VM1,VM2,INDEX,GAV,X1G,X2G,SD,NPTS)
      IF(F12*F12-.01*F1*F2)1,1,2
    1 VL1=F1
      VL2=F2
      R1=0.0
      GO TO 20
    2 A=.5*(F1+F2)
      B=SQRT((F1-F2)*(F1-F2)/4.0+F12*F12)
      BIG=A+B
      SMALL=A-B
      IF(F2-F1)3,3,4
    3 VL1=BIG
      VL2=SMALL
      GO TO 5
    4 VL2=BIG
      VL1=SMALL
    5 R1=(VL1-F1)/F12
   20 VL1=SQRT(VL1)
      VL2=SQRT(VL2)
      T11=1.0/SQRT(1.0+R1*R1)
      T21=R1*T11
      GAV=0.0
      X1G=0.0
      X2G=0.0
      PTS=NPTS
      H=2.*SD/(PTS-1.0)
      HH=H*H
      DO 10 I=1,NPTS
      VI=I-1
      XI1=VI*H-SD
      DO 10 J=1,NPTS
      VJ=J-1
      XI2=VJ*H-SD
      ARG1=T11*VL1*XI1-T21*VL2*XI2+VM1
      ARG2=T21*VL1*XI1+T11*VL2*XI2+VM2
      DUM=HH*G2(ARG1,ARG2,INDEX)*EXP(-.5*(XI1*XI1+XI2*XI2))
      GAV=GAV+DUM
      X1G=X1G+(ARG1-VM1)*DUM
   10 X2G=X2G+(ARG2-VM2)*DUM
      GAV=GAV*.159155
      X1G=X1G*.159155
      X2G=X2G*.159155
      RETURN
      END
```

```
      SUBROUTINE APRO(NU,X,Y,VMEAN,PHI,XBAR,FXBAR)
      DIMENSION X(1),Y(1)
      STAD=SQRT(PHI)
      N=NU-1
      K=1
      XBAR=0.0
      FXBAR=0.0
      DO 50 I=1,N
      IF (X(I+1)-X(I)) 5,50,5
    5 S=(Y(I+1)-Y(I))/(X(I+1)-X(I))
      B=(X(I+1)*Y(I)-X(I)*Y(I+1))/(X(I+1)-X(I))
      QA=(X(I+1)-VMEAN)/(1.414214*STAD)
      AQA=ABSF(QA)
      IF (AQA-5.) 70,60,60
   60 DERN=0.0
      GO TO 75
   70 DERN=1.128379*EXP(-(QA*QA))
   75 QB=(X(I)-VMEAN)/(1.414214*STAD)
      AQB=ABSF(QB)
      IF (AQB-5.) 90,80,80
   80 DERP=0.0
      GO TO 95
   90 DERP=1.128379*EXP(-(QB*QB))
   95 ALPH=1./(1.+.47047*AQA)
      BETA=1./(1.+.47047*AQB)
      ERFN=1.-ALPH*(.308428+ALPH*(-.084971+.66277*ALPH))*DERN
      ERFP=1.-BETA*(.308428+BETA*(-.084971+.66277*BETA))*DERP
      IF (QA-0.) 1,2,2
    1 ERFN=-ERFN
    2 IF(QB-0.) 3,4,4
    3 ERFP=-ERFP
    4 IF (K-1) 10,10,20
   10 XBXX=B+(S*VMEAN)
      XBAX=(XBXX*(1.+ERFN)*.5)-(STAD*S*DERN*.353553)
      FXBX=(S*PHI*.5*(1.+ERFN))-(XBXX*DERN*.353553*STAD)
      FXBA=(S*STAD*(X(I+1)-VMEAN)*DERN)*.353553
      FXBAX=FXBX-FXBA
      K=K+1
      GO TO 45
   20 IF (I-N) 30,40,40
   30 XBXX=B+(S*VMEAN)
      XBAX=(XBXX*(ERFN-ERFP)*.5)+(STAD*S*(DERP-DERN)*.353553)
      FXBX=(((XBXX*STAD)+S*STAD*(X(I)-VMEAN))*DERP)*.353553
      FXBA=(((-XBXX*STAD)-S*STAD*(X(I+1)-VMEAN))*DERN)*.353553
      FXBAX=(S*PHI*.5)*(ERFN-ERFP)+FXBA+FXBX
      GO TO 45
   40 XBXX=B+(S*VMEAN)
      XBAX=(XBXX*(1.-ERFP)*.5)+(STAD*S*DERP*.353553)
      FXBX=(S*PHI*.5*(1.-ERFP))+(XBXX*DERP*.353553*STAD)
      FXBA =(S*STAD*(X(I)-VMEAN)*DERP)*.353553
      FXBAX=FXBX+FXBA
   45 XBAR= XBAR+XBAX
      FXBAR=FXBAR+FXBAX
   50 CONTINUE
      RETURN
      END
```

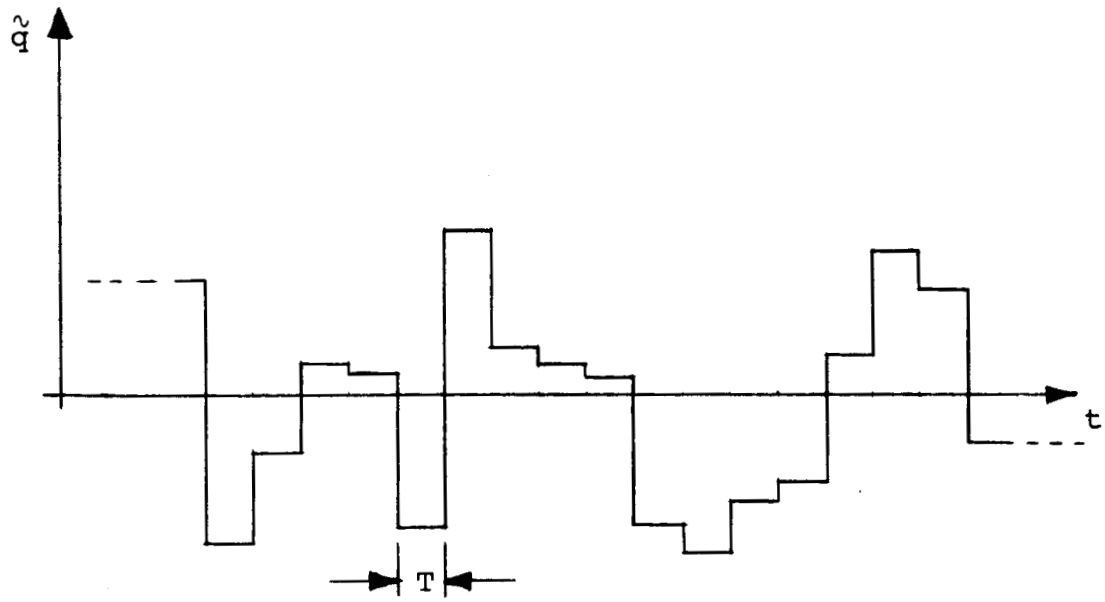
```
      SUBROUTINE POLY(NU,A,VMEAN,PHI,XBAR,FXBAR)
      DIMENSION A(1),B(15)
      N=NU+1
      STAD=SQRT(PHI)
      IF (VMEAN-0.0) 15,5,15
    5 DO 10 I=1,N
   10 B(I)=A(I)
      GO TO 40
   15 DO 35 L=1,N
      B(L)=0.0
      BIN=1.0
      VL=L-1
      DO 35 K=L,N
      VK=K-1
      IF (K-L) 30,25,30
   25 B(L)=A(K)
      GO TO 35
   30 B(L)=B(L)+A(K)*(VMEAN**( K- L))*BIN
   35 BIN=BIN*(VK+1.)/(VK+1.-VL)
   40 XBAR=0.0
      VMO=1.0
      DO 55 KO=1,N,2
      LO=KO-1
      VLO=LO
      UMO=KO-2
      VMO=UMO*VMO
      IF(KO-1) 50,45,50
   45 XBAX=B(KO)
      VMO=1.0
      GO TO 55
   50 XBAX=B(KO)*(STAD**VLO)*VMO
   55 XBAR=XBAR+XBAX
      FXBAR=0.0
      VMO=1.0
      DO 60 KE=2,N,2
      VKE=KE
      MO=KE-1
      UMO=MO
      VMO=UMO*VMO
      FXBAX=B(KE)*(STAD**VKE)*VMO
   60 FXBAR=FXBAR+FXBAX
      RETURN
      END
```

## APPENDIX D

## THE MONTE CARLO METHOD USED FOR COMPARISONS WITH THE APPROXIMATE SOLUTIONS

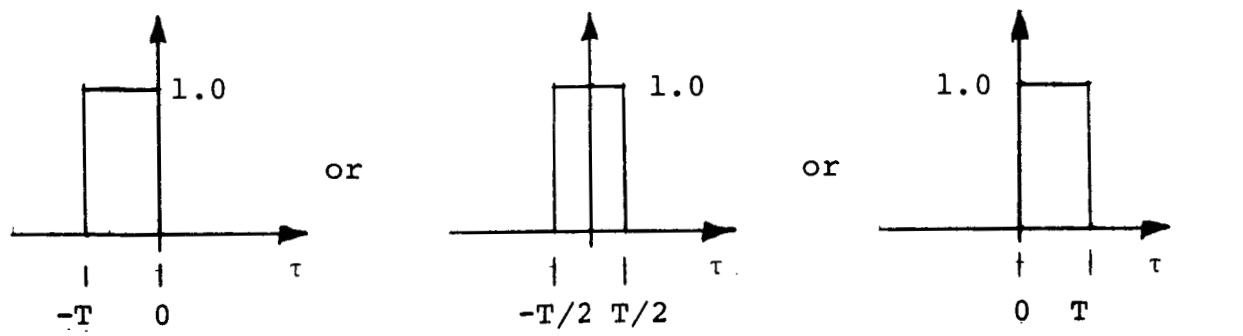
The Monte Carlo results used for comparisons in Section III were obtained by a very straight forward procedure. The system differential equations were integrated (on a digital computer) a large number of times (typically 100) with appropriate initial conditions and with a simulated "white noise" input obtained from a random number subroutine. Samples of all the state variables were accumulated in a manner to produce finite sample estimates of means and covariances at specified time intervals.

To produce "white noise" inputs and random initial conditions a random number generator based on the method of Reference 13 was used to produce a nearly gaussian distributed set of random numbers with a mean of zero, a standard deviation of 1.0, and a maximum deviation of 6.0. Tests on the number sequences showed successive samples to have very low correlation. The "white noise" was simulated with a random function of the type shown in the sketch.




The interval T is the same as the integration interval used to integrate the system equations.

The auto-correlation function $\overline{\tilde{q}(t)\tilde{q}(t+\tau)}$ for such a random time function might be




t at beginning of interval

t in center of interval

t at end of interval

so that such a function is not strictly stationary. If T is made very small compared to the "time constants" in the system and the amplitude of $\tilde{q}(t)$ is normalized by using a multiplying factor to keep the area of the auto-correlation function "pulse" equal to 1.0, then

$$\overline{\tilde{q}(t)\tilde{q}(t+\tau)} \simeq \overline{q(t)q(t+\tau)} = \delta(\tau)$$

The estimate $M_N$ used to obtain the mean $m_x$ of a variable x from N samples at a particular time is

$$M_N = \frac{1}{N} \sum_{i=1}^{N} x^{(i)} \tag{D-1}$$

where $x^{(i)}$ is the $i^{th}$ sample of x. The variance of this estimate

$$\text{Var}[M_N] = \frac{1}{N} \phi_{xx} \tag{D-2}$$

where $\phi_{xx}$ is the variance of x.

The estimate $V_N$ for the covariance $\phi_{xy} = \overline{(x-m_x)(y-m_y)}$ based on N samples was

$$V_N = \left(\frac{N}{N-1}\right)\left[\frac{1}{N} \sum_{i=1}^{N} x^{(i)}y^{(i)} - \frac{1}{N} \sum_{i=1}^{N} x^{(i)} \frac{1}{N} \sum_{j=1}^{N} y^{(j)}\right] \tag{D-3}$$

The variance of $V_N$ depends on higher moments of x and y, but if it is assumed that x and y are gaussian for convenience

$$\text{Var}[V_N] = \frac{1}{N-1}(\phi_{xy}^2 + \phi_{xx}\phi_{yy}) \tag{D-4}$$

To get the plus-or-minus, one standard deviation bands on the means and covariances in the figures of Section III, the results of (D-3) were used in (D-2) and (D-4). Although the gaussian assumption used in (D-4) was violated, it was felt that the results

would still be a good indication of the reliability of the Monte Carlo results.

Relative Computation Effort Between Monte Carlo and Covariance Equations.

One might ask whether the Monte Carlo approach could be used rather than approximate mean and covariance equation approach for the solution of problems involving random disturbances in nonlinear systems. The Monte Carlo approach is appealing because it is simple, easy to implement, and requires virtually no assumptions to be made about the nature of the system or its probability density function. It is flexible in the type of statistical data that can be obtained since one could estimate probability density functions from accumulated frequency distributions almost as easily as first and second moments.

A simple analysis will yield a measure of the relative computation times required for the approximate covariance equation approach and the Monte Carlo approach. Let n be the number of first order differential equations which describe the dynamic system. Then the total number of mean and covariance equations will be $n(n+3)/2$. If D is the average "factor of difficulty" of the mean and covariance equations over the system equations and $I_c$ is the integration interval required for the mean and covariance equations, the computation time for the covariance equation method can be written

$$T_c = C\left(\frac{n(n+3)}{2}\right)\frac{D}{I_c}$$

where C is a proportionality constant. Likewise, if N is the number of samples (or solutions) required by the Monte Carlo approach and $I_m$ is the integration interval size required for the Monte Carlo approach

$$T_M = C\frac{nN}{I_m}$$

The ratio

$$T_c/T_m = \left(\frac{n+3}{N}\right)\left(\frac{D}{2}\right)\left(\frac{Im}{I_c}\right)$$

To get an idea about N, one can find from (D-4) (an approximation) that the ratio of standard deviation in the estimate of $\phi_{xx}$ to $\phi_{xx}$ is about 0.14 for N=100. (To cut that in half N would have to go to 400.) Assume that N=100 is adequate for purposes of comparison.

The average factor of difficulty D might typically be from 2 to 4. On the other hand, the Monte Carlo approach might require a smaller interval size to properly simulate the "white noise". For purposes of comparison assume

$$\left(\frac{D}{2}\right)\left(\frac{I_m}{I_c}\right) = 1$$

With these assumptions

$$T_c/T_m \approx \frac{n+3}{100}$$

It appears from this that the covariance equation approach has

quite an advantage for low order systems ($n<10-20$) but that for very high order systems ($n\geq 50-100$) the Monte Carlo Approach is more economical.

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