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OPTIMUM STATION-SATELLITE CONFIGURATIONS FOR SIMULTANEOUS OBSERVATIONS TO SATELLITES

Kurt Lambeck

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Smithsonian Institution
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Cambridge, Massachusetts, 02138

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BIOGRAPHICAL NOTE

Mr. Lambeck graduated in geodesy from the University of New South Wales, Australia, in 1963. He studied at the Geodetic Institute, Delft, Holland, in 1964 and at the National Technical University of Athens in 1965. He is currently working at the Department of Surveying and Geodesy, Oxford University, England, and was Consultant Geodesist with the Smithsonian Astrophysical Observatory during the summer of 1966.

His interests lie in satellite geodesy, particularly in combining the results of different methods and of different observations.

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ABSTRACT

Simultaneous direction observations from two stations to satellite positions give a solution for the direction of the vector joining the stations. Such a solution is generally of lower accuracy in height than in azimuth; there exist, however, conditions governing the distribution of the observations that ensure equal accuracy in all directions and that are optimal from the viewpoint of the number of observations required. These conditions are derived and a simple formula is established for specifying *a priori* the accuracy of the direction joining the stations.

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Kurt Lambeck

1. INTRODUCTION

The simultaneous observation of directions to satellites from two or more stations enables the directions of the vectors joining these stations to be determined in an astronomical reference system. These directions may then be used as the basis of a more extensive three-dimensional triangulation network, giving a purely geometric solution to the problem of determining station positions and the shape of the earth.

The present distribution of the astrophysical observing stations in the Smithsonian Astrophysical Observatory tracking network means that, in the majority of cases, simultaneous observations between only two stations at a time are feasible, and that an unfavorable error propagation will generally exist through the triangulation formed by the directions between the observatories. Little can be done to improve the latter unless range measurements are introduced; but the single space directions do, nevertheless, provide very useful constraints in the dynamic solution of the earth's shape.

This work was supported in part by Grant No. 87-60 from the National Aeronautics and Space Administration.

With these factors in mind the optimum distribution of satellite positions for simultaneous observations from only two stations is sought, although the criteria developed in the subsequent sections will also be of value in assessing the optimum configuration for simultaneous observations from more than two stations.

The actual values of the space directions are only part of the required solution; precision estimates, which can be obtained by a judicial use of least-squres procedures, are also required. The magnitude and reliability of such estimates will depend on: (1) the assumptions made about the variance-covariance matrix of the original observed quantities; (2) the rigor of the variety of "corrections" that must be applied to the observations; and (3) the "geometry" of the solution, that is, the relative distribution of the satellite positions and the observing stations.

The last of these factors will be investigated here, assuming that the variance-covariance matrix of the observations is known and that the corrections in (2) have been applied.

Several terms, the definitions of which follow, have been introduced to assist in describing the geometry of the satellite-station configuration. (See also Figure 1.)

A. Station-station vector: the straight line joining the two observatories.

B. Common vertical plane: the plane containing the verticals at the two observatories. The assumption of a spherical earth inherent in this definition will suffice here.

C. Satellite plane: the plane defined by the simultaneous direction observations from the two stations to the satellite. Both the vertical plane and the satellite plane will contain the station-station vector.

D. Satellite-plane angle: Z, the angle made by the satellite plane with the vertical plane.

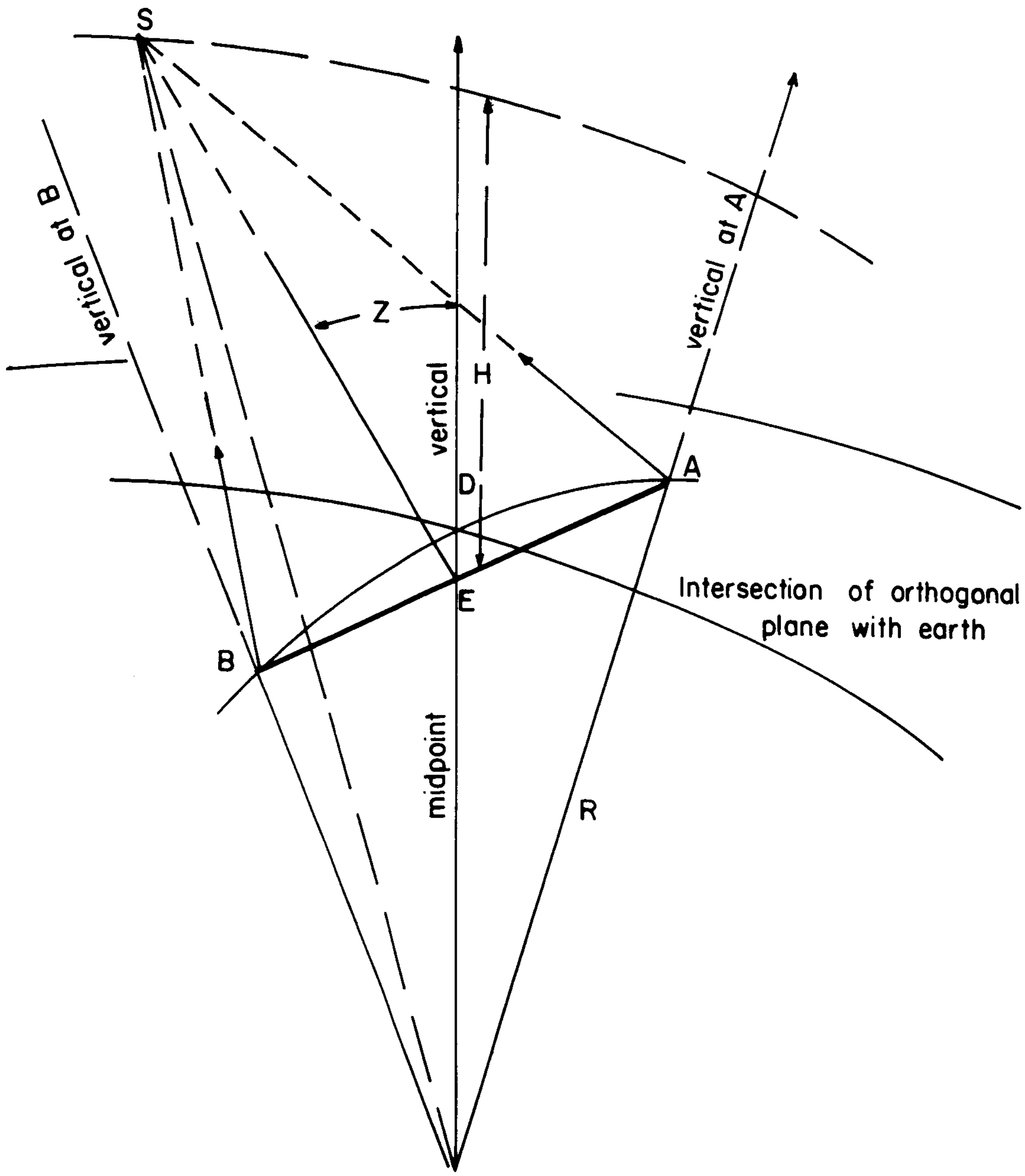


Figure 1. Station-satellite configuration for the case C =0; i.e., where the satellite positions S lie in the orthogonal plane. The vertical plane is the plane containing the verticals at A and B. The orthogonal plane is that plane passing through the point midway between A and B, and is normal to the vertical plane.

E. Midpoint vertical: the vertical passing through the point midway between the two observatories. This vertical will lie in the vertical plane.

F. Orthogonal plane: the plane that contains the midpoint vertical and is normal to the vertical plane. The perpendicular distance of the satellite from the orthogonal plane will be denoted by C.

G. Horizon distance: η, the angular distance subtended at the earth's center by the observing station and the subsatellite point.

H. Coverage area: the area on the earth's surface enclosed by the horizon distance.

The precision of the direction of the station-station vector can be described by projecting its variance-covariance matrix onto the orthogonal plane. This projection will then describe the precision in any direction on this plane. Two directions of particular interest are the direction parallel to the intersection of the vertical plane with the orthogonal plane and the direction in the orthogonal plane normal to the vertical plane. The latter will correspond with good accuracy to the accuracy in azimuth of the station-station vector, while the former, the component in the vertical plane, is readily converted to the precision in height of one station relative to the other.* The precision in azimuth will be denoted by σ_A^2, and that in the vertical plane, by σ_V^2. The correlation between these two components will be denoted by σ_{AV}. The variance-covariance matrix of the station-station vector may therefore be written as

$$\begin{bmatrix} \sigma_V^2 & \sigma_{AV} \\ \sigma_{VA} & \sigma_A^2 \end{bmatrix}$$

*Because of the earth's curvature the precision in the vertical plane should be multiplied by the distance between the stations and the secant of half the angle subtended by the two stations at the earth's center in order to obtain the relative height precision. For a distance of 6000 km between stations, the error introduced by neglecting the curvature is of the order of 15%.

and the corresponding weight matrix as

$$\begin{bmatrix} W_V^2 & W_{AV} \\ W_{VA} & W_A^2 \end{bmatrix} = \frac{1}{\sigma_V^2 \sigma_A^2 - (\sigma_{AV})^2} \begin{bmatrix} \sigma_A^2 & -\sigma_{AV} \\ -\sigma_{VA} & \sigma_V^2 \end{bmatrix} .$$

The correlation and the ratio σ_V^2/σ_A^2 will essentially be a function of the geometrical configuration of the two stations and satellite position, whereas the magnitude of the error distribution defined by the variance-covariance matrix will depend more on the number and precision of the original observations than on the geometry.

Clearly, the ideal configuration would be one that gives a zero correlation coefficient and $\sigma_V^2/\sigma_A^2 = 1$. The magnitude of such a circular distribution could then be decreased by increasing the accuracy and the number of the original observations.

In the following paragraphs the error distributions of the station-station vector obtained from hypothetical configurations will be investigated. From these results an attempt will be made to draw certain general rules for the optimum geometry of station-satellite configurations.

2. THE MATHEMATICAL MODEL

The model used to describe the simultaneous direction observations from two stations is based on the condition that these directions and the station-station vector must be coplanar. Thus, if the station-object unit vectors are denoted by $\vec{u}_1$ and $\vec{u}_2$ and the station-station unit vector by $\vec{u}_3$, the condition that must be satisfied is

$$\vec{u}_1 \cdot \vec{u}_2 \times \vec{u}_3 = 0 \quad ;$$

or, if u_i^1, u_i^2, and u_i^3 are the components of $\vec{u}_i$ in the x y z directions, respectively (these axes being orthogonal and defined by the earth's rotation axis and the Greenwich meridian),

$$\text{determinant} \begin{vmatrix} u_1^1 & u_1^2 & u_1^3 \\ u_2^1 & u_2^2 & u_2^3 \\ u_3^1 & u_3^2 & u_3^3 \end{vmatrix} = \Delta = 0 \quad . \tag{1}$$

If the declination is denoted by δ_i, the right ascension by α_i, and the Greenwich hour angle by θ, the u_i are given by

$$\vec{u}_i = \begin{bmatrix} u_i^1 \\ u_i^2 \\ u_i^3 \end{bmatrix} = \begin{bmatrix} \cos \delta_i \cos (\alpha - \theta)_i \\ \cos \delta_i \sin (\alpha - \theta)_i \\ \sin \delta_i \end{bmatrix} \quad . \tag{2}$$

Linearizing equation (1), using (2) and

$$\delta_i = \delta_i^0 + d\delta_i \quad ,$$

$$\bar{\alpha}_i \equiv (\cos \delta_i)\, \alpha_i = \bar{\alpha}^0 + d\bar{\alpha}_i \quad ,$$

gives

$$\sum_{j=1}^{3} (a_j\, d\delta_j + b_j\, d\bar{\alpha}_j) + \Delta^0 = 0 \quad , \tag{3}$$

where

$$\begin{aligned} a_j = \; & \sin \delta_j \;\cos \delta_{j+1} \;\sin \delta_{j+2} \;\sin \left[(\alpha - \theta)_j - (\alpha - \theta)_{j+1}\right] \\ & - \cos \delta_j \;\cos \delta_{j+1} \;\cos \delta_{j+2} \;\sin \left[(\alpha - \theta)_{j+1} - (\alpha - \theta)_{j+2}\right] \\ & + \sin \delta_j \;\sin \delta_{j+1} \;\cos \delta_{j+2} \;\sin \left[(\alpha - \theta)_{j+2} - (\alpha - \theta)_j\right] \end{aligned}$$

and

$$\begin{aligned} b_j = & - \cos \delta_{j+1} \;\sin \delta_{j+2} \;\cos \left[(\alpha - \theta)_j - (\alpha - \theta)_{j+1}\right] \\ & + \sin \delta_{j+1} \;\cos \delta_{j+2} \;\cos \left[(\alpha - \theta)_{j+2} - (\alpha - \theta)_j\right] \quad ; \end{aligned}$$

the indices j are cyclic. For j=1, 2, the δ_j^0 and α_j^0 are observed quantities, δ_i and α_i, whose variance-covariance matrix is assumed known. The $d\delta_j$ and $d\alpha_j$ are therefore corrections to observations and will be denoted by

$$d\delta_j = \varepsilon_{\delta_j} \quad ,$$

$$\overline{d\alpha}_j = \left(\cos \delta_j\right) \varepsilon_{\alpha_j} = \varepsilon_{\overline{\alpha}_j} \qquad j = 1, 2 \quad .$$

For $j = 3$ (the station-station vector), the δ_3^0 and α_3^0 are approximate values and the $d\delta_3$ and $d\alpha_3$ are corrections to them, designated by

$$d\delta_3 = \Delta\delta_3 \quad ,$$

$$\overline{d\alpha}_3 \equiv (\cos \delta_3)\, \Delta\alpha_3 = \overline{\Delta\alpha}_3 \quad .$$

Equation (3) now takes the form

$$\left(a_1\ b_1\ a_2\ b_2\right) \begin{bmatrix} \varepsilon_{\delta_1} \\ \varepsilon_{\overline{\alpha}_1} \\ \varepsilon_{\delta_2} \\ \varepsilon_{\overline{\alpha}_2} \end{bmatrix} + \left(a_3\ b_3\right) \begin{vmatrix} \Delta\delta_3 \\ \Delta\overline{\alpha}_3 \end{vmatrix} + \Delta^0 = 0 \quad . \tag{4}$$

Since each pair of simultaneous observations will yield such an equation, a solution for the $\Delta\delta_3$ and $\overline{\Delta\alpha}_3$ is possible. One set of simultaneous observations will not give a solution for the unknown $\Delta\delta_3$ and $\Delta\alpha_3$ and it is therefore meaningless to speak about the variance matrix of the station-station vector in this case. However, the inverse of this matrix, the weight matrix, does exist; it is given by

$$W_j = \begin{bmatrix} a_3 A a_3 & a_3 A b_3 \\ b_3 A a_3 & b_3 A b_3 \end{bmatrix} \overset{\text{per definition}}{=} \begin{bmatrix} W_{V_j}^2 & W_{A_j V_j} \\ W_{V_j A_j} & W_{A_j}^2 \end{bmatrix} \quad , \tag{5}$$

where

$$A = \left[\left(a_1^2 + b_1^2 + a_2^2 + b_2^2\right)\sigma^2\right]^{-1} \quad ,$$

assuming that the variance-covariance matrix of the observations is a diagonal matrix, the nonzero elements of which are σ^2.

If it is further assumed that there is no correlation between the observed quantities of different pairs of simultaneous observations, the weight matrix of the station-station vector determined from n sets of simultaneous observations is simply the sum of the individual weight matrices for each plane. That is,

$$W = \sum_{j=1}^{n} \begin{bmatrix} W_{V_j}^2 & W_{A_j V_j} \\ W_{V_j A_j} & W_{A_j}^2 \end{bmatrix} \quad .$$

This matrix will generally be nonsingular. For correlation-free observations, then, the total variance-covariance matrix is simply the inverse of the sum of the weight matrices of the individual satellite planes.

3. THREE HYPOTHETICAL STATION-SATELLITE CONFIGURATIONS

The three hypothetical configurations are:

A. Case 1. The satellites observed lie in the orthogonal plane.

B. Case 2. The satellite points lie in a plane that is parallel to the orthogonal plane and that passes through one of the observatories.

C. Case 3. The satellite points lie in a plane that is parallel to the orthogonal plane but passes through a point on the station-station vector extended a distance L, 2L, respectively, from two stations, L being the distance between the two stations.

The distance from the satellite to the orthogonal plane has been denoted by C, so that the three cases may be distinguished as $C = 0$, $C = L/2$, and $C = 3L/2$.

3.1 Case 1. C = 0.

The two stations are represented by A and B, and S is any satellite position on the orthogonal plane; D is the point where the midpoint vertical cuts the earth's surface, and E is the point on the station-station vector midway between A and B (see Figure 1). The distance ED will be a function of the distance L between the two stations, but this variable can be eliminated by measuring the height H of the satellite from E rather than from D. The two variables that define the shape and orientation of the satellite plane are therefore (for $C = 0$) the satellite-plane angle Z and the ratio L/H.

For various values of Z and L/H the equations of the satellite planes can be computed from equation (4), and the contributions of each of these planes to the weight in height, azimuth, and the correlation term can be determined. Denoting the weight in the vertical plane by W_V^2, that in azimuth by W_A^2, and the correlation between these two directions by W_{AV}, the variations of W_V^2, W_A^2, and W_{AV} with Z and L/H are given in Figures 2a, b, and c.

For L/H approaching infinity (that is, the satellite is midway between the two stations), both W_V^2 and W_A^2 approach the limiting value 2, as would be expected since this is equivalent to measuring directly the station-station vector direction twice.

For two planes of equal satellite-plane angles, and equal L/H but on opposite sides of the vertical plane, the W_V^2 and W_A^2 are the same, although the correlation will be of opposite sign. Thus, any pair of such planes will yield correlation-free components of the station-station vector.

Because the integral of the weight functions between arbitrary limits of Z represents the total weight of satellite planes distributed uniformly with respect to Z, those limits of Z required to make $\Sigma W_V^2/\Sigma W_A^2 = 1$ may be solved for. The maximum value of Z_{max} will generally be determined from intervisibility and refraction considerations, and the corresponding minimum value (Z_{min}) can therefore be determined by integrating the weight functions over the range $Z_{max} - Z_{min}$, imposing the condition $\Sigma W_V^2/\Sigma W_A^2 = 1$. Such integrations for variable Z_{max} and L/H have been performed (see Table 1). The integrations have been carried out numerically, and the accuracy of the Z_{min} is estimated to be of the order 0.5°. Thus, for all practical purposes, the relationship between Z_{max} and Z_{min} is independent of L/H. Mean values of Z_{min} are tabulated in the last column of Table 1. The zenith distance of the satellite as viewed from one of the observatories will, on the other hand, be dependent on both L and H because of the earth's curvature.

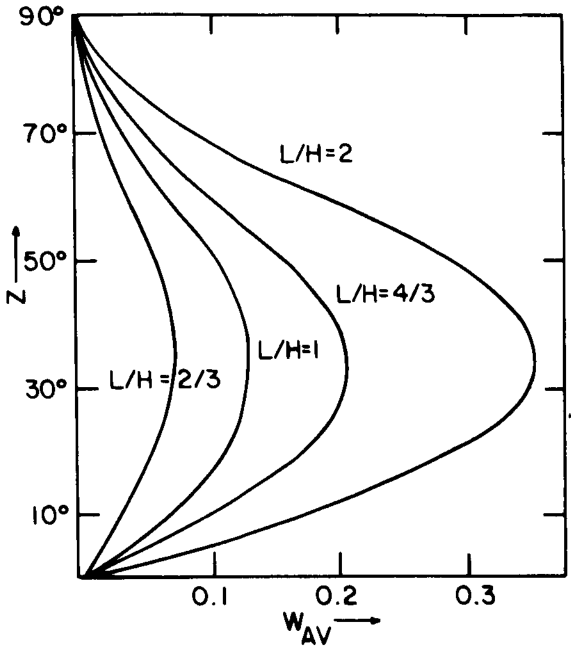


Figure 2a. Weight component in the vertical plane.

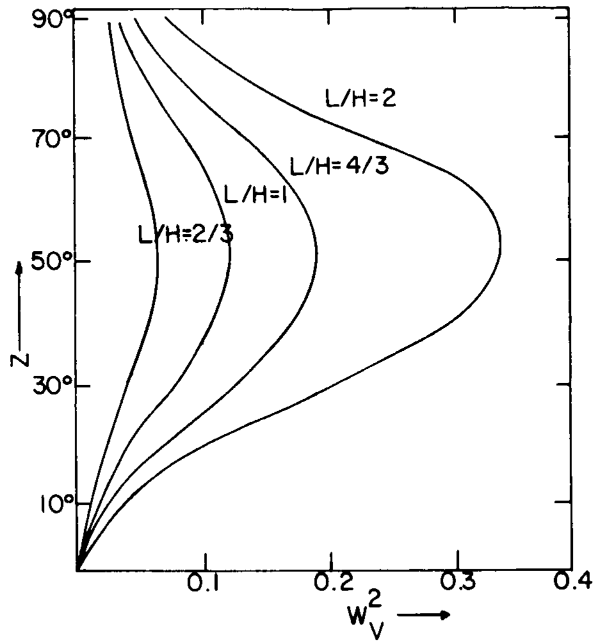


Figure 2c. Correlation terms between the components in the vertical plane and in azimuth.

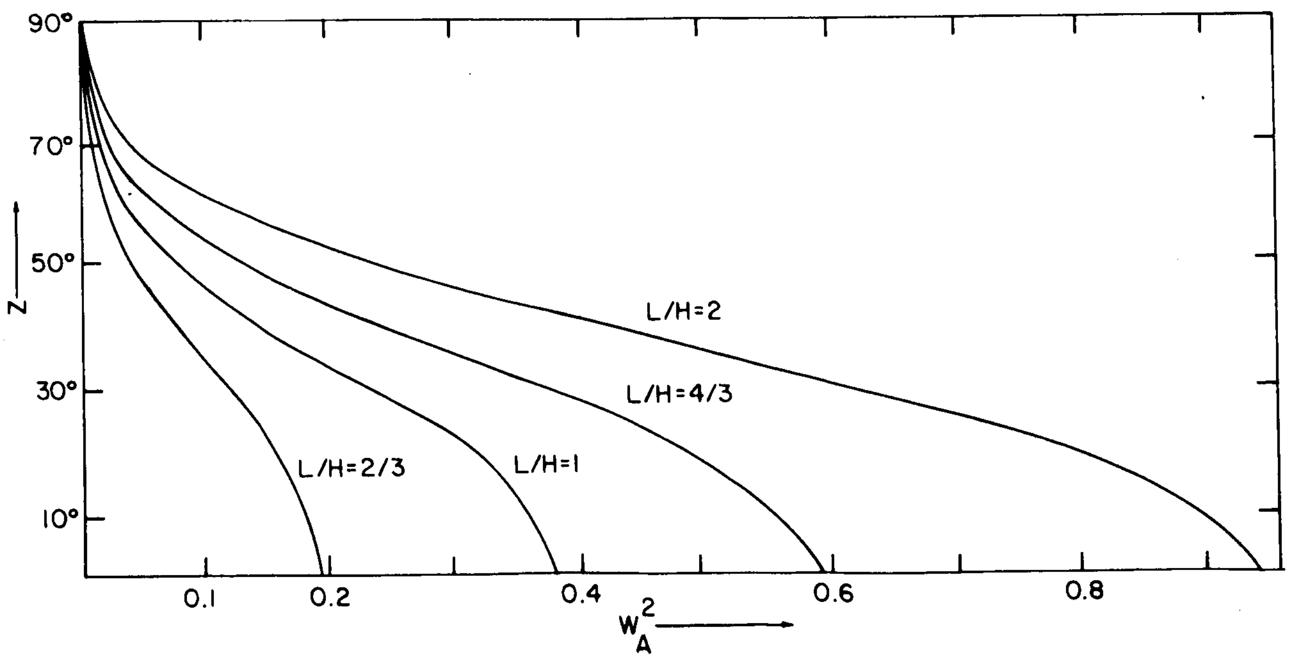


Figure 2b. Weight component in azimuth as a function of the satellite-plane angle Z and the ratio L/H.

Table 1. Values for Z_{min} corresponding to variable Z_{max} and L/H for the case C = 0

Z_{max} \ L/H	2	4/3	1	2/3	$(Z_{min})_{mean}$
80	23.0	24.7	24.0	23.5	23.8
70	27.0	28.0	26.5	27.0	27.4
60	32.5	33.5	32.5	32.0	32.5
50	40.0	41.5	40.0	41.0	40.6
45	45.0	45.0	45.0	45.0	45.0

3.2 Case 2. C = L/2

The procedure followed here is the same as for case 1, except the observations are assumed to have been made to objects lying in a plane that is parallel to the orthogonal plane and that passes through one of the stations. The height H is defined now as the distance, measured in this plane, from the station to a satellite position on the intersection of this plane with the vertical plane (see Figure 3). This again enables the position of the satellite to be defined by the two variables Z and L/H.

The weight functions W_V^2, W_A^2, and W_{AV} are given in Figures 4a, b, and c, respectively.

Table 2 gives the Z_{min} for variable Z_{max} and L/H and shows that they can be considered to be independent of the latter.

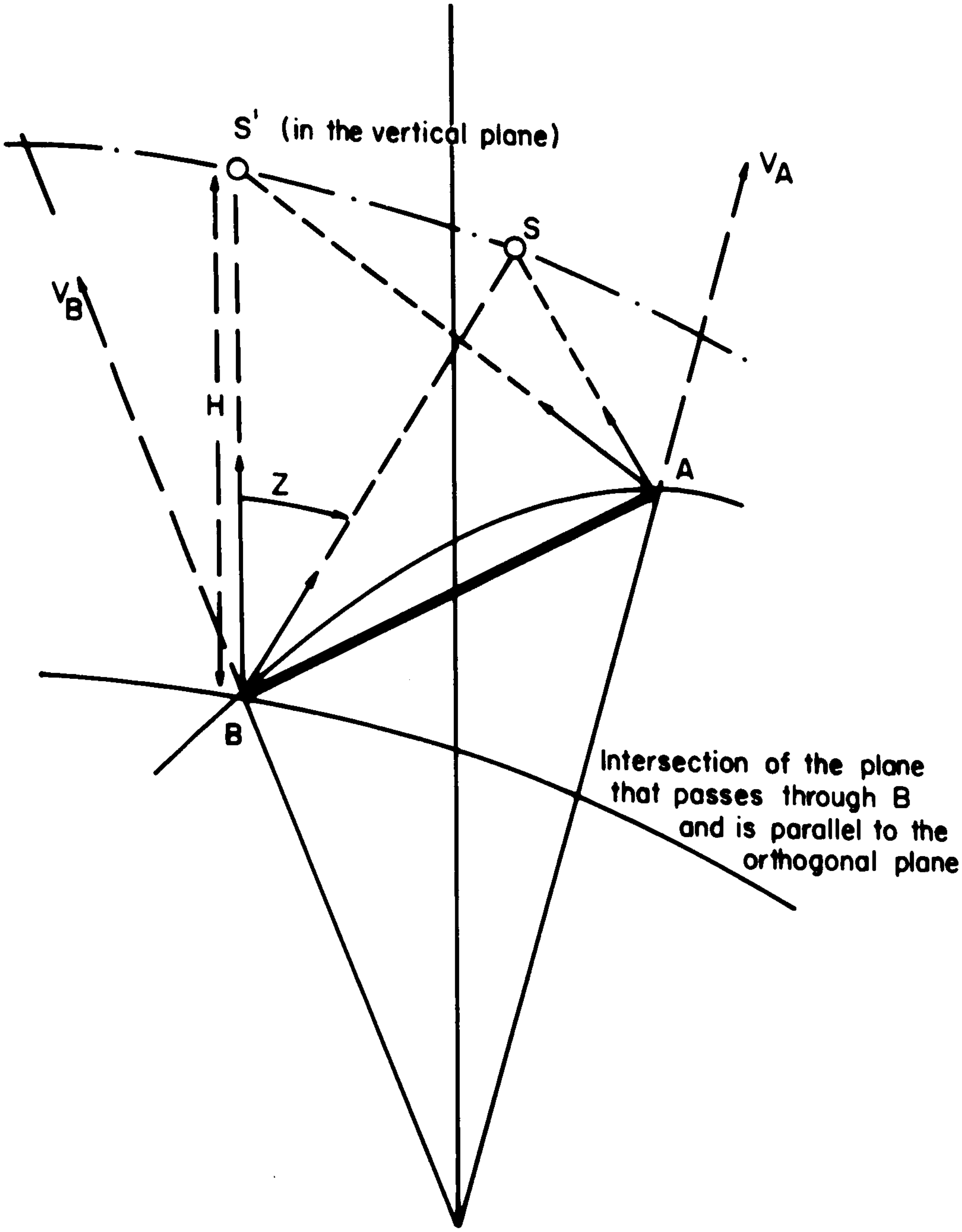


Figure 3. Station-satellite configuration for the case C = L/2; i.e., the satellite positions lie in a plane that passes through one of the stations (B) and is parallel to the orthogonal plane. (S′ is a satellite position in the vertical plane, and S is any other position.)

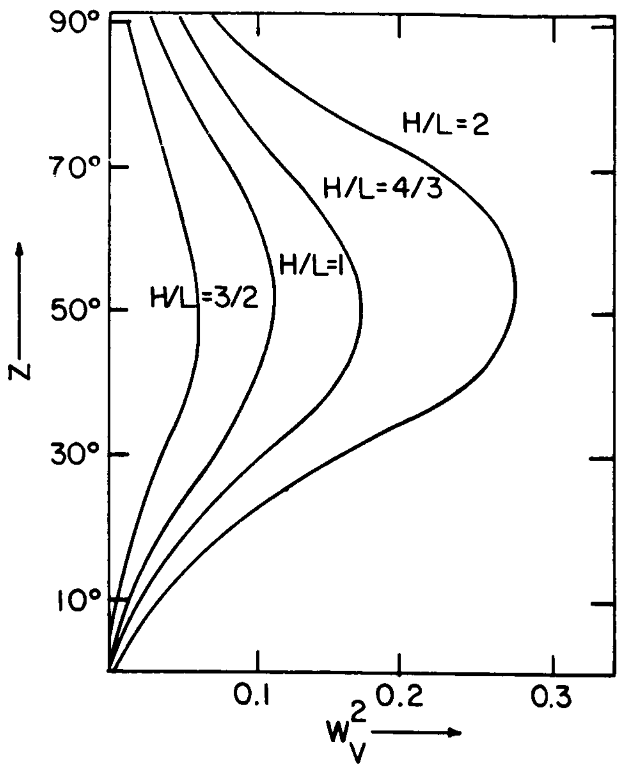


Figure 4a. Weight component in the vertical plane.

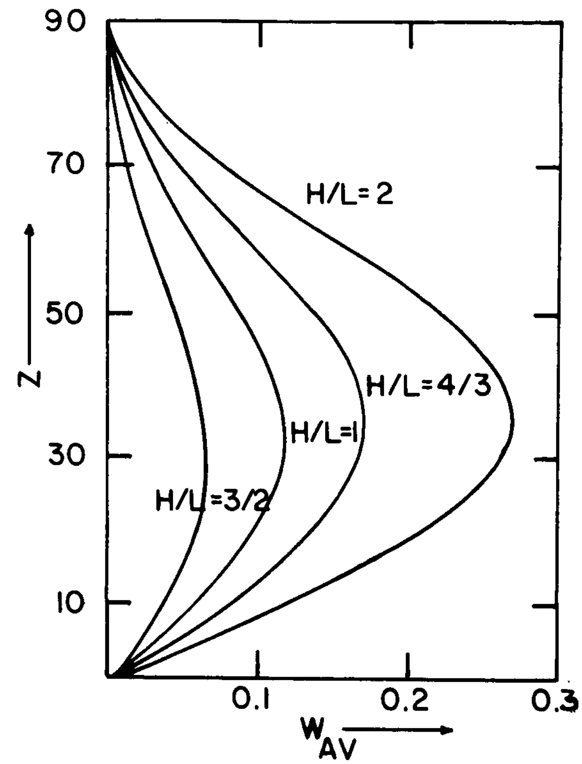


Figure 4c. Correlation terms between the components in the vertical plane and in azimuth.

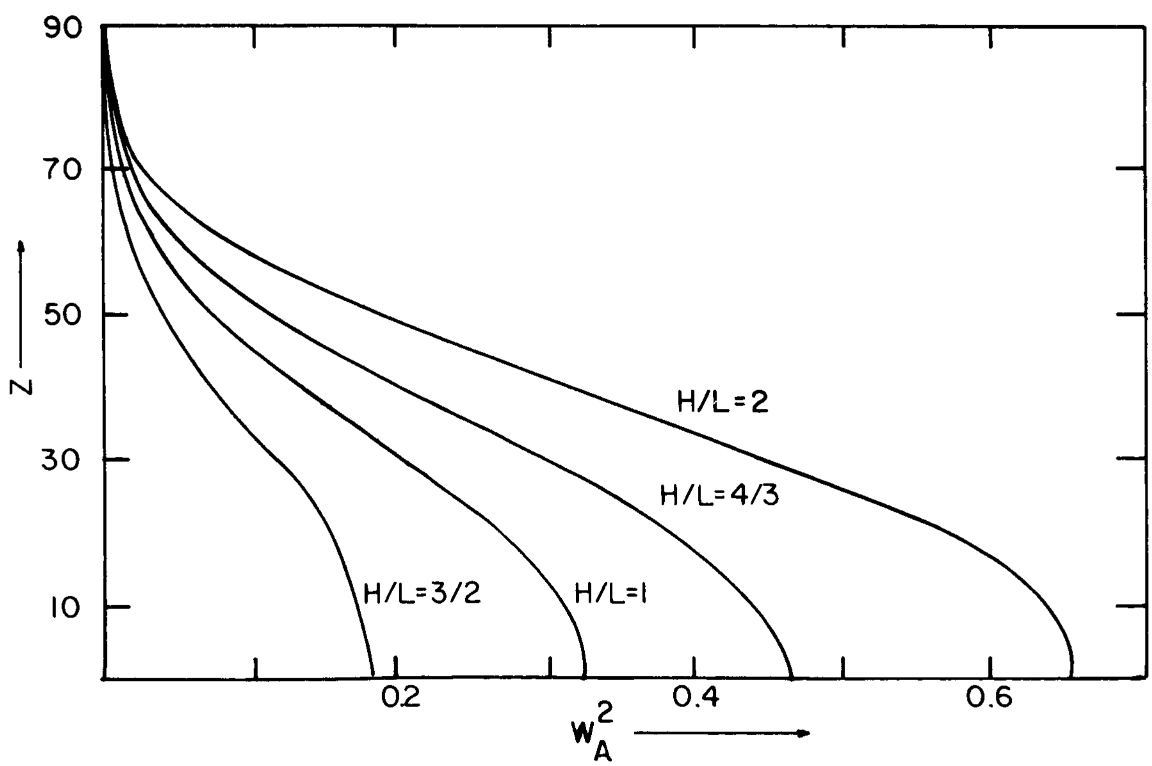


Figure 4b. Weight component in azimuth.

Table 2. Values for Z_{min} corresponding to variable Z_{max} and L/H for the case C = L/2.

Z_{max} \ L/H	2	4/3	1	2/3	$(Z_{min})_{mean}$
80	23.0	23.0	23.5	24.0	23.4
70	26.0	26.0	26.5	27.0	26.4
60	31.5	31.7	31.5	32.0	31.7
50	38.5	39.5	39.0	40.0	39.3
45	45.0	45.0	45.0	45.0	45.0

As in the previous case, two satellite positions forming mirror images about the vertical plane have weight functions that differ only in the sign of the correlation terms. Also, the weight functions for two objects forming mirror images about the orthogonal plane will be identical.

3.3 Case 3. C = 3L/2

In this case the hypothetical satellite positions are assumed to lie in a plane that is also parallel to the orthogonal plane but passes through a point F on the station-station vector extended a distance L beyond the nearest station. The definition of height is now the distance from F to an object lying in this plane and in the vertical plane (see Figure 5). Figures 6a, b, and c give the weight functions W_V^2, W_A^2, and W_{AV}, respectively, while the relationship between Z_{max} and Z_{min} is tabulated in Table 3. The properties of the weight functions of mirror-image objects given for case 2 (either about the vertical plane or about the orthogonal plane) are valid here as well.

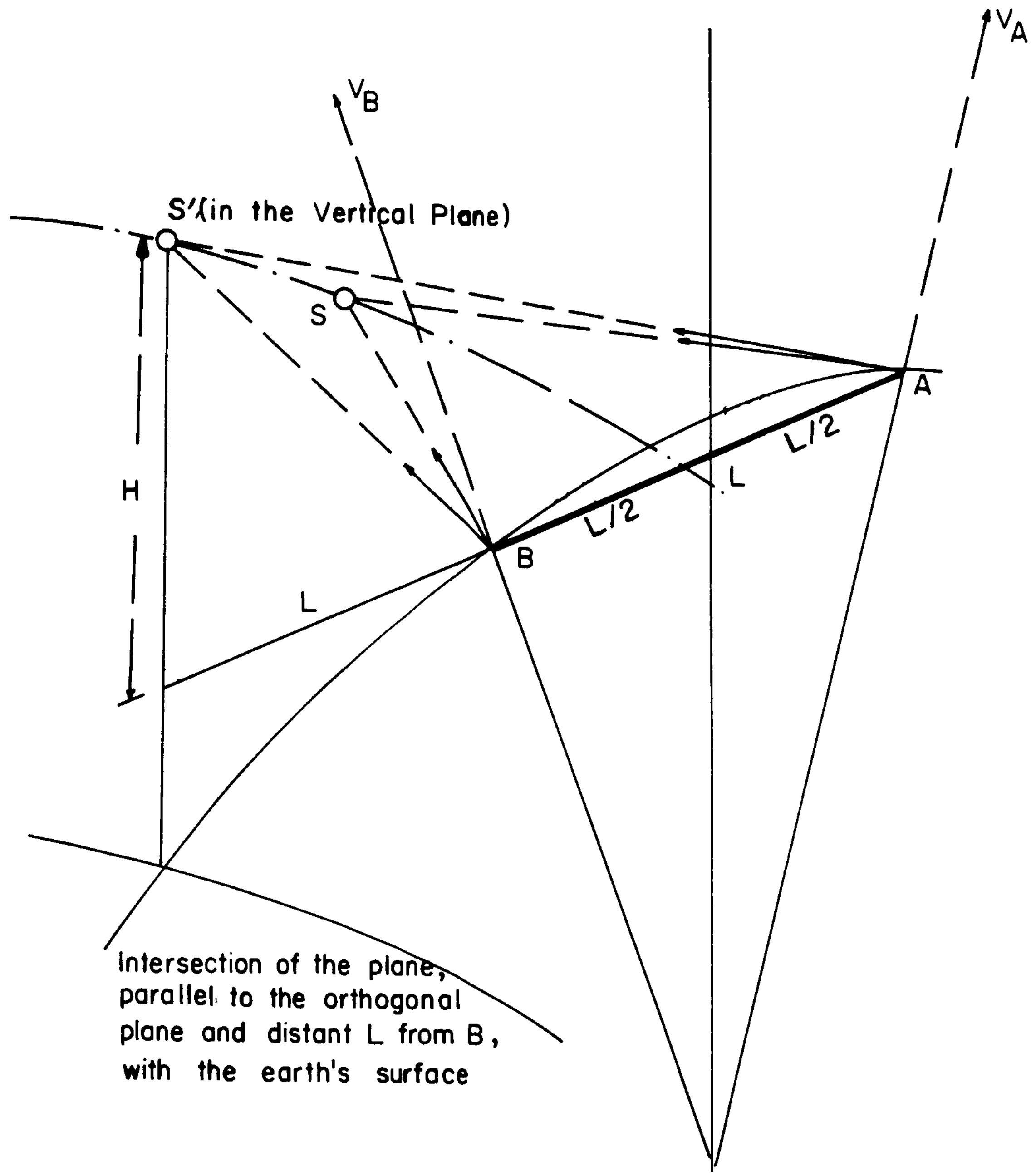


Figure 5. Station-satellite configuration for the case $C = 3L/2$; i.e., where the satellite positions lie in a plane that is parallel to the orthogonal plane and cuts the line AB at a distance L from the nearest observatory. (S′ is a satellite position in the vertical plane.)

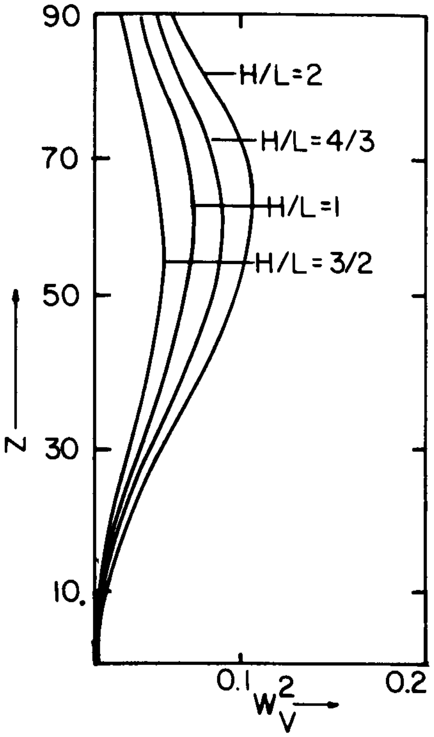


Figure 6a. Weight component in the vertical plane.

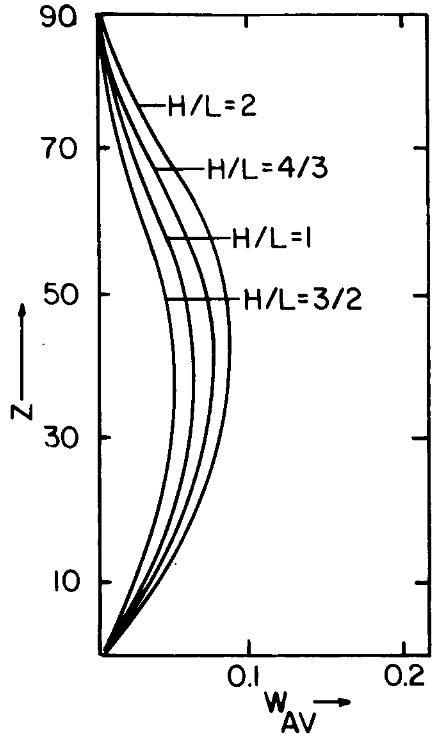


Figure 6c. Correlation terms between the components in the vertical plane and in azimuth.

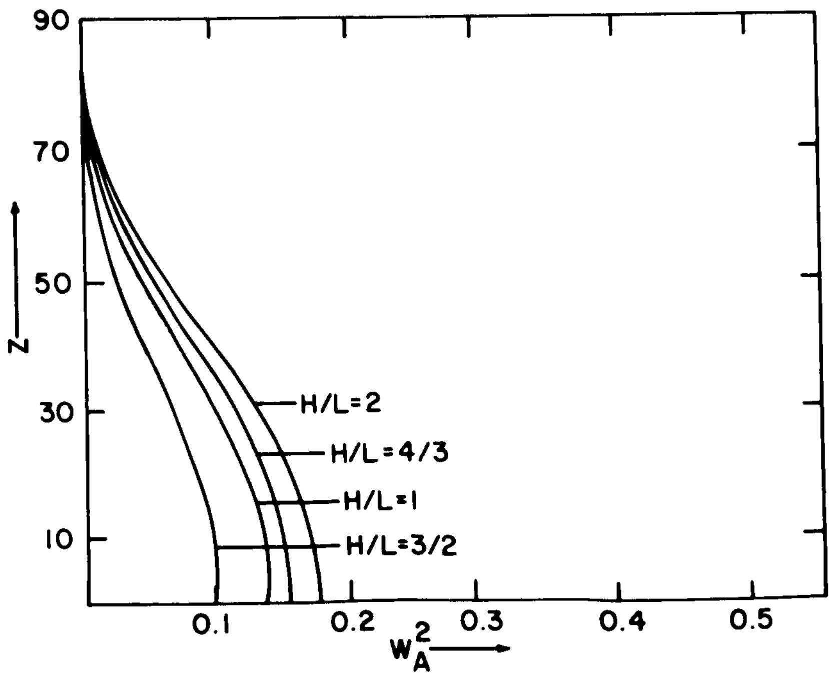


Figure 6b. Weight component in azimuth.

Table 3. Values for Z_{min} corresponding to variable Z_{max} and L/H for the case C = 3L/2

Z_{max} \ L/H	2	4/3	1	2/3	$(Z_{min})_{mean}$
80	18.0	19.7	19.2	21.5	19.6
70	23.0	24.0	24.7	26.0	24.4
60	29.5	31.0	31.0	31.5	30.8
50	39.0	40.0	39.5	39.0	39.3
45	45.0	45.0	45.0	45.0	45.0

3.4 Remarks on the Three Cases

Figure 7 summarizes the results of Figures 2, 4, and 6 in an alternative form.

The relationships between the Z_{max} and Z_{min} for the three cases indicate a marked independence of both C and L/H, particularly since intervisibility and refraction will generally impose a limit of $Z_{max} > 70°$, and the error introduced by ignoring the dependence on both C and L/H will not exceed about 2°. Mean values for Z_{min} and their estimated accuracy are given in Table 4 and Figure 8. This dependence on the satellite-plane angle alone therefore means that the apparent ambiguity introduced by the different definitions of H in the three cases considered is of no consequence.

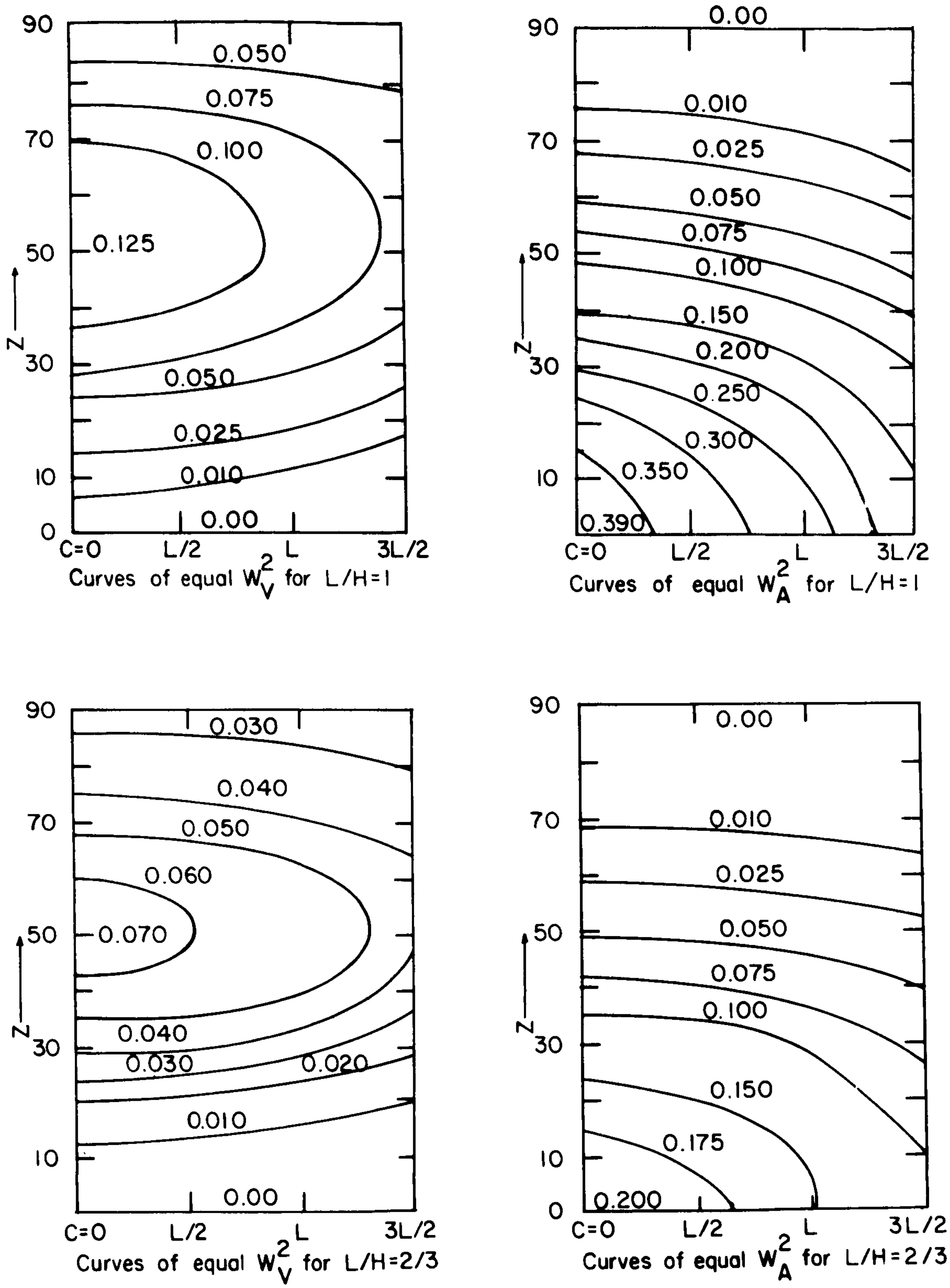
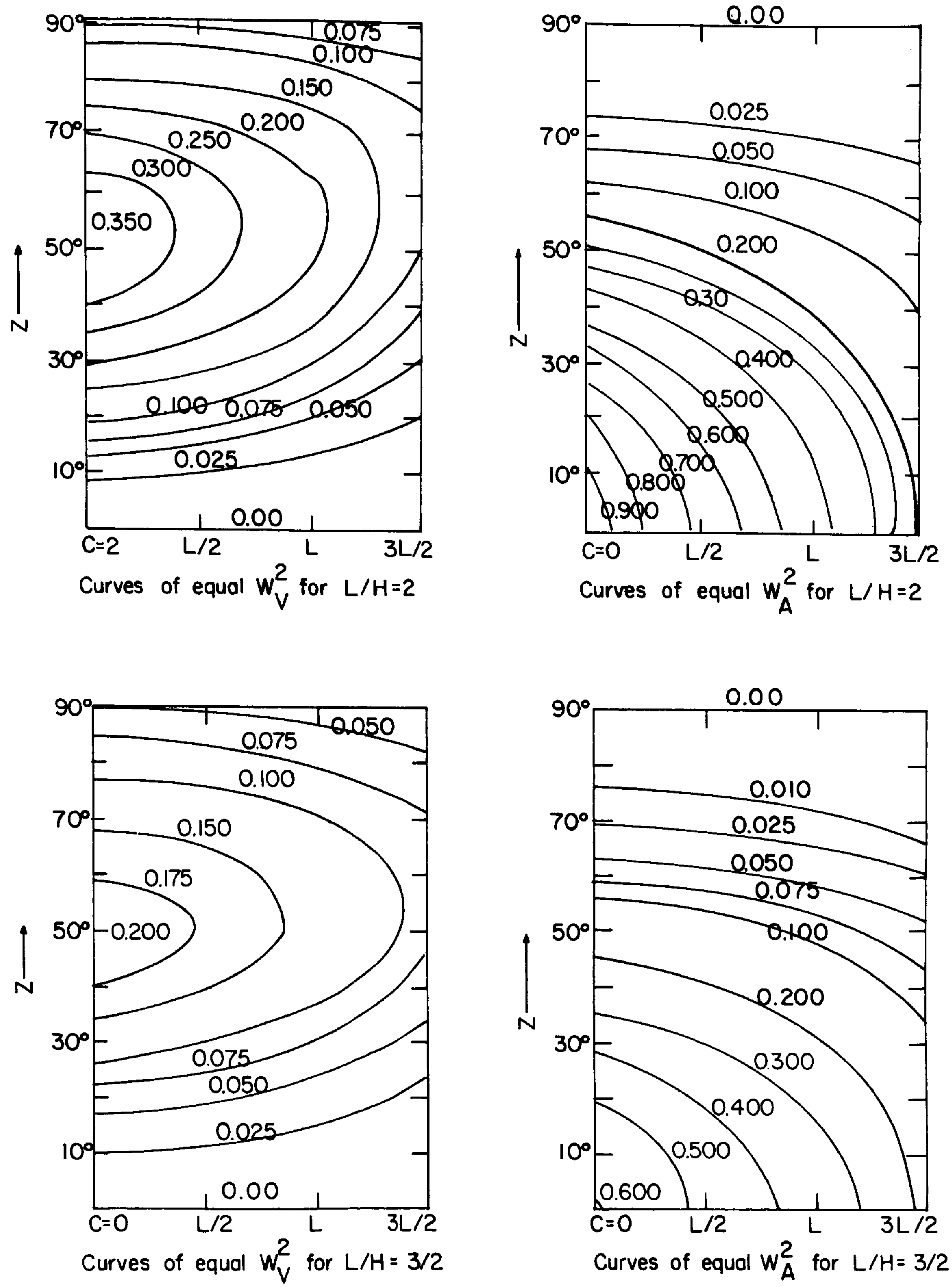


Figure 7. W_V^2 and W_A^2 as functions of Z, L/H, and C.



Curves of equal W_V^2 for L/H=2

Curves of equal W_A^2 for L/H=2

Curves of equal W_V^2 for L/H= 3/2

Curves of equal W_A^2 for L/H= 3/2

Figure 7. (continued)

Table 4. Mean values of Z_{min} for variable Z_{max}. The value Z_{min} is independent of both L/H and C. The last column gives the estimates of the accuracy of the Z_{min}

Z_{max}	Z_{min}	$\sigma_{Z_{min}}$
80	22.3	4.6
70	26.1	2.1
60	31.7	1.0
50	39.8	0.7
45	45.0	0.0

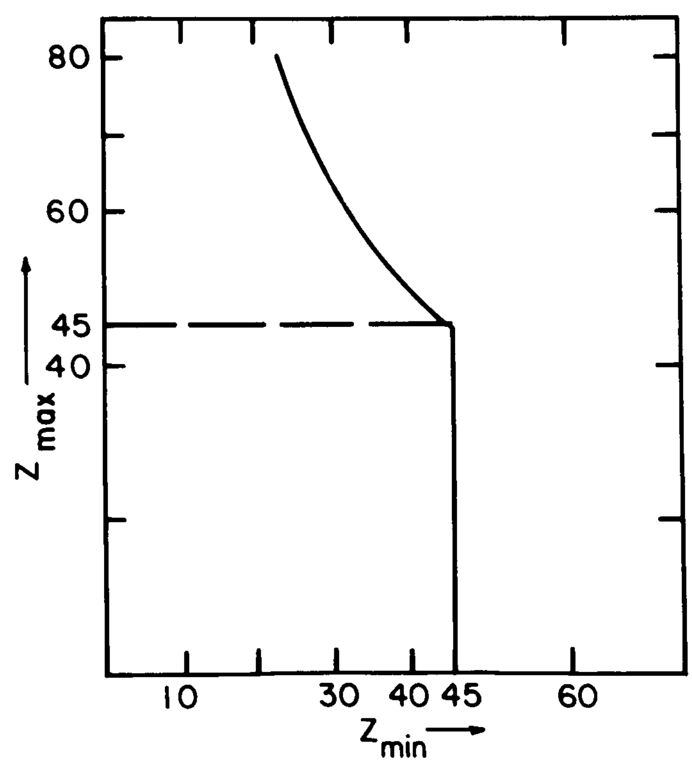


Figure 8. The limits of Z between which the satellite planes should be equally distributed if the conditions that $\Sigma W_V^2/\Sigma W_A^2 = 1$ is to be satisfied.

On the other hand, Z_{max} is a function of both C and L/H, so that the limits of Z between which the observations must lie vary with these two parameters. The relationship is, however, in all cases the simple one depicted in Figure 8.

Two tests have been made that verify these remarks. In the first test the maximum satellite-plane angle was taken as 60°, and the corresponding Z_{min} therefore was 31°. The observations were equally spaced at intervals of $\delta Z = 2°$ and randomly distributed with respect to L/H and C. Figure 9a shows the distribution of the satellite points. From Figures 2, 4, and 6 the total weights of the station-station vector were determined (see Table 5).

Table 5. The distribution of observations with respect to Z, C, and L/H for test 1. Columns 4, 5, and 6 give the weights in the vertical plane, in azimuth, and the correlation terms, respectively. For the total weights, $\Sigma W_V^2 / \Sigma W_A^2 = 1.01$.

Z	Case	L/H	W_V^2	W_A^2	W_{AV}
60	3	2/3	0.047	0.015	0.026
58	1	2	0.335	0.129	0.195
56	2	2/3	0.062	0.026	0.036
54	2	1	0.112	0.062	0.082
52	3	4/3	0.065	0.049	0.066
50	3	2/3	0.046	0.030	0.039
48	2	2	0.272	0.206	0.236
46	3	1	0.060	0.053	0.057
44	2	2	0.258	0.254	0.252
42	1	2	0.318	0.384	0.350
40	1	4/3	0.168	0.238	0.208
38	3	1	0.047	0.076	0.060
36	2	4/3	0.128	0.242	0.172
34	2	2/3	0.043	0.096	0.065
32	3	1	0.035	0.111	0.058
		Sum	1.996	1.971	1.902

For the second test the observations were assumed to be distributed at 1° intervals of Z, and randomly with respect to L/H and C, between the limits $Z_{max} = 70°$ and the corresponding $Z_{min} = 26°$ (see Figure (9b)). The data are presented in Table 6. The results of both tests speak for themselves.

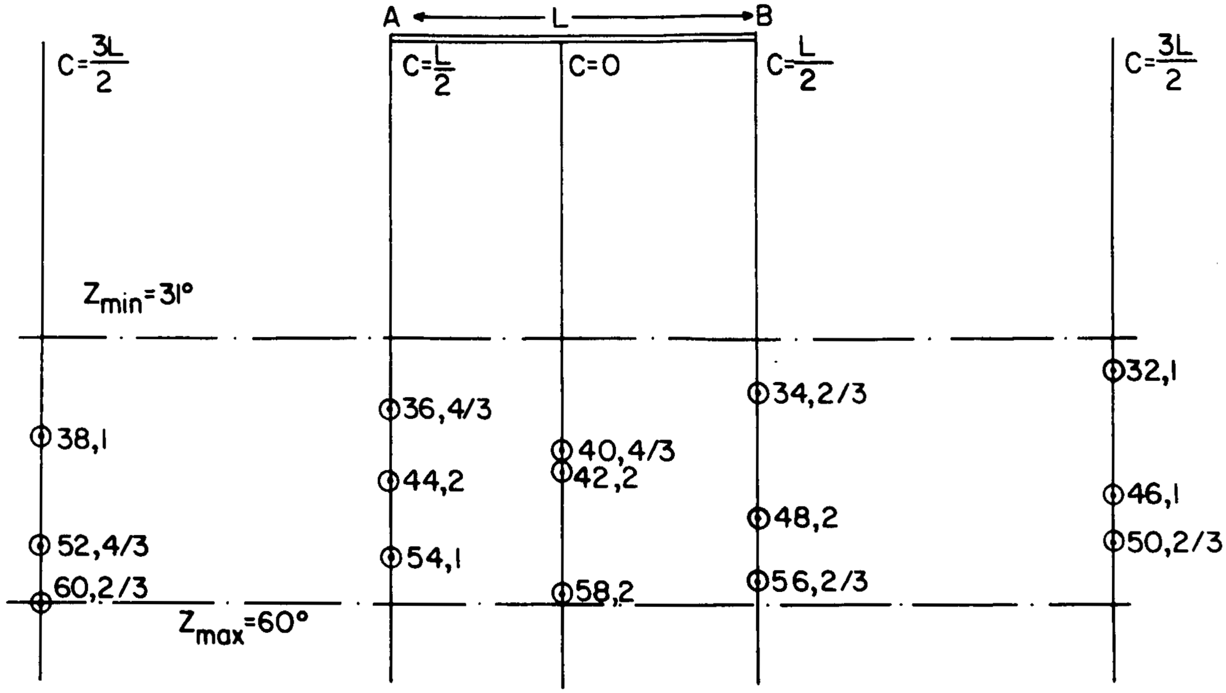


Figure 9a. Distribution of observations for test 1. The first number behind each satellite position (⊙) refers to Z and the second number to L/H.

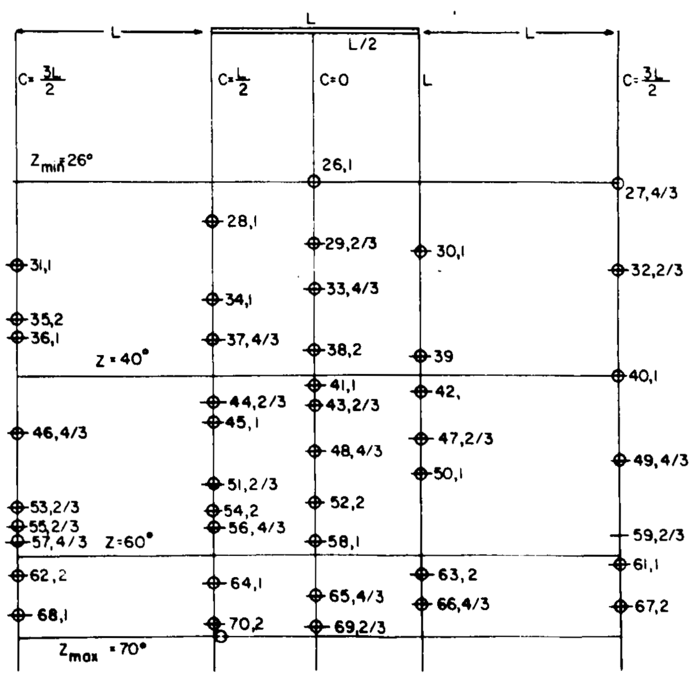


Figure 9b. Distribution of observations for test 2.

Table 6. Results of test 2, $\Sigma W_V^2 / \Sigma W_A^2 = 1.03$

Z	Case	L/H	W_V^2	W_A^2	W_{AV}	Z	Case	L/H	W_V^2	W_A^2	W_{AV}
70	2	2	0.218	0.025	0.079	47	2	2/3	0.060	0.053	0.052
69	1	2/3	0.046	0.046	0.019	46	3	4/3	0.072	0.065	0.072
68	3	1	0.065	0.065	0.026	45	2	1	0.106	0.100	0.106
67	3	2	0.110	0.018	0.047	44	2	2/3	0.056	0.060	0.254
66	2	4/3	0.142	0.030	0.065	43	1	2/3	0.062	0.066	0.065
65	1	4/3	0.163	0.036	0.134	42	2	4/3	0.155	0.184	0.162
64	2	1	0.098	0.024	0.050	41	1	1	0.112	0.133	0.130
63	2	2	0.257	0.063	0.132	40	3	1	0.050	0.069	0.060
62	3	2	0.112	0.032	0.060	39	2	2	0.228	0.313	0.271
61	3	1	0.068	0.020	0.037	38	1	2	0.282	0.455	0.366
60	1	2/3	0.056	0.018	0.030	37	2	4/3	0.132	0.232	0.170
59	2	2/3	0.046	0.015	0.027	36	3	2/3	0.044	0.080	0.059
58	1	1	0.120	0.046	0.075	35	3	1	0.061	0.114	0.084
57	3	4/3	0.087	0.030	0.057	34	2	1	0.082	0.175	0.122
56	2	4/3	0.168	0.073	0.112	33	1	2/3	0.138	0.360	0.212
55	3	2/3	0.047	0.023	0.033	32	3	1	0.023	0.070	0.083
54	2	2	0.281	0.146	0.200	31	3	4/3	0.058	0.095	0.057
53	3	2/3	0.041	0.025	0.035	30	2	1	0.072	0.202	0.120
52	1	2	0.350	0.219	0.273	29	1	2/3	0.034	0.125	0.070
51	2	2/3	0.061	0.040	0.046	28	2	1	0.064	0.220	0.116
50	2	1	0.111	0.073	0.094	27	3	4/3	0.030	0.126	0.062
49	3	4/3	0.076	0.056	0.069	26	1	1	0.062	0.275	0.126
48	1	4/3	0.191	0.161	0.172			Sum	4.897	4.759	4.691

The three cases investigated have all shown that the weight functions of two satellite points forming mirror images about the vertical plane are identical except for the signs of the correlation terms. Similarly, the weight functions of two objects forming mirror images about the orthogonal plane are identical in all respects. Thus, the weight functions of two points located symmetrically about the midpoint vertical will differ only in the signs of the correlation coefficients.

The importance of observing objects on both sides of the vertical plane is also illustrated by the two tests, as both yield almost singular solutions for the station-station vector despite the fact that $\Sigma W_V^2 / \Sigma W_A^2 = 1$. The removal of the correlation between the two directions can be achieved by matching any one satellite position by a second satellite position on the opposite side of the vertical plane, which either is a mirror image of the first about this plane or is symmetrical with the first about the midpoint vertical.

3.5 Intervisibility Requirements

A further condition that the satellite positions must satisfy is that they be visible from both stations and sufficiently elevated above the horizon to reduce uncertainties in atmospheric refraction to a minimum. This requires a relation between the zenith distance z and the satellite height H, namely,

$$\sin (z - \eta) = R \sin z / (R + H) \quad ,$$

where R is the earth's radius and η the horizon distance. Figure 10 shows this relationship.

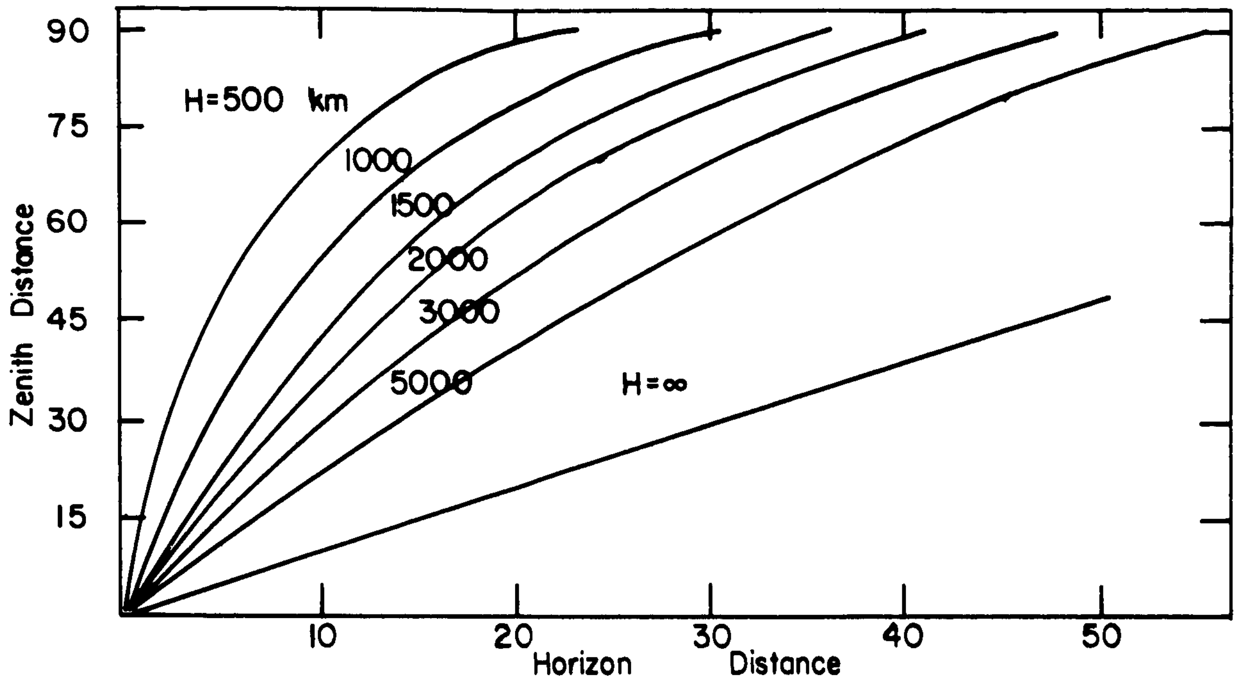


Figure 10. Monogram describing the relationship between the Zenith distance and height of a satellite.

4. DETERMINATION OF THE OPTIMUM REGION IN THE COMMON COVERAGE AREA

The horizon distance defines the common coverage area, and the intersection of any plane ($C = C_i$) parallel to the orthogonal plane with the limits of this area gives the maximum subsatellite distance $(\eta_{max})_i$ from the station-station vector for the case $C = C_i$. Using the appropriate satellite height, $(Z_{max})_i$ corresponding to $(\eta_{max})_i$ is derived using Figure 10, and consequently $(Z_{min})_i$ is determined from Figure 8. This last quantity is transformed into a subsatellite distance $(\eta_{min})_i$, again from Figure 10. Then $(\eta_{max})_i$ and $(\eta_{min})_i$ specify the limits of the subsatellite distances between which the subsatellite points must lie for the case $C = C_i$. If this procedure is repeated for several values of η, then the areas in which the subsatellite points must be distributed in order to obtain the optimum solution for the station-station vector are defined.

Note that for $Z_{max} = 45°$, $Z_{min} = 45°$; at this point C will attain its maximum value for subsatellite points in the optimum parts of the coverage area. The quantity C_{max} will be a function of both L and H, as well as of the maximum zenith distance z_{max} at which it is desired to observe the satellite. Values for C_{max}, derived empirically, are tabulated in Table 7, and will be used below for estimating *a priori* the magnitude of the variance matrix of the station-station vector.

Table 7. C_{max} and $(W^2/n)_z$ as a function of L, H, and z_{max}

L/H	L (km)	H (km)	z_{max}	C_{max}	$(W^2/n)_z$
2	1000	500	75°	0.612	0.270
	2000	1000		0.337	0.285
	3000	1500		0.221	0.292
	4000	2000		0.131	0.296
2	1000	500	67°	0.350	0.285
	2000	1000		0.137	0.296
2	1000	500	60°	0.125	0.296
				Mean	0.292
4/3	1000	750	75°	0.850	0.160
	2000	1500		0.568	0.165
	4000	3000		0.206	0.168
4/3	1000	750	67°	0.450	0.166
	2000	1500		0.294	0.167
	4000	3000		0.078	0.169
4/3	1000	750	60°	0.250	0.168
	2000	1500		0.081	0.169

L/H	L (km)	H (km)	z_{max}	C_{max}	$(W^2/n)_z$
				Mean	0.167
1	1000	1000	75°	1.15	0.097
	2000	2000		0.74	0.102
	3000	3000		0.53	0.105
	4000	4000		0.31	0.107
1	1000	1000	67°	0.687	0.103
	2000	2000		0.369	0.106
	3000	3000		0.271	0.108
1	1000	1000	60°	0.35	0.107
	2000	2000		0.19	0.108
	3000	3000		0.092	0.109
				Mean	0.106
2/3	1000	1500	75°	1.65	0.052
	2000	3000		1.025	0.056
	3000	4500		0.629	0.061
2/3	1000	1500	67°	1.125	0.057
	2000	3000		0.637	0.061
	3000	4500		0.375	0.063
2/3	1000	1500		0.638	0.061
	2000	3000		0.375	0.063
	3000	4500		0.208	0.065
				Mean	0.060

5. PERMISSIBLE TOLERANCES IN THE MATCHING OF PAIRS OF OBSERVATIONS

The methods of matching observations suggested above impose rather stringent conditions on the distribution of the simultaneous satellite observations, but as an exact circular error distribution of the station-station vector is not absolutely essential, some tolerance in the matching would be in order.

The degree of correlation between the directions in the vertical and in azimuth is defined by the correlation coefficient K as

$$K = - \frac{W_{VA}}{\left(W_V^2 \ W_A^2\right)^{1/2}}$$

and will be a function of the ratio L/H, Z, and C; any changes, d(H/L), dZ, dC, will affect K by an amount

$$dK = \left[\left(\frac{\partial K}{\partial (H/L)}\right)^2 \left(\frac{dH}{L}\right)^2 + \left(\frac{\partial K}{\partial Z}\right)^2 (dZ)^2 + \left(\frac{\partial K}{\partial C}\right)^2 (dC)^2\right]^{1/2} ,$$

$$\underset{=}{\text{per definition}} \left(dK_1^2 + dK_2^2 + dK_3^2\right)^{1/2} .$$

If δK is the maximum value that the correlation coefficient may have without becoming significant, and the influences of dH/L, dZ, and dC upon δK are assumed equal, then

$$dK_1 = dK_2 = dK_3 < \delta K/\sqrt{3} \quad .$$

In the following analysis a limiting value of $\delta K = 0.2$ has been used, so that

$$dK_i = 0.115 \quad , \qquad\qquad i = 1, 2, 3 \quad .$$

For large numbers of observations the conditions for optimum geometry derived above are approximately equivalent to subsatellite points distributed evenly with respect to area in the part of the common coverage area defined by the $Z_{max} - Z_{min}$ criteria. The question of what are the permissible tolerances in Z and C therefore ceases to be important for satellites of approximately equal heights.

5.1 Permissible Tolerance in Height of a Pair of Matching Observations

Differentiating the correlation coefficient with respect to (H/L) gives

$$\frac{\partial K}{\partial(H/L)} = \left[\frac{-1}{W_V W_A}\,\frac{\partial W_{VA}}{\partial(H/L)} + \frac{1}{2}\,\frac{W_{VA}}{W_V^3\, W_A}\,\frac{\partial W_V^2}{\partial(H/L)} + \frac{1}{2}\,\frac{W_{VA}}{W_V W_A^3}\,\frac{\partial W_A^2}{\partial(H/L)}\right] d\,\frac{H}{L} \quad ,$$

and for a change of dH in H

$$dK_1 = \left\{\left[\frac{1}{W_V W_A}\,\frac{\partial W_{VA}}{\partial(H/L)}\right]^2 + \frac{1}{4}\left[\frac{W_{VA}}{W_V^3\, W_A}\,\frac{\partial W_V^2}{\partial(H/L)}\right]^2 + \frac{1}{4}\left[\frac{W_{VA}}{W_V\, W_A^3}\,\frac{\partial W_A^2}{\partial(H/L)}\right]^2\right\}^{1/2} \frac{H}{L}\,\frac{dH}{H} \quad ,$$

or

$$dK_1 = A_1 \frac{dH}{H} \quad .$$

For $\delta K = 0.2$,

$$\frac{dH}{H} < \frac{0.115}{A_1} \quad .$$

The terms of A_1 can be evaluated numerically using the weight functions given in Figures 2, 4, and 6. Table 8 summarizes the values of A_1 for $H = 0.62$ L and 1.25 L, and for $C = 0$, L/2, and 3L/2. To within about 10% the values of A_1 are independent of Z, and on this assumption dH/H as a function of C and H/L only, is given in Figure 11.

Table 8 . Coefficients A_1 as a function of H/L, C, and Z

Case	H/L \ Z	10	30	50	70	Mean	σ_{mean}
C = 0	0.62	1.3	1.5	1.7	1.6	1.5	0.2
	1.25	2.1	1.9	1.9	2.4	2.1	0.2
C = L/2	0.62	1.1	1.3	1.4	1.3	1.3	0.1
	1.25	2.0	1.9	1.8	1.9	1.9	0.1
C = 3L/2	0.62	0.6	0.5	0.9	0.7	0.7	0.2
	1.25	0.8	0.9	1.0	1.4	1.0	0.3

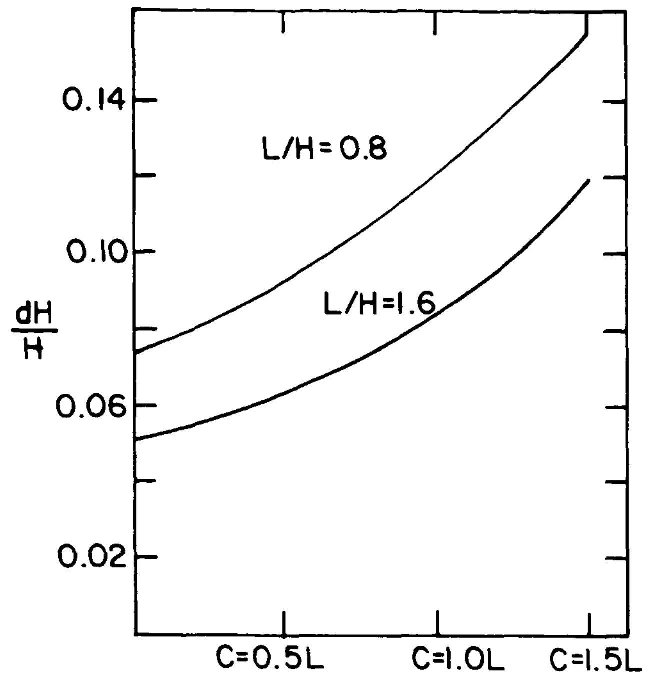


Figure 11 Tolerance in H/L permissible in matching pairs of observations.

5.2 Permissible Tolerance in Satellite-Plane Angle in Matching Observations

A difference of dZ in the satellite-plane angles of two objects that otherwise would form mirror images about the vertical plane will introduce a correlation coefficient of magnitude

$$dK_2 = \left[\left(\frac{1}{W_V W_A} \frac{\partial W_{VA}}{\partial Z}\right)^2 + \frac{1}{4}\left(\frac{W_{VA}}{W_V^3 W_A} \frac{\partial W_V^2}{\partial Z}\right)^2 + \frac{1}{4}\left(\frac{W_{VA}}{W_V W_A^3} \frac{\partial W_V^2}{\partial Z}\right)^2\right]^{1/2} dZ$$

$$= A_2 \, dZ \quad .$$

As before, the A_2 can be evaluated directly from Figures 2, 4, and 6. Table 9 gives values of A_2 for L/H = 2 and 2/3, and for C = 0, L/2, and 3L/2. These results suggest that A_2 can be treated as independent of both C and L/H. For the total magnitude of the correlation coefficient δK to be less than 0.2,

$$dZ_2 < \frac{0.115}{A_2} \quad .$$

By use of the mean values of A_2 from Table 9, the tolerance in Z permissible in matching any two sets of simultaneous observations is computed. The results are given in Figure 12.

Table 9. Coefficients A_2 as functions of H, C, and Z. The bottom row gives the values of A_2 averaged over constant Z

Case	H \ Z	10	30	50	70
C = 0	L/2	0.12	0.03	0.04	0.12
	3L/2	0.16	0.05	0.04	0.15
C = L/2	L/2	0.13	0.04	0.04	0.12
	3L/2	0.14	0.04	0.05	0.12
C = 3L/2	L/2	0.15	0.07	0.06	0.14
	3L/2	0.14	0.07	0.06	0.14
	Mean	0.14	0.05	0.05	0.13

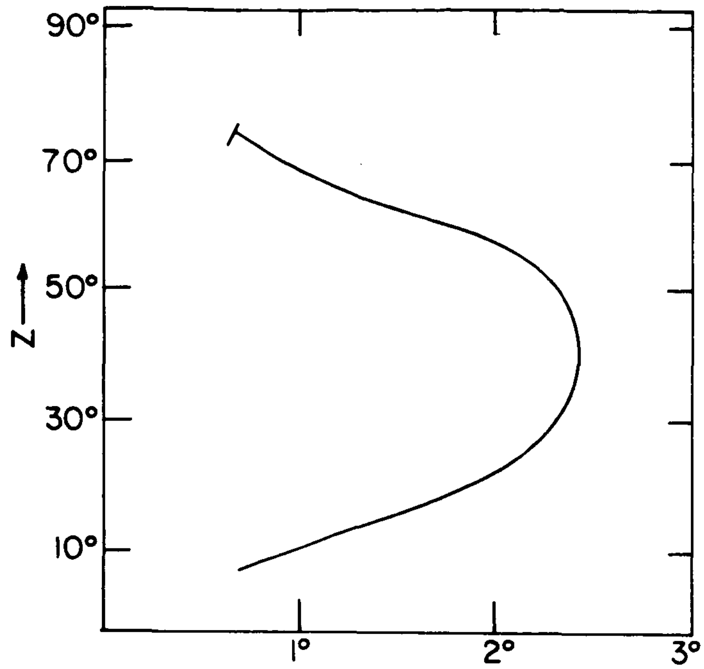


Figure 12. Tolerance in Z permissible in matching pairs of observations.

5.3 Tolerance in Matching Pairs of Observations with Respect to C

The correlation introduced by a difference dC in two otherwise mirror-image (about the vertical plane) satellite positions will be

$$dK_3 = \left[\left(\frac{1}{W_V W_A}\,\frac{\partial W_{AV}}{\partial C}\right)^2 + \frac{1}{4}\left(\frac{W_{VA}}{W_V^3 W_A}\,\frac{\partial W_V^2}{\partial C}\right)^2 + \frac{1}{4}\left(\frac{W_{VA}}{W_V W_A^3}\,\frac{\partial W_A^2}{\partial C}\right)^2\right]^{1/2} dC$$

$$= A_3 \, dC \quad .$$

Values of A_3 for $L/H = 2$, 1, and 2/3 and for $C = 0$, $L/4$, and L are tabulated in Table 10. No simple relation appears to exist between the three variables, but as such a relationship is desirable for the sake of simplicity and the accuracy requirements are not very stringent, A_3 will be considered independent of Z. The last three columns of Table 10 give, respectively, the mean values of A_3, their standard deviations σ_{A_3}, and the ratio σ_{A_3}/A_3. The error introduced, therefore, is of the order of 25%, but this may be accounted for when computing the limits of dC by introducing an extra factor of 1.25. That is,

$$dC < \frac{0.092}{A_3} \quad .$$

The dC as a function of C and the L/H based on this expression are given in Figure 13.

Table 10. Coefficients A_3 as functions of H, C, and Z. Column 7 gives the mean values of A_3 averaged over Z, column 8 gives the standard deviations of the mean, and the last column gives the ratio $\sigma_{mean}/(A_3)_{mean}$

Case	H \ Z	10	30	50	70	Mean $(A_3)_{mean}$	σ_{mean}	$\sigma_{mean}/(A_3)_{mean}$
C = 0	L/2	0.74	0.65	0.56	0.39	0.59	0.15	0.25
	L	0.31	0.23	0.24	0.38	0.29	0.07	0.24
	3L/2	0.30	0.19	0.26	0.25	0.25	0.05	0.20
C = L/4	L/2	0.85	0.74	0.62	0.51	0.68	0.15	0.22
	L	0.40	0.23	0.28	0.41	0.34	0.09	0.26
	3L/2	0.34	0.22	0.29	0.47	0.32	0.11	0.34
C = L	L/2	1.33	1.29	1.17	0.79	1.14	0.24	0.21
	L	1.01	0.88	0.61	0.41	0.73	0.27	0.36
	3L/2	0.75	0.70	0.49	0.58	0.60	0.15	0.25

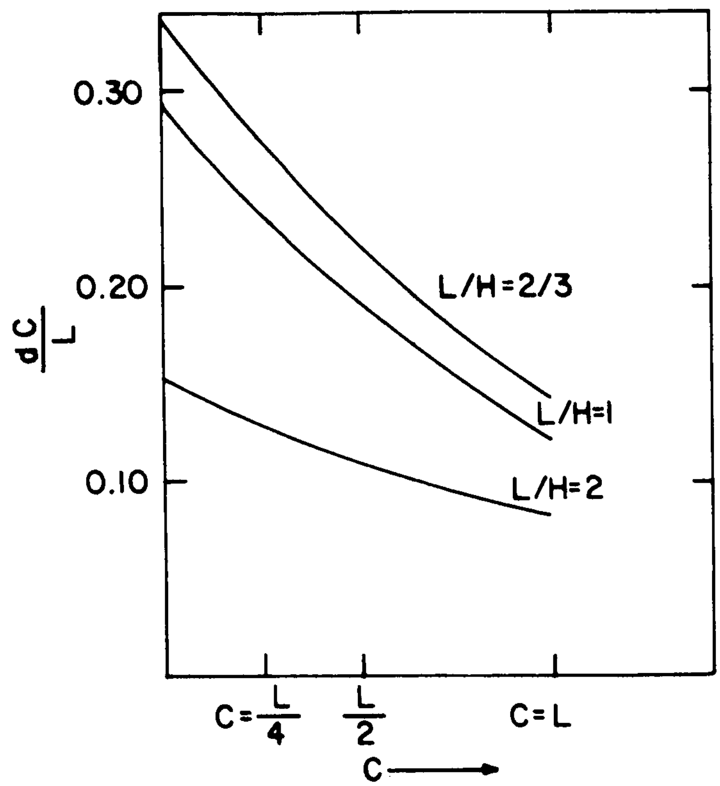


Figure 13. Permissible tolerance in C in matching pairs of observations The tolerances in Figures 11, 12 and 13 are based on the correlation coefficient not to exceed 0. 20.

6. MAGNITUDES OF W_V^2 AND W_A^2

It is evident from Figures 2, 4, and 6 or from Figure 7 that the closer the matched pairs lie to the orthogonal plane and the larger the ratio L/H, the smaller will be the magnitudes of the resultant error ellipse, although the shape of this error function will be independent of both C and L/H.

From these figures the total weight W^2 of n observations distributed evenly between the appropriate limits of Z can be computed for any value of C, L/H, and Z_{max}. Typical results are given in Table 11. The last column gives the mean weight $(W^2/n)_Z$ of n observations distributed between the limits of Z. Most noticeable is that for constant C and L/H, $(W^2/n)_Z$ is independent of the value of Z_{max}. Figure 14 gives $(W^2/N)_Z$ as a function of C and L/H.

Now, from Table 7 and Figure 14, $(W^2/n)_z$ as a function of z_{max}, L, H, and C_{max} can be computed, it being implied that the conditions of equal variance in all directions are imposed at all times. Such computations have been made, and the results are shown in the last column of Table 7.

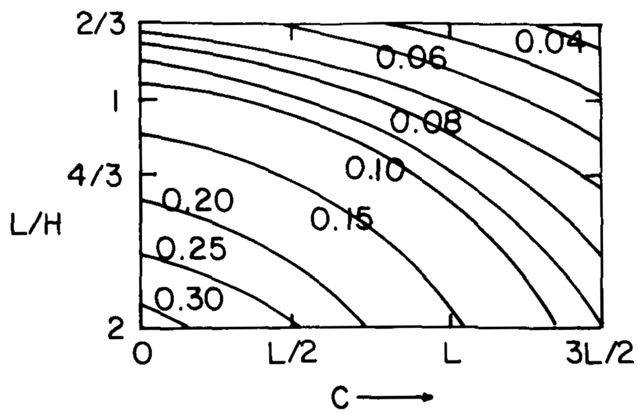


Figure 14. $(W^2/n)_Z$ as a function of C and L/H. W^2 is the total weight of n observations distributed between the limits of Z_{max} and Z_{min} such that $W^2 = \Sigma W_V^2 = \Sigma W_A^2$.

Table 11. $\Sigma W_V^2/n$ as a function of C, L/H, and Z_{max}; n is the number of observations. The Z_{min} corresponding to each Z_{max} value is chosen so that $\Sigma W_V^2 = \Sigma W_A^2 = W^2$

Case	L/H	Z_{max}	n	W^2	$(W^2/n)_Z$
C = 0	2	70	20	5.64	0.28
		60	14	4.14	0.29
		50	6	2.00	0.33
		45	2	0.67	0.30
	4/3	70	20	2.99	0.15
		60	14	2.10	0.15
		50	6	1.12	0.19
		45	2	0.38	0.19
	1	70	20	2.20	0.11
		60	14	1.16	0.08
		50	6	0.68	0.11
		45	2	0.24	0.12
	2/3	70	20	1.14	0.07
		60	14	0.86	0.06
		50	6	0.37	0.06
		45	2	0.12	0.06
C = L/2	2	70	20	4.61	0.23
		60	14	3.45	0.25
		50	6	1.50	0.25
		45	2	0.50	0.25
	4/3	70	20	2.91	0.15
		60	14	2.19	0.16
		50	6	0.95	0.16
		45	2	0.32	0.16
	1	70	20	1.97	0.10
		60	14	1.46	0.10
		50	6	0.62	0.10
		45	2	0.21	0.10
	2/3	70	20	1.04	0.05
		60	14	0.78	0.06
		50	6	0.34	0.06
		45	2	0.11	0.06
C = 3L/2	2	70	20	1.60	0.08
		60	14	1.06	0.08
		50	6	0.51	0.09
		45	2	0.17	0.09

Table 11. (Cont.)

Case	L/H	Z_{max}	n	W^2	$(W^2/n)_Z$
C = 3L/2 (cont.)	4/3	70	20	1.31	0.07
		60	14	0.97	0.07
		50	6	0.43	0.07
		45	2	0.14	0.07
	1	70	20	1.08	0.05
		60	14	0.80	0.06
		50	6	0.34	0.06
		45	2	0.11	0.06
	2/3	70	20	0.73	0.04
		60	14	0.55	0.04
		50	6	0.24	0.04
		45	2	0.08	0.04

The $(W^2/n)_z$ differ from the $(W^2/n)_Z$ only in that the former are for a specific value of C, namely C_{max}, whereas the latter refer to arbitrary values of C. The results for $(W^2/n)_z$ indicate that this quantity, for all practical purposes, is independent of z_{max} and C_{max} and dependent on L/H only. This may be seen from an inspection of the appropriate data. For, in Figure 14 $(W^2/n)_Z$ varies most rapidly with C when L/H is large, while Table 7 indicates that for large L/H, C_{max} is relatively small. Similarly, for small L/H, Table 7 indicates relatively larger values of C_{max}, but Figure 14 indicates that $(W^2/n)_Z$ varies less rapidly with C.

The mean values of $(W^2/n)_z$, W^2/n, drawn in Figure 15 as a function of L/H indicate a linear relationship between L/H and W^2/n, given by

$$W^2/n = 0.19\left(\frac{L}{H}\right) - 0.08 \qquad \left(0.5 < \frac{L}{H} \leq 2.0\right) .$$

If satellites over a wide range of heights are used to determine the station-station vector, the mean value of W^2/n is of the order 0.17. The variance matrix of this vector is therefore given by

$$\begin{bmatrix} \sigma_V^2 & \sigma_{AV} \\ \sigma_{VA} & \sigma_A^2 \end{bmatrix} = \frac{\sigma_{s.0.}^2}{0.17n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

where $\sigma_{s.0.}^2$ is the variance of a single direction. This expression is evaluated in Figure 16 for variable $\sigma_{s.0.}^2$ and n.

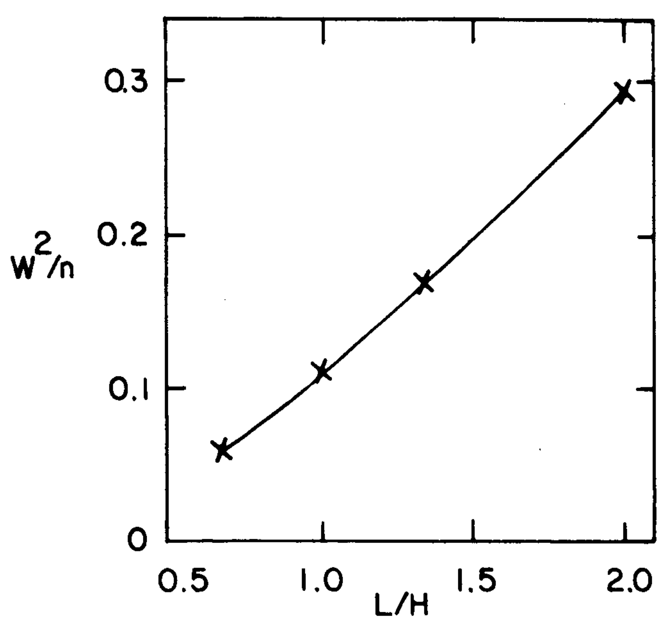


Figure 15. W^2/n as a function of L/H.

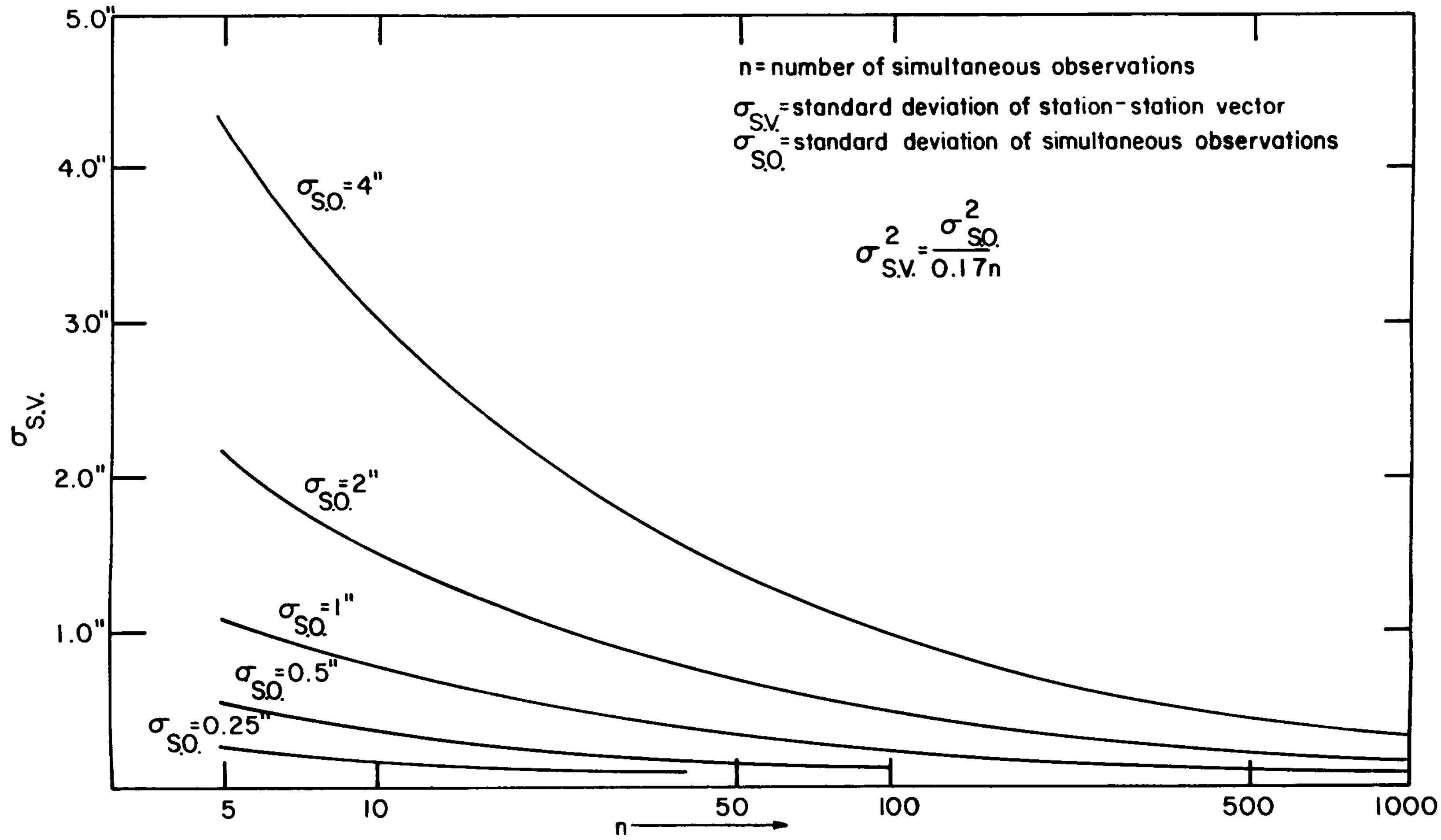


Figure 16. Accuracy of the direction of the station-station vector derived from n pairs of simultaneous observations.

7. SUMMARY OF RESULTS

If an optimum solution for the station-station vector is to be obtained, the distribution of simultaneous observations must be such that:

A. The satellite planes are equally distributed with respect to Z between the limits Z_{max} and Z_{min} established in Table 4, to ensure that the total weight in the vertical plane will equal that in azimuth.

B. Any satellite position is to be matched by a second satellite position on the opposite side of the vertical plane such that the positions either form mirror images about the vertical plane or lie symmetrically about the mid-point vertical. The tolerances permissible in such matchings have been indicated in Figures 11, 12, and 13; but, as mentioned above, when large numbers of observations are considered, these limits will cease to be of importance. What will still be important is that there are equal numbers of observations on both sides of the vertical plane.

C. Figure 15 clearly indicates the preference that should be given to configurations that yield large values of L/H. The ratio L/H = 2, for example, is clearly three times better than the ratio L/H = 1. However, the use of larger values of L/H does decrease the area of the optimum region in which the subsatellite points should lie, therefore decreasing the frequency with which any satellite may be observed. If only a few precise pairs of simultaneous observations are required, it will be preferable to select those satellite positions with optimum values for L/H. This would, for example, be the case for wild-BC4 observations. Each plate furnishes a spatial direction with a standard deviation of about 0.″4, and as few as six pairs of plates give an equally precise determination for the direction of the station-station vector.

In the event the individual satellite directions are of a lower precision, a larger number of simultaneous observations will be required. For example, with Baker-Nunn photography a single synthetic simultaneous direction has a precision of about 1'' of arc. To determine the direction of the station-station vector to within 0''.4 therefore requires at least 40 pairs of simultaneous observations. Now it may be preferable — on a time scale — to observe all possible satellite positions that satisfy the two criteria summarized above, but with no conditions imposed on the ratio L/H. The increase in the frequency of observations will more than compensate for the fact that the contributions of some of these observations to the total variance of the station-station vector may be less than if the conditions on L/H were imposed.

D. The accuracy of the direction of the station-station vector is given by

$$\begin{bmatrix} \sigma_V^2 & \sigma_{AV} \\ \sigma_{VA} & \sigma_A^2 \end{bmatrix} = \frac{\sigma_{s.0.}^2}{\left[0.19\left(\frac{L}{H}\right) - 0.08\right]n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

E. Throughout, it has been assumed that all observations are of equal accuracy. This need not be the case, however, when the observations are are close to the horizon. Decreasing the zenith distance at which the satellite is to be observed will reduce such uncertainties, and the above criteria will still be valid, although the frequency with which a particular satellite can be observed will be further decreased.

The results are illustrated in Section 8.

8. TWO EXAMPLES

The two cases are for the astrophysical observing stations at Organ Pass, New Mexico (9001) and Jupiter, Florida (9010). The chord distance between these stations is approximately 2600 km. The satellites considered are 60 009 02 (Echo 1 Rocket) and 61 028 01 (Midas 4). Their respective heights are 1500 and 3500 km, and both are in near-polar orbits.

60 009 02. (See Figure 17.) The maximum zenith distance of observation is 75°, and the corresponding horizon distance is 20°.

For $C = 0$; $\eta_{max} = 15°$, $Z_{max} = 65°$, $Z_{min} = 30°$, $\eta = 5.5°$

$C = L/4$; $\eta_{max} = 12.5°$, , $Z_{max} = 51°$, $Z_{min} = 38°$, $\eta_{min} = 8.5°$

and for $Z_{max} = 45°$, $Z_{min} = 45°$, $\eta = 10.5°$

61 028 01. (See Figure 18.) The maximum zenith distance of observation is 75°, and the corresponding horizon distance is 37°.

For $C = 0$; $\eta_{max} = 35°$, $Z_{max} = 70°$, $Z_{min} = 25°$, $\eta_{min} = 90°$.

$C = L/2$; $\eta_{max} = 26.5°$, $Z_{max} = 60°$, $Z_{min} = 31°$, $\eta_{min} = 11.5°$

and for $Z_{max} = 45°$, $Z_{min} = 45°$, $\eta = 18.5°$.

Both cases are plotted on a sterographic projection. The areas of optimum positions for $z_{max} = 65°$ and 60° are also indicated.

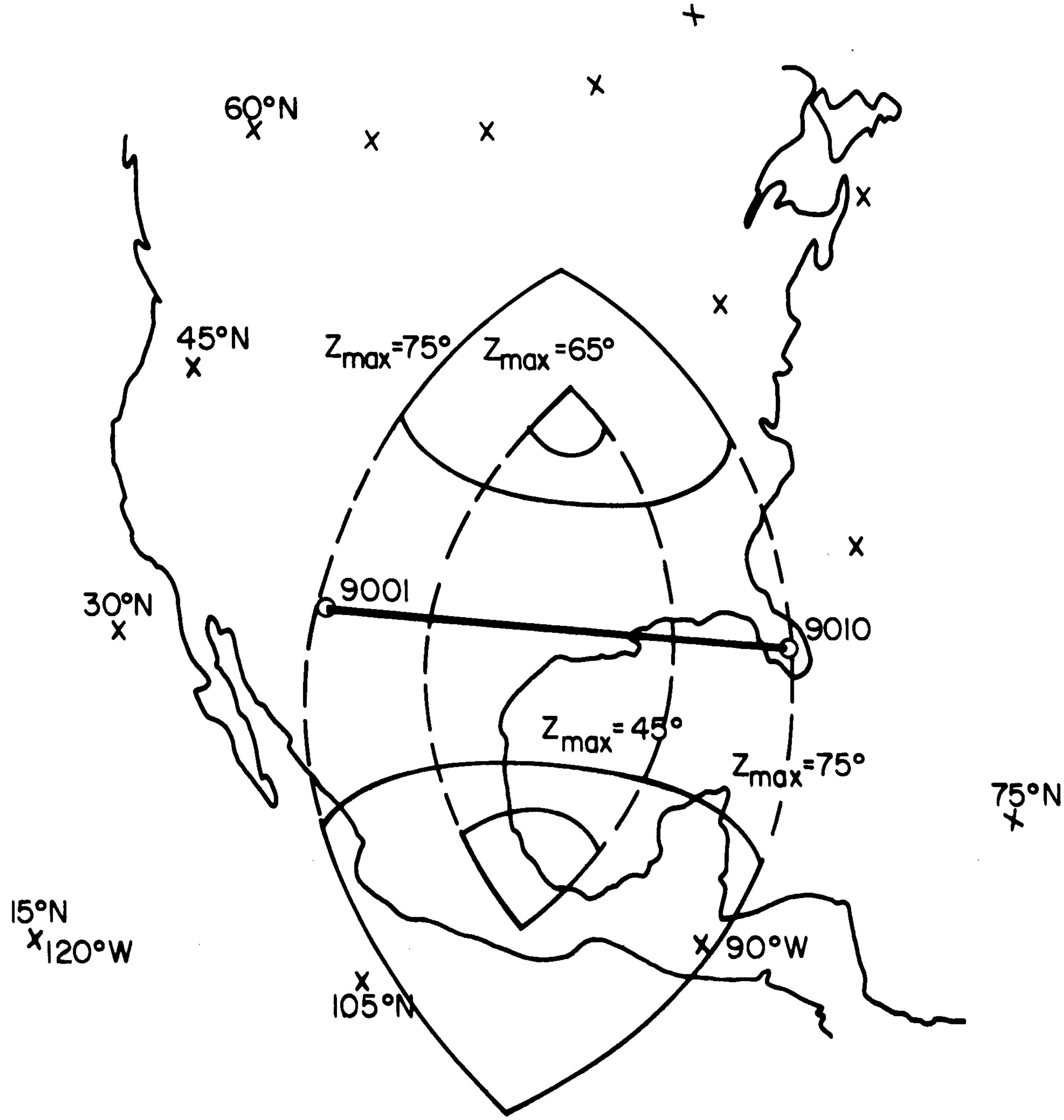


Figure 17. Optimum areas in which subsatellite points should be distributed evenly with respect to area for Satellite Echo 1 Rocket (height 1500 km) and two different maximum zenith distances.

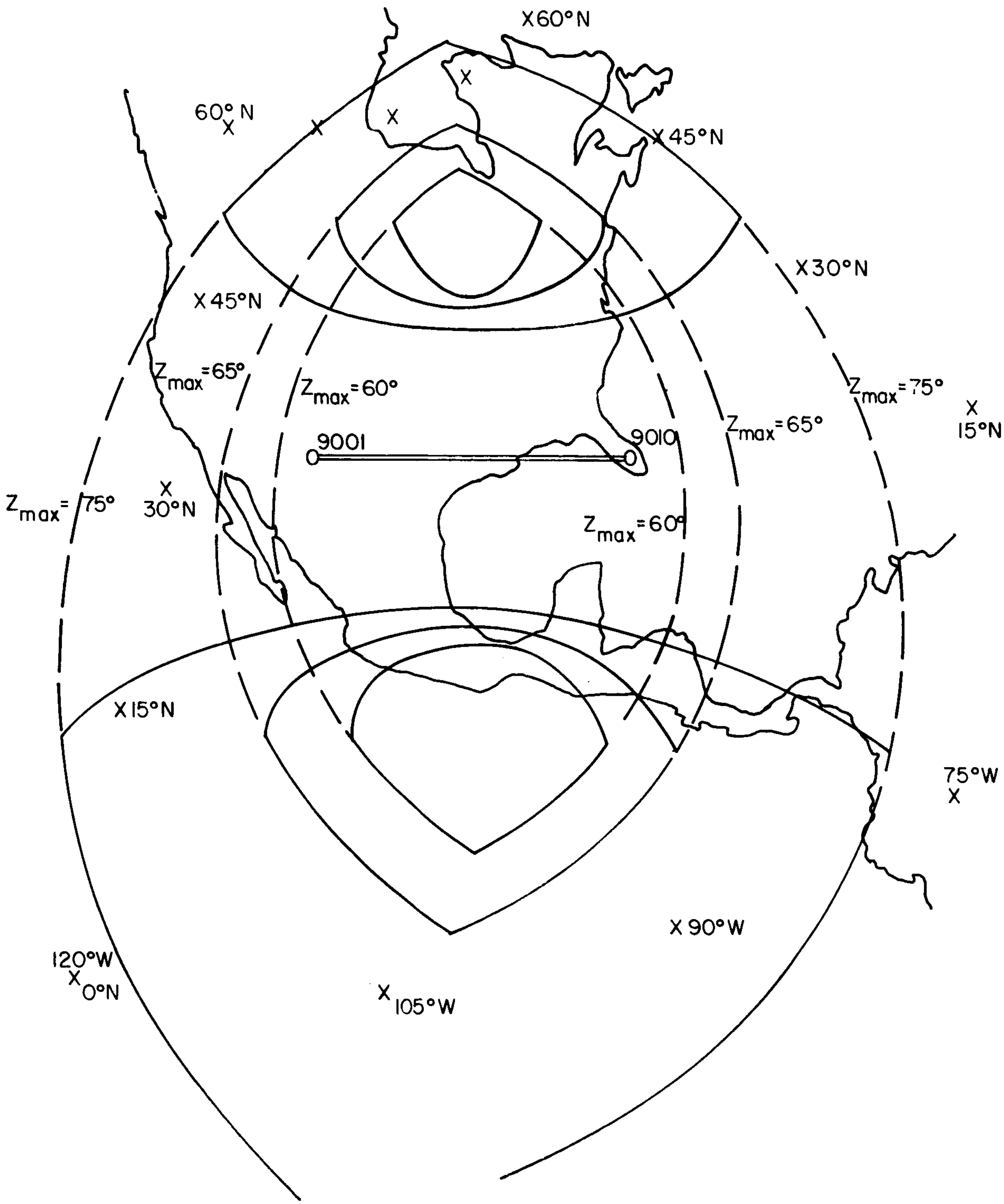


Figure 18. Optimum areas in which subsatellite points should be distributed evenly with respect to area for Satellite Midas 4 (height 3500 km) and for three different maximum zenith distances.