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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS

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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS

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FOREWORD

This report describes the research carried out under Contract NAS 8-20306, "Study On Determining Stability Domains for Nonlinear Dynamical Systems," during the period 1 May 1966 to 1 February 1967, by the Research Department of Grumman Aircraft Engineering Corporation, Bethpage, New York 11714, for the NASA George C. Marshall Space Flight Center, Huntsville, Alabama.

The objective of this study was to do research in techniques for determining exactly or with good approximation the domain of stability of nonlinear dynamical systems. The goal of the longer study, of which this is the first part, is to develop techniques for analytically determining the stability properties of booster guidance schemes in order to compare competing schemes with respect to their ability to compensate for off-nominal conditions. Mr. Commodore C. Dearman, Jr. of the Aero-Astrodynamics Laboratory was responsible for initiating this study program and acted as the technical representative for NASA. We are grateful to Mr. Dearman for sponsoring this study and for his encouragement.

The study was conducted by Dr. Gunther R. Geiss (principal investigator), Dr. John V. Abbate, and Messrs. James Alberi, Dushan Boyanovitch, Robert McGill, David Rothschild, and Gerald E. Taylor. The authors are indebted to Mr. McGill for his conceptual contributions to the computational aspects of this study, and for contributing the closed-loop guidance example that is studied herein. We are indebted to Messrs. McGill and Taylor for making the Min-All algorithm available for this study, and to Messrs. Taylor, Alberi, and Rothschild for conducting the numerical experiments and for numerous suggestions regarding the development of the estimation procedure.

ABSTRACT

This report presents the results of research carried out under Contract NAS 8-20306, "Study On Determining Stability Domains for Nonlinear Dynamical Systems," for NASA George C. Marshall Space Flight Center. A numerical procedure for obtaining an optimal quadratic estimate of the domain of attraction of an equilibrium solution to a quasi-linear autonomous differential equation is developed and evaluated. A procedure due to Liapunov for determining the Liapunov functions that yield exact information on the temporal behavior of linear systems is reviewed and the implications of its extension to quasi-linear systems are discussed. A simple closed-loop guidance system is analyzed and the unique features of its stability properties are illustrated as possible characteristics of more complex guidance systems. The equations of the Iterative Guidance Mode are reviewed to illustrate the problems that are fundamental to the stability analysis of such a system. Conclusions of the study and recommendations for further research are presented.

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I. INTRODUCTION

The concepts of Liapunov's direct method for the stability analysis of nonlinear dynamical systems has generated a continually expanding research program aimed at finding analytical stability analysis techniques applicable to highly complex physical systems. The research effort described in this report has been directed toward several aspects of the stability problem pertinent to the analysis of space vehicle guidance systems.

Present day guidance systems are described for finite intervals of time by nonlinear, nonautonomous differential equations. Moreover, the control laws for these systems are, in some cases, generated by iterative procedures. Such a system is far more complicated than any system which has been successfully analyzed by current state of the art stability analysis techniques other than simulation. Before an analysis of an actual guidance system can be undertaken, it is necessary to review and expand our knowledge of several fundamental aspects of the over-all problem of stability.

Our effort in this study was focused on: a) effective use of present techniques and the development of new techniques for determining the domain of stability (exactly or approximately) of high order nonlinear systems; b) numerical means for implementing these techniques; c) the relationship of Liapunov stability to finite time stability; d) stability analysis of nonautonomous systems; e) formulation of mathematical models of guidance schemes; and f) analysis of a simplified time-dependent closed-loop guidance system. Basic research has been initiated in each of these areas and the preliminary results, reported herein, should serve as the foundations of an expanded research program ultimately leading to successful stability analyses of booster guidance systems and the large class of related systems.

Section II of this report describes the formulation and development of a numerical algorithm for determining an "optimal" quadratic estimate of the domain of attraction of an equilibrium solution of a quasi-linear differential equation. The estimate is optimal in the sense of largest enclosed volume and is based upon the use of a quadratic form Liapunov function. This section also describes the numerical experiments performed with this algorithm and the conclusions drawn from them.

Section III describes a procedure due to Liapunov for calculating the Liapunov functions that determine the exact temporal behavior of a linear system. Hopefully, the procedure can be extended to quasi-linear systems, in which case it can be used for evaluating the stability of finite time systems.

Section IV describes a simple closed-loop time dependent quidance system derived from Zermelo's problem. An analysis of the autonomous approximation to this system is presented and an approximate method for analyzing the actual nonautonomous system is described.

In Section V, we review the equations describing the Iterative Guidance Mode and make some comments concerning the stability analysis of such a system.

The last section (VI) presents the conclusions drawn from this study and some recommendations for further research.

II. OPTIMAL QUADRATIC ESTIMATION OF THE DOMAIN OF ATTRACTION

A. Problem Formulation

Briefly, the procedure, which was first described in Ref. 1 and is studied here, is based upon choosing the quadratic form Liapunov function that yields the largest estimate of the domain of attraction for the given motion and system of equations. In particular, assume that the system is of the form

$$\dot{x} = Ax + f(x) \quad , \quad x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad , \quad f(0) = 0 \quad , \quad A \text{ stable;} \tag{1}$$

i.e., it is n-dimensional, autonomous, quasi-linear, and stable. As a result of these assumptions the quadratic Liapunov function V,

$$V(x) = x^T P x \quad , \quad P > 0 \quad , \tag{2}$$

will have as its time derivative

$$\dot{V}(x) = - x^T Q x + 2x^T P f(x) \quad , \tag{3}$$

where Q is determined from the Liapunov equation

$$- Q = A^T P + PA \quad . \tag{4}$$

If Q is chosen to be positive definite, then P will be positive definite as a result of A being stable and $\dot{V}$ will be negative in the region

$$D: \left(x \;\middle|\; \frac{\|f(x)\|}{\|x\|} < \frac{\lambda^{min}(Q)}{2\lambda^{max}(P)} \right) \quad , \tag{5}$$

where $\lambda^{min}(Q)$ and $\lambda^{max}(P)$ are, respectively, the minimum eigenvalue of Q and the maximum eigenvalue of P.

According to LaSalle and Lefschetz (Ref. 2) an estimate of the domain of attraction of the equilibrium solution $x(t) = 0$ of Eq. (1)* is given by

$$\Omega_\ell : \left(x \mid V(x) < \ell \quad , \quad \dot{V}(x) < 0 \right) \tag{6}$$

if Ω_ℓ is bounded. Thus, relative to this choice of $V(x)$, i.e., the choice of Q, the best estimate is obtained by defining the set E as

$$E: \left(x \mid \dot{V}(x) = 0 \quad , \quad x \neq 0 \right) \tag{7}$$

and then choosing ℓ to be

$$\ell = \min_{x \in E} V(x) \quad . \tag{8}$$

Then, the optimal choice of Q from the set of all positive definite $n \times n$ matrices, denoted Q^0, is defined by

$$J(Q^0) = \max_{Q > 0} J(Q) \quad , \tag{9}$$

where

$$J(Q) = \ell^{n/2} \left(\prod_{i=1}^{n} \lambda_i(P) \right)^{-\frac{1}{2}} = \left(\frac{\ell^n}{\det P} \right)^{\frac{1}{2}} \quad . \tag{10}$$

This definition of Q^0 [Eq. (9)] will yield the best estimate in terms of enclosed volume, of the domain of attraction under the constraint that $V(x)$ be a positive definite quadratic form. Thus, an optimal estimate of the domain of attraction with respect to quadratic form Liapunov functions can be obtained via a numerical algorithm that solves Eqs. (8) and (9).

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*N.B. Hereafter it will be understood that we are concerned with the equilibrium solution $x(t) = 0$ of Eq. (1).

B. Development of the Numerical Algorithm

Before the quantities ℓ and Q^0 can be computed from Eqs. (8) and (9), four problems must be resolved, viz.:

1) How to generate the set of positive definite $n \times n$ matrices from which candidate Q matrices are chosen.

2) How to solve the Liapunov equation [Eq. (4)] for P given A and Q.

3) How to handle the constraints implied in Eqs. (8) and (9), viz, that x be an element of set E and that Q be positive definite.

4) How to efficiently compute the minimum of a function.

1. Parameterization of the Set of Positive Definite Matrices

The generation of the set of positive definite $n \times n$ matrices can be carried out by resorting to the brute force approach of forming an arbitrary $n \times n$ symmetric matrix and then applying the determinantal test (Ref. 3) to determine if it is positive definite. This procedure requires the arbitrary choice of $\frac{n(n+1)}{2}$ matrix elements and then the evaluation of the determinants of the n-principal minors of the matrix. However, it does not provide information on how to correct a candidate matrix that fails the test for positive definiteness. Therefore, it would be desirable to generate the matrix by a procedure that guarantees the matrix is positive definite and spans the entire set of positive definite matrices. In this section, we develop such a procedure based upon the work of Murnaghan (Ref. 4) on the parameterization of the group of unitary matrices.

It is well known (Ref. 3) that all real symmetric matrices are orthogonally similar to a diagonal matrix, and that all positive definite (pd) matrices are then orthogonally similar to a diagonal matrix with positive diagonal elements; i.e., let Q be pd, then

$$Q = S^T \Lambda S \quad , \tag{11}$$

where

$$\Lambda = \text{diag}\left\{\lambda_1, \lambda_2, \ldots, \lambda_n\right\}$$

$$\lambda_i > 0 \quad , \quad i = 1, 2, \ldots, n \tag{12}$$

$$S^T S = I \quad .$$

Thus, the parameterization of all pd matrices Q is reduced to the parameterization of the group of orthogonal matrices S.

In Ref. 4, Murnaghan proves that the parameterization of the group of $n \times n$ unitary matrices U is accomplished by the factorization

$$U = D \left[\prod_{k=1}^{n-1} U_{n-k}\right] \quad , \tag{13}$$

where

$$D = \text{diag}\left\{e^{i\delta_1}, e^{i\delta_2}, \ldots, e^{i\delta_{n-1}}, e^{i\varphi_n}\right\} \quad , \tag{14}$$

$$U_k = \left[\prod_{\ell=k+1}^{n-1} U_{k\ell}(\theta_\mu, \sigma_\rho)\right]\left[U_{kn}(\varphi_k, \sigma_\gamma)\right] \quad , \tag{15}$$

$$\gamma = \frac{(2n - k)(k - 1)}{2} + 1 \quad ,$$

$$\rho = \frac{(2n - k)(k - 1)}{2} + 1 + n - \ell \quad ,$$

$$\mu = \frac{(2n - k - 2)(k - 1)}{2} + (n - \ell) \quad ,$$

$$U_{k\ell}(\theta,\sigma) = (u_{ij}) : \begin{cases} u_{ii} = 1 , \; i \neq k,\ell \\ u_{kk} = \cos\,\theta \\ u_{\ell\ell} = \cos\,\theta \\ u_{ij} = 0 , \; i \neq j , \; i,j \neq k,\ell \\ u_{k\ell} = - e^{-i\sigma} \sin\,\theta \\ u_{\ell k} = + e^{+i\sigma} \sin\,\theta , \end{cases} \tag{16}$$

$$-\pi \leq \phi < \pi , \quad -\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2} , \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} , \quad -\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2} .$$

The factorization of the group of orthogonal matrices is immediately obtained by requiring U to be real; i.e., $\delta = \sigma = 0$, $\varphi_n = \pm \pi$, $-\pi \leq \varphi_k < \pi$, $k \neq n$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. In particular,

$$S = D_1 \left[\prod_{k=1}^{n-1} S_{n-k} \right] , \tag{17}$$

$$D_1 = \mathrm{diag} \left\{ 1, \ldots, 1, \pm 1 \right\} , \tag{18}$$

$$\begin{gathered} S_k = \left[\prod_{\ell=k+1}^{n-1} S_{k\ell}(\theta_\mu) \right] \left[S_{kn}(\varphi_k) \right] , \quad S_{k\ell}(\theta_\mu) = U_{k\ell}(\theta_\mu, 0) \\ \mu = \frac{(2n - k - 2)(k - 1)}{2} + n - \ell . \end{gathered} \tag{19}$$

This factorization contains $\frac{(n - 1)(n - 2)}{2}$ thetas and n phis, or a total of $\frac{n(n - 1)}{2} + 1$ parameters. The n lambdas in Eq. (12) raise the number of parameters to $\frac{n(n + 1)}{2} + 1$ — one more than required. Thus if we restrict S to be a rotation matrix (i.e., choose $\varphi_n = 0$), the number of parameters will be $\frac{n(n + 1)}{2}$, the number required to represent an arbitrary symmetric matrix. The choice $\varphi_n = 0$ is intuitivaly motivated by the consideration that we wish to rotate and scale the ellipsoid associated with the quadratic form formed from the pd matrix and do not want to reflect coordinates or change the handedness of the coordinate system.

The factorization of a pd matrix of dimension three is thus given by

$$P = S^T \Lambda S \quad , \tag{20}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad , \tag{21}$$

$$\lambda_1, \lambda_2, \lambda_3 > 0 \quad ,$$

and

$$S = S_2 S_1 = S_{23}(\varphi_2)\; S_{12}(\theta_1)\; S_{13}(\varphi_1) \quad , \tag{22}$$

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\varphi_2 & -s\varphi_2 \\ 0 & s\varphi_2 & c\varphi_2 \end{pmatrix} ,$$

$$S_{12} = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (23)$$

$$S_{13} = \begin{pmatrix} c\varphi_1 & 0 & -s\varphi_1 \\ 0 & 1 & 0 \\ s\varphi_1 & 0 & c\varphi_1 \end{pmatrix} ,$$

$$-\pi \leq \varphi_1 < \pi \ , \ -\pi \leq \varphi_2 < \pi \ , \ -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2} \ ,$$

$$c\varphi_1 = \cos \varphi_1 \ , \ s\varphi_1 = \sin \varphi_1 \ .$$

Thus, it is clear that by using this representation under the restrictions

$$\begin{aligned} &\lambda_i > 0 \quad , \quad i = 1, 2, \ldots, n \\ &-\pi \leq \varphi_i < \pi \quad , \quad i = 1, 2, \ldots, n - 1 \qquad (24) \\ &-\frac{\pi}{2} \leq \theta_i \leq \frac{\pi}{2} \quad , \quad i = 1, 2, \ldots, \frac{(n-1)(n-2)}{2} \end{aligned}$$

the candidate Q matrices are guaranteed to be positive definite.

2. Solution of the Liapunov Equation

The Liapunov equation, Eq. (4), viz.,

$$A^T P + PA = - Q \quad , \tag{4}$$

where P and Q are $n \times n$ symmetric matrices and A is a stable $n \times n$ matrix, can be solved for P as follows: Assume that

$$(\lambda - \lambda_i)^{\alpha_i} \quad , \quad i = 1, 2, \ldots, r \ , \ \sum_{i=1}^{r} \alpha_i = n \tag{25}$$

are the elementary divisors of A (and thus A^T) over $\mathbb{C}$, the field of complex numbers. Then there exist matrices U, V such that

$$A = U^{-1}\tilde{A}_1 U \tag{26}$$

and

$$A^T = V^{-1}\tilde{A}_1 V \tag{27}$$

where $\tilde{A}_1$ is the Jordan normal form for A (and thus A^T), i.e.,

$$\tilde{A}_1 = \text{diag} \left\{ \lambda_1 I_{\alpha_1} + N_{\alpha_1} \ , \ \lambda_2 I_{\alpha_2} + N_{\alpha_2} \ , \ \ldots \ , \ \lambda_r I_{\alpha_r} + N_{\alpha_r} \right\} \tag{28}$$

$$I_{\alpha_\mu} = (\delta_{ij}) \quad , \quad i,j = 1, 2, \ldots, \alpha_\mu \tag{29}$$

$$N_{\alpha_\mu} = (\delta_{i+1,j}) \quad , \quad i,j = 1, 2, \ldots, \alpha_\mu \tag{30}$$

and δ_{ij} is the Kronecker delta. The Liapunov equation then becomes

$$\tilde{A}_1 Y + Y\tilde{A}_1 = - D \quad , \tag{31}$$

where

$$Y = VPU^{-1} \tag{32}$$

and

$$D = VQU^{-1} \tag{33}$$

In Ref. 5, Ma gives a finite series solution for the matrix equation

$$\tilde{A}X - X\tilde{B} = C \quad , \tag{34}$$

where $\tilde{A}$ and $\tilde{B}$ are in Jordan normal form. Thus, via the identification $X = Y$, $\tilde{A} = \tilde{A}_1$, $\tilde{B} = -\tilde{A}_1$ and $C = -D$, the solution to Eq. (31) is obtained from Ma's solution to Eq. (34). The solution is:

$$Y_{ij} = - \sum_{n=0}^{\alpha_i+\alpha_j-2} (\lambda_i + \lambda_j)^{-(n+1)} (-1)^n \sum_{\sigma+\tau=n} \frac{n!}{(n-\tau)!\tau!} N^{\sigma}_{\alpha_i} D_{ij} N^{\tau}_{\alpha_j} \quad , \tag{35}$$

where Y_{ij}, and D_{ij} are the ij elements of the partitions of Y and D which are the same as the partition of $\tilde{A}_1$. Finally, we obtain P as

$$P = V^{-1}YU \tag{36}$$

from Eq. (32). Note that since A is assumed stable, all λ_i will have negative real parts and

$$\lambda_i + \lambda_j \neq 0 \qquad i,j = 1, 2, \ldots, r \quad . \tag{37}$$

Thus, only the first case of Ma's solution, Eq. (35), need be considered here. Further, if A is of simple structure, then the solution is

$$y_{ij} = \frac{-d_{ij}}{\lambda_i + \lambda_j} \quad , \quad i,j = 1, 2, \ldots, n \quad , \tag{38}$$

where y_{ij} and d_{ij} are elements of Y and D, respectively.

3. Penalty Function Formulation for Constraints

To compute ℓ and Q^0 via Eqs. (8) and (9), we require a method for handling the respective constraints: $x \in E$, i.e., x such that $\dot{V}(x) = 0$ and $x \neq 0$; and Q is positive definite. Since the algorithm to be used to compute Eqs. (8) and (9) has been designed for unconstrained problems, we will use a device due to Courant (Ref. 6) which is called the penalty function. This approach to constrained extremal problems has been successfully used in optimal control and variational settings, e.g., see Kelley (Ref. 7), and McGill (Ref. 8).

Consider Eq. (8), which may be written:

$$\ell = \min_{x \neq 0} V(x) \quad \text{on} \quad \dot{V}(x) = 0 \quad . \tag{8'}$$

A penalty function formulation of the same problem is

$$\ell = \min_{x} \left(V(x) + K_1^2 \dot{V}^2(x) \, g(x) \right) \tag{39}$$

or

$$\ell = \min_{x} \left(V(x) + K_1^2 \dot{V}^2(x) + K_2^2 h(x) \right) \quad , \tag{40}$$

where $\dot{V}^2(x)$ is the penalty associated with not meeting the constraint $\dot{V}(x) = 0$ and $g(x)$ or $h(x)$ are designed to penalize choices of x close to or equal to zero, e.g.,

$$h(x) = \frac{1 + \|x\|^{2m}}{\|x\|^{2m}} \quad , \quad m = 1, 2, \ldots \tag{41}$$

In view of the parameterization we have developed for Q, Eq. (9) may be rewritten as:

$$J(Q^0) = \max_{\substack{\lambda_i(Q) > 0 \\ -\pi \leq \varphi_K < \pi \\ -\frac{\pi}{2} \leq \theta_j \leq \frac{\pi}{2}}} J(Q) = \max_{\substack{\lambda_i(Q) > 0 \\ -\pi \leq \varphi_K < \pi \\ -\frac{\pi}{2} \leq \theta_j \leq \frac{\pi}{2}}} \left(\frac{\ell^n}{\det P}\right)^{\frac{1}{2}} . \tag{9'}$$

A penalty function formulation of the corresponding minimum problem (our numerical algorithm was written for minimum problems) is:

$$J^{-1}(Q^0) = \min_{\lambda_i, \varphi_K', \theta_j'} \left\{ \left(\frac{\det P}{\ell^n}\right)^{\frac{1}{2}} + K_3^2 G(\lambda_1, \lambda_2, \ldots, \lambda_n) \right\} \tag{42}$$

where

$$\varphi_K = -\pi + \varphi_K' \bmod 2\pi \quad , \quad -\infty < \varphi_K' < \infty \tag{43}$$

$$\theta_j = \frac{1}{2}(-\pi + \theta_j' \bmod 2\pi) \quad , \quad -\infty < \theta_j' < \infty \tag{44}$$

and

$$G(\lambda_1, \ldots, \lambda_n) = \begin{cases} 1, & \text{any } \lambda_i \leq \epsilon , \quad i = 1, 2, \ldots, n \\ 0, & \text{all } \lambda_i > \epsilon , \quad i = 1, 2, \ldots, n \end{cases} \tag{45}$$

In the above penalty formulations, the constants K_1, K_2, K_3 are chosen large enough to assure meeting the respective constraints to the required accuracies, and ϵ is chosen to define a "forbidden" neighborhood of zero. Details of the actual penalty formulations used in the numerical experiments will be given in a later section.

4. The Min-All Algorithm

The numerical algorithm being used to compute solutions to Eq. (8) and Eq. (9) via Eq. (40) and Eq. (42) is being developed at Grumman by McGill and Taylor and is based upon the work of Davidon (Ref. 9) and Fletcher and Powell (Ref. 10). The algorithm utilizes a modified gradient search concept and proceeds as follows:

To find the minimum over all x of $f(x)$, where $x^T = (x_1, \ldots, x_n)$ and $f(x)$ is a scalar function, choose an initial point x_0, and an arbitrary $n \times n$ positive definite symmetric matrix H_0 (e.g., the identity matrix). Then, let

$$s_k = - H_k f_k' \quad , \tag{46}$$

where

$$f_k' = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T \quad , \tag{47}$$

and find $\alpha_k > 0$ such that $f(x_k + \alpha_k s_k)$ is minimum with respect to α_k. Now, let

$$x_{k+1} = x_k + \alpha_k s_k \quad , \tag{48}$$

and compute $f(x_{k+1})$ and $f'(x_{k+1})$. Define

$$y_k = f_{k+1}' - f_k' \quad , \tag{49}$$

and then compute H_{k+1} as follows

$$H_{k+1} = H_k + A_k + B_k \quad , \tag{50}$$

where

$$A_k = (s_k^T y_k)^{-1} \alpha_k s_k s_k^T \tag{51}$$

and

$$B_k = - (y_k^T H_k y_k)^{-1} H_k y_k y_k^T H_k \quad . \tag{52}$$

This procedure is repeated until

$$\|x_{k+1} - x_k\| < \epsilon \quad , \quad 0 < \epsilon \ll 1 \quad . \tag{53}$$

C. The Numerical Experiments

The numerical experiments dealt with two well known equations, viz., the Duffing equation with damping, i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - x_2 - x_1 + 0.04\, x_1^3 \end{aligned} \tag{54}$$

and the Van der Pol equation with unstable limit cycle, i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - x_2(1 - x_1^2) - x_1 \quad . \end{aligned} \tag{55}$$

The Duffing equation was chosen because the estimate of the domain of attraction could be obtained analytically (see Ref. 11) and thus a reliable check of the numerical results was available. The Van der Pol equation was chosen because the domain of attraction of the zero solution is well documented (the interior of the limit cycle) and is far from a quadratic curve. Thus, it would be a good test of the conservativeness of the estimate.

1. Details of the Experiments

Both of the equations whose domains of attraction of their zero solutions are to be estimated are second order and they both have the same linear part. As a result, the parameterization of Q is:

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta & (\lambda_2 - \lambda_1)\sin\theta\cos\theta \\ (\lambda_2 - \lambda_1)\sin\theta\cos\theta & \lambda_1 \sin^2\theta + \lambda_2\cos^2\theta \end{pmatrix} \tag{56}$$

$$\lambda_1, \lambda_2 > 0 \quad , \quad -\pi \leq \theta < \pi$$

and the Liapunov equation for P is

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = -Q\ . \tag{57}$$

Since P is a 2×2 matrix, it can be determined directly without recourse to the method of solution presented earlier. The equations implied by Eq. (57) are:

$$\begin{aligned} p_{12} + p_{21} &= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta \\ +p_{11} - p_{12} - p_{22} &= (\lambda_1 - \lambda_2)\sin\theta\cos\theta \\ p_{12} + p_{21} - 2p_{22} &= -(\lambda_1 \sin^2\theta + \lambda_2\cos^2\theta) \\ p_{12} &= p_{21} \end{aligned} \tag{58}$$

and their solution is:

$$p_{11} = \frac{1}{2}\Big(\lambda_1(\cos^2\theta + 1) + \lambda_2(\sin^2\theta + 1)\Big) + (\lambda_2 - \lambda_1)\sin\theta\cos\theta$$

$$p_{12} = p_{21} = \frac{1}{2}(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta) \qquad (59)$$

$$p_{22} = \frac{1}{2}(\lambda_1 + \lambda_2)$$

Further, one can scale all the equations by an arbitrary constant without affecting the solution. For simplicity we choose λ_1^{-1} as the scale factor and obtain the form in which the equations were programmed:

$$p_{11}' = \frac{p_{11}}{\lambda_1} = \frac{1}{2}\Big(\cos^2\theta + 1 + \lambda(\sin^2\theta + 1)\Big) + (\lambda - 1)\sin\theta\cos\theta$$

$$p_{12}' = \frac{p_{12}}{\lambda_1} = \frac{1}{2}(\cos^2\theta + \lambda\sin^2\theta)$$

$$p_{22}' = \frac{p_{22}}{\lambda_1} = \frac{1}{2}(1 + \lambda) \qquad (60)$$

$$\lambda = \frac{\lambda_2}{\lambda_1} > 0 \quad , \quad \theta = -\pi + \theta' \bmod 2\pi \quad , \quad -\infty < \theta' < \infty$$

The Liapunov function in all cases, except where noted, was

$$V = \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 \ . \qquad (61)$$

Its derivatives with respect to the Duffing equation and the Van der Pol equation have the same quadratic part and are given respectively by:

$$\dot{V}_D = -\left[\frac{1}{2}(\cos^2\theta + \lambda\sin^2\theta)x_1^2 + \Big((\lambda - 1)\sin\theta\cos\theta\Big)x_1x_2 + \frac{1}{2}(\sin^2\theta + \lambda\cos^2\theta)x_2^2\right] + 0.04\,p_{12}x_1^4 + 0.04\,p_{22}x_1^3x_2 \tag{62}$$

$$\dot{V}_P = -\left[\qquad\qquad\right] + p_{12}x_1^3x_2 + p_{22}x_1^2x_2^2 \tag{63}$$

In both cases, the terms of order higher than second are of fourth order and thus, a candidate $g(x)$ in Eq. (39) is:

$$g(x) = \|x\|^{-2m} \quad , \quad m = 4, 5, \ldots \tag{64}$$

2. Chronicle of Experiments and Results

The earliest experiments done using the approach described above dealt with the Duffing equation and the determination of ℓ for given λ, θ (see Ref. 1). The early experiments performed under this contract investigated the problem of computing ℓ near Q^0, which in this case is on the boundary of the allowed λ, θ, viz., $\lambda^0 = 0$, $\theta^0 = 0$. It was determined that ℓ could be calculated sufficiently near Q^0 for our purposes, i.e., $J(Q)$ to within 10 percent of $J(Q^0)$; however, there was repeated difficulty with the algorithm computing to the global minimum, $\ell = 0$ at $x = 0$. At this stage we were using Eq. (39) with $g(x) = 1$. Experiments were begun with $g(x)$ as defined in Eq. (64); however, the results were unsatisfactory since this form for $g(x)$ reduced the effective value of K_1^2 for $|x| > 1$ and thus the accuracy with which the constraint was met depended on the computed x. The function $g(x)$ was then changed to:

$$g(x) = \frac{1 + \|x\|^{2m}}{\|x\|^{2m}} \quad , \quad m = 4, 5, \ldots \tag{65}$$

This formulation did remove the global minimum without materially affecting the effective value of K_1^2, but it also introduced local minima whose locations and values were unknown. Thus, this approach was abandoned since it appeared better to compute to a well known unwanted solution than to a poorly known unwanted solution. To remove the dependence of the algorithm upon the initial search point, another Min-All loop was incorporated to obtain a point on the constraint $\dot{V}(x) = 0$. This was done via computing on the expression

$$0 = \min_{x} \left((\|x\|^2 - c^2)^2 + \dot{V}^2(x) \right) \quad . \tag{66}$$

Here, one stepped the value c^2 until zero was the computed minimum and thus a point of intersection of the circle of radius c and the curve $\dot{V}(x) = 0$. This was then modified to make use of the actual $V(x)$, viz.,

$$0 = \min_{x} \left((V(x) - c^2)^2 + \dot{V}^2(x) \right) \quad , \tag{67}$$

where, again, c^2 is stepped until an intersection of $V(x) = c^2$ and $\dot{V}(x) = 0$ is obtained. This procedure removed the dependence on the initial search point.

The problem of computing Q^0 via Eq. (42) was then run with the determination of the initial point per Eq. (67) and ℓ per Eq. (39) with $g(x) = 1$ as inner loops. This worked well for the Duffing problem although some difficulties were encountered when λ was close to the boundary $\lambda = 0$. It did not work well at all for the Van der Pol problem and in both problems this approach was believed to be very time consuming.

At this point our attention was focused on the Van der Pol equation and an attempt was made to reformulate the problem such that it becomes a simultaneous minimization over x and λ, θ. The new formulation is:

$$J^{-1}(Q^0) + K_1^2 \ell = \min_{x,\lambda,\theta} \left\{ \left(\frac{\det P(\lambda,\theta)}{\left(V(x,\lambda,\theta)\right)^n} \right)^{\frac{1}{2}} + K_1^2 V(x,\lambda,\theta) + K_2^2 \dot{V}^2(x,\lambda,\theta) + K_3^2 G(\lambda) \right\} \tag{68}$$

The first term is the reciprocal of the area of the figure given by

$$V = x^T P(\lambda,\theta)x = c \tag{69}$$

that has parameters λ,θ and that passes through the point x. The next two terms are the penalty terms for minimizing V on $\mathring{V} = 0$ and the last term is the penalty for nonpositive lambda. When $G(\lambda)$ was as given in Eq. (45) with $\epsilon = 0$, very poor convergence resulted and very small areas, compared to results obtained by a manual trial and error search, were obtained. Epsilon was then chosen positive and when it reached the known optimal value of λ,λ^0, the convergence improved. This is, however, hardly practical.

The problem was again modified. The first term of Eq. (68) was changed to

$$K_4^2 \left(\left(\frac{\left(V(x,\lambda,\theta)\right)^n}{\det P(\lambda,\theta)} \right)^{\frac{1}{2}} - c_A^2 \right)^2 \tag{70}$$

(Note that the left side of Eq. (68) is no longer the minimum value of the new expression.) Here, the algorithm converged when $c_A^2 > 2.1$, which is near the optimal value. As c_A^2 approached 2.1 from above 2.3 the convergence of the algorithm deteriorated rapidly.

A problem that had been apparent throughout earlier experimentation with the Van der Pol equation again appeared. There are four branches to the $\dot{V}_P = 0$ locus and they are symmetrical in pairs (see Fig. 1). Thus, there are four local minima which are equivalent, for our purposes, in pairs and the algorithm shows no knowledge of the existence of the other three minima when it is converging on the fourth. This introduces a significant problem, viz., how to determine when the desired result has been computed. The only approach that is apparent now is to eliminate the computed minimum and try to compute a lesser minimum; however, this is usually not successful because new meaningless minima are introduced in the process of removing a minimum. The alternative is to reformulate the problem completely so that the desired result is the global minimum or the only minimum of the new problem. It is not at all apparent how one does this. However, the indicated sensitivity of the convergence also suggests that a significant effort should be devoted to formulating a new minimum problem that will yield the desired results.

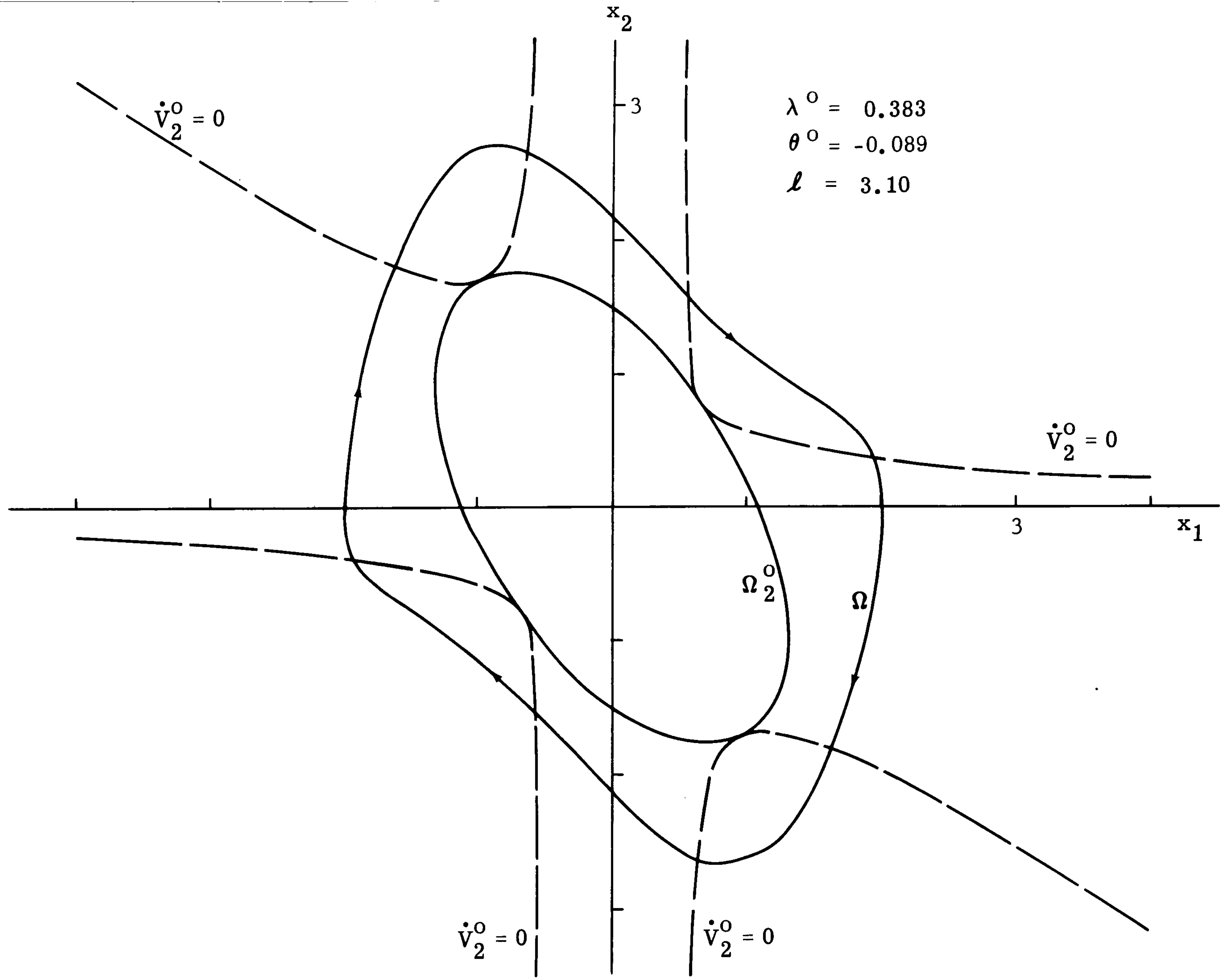

Fig. 1 Comparison of the Actual Domain of Attraction, Ω, the Optimal Quadratic Estimate, Ω_2^o, and the Loci $\dot{V}_2^o = 0$, for the Van der Pol Equation, Eq (55)

3. Higher Order Estimates

Throughout the experimentation with the Van der Pol equation, it was apparent that the "optimal" quadratic estimate of the domain of attraction was considerably smaller (roughly 50 percent) than the actual domain and that they had no boundary points in common. Thus, we tried to evolve a technique for getting a better higher order estimate.

The Zubov method (Ref. 12) is a method of finding the exact domain of attraction by solving a partial differential equation for the Liapunov function. Zubov showed that his equation could be solved using a series of homogeneous polynomials and that a truncated series solution would provide an estimate of the domain of attraction. There are two principal difficulties in using the Zubov method: 1) there is an arbitrary function that must be chosen properly; and 2) the convergence of the series solution is very nonuniform in the sense that successively better estimates are not necessarily obtained by including higher order terms.

Our experimentation with this method was based on the hope that the arbitrary function could be derived from our "optimal" quadratic Liapunov function and that it might improve the convergence of the series solution, in fact, make each estimate successively better.

The form of the Zubov equation we dealt with was:

$$\frac{dV}{dt} = \nabla V \cdot \left(Ax + f(x)\right) = -\varphi(x)\left(1 + \left(Ax + f(x)\right)^T \left(Ax + f(x)\right)\right) \tag{71}$$

where V is the Liapunov function and $\varphi(x)$ is the arbitrary function which can be taken to be a quadratic form. Our procedure was to take the series solution to be in the form

$$V(x) = \sum_{i=2}^{\infty} V_i(x) \quad , \quad V_i(cx) = c^i V(x) \tag{72}$$

which is a series of homogeneous polynomials beginning with second order, and to choose $-\varphi(x)$ to be the quadratic part of the time derivative of the "optimal" quadratic Liapunov function, viz.,

$$\varphi(x) = x^T Q^0 x \quad . \tag{73}$$

Equation (71) is then solved for $V_4(x)$, which is the next term in the series for both the Duffing and Van der Pol equations. The new Liapunov function is then $V_2 + V_4$ and the estimate of the domain of attraction D_4 is then obtained as

$$D_4 : \left(x | V_2(x) + V_4(x) < \ell_4\right) \quad , \tag{74}$$

where

$$\ell_4 = \min_{x \in E_1} \left(V_2(x) + V_4(x)\right) \tag{75}$$

$$E_1 : \left(x | \dot{V}_2(x) + \dot{V}_4(x) = 0\right) \quad . \tag{76}$$

The results were unsatisfactory in that although there was a significant improvement in the fourth order estimate for the Van der Pol equation, there was a degradation in the case of the Duffing equation (see Figs. 2 and 3). Thus, using the "optimal" quadratic Liapunov function to determine the arbitrary φ function does not produce an improved estimate in all cases.

4. Conclusions

A procedure for computing the "optimal" quadratic estimate of the domain of attraction of an equilibrium solution of a quasi-linear differential equation has been developed. Experiments have shown that the computation of an estimate, given a set of parameters, [computing ℓ Eq. (8)] is a reasonably robust process; however, the computation of the optimal parameters [Q^0 of Eq. (9)] is, at present, a very sensitive process. The problem that seems to be fundamental in both processes is that of computing local minima of a function of many variables. A possible blessing in our formulation is that there is no unique formulation for the function we minimize to solve our problem. Thus, continued experimentation should lead to the function which makes the process more robust. The ideal situation in all events would be to have the solution to our problem be the sole minimum of the function, but it is not clear how one constructs such a function.

The procedure developed and examined here does have the advantages of: 1) relative insensitivity to dimension; 2) not requiring any matrix inversions; and 3) providing an estimate which is relatively easy to visualize (a hyperellipsoid).

Combination of the Zubov approach and the "optimal" quadratic Liapunov function does not always yield a better higher order estimate of the domain of attraction.

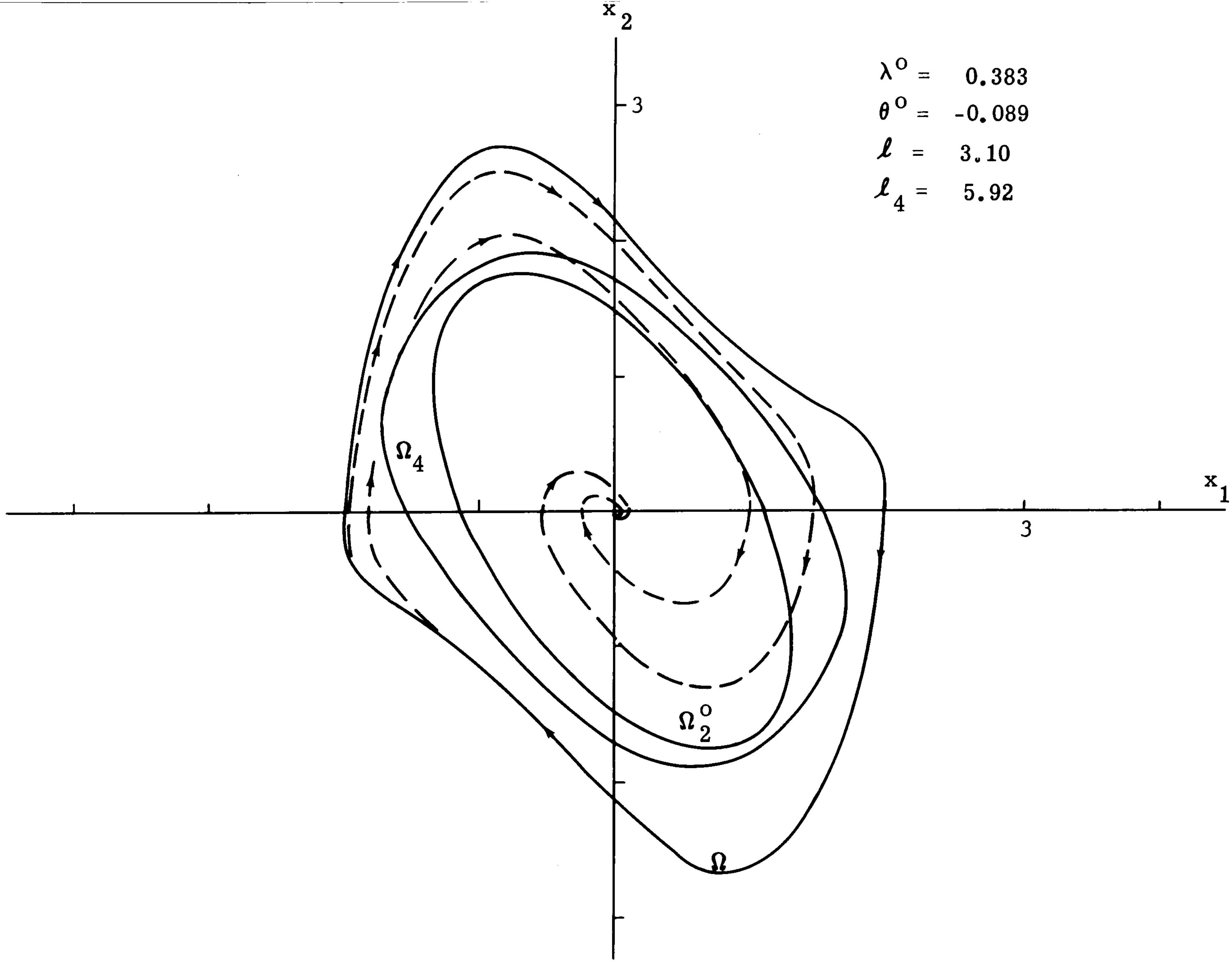

Fig. 2 Comparison of Actual Domain of Attraction, Ω, with the Optimal Quadratic Estimate, Ω_2^o, the Fourth Order Zubov Estimate, Ω_4, and Typical System Trajectories for the Van der Pol Equation, Eq (55)

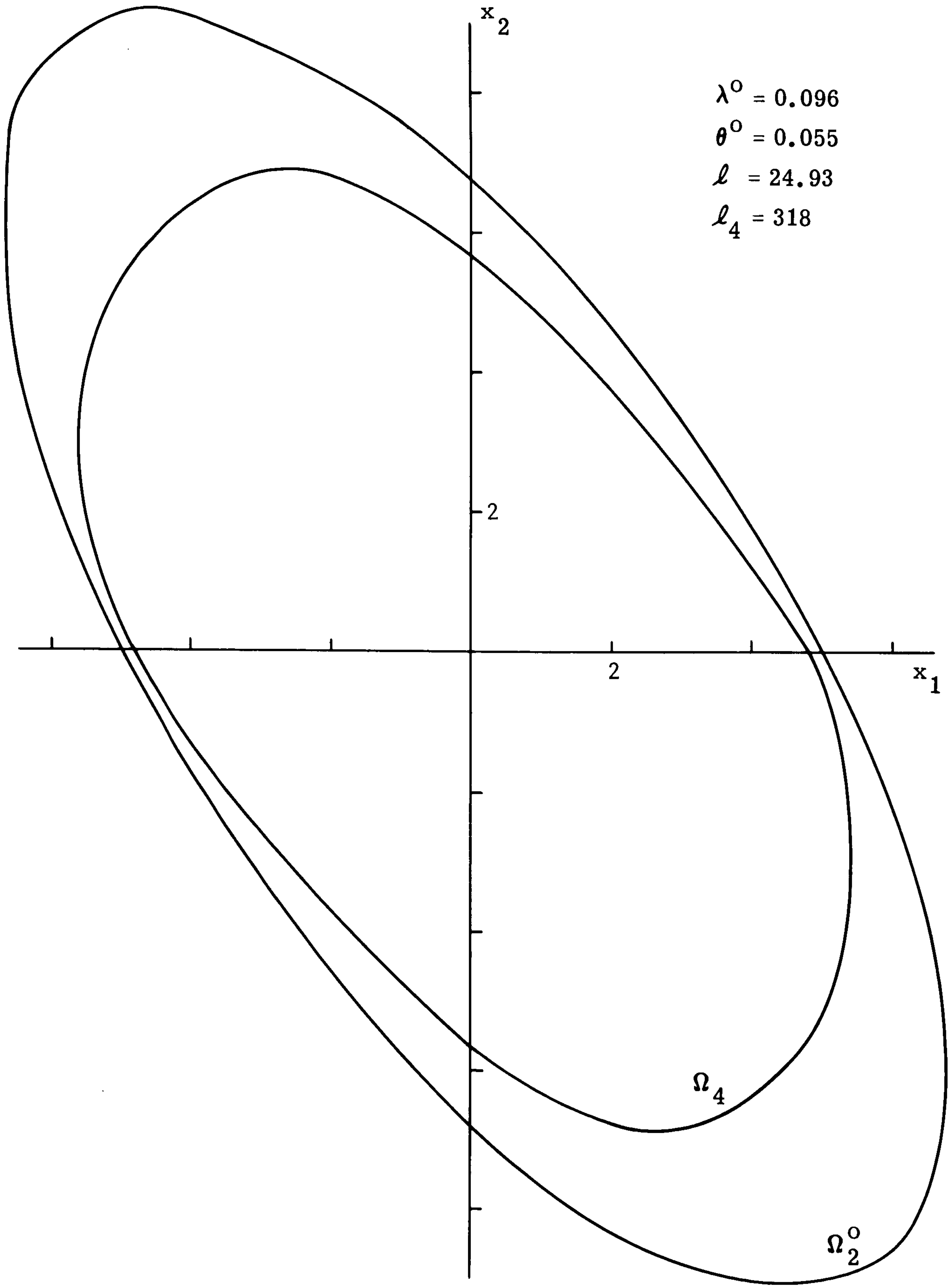

Fig. 3 Comparison of the Optimal Quadratic Estimate, Ω_2^o, and the Fourth Order Zubov Estimate, Ω_4, for the Duffing Equation, Eq (54)

III. ESTIMATION OF TEMPORAL BEHAVIOR

Since booster guidance systems are operative over finite intervals of time, stability analyses of such systems must be based on short-time considerations. The system must be guided from an initial state at time t_o to a final state at time $t_o + T$. If the system is perturbed from its path, it must reach some prescribed neighborhood of the final state within the interval of time T if it is to be considered stable.

Several recent papers have dealt with the problem of estimating the transient response of a class of linear or linearized nonlinear systems from the Liapunov stability equations, Refs. 13 and 14. If a Liapunov function, $V(x)$, exists, upper and lower bound estimates on the rate of decay can be obtained from

$$\begin{aligned} \eta_u &= \max_x \frac{\left(-\dot{V}(x)\right)}{V(x)} \\ \eta_\ell &= \min_x \frac{\left(-\dot{V}(x)\right)}{V(x)} , \end{aligned} \tag{77}$$

where η_u and η_ℓ, respectively, give the upper and lower bounds on the smallest and largest time constants of the system. However, the quality of these estimates depends on the choice of function $V(x)$. These papers do not provide information about how the Liapunov function should be chosen to provide the best estimates. On the other hand, Liapunov has shown that for a linear autonomous system, the Liapunov function which gives the exact rate of response of the system can be determined, Ref. 15. In essence, the methods proposed in Refs. 13 and 14, seek to find an upper and lower bound on the ratio of some Liapunov function and its derivative; the method proposed by Liapunov determines a function, or set of functions, such that the ratio of the function and its derivative is constant for all x.

Consider the n^{th} order system of linear autonomous differential equations

$$\frac{dx}{dt} = A\,x \tag{78}$$

where x is an $n \times 1$ column vector and A is an $n \times n$ constant matrix. It is desired to find a homogeneous function $V(x) = \Sigma\, P^{(m_1, m_2, \ldots, m_n)} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ of degree m such that

$$\nabla V \cdot \dot{x} = \sum_{s=1}^{n} \left(a_{s1}x_1 + a_{s2}x_2 \quad \ldots + a_{sn}x_n\right) \frac{\partial V}{\partial x_s} = \gamma V \quad , \tag{79}$$

where γ is a constant. By equating coefficients of the same products of the form $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ a set of N linear homogeneous equations result where $N = \frac{n(n+1) \ldots (n+m-1)}{m!}$. The equations are solved by finding the set of eigenvalues γ_i and the set of eigenvectors P_i which determine the set of Liapunov functions $V_i(x)$. Liapunov has shown that the eigenvalues γ_i can be determined directly from the expression

$$\gamma_i = m_{i1}\lambda_1 + m_{i2}\lambda_2 + \ldots + m_{in}\lambda_n \quad , \tag{80}$$

where the λ_i are the eigenvalues of the matrix A and the m_i are the sets of all nonnegative integers satisfying the relation

$$m_{i1} + m_{i2} + \ldots + m_{in} = m \quad . \tag{81}$$

For each eigenvalue γ_i and the corresponding Liapunov function $V_i(x)$, the following differential equation results:

$$\frac{dV_i}{dt} = \gamma_i V_i \quad . \tag{82}$$

Consequently,

$$V_i(t) = V_i(0)\, e^{+\gamma_i t} \quad . \tag{83}$$

Liapunov's method provides an exact method for determining the transient response of a linear system. Of course, the importance of the method is not its application to linear systems, but its possible extension so that the transient response of quasi-linear systems may be determined with more accuracy than the estimation procedures of Refs. 13 and 14. (The methods proposed in Refs. 13 and 14 are approximate even for the linear case.) It is also believed that further study of Liapunov's method for finding Liapunov functions will be useful in developing efficient techniques for obtaining better estimates of domains of attraction.

IV. ANALYSIS OF A SYSTEM WITH TIME DEPENDENT CLOSED-LOOP GUIDANCE

Booster guidance control systems are described for finite intervals of time by nonlinear, nonautonomous differential equations with control laws which are in some cases determined by iterative procedures. Because the standard well known techniques of stability analysis are not readily applicable to such systems, new techniques, or variations of present Liapunov methods are being sought.

Initial research in this area has consisted of determining the stability of a simple control system that contains characteristics representative of the more complex booster guidance systems. Specifically, we are concerned with guiding the motion of a particle moving at constant speed in a plane in the presence of a constant disturbance. The dynamics of the system are given by

$$\dot{x}_1 = v \cos u$$
$$\dot{x}_2 = V_o + v \sin u \quad , \tag{84}$$

where v is the magnitude of the velocity of the particle relative to the disturbance, V_o is the disturbance and u is the direction of the velocity. The control law $u(t)$ is such that the particle is guided from the initial point (x_1^o, x_2^o) to the final point (x_1^f, x_2^f) in minimum time in the face of disturbances in the initial conditions. The problem has been made more specific by letting $v = 1$, $V_o = 1/2$, $x_1^o = x_2^o = 0$ and $x_1^f = 2$, $x_2^f = 1$.

In the absence of any disturbance in the initial conditions, the optimum control law for minimum time is $u^*(t) = 0$, with the corresponding trajectory $x_1^* = t$, $x_2^* = t/2$, and nominal time $T_f = 2$. The initial disturbance is assumed to be randomly distributed with a bivariate normal distribution of errors in initial conditions with mean value 0 and standard deviation 0.1. The control law is assumed to be linear, time varying, and of the form

$$u(t) = u^*(t) + p_1(t) \left[x_1(t) - x_1^*(t)\right] + p_2(t) \left[x_2(t) - x_2^*(t)\right] \quad , \tag{85}$$

where

$$p_1 = p_{10} + p_{11}t \quad ,$$

and (86)

$$p_2 = p_{20} + p_{21}t \quad .$$

For the specific problem considered, with

$$\begin{aligned} p_{10} &= 0.153 \qquad p_{11} = 0.090 \\ p_{20} &= -0.305 \qquad p_{21} = -0.195 \quad , \end{aligned} \tag{87}$$

the mean square miss of the target point (2,1) is 0.00139; the resulting expected final time is 2.00287.

The stability problem for the system described above may be stated as follows: From what set of initial states (x_1^o, x_2^o) will the particle reach a point in some ϵ-neighborhood of the final state (2,1), i.e.,

$$(x_1^f - 2)^2 + (x_2^f - 1)^2 < \epsilon \quad , \tag{88}$$

subject to the constraint $T_f \leq M$, where M is some constant.

To determine any characteristic of the problem that might be useful in the development of a general stability technique, an analysis of the system response was undertaken. It was initially assumed that the control law $u(t)$ was time invariant, i.e.,

$$u(t) = p_{10}\left(x_1(t) - x_1^*(t)\right) + p_{20}\left(x_2(t) - x_2^*(t)\right) \quad . \tag{89}$$

Thus, the system is described by

$$\begin{aligned} \dot{x}_1(t) &= \cos\left[p_{10}\left(x_1(t) - t\right) + p_{20}\left(x_2(t) - t/2\right)\right] \\ \dot{x}_2(t) &= \frac{1}{2} + \sin\left[p_{10}\left(x_1(t) - t\right) + p_{20}\left(x_2(t) - t/2\right)\right] \quad . \end{aligned} \tag{90}$$

Under the translation

$$y_1(t) = x_1(t) - t$$
$$y_2(t) = x_2(t) - t/2 \quad , \tag{91}$$

Eqs. (90) become

$$\mathring{y}_1 = -1 + \cos\left[p_{10}y_1 + p_{20}y_2\right]$$
$$\dot{y}_2 = \sin\left[p_{10}y_1 + p_{20}y_2\right] \tag{92}$$

with the equilibrium solution $y_1 = y_2 = 0$.

The loci of equilibrium points for Eqs. (92) are given by

$$y_2 = -\frac{p_{10}}{p_{20}}\,y_1 + \frac{2n\pi}{p_{20}} \quad , \quad n = 1, 2, \ldots \quad . \tag{93}$$

Thus, if the initial point $(y_1^o = x_1^o \, , \, y_2^o = x_2^o)$ lies on the locus, the solution for $y_1(t)$ and $y_2(t)$ will be constant in time, and the trajectories in the x_1, x_2-plane will be given by

$$x_1(t) = t + x_1(0)$$
$$x_2(t) = t/2 + x_2(0) \quad . \tag{94}$$

These trajectories are parallel to the optimum trajectory and do not converge to the target point.

To find the complete trajectories in the y_1,y_2-plane, let

$$z(t) = p_{10}y_1(t) + p_{20}y_2(t) \quad , \tag{95}$$

then from Eqs. (92)

$$\dot{z}(t) = p_{10}\dot{y}_1(t) + p_{20}\dot{y}_2(t) = -p_{10} + p_{10}\cos z + p_{20}\sin z \quad . \tag{96}$$

Solving for $z(t)$,

$$z(t) = 2\tan^{-1}\left[\frac{p_{20}}{p_{10}}\frac{1}{1 - ke^{-p_{20}t}}\right] \quad , \tag{97}$$

where

$$k = 1 - \frac{1}{\frac{p_{10}}{p_{20}}\tan\frac{z(0)}{2}} \quad . \tag{98}$$

By substituting Eq. (97) into Eqs. (92), and integrating,

$$y_1(t) = \frac{2p_{10}}{p_{10}^2 + p_{20}^2}\left[\tan^{-1}\frac{p_{10}}{p_{20}}\left(ke^{-p_{20}t} - 1\right) - \tan^{-1}\frac{p_{10}}{p_{20}}(k-1)\right]$$

$$+ \frac{p_{20}}{p_{10}^2 + p_{20}^2}\ln\frac{\left((k-1)^2 + \frac{p_{20}^2}{p_{10}^2}\right)e^{-2p_{20}t}}{\left(ke^{-p_{20}t} - 1\right)^2 + \frac{p_{20}^2}{p_{10}^2}} + y_1(0) \tag{99}$$

$$y_2(t) = \frac{2p_{20}}{p_{10}^2 + p_{20}^2}\left[\tan^{-1}\frac{p_{10}}{p_{20}}\left(ke^{-p_{20}t} - 1\right) - \tan^{-1}\frac{p_{10}}{p_{20}}(k-1)\right]$$

$$- \frac{p_{20}}{p_{10}^2 + p_{20}^2} \ln \frac{\left((k-1)^2 + \frac{p_{20}^2}{p_{10}^2}\right) e^{-2p_{20}t}}{\left(ke^{-2p_{20}t} - 1\right)^2 + \frac{p_{20}^2}{p_{10}^2}} + y_2(0) \quad . \qquad \text{(99) (Cont.)}$$

It is evident from Eqs. (98) and (99) that if the initial point is on the equilibrium locus, Eqs. (99) reduce to

$$y_1(t) = y_1(0) \quad , \quad y_2(t) = y_2(0) \quad .$$

It can also be seen from Eqs. (99) or Eq. (97) that if the initial point $\left(y_1(0), y_2(0)\right)$ does not lie on the equilibrium locus, and if p_{20} is real and negative, then the final point $\left(y_1(\infty), y_2(\infty)\right)$ will lie on the equilibrium locus. Thus if p_{20} is real and negative, every solution $y(t)$ tends to the set of points where

$$y_2 = - \frac{p_{10}}{p_{20}} y_1 + \frac{2n\pi}{p_{20}} \quad , \quad n = 1, 2, \ldots \quad .$$

An approximate analysis of the response of the system when the control law is time varying, as given by Eqs. (85) and (86), can be carried out if $p_1(t)$ and $p_2(t)$ are approximated by "staircase" functions such that

$$\begin{aligned} p_1(nT) &= p_{10} + p_{11}nT \\ p_2(nT) &= p_{20} + p_{21}nT \quad . \end{aligned} \qquad (100)$$

In the n^{th} interval, i.e., $(n-1)T < t < nT$,

$$\begin{aligned} z_n(\tau) &= (p_{10} + p_{11}nT)\, y_{1n}(\tau) + (p_{20} + p_{21}nT)\, y_{2n}(\tau) \\ &= 2 \tan^{-1} \left[\frac{p_2(nT)}{p_1(nT)} \cdot \frac{1}{1 - k_n e^{-p_2(nT)\tau}} \right] \end{aligned} \qquad (101)$$

for $0 \leq \tau \leq T$. The constant k_n is given by

$$k_n = 1 - \frac{1}{\dfrac{p_1(nT)}{p_2(nT)} \tan \dfrac{z_{n-1}(T)}{2}} \quad . \tag{102}$$

By using Eqs. (101) and (102), the following linear difference equation for k_n results:

$$k_n - F_n k_{n-1} = \frac{p_{11}p_{20} - p_{10}p_{21}}{p_1(nT)\; p_2[(n-1)T]} \quad , \tag{103}$$

where

$$F_n = \frac{p_1[(n-1)T]\; p_2(nT)}{p_1(nT)\; p_2[(n-1)T]}\, e^{-p_2[(n-1)T]T} \quad . \tag{104}$$

Solving Eq. (103) yields

$$k_n = \left[\prod_1^n F_i\right]\left[k_0 + \sum_1^n \frac{R_i}{\prod_1^i F_j}\right] \tag{105A}$$

or

$$k_n = \left[\frac{p_{10}p_2(nT)}{p_{20}p_1(nT)}\, e^{-\left(p_{20} + \frac{p_{21}(n-1)T}{2}\right)nT}\right] \cdot \left[k_0 + \frac{p_{20}(p_{11}p_{20} - p_{10}p_{21})}{p_{10}} \right. \tag{105B}$$

$$\left. \sum_{i=1}^{n} \frac{e^{\left(p_{20} + p_{21}\frac{(i-1)T}{2}\right)iT}}{p_2(nT)\; p_2\left((n-1)T\right)}\right] \quad .$$

The approximate location of the particle at any instant of time is then specified by

$$y_{in}(\tau) = P_{in}(\tau) + y_{io}(0) + \sum_{j=1}^{n-1} P_{ij}(1) \ , \ i = 1, 2 \ , \qquad (106)$$

where

$$P_{in}(\tau) = \frac{2p_i(nT)}{p_1^2(nT) + p_2^2(nT)} \left[\tan^{-1} \frac{p_1(nT)}{p_2(nT)} \left(k_n e^{-p_2(nT)\tau} - 1 \right) - \tan^{-1} \frac{p_1(nT)}{p_2(nT)} (k_n - 1) \right]$$

$$- (-1)^i \frac{p_2(nT)}{p_1^2(nT) + p_2^2(nT)} \ln \left\{ \frac{(k_n - 1)^2 + \dfrac{p_2^2(nT)}{p_1^2(nT)}}{\left(k_n e^{-p_2(nT)\tau} - 1 \right)^2 + \dfrac{p_2^2(nT)}{p_1^2(nT)}} \right\} e^{-2p_2(nT)\tau} \qquad (107)$$

The work carried out to date has essentially consisted of analyzing the given system and has resulted in Eqs. (106) and (107). It will now be necessary to study these equations along with the trajectories that result from a computer simulation in order to determine characteristics that will enable development of a stability analysis technique that does not require a solution of the system equations and will be applicable to booster guidance systems. To achieve this, research efforts should be directed toward examining the possibility of applying standard Liapunov stability techniques over each interval $(n-1)T \leq t \leq nT$, and using the information obtained from these intervals to conclude stability for the entire interval $0 \leq t \leq T_f$. For example, a possible approach to the problem described above would be to determine a set of Liapunov functions $V_n(x)$, where $V_n(x)$ is the Liapunov function for the system described in the interval $nT < t < (n + 1)T$. The system described in the n^{th} interval would be analyzed for stability without considering the finite length of the interval and the domain of asymptotic stability implied by each function $V_n(x)$ would be determined. The region in which the system state lies at time $(n + 1)T$ could be determined using the methods of Refs. 13 and 14 or Liapunov's method which is described earlier in this report. If, for all n, this region lies within the domain of asymptotic stability for the system described in the $(n + 1)^{st}$ interval, and if this region is smaller than the domain of asymptotic stability for the system described in the n^{th} interval, finite time stability could be concluded.

V. THE ITERATIVE GUIDANCE MODE

The booster launch vehicle using the iterative guidance scheme described by Chandler and Smith (Ref. 16) may be regarded as a non-linear, sampled-data system. A model representing the system for the two dimensional spherical earth configuration may be formulated in state space.

It is convenient to describe the system in a η,ξ coordinate system such that the η-axis is the earth centered plumb line and the total gravity force at the cut-off point is in the η-direction. The equations of motion of the vehicle are then given by

$$\ddot{\xi} = a(t) \cos \lambda + g_\xi$$
$$\ddot{\eta} = a(t) \sin \lambda + g_\eta \quad , \tag{108}$$

where $a(t)$ is the longitudinal vehicle acceleration given by $a(t) = V_{ex} \frac{1}{\tau - t}$; λ is the steering angle; g_ξ and g_η are gravity components. The remaining terms in Eq. (108) are defined by

V_{ex} = exhaust velocity

m_o = initial mass

$\dot{m}$ = mass flow rate

$\tau = \dfrac{m_o}{\dot{m}}$

t = time.

The steering law λ is computed to guide the vehicle from some initial state $\left(\xi(0), \eta(0), \dot{\xi}(0), \dot{\eta}(0)\right)$ to some final state $(\xi_f, \eta_f, \dot{\xi}_f, \dot{\eta}_f)$. In the problem considered here, only the final altitude, η_f, and the final velocity $(\dot{\xi}_f, \dot{\eta}_f)$ are specified; the range $\left(\eta_f - \eta(0)\right)$ is unspecified. The cut-off point is determined by the amount of time, T_c, needed to achieve the final state and consequently is dependent on both the initial and final states. In the case of the range coordinate, $\eta(0)$ depends on T_f. ($\eta_f = 0$ as a result of the coordinate system chosen.)

The steering law was derived on the assumption of a flat earth with a uniform gravity field (Refs. 17 and 18), and is of the form

$$\lambda = \tilde{\lambda} - (K_1 + K_2 t) \quad , \tag{109}$$

where $\tilde{\lambda}$, K_1, and K_2 are constants dependent on the initial and final states. To approximate the flat earth assumption in the case of the spherical earth, gravity is assumed to be constant over the flight path and equal to the average of the values of gravity in the initial and final states.

In order that the system be adaptive and adjust for deviations from the nominal path which result from the approximations made and external disturbances, the state of the system is evaluated at discrete sampling instants, $t = nT$, and the instantaneous state at the sampling instants and the required final state are used to determine the time to cut-off, $T_c(nT)$; the average gravity components $\bar{g}_\xi(nT) = \frac{1}{2} g_\xi(nT)$ and $\bar{g}_\eta(nT) = \frac{1}{2}[g_\eta(nT) + g_\eta\left(T_f(nT)\right)]$; and the steering law $\lambda(nT)$ necessary to guide the system from the instantaneous state to the final state. During the interval $nT < t < (n+1)T$, the steering law is given by

$$\lambda(nT) = \tilde{\lambda}(nT) - K_1(nT)\,[1 - B(nT)(t - nT)] \tag{110}$$

where

$$\tilde{\lambda}(nT) = \tan^{-1} \frac{\dot{\eta}_f - \dot{\eta}(nT) + \bar{g}_\eta(nT)}{\dot{\xi}_f - \dot{\xi}(nT) + \bar{g}_\xi(nT)} \tag{111A}$$

$$B(nT) = \frac{L(nT)}{\tau L(nT) - T_c(nT)} \tag{111B}$$

$$K_1(nT) = \frac{\eta_f - \left[\eta(nT) + \dot{\eta}(nT)T_c(nT) + \frac{1}{2}\,\bar{g}_\eta(nT)T_c^2(nT)\right] + s(nT)\sin\tilde{\lambda}(nT)}{\left[s(nT) - B(nT)Q(nT)\right]\,\cos\,\tilde{\lambda}(nT)} \tag{111C}$$

$$L(nT) = \ln \frac{\tau - nT}{\tau - T_f(nT)} \tag{111D}$$

$$s(nT) = V_{ex} \left[\left(\tau - T_f(nT) \right) \ln \frac{\tau - nT}{\tau - T_f(nT)} - T_c(nT) \right] \tag{111E}$$

$$Q(nT) = \tau s(nT) - \frac{T_f^2(nT)}{2} - (nT) T_f(nT) + \frac{(nT)^2}{2} \tag{111F}$$

$$T_f(nT) = nT + T_c(nT)$$

To obtain Eq. (108) in state space form, let $x_1 = \xi$, $x_2 = \dot{\xi}$, $x_3 = \eta$, $x_4 = \dot{\eta}$. The system dynamics then become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(t) \cos \lambda + g_\xi \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= a(t) \sin \lambda + g_\eta \quad , \end{aligned} \tag{112}$$

where $\lambda = \tilde{\lambda} - K_1(1 - Bt)$ and $\tilde{\lambda}$, K_1 and B are given by Eqs. (111). Equation (112) can be rewritten as

$$\dot{x} = F(x, x_f, t) \quad .$$

The system described in this section is in many ways similar to the time dependent closed-loop system described in the previous section. Both systems are nonautonomous and nonlinear and the form of the system dynamic equations for each case are similar. However, the steering angle determined by the iterative guidance scheme is far more complicated than the time dependent steering angle described in Sec. IV.

We believe that a suitable stability analysis technique applicable to guided space vehicles using the iterative guidance mode is not imminent. This is primarily due to the finite time operation, and the lack of meaning of the comparison of disturbed and nominal trajectories which is fundamental to standard stability analyses. Indeed, the whole concept of stability must be examined and redefined in order to have meaning for such systems. Once this has been accomplished, results for the stability analysis of systems such as that described in the previous section will be forthcoming and will serve as the basis for the analysis of the more complicated systems.

VI. CONCLUSIONS AND RECOMMENDATIONS

The results we have obtained to date indicate that one can certainly obtain an estimate of the domain of attraction of an equilibrium solution of a quasi-linear dynamical system and it seems feasible to compute an optimal quadratic estimate. One may also, in some cases, improve an optimal quadratic estimate by using the Zubov method to obtain a higher order estimate. The three problems which seem to be fundamental to these processes and which require more research are: 1) how to successively compute the minima of a function; 2) how to formulate a function whose global minimum or only minimum is the solution to this optimal estimation problem; and 3) how to compute successively better higher order estimates.

A review of Liapunov's work led to rediscovery of his method for establishing the exact temporal behavior of a linear system which contrasts sharply with the approximate methods that have been advanced in the recent literature. If this method can be effectively applied to estimating the temporal behavior of quasi-linear systems, then we will have an approach to the stability analysis of finite time systems in the sense of estimating the amount of contraction between a set of initial states and the corresponding set of final states.

The analysis of a simple closed-loop guidance system and a review of the equations of the Iterative Guidance Mode point out that the most serious problem that must be solved in order to reach the long term goal of this study is the formulation of what stability means in a finite time process whose goal is to reach a point in space, not to follow or remain near a particular path. The other obvious problem is the development of techniques for analyzing nonautonomous, nonlinear, finite time systems with respect to the stability definition that is evolved for these systems.

In summary, we have examined some of the theoretical, computational and practical aspects of the problem of evaluating the stability of guided space vehicles, and we have indicated what appear to be the fundamental problems and some likely approaches to their solution. Yet, we have only scratched the surface of an important and apparently very difficult problem that will require a great deal more research.

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