

**LYAPUNOV FUNCTIONS FROM AUXILIARY
EXACT DIFFERENTIAL EQUATIONS**

By Edwin Kinnen and Chiou-Shiun Chen

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Lyapunov Functions From Auxiliary
Exact Differential Equations

Edwin Kinnen

Chiou-Shiun Chen

University of Rochester

Rochester, New York

Abstract

A Lyapunov function is the solution to an exact differential equation described in the phase space of the corresponding nonlinear differential equation. In this report, we consider this idea in reverse and discuss the problem of finding an auxiliary exact differential equation from a given nonlinear differential equation such that its solution, obtained by integration, is a Lyapunov function for the original differential equation. The generalization afforded by this perspective is sufficiently broad to include published techniques for developing Lyapunov functions as reasonable attempts to solve the reformulated problem. It is shown that most of the Lyapunov functions appearing in the literature can be derived from various ways of developing this auxiliary differential equation. New methods for obtaining Lyapunov functions are also evident within the context of an auxiliary exact differential equation.

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CONTENTS

I.	Introduction	1
II.	Exact Differential Equations	2
III.	Lyapunov Functions as Solutions to Exact Differential Equations	10
IV.	Methods of Obtaining the Vector \underline{h}	11
V.	Alternate Auxiliary Exact Differential Equations	24
VI.	Discussion	25
	References	27

LYAPUNOV FUNCTIONS FROM AUXILIARY

EXACT DIFFERENTIAL EQUATIONS

I. INTRODUCTION

Stability characteristics of an equilibrium point of an autonomous nonlinear differential equation are often determined by the direct or second method of Lyapunov. In its simplest form, the direct method assumes a sign definite scalar function in the region of space including the singular point such that projections of the differential equation solutions onto the surface described by the scalar function can be examined as a function of time. If the value of the scalar function when constrained to change according to the differential equation is found to have a negative time derivative, for example, then solutions around the equilibrium point are known to be asymptotically stable. This method has the principal advantage of circumventing the problems of solving nonlinear differential equations. But the advantage may be lost unless the scalar functions required by the method can be found more easily than the original solutions. Scalar functions with the necessary properties to establish Lyapunov stability characteristics are called Lyapunov functions.

In most explanations of the direct method of Lyapunov, the initial selection of a scalar function for a possible Lyapunov function is not identified theoretically with any particular differential equation. Certain functions have been found to be useful for classes of differential equations; these are often simply stated and then shown to be Lyapunov functions for selected examples. In practice, when an unfamiliar differential equation is encountered, these same functions are usually chosen as the basis for a trial and error search procedure.

Thus, while the direct method of Lyapunov has theoretical significance, its practical usefulness is restricted to differential equations for which satisfactory Lyapunov functions are known or can be obtained through a definitive procedure. Few reports, however, describe procedures that are useful beyond the illustrative examples. It is reasonable to ask then if a more direct procedure for developing Lyapunov functions can be found, one which depends in its details on the specific differential equation of concern. Intuitively, as classes of differential equations are considered which imply increasingly more analytically complex solutions, we would expect to find Lyapunov functions from more limited classes of scalar functions, those implying correspondingly complex analytical descriptions. The use of an auxiliary differential equation derived from the nonlinear differential

equation to find a Lyapunov function, as described in the following sections, appears to be in accord with this philosophy.

Lyapunov functions are a class of scalar functions representing single-valued, nested, closed surfaces in the phase space of the differential equation. Exact differential equations exist for the descriptions of these surfaces. Conversely, if exact differential equations describing the closed contours of these surfaces are known, the scalar functions can be obtained directly by integration. This report describes procedures for developing a Lyapunov function by initially obtaining an auxiliary exact differential equation from the nonlinear differential equation. The procedures and details are not unique but are shown to be inclusive of many others described in the literature. Insight is also given for alternate procedures. A subsequent report will extend the work, specifically describing a more direct way of obtaining the vector \underline{h} that is used to convert the given nonlinear differential equation into an auxiliary exact differential equation.

II. Exact Differential Equations

Consider a set of n first order, autonomous differential equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}(t)), \quad = \frac{d}{dt} \quad (1)$$

where $\underline{x} = [x_1, x_2, \dots, x_n]^t$ and $\underline{f} = [f_1, f_2, \dots, f_n]^t$ are n dimensional column vectors, and all $f_i = f_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$, together with their first partial derivatives are defined and continuous in some domain Ω of space E_n . Assume that an equilibrium point exists at the origin, $\underline{x} = \underline{0}$ also in Ω . If we define

$$g_i = f_1 + f_2 + \dots + f_{i-1} - f_{i+1} - \dots - f_n \quad (2)$$

then

$$\sum_{i=1}^n g_i \dot{x}_i = \langle \underline{g}, \dot{\underline{x}} \rangle = 0$$

or $\langle \underline{g}, d\underline{x} \rangle = 0.$ (3)

Equation (3) is recognized as a differential equation representing the solution trajectories of (1) in phase space. Consequently a solution to (1) is a solution to (3). Note also that (3) is only one possible differential equation that can be derived from linear and nonlinear combinations of (1).

All g_i are defined and continuous in the domain Ω of E_n . Equation (3) is said to be exact differential equation in Ω if there is some single-valued differentiable function $U(\underline{x})$, defined and continuous, together with its first partial derivatives in some neighborhood of every point of \underline{x} in Ω , such that [1,2]

$$dU(\underline{x}) = \langle \underline{g}, d\underline{x} \rangle$$

or
$$\frac{dU(\underline{x})}{dt} = \langle \nabla U(\underline{x}), \dot{\underline{x}} \rangle = \langle \underline{g}, \dot{\underline{x}} \rangle.$$
 (4)

The function $U(\underline{x})$, is called the first integral of equation (3) [3].

Expanding (4)

$$\sum_{i=1}^n \frac{\partial U(\underline{x})}{\partial x_i} \dot{x}_i = \sum_{i=1}^n g_i \dot{x}_i,$$

it follows that

$$\frac{\partial U(\underline{x})}{\partial x_i} = g_i, \quad i = 1, 2, \dots, n, \quad (5a)$$

and consequently

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad i, j = 1, 2, \dots, n, \quad (5b)$$

provided the first partial derivatives of g_i are continuous

[2]. Equations (5) are therefore necessary for the exactness of (3). Since (4) also defines \underline{g} as the gradient of the single-valued scalar function $U(\underline{x})$,

$$U(\underline{x}) = \int_C \langle \underline{g}, d\underline{x} \rangle = \int_C \langle \nabla U, d\underline{x} \rangle \quad (6)$$

and is independent of any path C contained in the domain where $U(\underline{x})$ is defined [4]. It follows, therefore, that (5) is also a sufficient condition for the exactness of the differential equation (3).

If equation (3) is not exact, then by definition the first integral does not exist. We can add a function $h(\underline{x})$ to $\underline{g}(\underline{x})$ such that $\langle \underline{g} + \underline{h}, d\underline{x} \rangle$ is an exact differential equation representing some scalar function $U(\underline{x})$. * Then also $\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = 0$, where $\dot{\underline{x}} = \underline{f}(\underline{x})$, represents the exact differential equation of a function whose time derivative is

$$\frac{dU(\underline{x})}{dt} = \langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \langle \underline{g} + \underline{h}, \underline{f} \rangle = \langle \underline{h}, \underline{f} \rangle. \quad (7)$$

*The possibility of using an integrating factor to obtain an exact differential equation, as discussed in most elementary differential equation texts, has not been found useful. This multiplicative factor does not appear readily extendable to equations of higher order.

For an \underline{h} such that (7) is an exact differential equation,

$$\frac{\partial U}{\partial x_i} = g_i + h_i \quad i = 1, 2, \dots, n,$$

and

$$\frac{\partial (g_i + h_i)}{\partial x_j} = \frac{\partial (g_j + h_j)}{\partial x_i}, \quad i, j = 1, 2, \dots, n.$$

The function $U(\underline{x})$ can also be evaluated by a line integral, since $\nabla U = \underline{g} + \underline{h}$. Assuming an \underline{h} such that U , the solution to (7), is a single-valued, continuously differentiable function in Ω , the line integral

$$\int_c \langle \nabla U, d\underline{x} \rangle = \int_c \langle \underline{g} + \underline{h}, d\underline{x} \rangle$$

is independent of the path [4]. Thus for any path in Ω , say from $\underline{0}$ to \underline{x} ,

$$U(\underline{x}) - U(\underline{0}) = \int_{\underline{0}}^{\underline{x}} \langle \underline{g}(\underline{y}) + \underline{h}(\underline{y}), d\underline{y} \rangle. \quad (8)$$

If the path is selected so that the integration progresses along one coordinate at a time, (8) becomes

$$\begin{aligned}
U(\underline{x}) &= \int_0^{x_1} \{g_1(y_1, x_2, \dots, x_n) + h_1(y_1, x_2, \dots, x_n)\} dy_1 \\
&+ \int_0^{x_2} \{g_2(0, y_2, x_3, \dots, x_n) + h_2(0, y_2, x_3, \dots, x_n)\} dy_2 \\
&+ \dots \\
&+ \int_0^{x_n} \{g_n(0, 0, \dots, 0, y_n) + h_n(0, 0, \dots, 0, y_n)\} dy_n.
\end{aligned} \tag{9}$$

Alternatively, the first integral $U(\underline{x})$ can be evaluated by the following procedure.

(i) For $n = 2$, we have

$$\frac{\partial U}{\partial x_1} = g_1(\underline{x}) + h_1(\underline{x})$$

and

$$\frac{\partial U}{\partial x_2} = g_2(\underline{x}) + h_2(\underline{x}).$$

Integrating the first equation with respect to x_1 ,

$$U = \int (g_1 + h_1) dx_1 + F_1(x_2).$$

To find $F_1(x_2)$, consider the partial derivative of the latter equation with respect to x_2 :

$$\frac{\partial U}{\partial x_2} = \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1 + \frac{dF_1(x_2)}{dx_2} ,$$

$$\text{or } \frac{dF_1(x_2)}{dx_2} = \frac{\partial U}{\partial x_2} - \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1 = g_2 + h_2 - \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1 .$$

$$\text{Therefore } F_1(x_2) = \int \{g_2 + h_2 - \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1\} dx_2 ,$$

and finally

$$U = \int (g_1 + h_1) dx_1 + \int \{g_2 + h_2 - \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1\} dx_2 . \quad (10a)$$

$$(ii) \text{ For } n = 3, \quad \frac{\partial U}{\partial x_1} = g_1(\underline{x}) + h_1(\underline{x})$$

$$\text{and } \frac{\partial U}{\partial x_2} = g_2 + h_2, \quad \frac{\partial U}{\partial x_3} = g_3 + h_3 .$$

Following the same method as in (i), the first equation can be integrated

$$U = \int (g_1 + h_1) dx_1 + F_1(x_2, x_3) .$$

Differentiating this with respect to x_2 and combining with the second of the above three equations,

$$\frac{\partial F_1}{\partial x_2} = g_2 + h_2 - \frac{\partial}{\partial x_2} \int (g_1 + h_1) dx_1 ,$$

or
$$F_1 = \int [g_2 + h_2 - \frac{\partial}{\partial x_2} \int (g_1+h_1)dx_1]dx_2 + F_2(x_3).$$

This is substituted into the equation for $U(\underline{x})$, which can then be differentiated with respect to x_3 and combined with the third of the original three equations. Solving for $\frac{\partial F_2}{\partial x_3}$, integrating with respect to x_3 , and substituting back into $U(\underline{x})$, it follows that

$$\begin{aligned} U = & \int (g_1+h_1)dx_1 + \int [(g_2+h_2) - \frac{\partial}{\partial x_2} \int (g_1+h_1)dx_1]dx_2 \\ & + \int \{(g_3+h_3) - \frac{\partial}{\partial x_3} \int (g_1+h_1)dx_1 \\ & - \frac{\partial}{\partial x_3} \int [(g_2+h_2) - \frac{\partial}{\partial x_2} \int (g_1+h_1)dx_1]dx_2\}dx_3. \end{aligned} \quad (10b)$$

This procedure can be extended similarly to develop the first integral for higher dimensional cases.

Equation (10b) simplifies significantly if the bracketed quantity in the last integral is independent of x_3 . Then, for $n = 3$,

$$\begin{aligned} U = & \int (g_1+h_1)dx_1 + \int [(g_2+h_2) - \frac{\partial}{\partial x_2} \int (g_1+h_1)dx_1]dx_2 \\ & + \int [(g_3+h_3) - \frac{\partial}{\partial x_3} \int (g_1+h_1)dx_1]dx_3. \end{aligned}$$

It can be proved for any n , that if

$$(g_i+h_i) - \frac{\partial}{\partial x_i} \int (g_1+h_1) dx_1, \quad i = 2, 3, \dots, n,$$

is a function of x_i only,

$$U(\underline{x}) = \int (g_1+h_1) dx_1 + \int [(g_2+h_2) - \frac{\partial}{\partial x_2} \int (g_1+h_1) dx_1] dx_2$$

+ ...

$$+ \int [g_n+h_n) - \frac{\partial}{\partial x_n} \int (g_1+h_1) dx_1] dx_n. \quad (11)$$

III. Lyapunov Functions as Solutions to Exact Differential Equations

In the last section, it is stated that we can add a vector \underline{h} to the differential equation $\langle \underline{g}, \dot{\underline{x}} \rangle = 0$ such that $\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \dot{U}$ is an exact differential equation with a solution given by (9) or (10). It remains then to select \underline{h} so that this solution has the characteristics of a Lyapunov function* with respect to the differential equation (1), i.e., that $\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \dot{V}$ be an auxiliary exact differential equation to (1).

Therefore, given equations (1) and (2), if we can find an \underline{h} which insures that

(A) $\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \dot{V}$ is an exact differential equation,

(B) $\langle \underline{h}, \underline{f} \rangle = \frac{dV(\underline{x})}{dt}$ is at least negative semidefinite, and

(C) the first integral, $V(\underline{x})$, is positive definite,

then $V(\underline{x})$ is a Lyapunov function with respect to (1).

The necessary and sufficient condition for exactness given by (5) is equivalent to the condition that

* A scalar function $V(\underline{x})$ with the properties that:

(i) $V(\underline{x})$ is continuous together with its first partial derivatives in a region Ω about the origin,

(ii) $V(\underline{0}) = 0$, $V(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, and

(iii) $\frac{dV(\underline{x})}{dt} \leq 0$ in Ω ,

is called a Lyapunov function [5,8], and similarly if $\dot{V} > 0$ in Ω .

(A') the matrix

$$\left[\frac{\partial(g_i+h_i)}{\partial x_j} \right] = \left[\begin{array}{cccc} \nabla(g_1+h_1) & \nabla(g_2+h_2) & \dots & \nabla(g_n+h_n) \end{array} \right]$$

be symmetric.

The existence of an \underline{h} satisfying the three conditions (A) or (A') - (C) is both necessary and sufficient for the existence of a Lyapunov function for (1). The proof of sufficiency follows from definitions. Conversely, if a Lyapunov function, $V_1(\underline{x})$, exists for (1), conditions (B) and (C) are satisfied. Condition (A) is satisfied if we allow

$$h_i = \frac{\partial V_1(\underline{x})}{\partial x_i} - g_i ; \quad (12)$$

thus the necessity.

IV. Methods of Obtaining the Vector \underline{h}

The problem of finding a Lyapunov function for equation (1) has been restated as a problem of determining the components of a vector \underline{h} for the auxiliary differential equation (7) such that conditions (A) or (A') - (C) are satisfied. A variety of methods can be considered to aid this search. Some of these are summarized below and are readily identified with the familiar techniques for developing Lyapunov functions that are described in the literature.

(i) The simplest case exists when condition (A) is satisfied for all $h_i = 0$. This implies that (3) is exact. Condition (B) is then satisfied and only (C) needs to be considered. If (9) or (10) results directly in a function that satisfies (C), it is a Lyapunov function with respect to (1). In this case,

the first integral is also the phase plane solution to the differential equation (1). This procedure has been called the method of (sign definite) first integrals [5,6]. If, however, the first integral does not satisfy condition (C), i.e., is not sign definite, an alternate procedure is required, e.g., $h \neq 0$.

Example. Consider
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\end{aligned}$$

From (2), $g_1(\underline{x}) = x_1$ and $g_2(\underline{x}) = x_2$. Condition (A') is satisfied with $h_1 = h_2 = 0$. The first integral is

$$V(\underline{x}) = \int_0^{x_1} g_1(y_1, x_2) dy_1 + \int_0^{x_2} g_2(0, y_2) dy_2 = \frac{x_1^2}{2} + \frac{x_2^2}{2}$$

which is positive definite. Therefore $V(\underline{x}) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$ is

a Lyapunov function, $\dot{V}(\underline{x}) = 0$, and the equilibrium point $(0,0)$ is stable. As $V \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$, this equilibrium point is also globally stable.

(ii) If equation (3) is not exact, all h_i cannot be set to zero as in (i). With or without trying to differentiate between g and h , we could proceed, for example, to select $\nabla V = (g+h)$ by trial and error to satisfy both (A') and (B). If this is done, (C) can then be considered. If (C) is not satisfied, the procedure is repeated for another choice of h or ∇V , etc. While an analogous procedure for searching for a Lyapunov function has not been previously suggested, it is evident from this restatement of the problem in terms of an auxiliary exact differential equation. The dependency of this particular procedure on intuition and a quasi-random selection of undetermined constants is no greater than other accepted procedures.

Example:
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1)\end{aligned}$$

where $a > 0$, $g(0) = 0$, and $x_1 g(x_1) > 0$ for $x_1 \neq 0$. Then

$$g_1 = ax_2 + g(x_1), \quad g_2 = x_2.$$

Since $\frac{\partial g_1}{\partial x_2} \neq \frac{\partial g_2}{\partial x_1}$, h_i cannot be zero.

Somewhat arbitrarily choose

$$g_1 + h_1 = g(x_1), \text{ or } h_1 = -ax_2,$$

and $g_2 + h_2 = x_2$, or $h_2 = 0$.

Condition (A') is satisfied, and condition (B) is satisfied as

$$\langle \underline{h}, \underline{f} \rangle = -ax_2^2.$$

To check condition (C),

$$V(\underline{x}) = \frac{x_2^2}{2} + \int_0^{x_1} g(x_1) dx_1,$$

which is positive definite. $V(\underline{x})$ therefore is a Lyapunov function.

(iii) Rather than attempt to select $\nabla V = (\underline{g} + \underline{h})$ in its entirety as in (ii), we could proceed in the same manner except by

trying to find the components of \underline{h} only. Alternately we might simplify the effort by separating out the nonlinear components of ∇V . From the form of (3), it is noted that each g_i in general includes both linear and nonlinear terms in the dependent variables. Therefore define

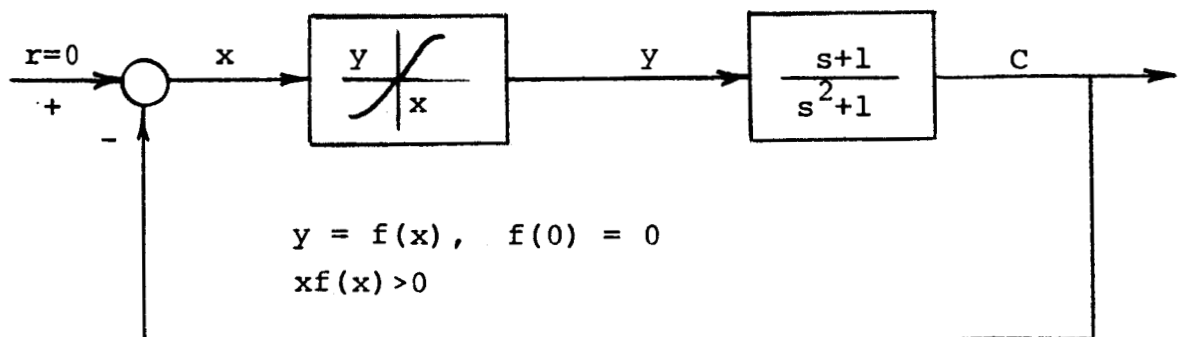
$$\underline{g} \stackrel{\Delta}{=} \underline{g}_l + \underline{g}_n, \quad (13)$$

where \underline{g}_l contains only linear terms. If $\underline{h} = \underline{h}_l + \underline{h}_n$ is similarly defined, the procedure in (ii) could be followed by separately selecting a linear and nonlinear part of ∇V . Equations (8) or (9) can then be written as

$$V(\underline{x}) = \int_0^{\underline{x}} \langle \underline{g}_l + \underline{h}_l, d\underline{x} \rangle + \int_0^{\underline{x}} \langle \underline{g}_n + \underline{h}_n, d\underline{x} \rangle. \quad (14)$$

The first of the two integrals can be resolved into a quadratic form, and (14) is recognized as the familiar quadratic plus integral form [5,7,8]. If the nonlinearity of the system is known analytically, it may be possible to evaluate the second integral. Otherwise the satisfaction of condition (C) can be considered, in part, directly through the integral characteristics of the nonlinearity [5,7,8].

Example. Consider the second order nonlinear control system



The differential equation describing the system has been shown to be [5]

$$\ddot{x} + x + f(x) + f'(x)\dot{x} = 0.$$

This can be written in the form

$$\dot{x}_1 = x_2 - f(x_1)$$

$$\dot{x}_2 = -x_1 - f(x_1)$$

with an equilibrium point noted at $x_1 = x_2 = 0$. In this case (2) becomes $g_1 = x_1 + f(x_1)$ and $g_2 = x_2 - f(x_1)$; then

$$\underline{g}_l + \underline{g}_n = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f(x_1) \\ -f(x_1) \end{bmatrix}.$$

Choosing $\underline{h}_l + \underline{g}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f(x_1) \end{bmatrix},$

$$\nabla V = \underline{g} + \underline{h} = \begin{bmatrix} x_1 + f(x_1) \\ x_2 \end{bmatrix}$$

satisfies (A'). For condition (B),

$$\langle \underline{h}, \dot{\underline{x}} \rangle = -x_1 f(x_1) - f^2(x_1),$$

which is negative semidefinite. The first integral, equation (14), is

$$V(\underline{x}) = \frac{1}{2}(x_1^2 + x_2^2) + \int_0^{x_1} f(x_1) dx_1$$

which is positive definite. Therefore, a Lyapunov function has been derived and, since $V(\underline{x}) \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$, the equilibrium point is globally stable.

If the characteristic of the nonlinearity is given analytically, for example, $f(x_1) = x_1^3$,

then
$$V(\underline{x}) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{x_1^4}{4},$$

and
$$\dot{V}(\underline{x}) = \langle h, f \rangle = -x_1 f(x_1) - f_1^2(x_1) = -x_1^4 - x_1^6.$$

(iv) Alternately we can write $\underline{g} + \underline{h}$ in the exact differential equation (7), for example, as

$$\begin{aligned} g_1 + h_1 &= a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \\ &\vdots \\ g_i + h_i &= a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \\ &\vdots \\ g_n + h_n &= a_{n1} x_1 + a_{n2} x_2 + \dots + 2x_n \end{aligned} \tag{15}$$

where the a_{ij} have a constant part plus a function of

- x_1, x_2, \dots, x_{n-1} , i.e., $a_{ij} = a_{ijk} + a_{ijv}$. The a_{ii} may further be limited such that $a_{iik} > 0$ and $a_{iiv} = a_{iiv}(x_i)$
- to facilitate satisfying condition (C). Undetermined constants in (15) could be selected to first satisfy (B) and then the remaining terms to satisfy (A'). Lastly condition (C) would be checked following the use of (9) or (10). This procedure corresponds essentially to a familiar variable gradient method [9].

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - x_1^3.\end{aligned}$$

From (15), the gradient is assumed to be

$$\underline{g} + \underline{h} = \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + 2x_2 \end{bmatrix}, \text{ and } \underline{g} = \begin{bmatrix} x_2 + x_1^3 \\ x_2 \end{bmatrix}.$$

$$\text{Then } \underline{h} = (\underline{g} + \underline{h}) - \underline{g} = \begin{bmatrix} (\alpha_{11} - x_1^2)x_1 + (\alpha_{12} - 1)x_2 \\ \alpha_{21}x_1 + x_2 \end{bmatrix}.$$

Condition (B) requires that $\langle \underline{h}, \underline{f} \rangle$ be at least negative semi-definite:

$$\langle \underline{h}, \underline{f} \rangle = -\alpha_{21}x_1^4 + (\alpha_{12} - 2)x_2^2 + (\alpha_{11} - \alpha_{21} - 2x_1^2)x_1x_2.$$

Somewhat arbitrarily let $\alpha_{11} - \alpha_{21} - 2x_1^2 = 0$, $\alpha_{12} = 1$ and $\alpha_{21} > 0$.

$$\text{Thus } \underline{g} + \underline{h} = \begin{bmatrix} \alpha_{21}x_1 + 2x_1^3 + x_2 \\ \alpha_{21}x_1 + 2x_2 \end{bmatrix} .$$

To satisfy condition (A'), the matrix

$$\left[\frac{\partial (g_i + h_i)}{\partial x_j} \right] = \begin{bmatrix} \alpha_{21} + 6x_1^2 & \alpha_{21} \\ 1 & 2 \end{bmatrix}$$

must be symmetric; so let $\alpha_{21} = 1$. The first integral, from (9), is

$$\begin{aligned} V(\underline{x}) &= \int_0^{x_1} (y_1 + 2y_1^3 + x_2) dy_1 + \int_0^{x_2} 2y_2 dy_2 \\ &= \frac{x_1^2}{2} + \frac{x_1^4}{2} + x_1x_2 + x_2^2 = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_2^2 + x_1^4) \end{aligned}$$

and $\dot{V}(\underline{x}) = \langle \underline{h}, \underline{f} \rangle = -x_2^2 - x_1^4.$

Therefore $V(\underline{x})$ is a Lyapunov function for the example and the equilibrium point is (globally) asymptotically stable.

(v) Consider \underline{h} in (7) separated into two parts $\underline{h}_1 + \underline{h}_2$, where \underline{h}_1 is selected specifically to satisfy (A') only. \underline{h}_2 might then be chosen with respect to condition (B) such that the exactness condition (A') is not effected. Finally condition (C) is examined. If (C) cannot be satisfied, an alternate \underline{h}_2 might be considered. This procedure is a generalization of a method described by Infante and by Walker, although their objectives may be less apparent to a casual reader [10, 11]. In their terminology the "new nearby system" is equivalent to equation (7), where

$$\langle \underline{g} + \underline{h}, \dot{\underline{x}} \rangle = \langle \underline{g} + \underline{h}_1 + \underline{h}_2, \dot{\underline{x}} \rangle.$$

Example: Consider

$$\begin{aligned} \dot{x}_1 &= x_2 & &= f_1(\underline{x}) \\ \dot{x}_2 &= -x_1 - \epsilon(1-x_1^2)x_2 & &= f_2(\underline{x}), \quad \epsilon > 0. \end{aligned}$$

Then
$$g_1 = -f_2 = x_1 + \epsilon(1-x_1^2)x_2,$$

and
$$g_2 = f_1 = x_2.$$

The original equation can be rewritten as

$$g_1 \dot{x}_1 + g_2 \dot{x}_2 = 0 \tag{16}$$

which is not exact.

Adding
$$\underline{h}_1 = \begin{bmatrix} 0 \\ \epsilon x_1 \left(1 - \frac{x_1^2}{3}\right) \end{bmatrix}$$
 to \underline{g} will make (16) exact:

$$\langle \underline{g} + \underline{h}_1, \dot{\underline{x}} \rangle = g_1 \dot{x}_1 + [g_2 + \epsilon x_1 \left(1 - \frac{x_1^2}{3}\right)] \dot{x}_2 = 0.$$

But condition (B) is not satisfied, i.e.,

$$\begin{aligned} \langle \underline{h}_1, \underline{f} \rangle &= \epsilon x_1 \left(1 - \frac{x_1^2}{3}\right) \{-x_1 - \epsilon(1 - x_1^2)x_2\} \\ &= -\epsilon^2 x_1 x_2 (1 - x_1^2) \left(1 - \frac{x_1^2}{3}\right) - \epsilon x_1^2 \left(1 - \frac{x_1^2}{3}\right) \end{aligned}$$

is indefinite. Therefore add \underline{h}_2 to $\underline{g} + \underline{h}_1$, subject to $\frac{\partial h_{21}}{\partial x_2} - \frac{\partial h_{22}}{\partial x_1} = 0$, with the purpose of satisfying condition (B).

$$\begin{aligned} \text{As } \langle \underline{h}_1 + \underline{h}_2, \underline{f} \rangle &= -\epsilon^2 x_1 x_2 (1 - x_1^2) - \epsilon x_1^2 \left(1 - \frac{x_1^2}{3}\right) \\ &\quad + h_{21} x_2 - h_{22} [x_1 + \epsilon(1 - x_1^2)x_2], \end{aligned}$$

we could choose $h_{21} = \epsilon^2 x_1 (1 - x_1^2) \left(1 - \frac{x_1^2}{3}\right)$ and $h_{22} = 0$.

$$\text{Then } \dot{V} = -\epsilon^2 \left(1 - \frac{x_1^2}{3}\right),$$

which is negative semidefinite within the region $x_1^2 = 3$. To evaluate the first integral,

$$\underline{g} + \underline{h}_1 + \underline{h}_2 = \begin{bmatrix} x_1 + \epsilon(1 - x_1^2)x_2 + \epsilon^2 x_1 (1 - x_1^2) \left(1 - \frac{x_1^2}{3}\right) \\ x_2 + \epsilon x_1 \left(1 - \frac{x_1^2}{3}\right) \end{bmatrix}$$

$$\begin{aligned} \text{and } V(\underline{x}) &= \frac{x_1^2}{2} + \epsilon \left(x_1 - \frac{x_1^3}{3}\right) x_2 + \epsilon^2 \left(\frac{x_1^2}{2} - \frac{x_1^4}{3} + \frac{x_1^6}{18}\right) + \frac{x_2^2}{2} \\ &= \frac{1}{2} \{x_1^2 + [x_2 + \epsilon(x_1 - \frac{x_1^3}{3})]^2\}, \end{aligned}$$

which is positive definite.

(vi) If one intuitively selects a first integral function so as to satisfy (C) and if the second partial derivatives exist, then (A') is also satisfied and condition (B) only remains to be considered. If the result is not satisfactory, another first integral can be tried. While this approach corresponds to the method of guessing that so often confuses the neophyte, it may also be considered as the basis of the more sophisticated method of squaring proposed by Krasovskii [5].

Example. Consider: $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{f}(0) = \underline{0}$, where \underline{f} has continuous first partial derivatives. Arbitrarily let the first integral be the positive definite function $V(\underline{x}) = \langle \underline{f}(\underline{x}), \underline{f}(\underline{x}) \rangle$. To check condition (B), $\dot{V} = \langle \underline{f}, \dot{\underline{f}} \rangle + \langle \dot{\underline{f}}, \underline{f} \rangle = \langle \underline{f}, \underline{F} \dot{\underline{x}} \rangle + \langle \underline{F} \dot{\underline{x}}, \underline{f} \rangle$

where $\underline{F} = \frac{\partial f_i}{\partial x_j}$.

Then $\dot{V} = \langle \underline{f}, \underline{F}\dot{\underline{x}} \rangle + \langle \dot{\underline{x}}, \underline{F}^t \underline{f} \rangle = \langle \underline{f}, \hat{\underline{F}} \underline{f} \rangle$, where $\hat{\underline{F}} = \underline{F} + \underline{F}^t$.

Condition (B) then requires that $-\hat{\underline{F}}$ be positive definite in the neighborhood of 0, corresponding to a statement of a theorem due to Krasovskii.

(vii) Using a somewhat different technique from those already mentioned, we might begin initially by selecting a function $\dot{V}(\underline{x})$ to satisfy condition (B) but also to allow a solution for $V(\underline{x})$ to be obtained directly by solving the partial differential equation

$$\langle \nabla V, \underline{f} \rangle = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i = \dot{V}(\underline{x}) \quad (17)$$

If this can be done and condition (C) is satisfied for the solution, $V(\underline{x})$, a Lyapunov function results and (A') is satisfied. If (C) is not satisfied by the solution, a more discerning choice of $\dot{V}(\underline{x})$ is required. This is the method proposed by Zubov [5, 12]. In effect, Zubov's method requires one to select the entire scalar function $\langle \underline{h}, \underline{f} \rangle = V (= \phi)$ such that both conditions (A) and (B) are satisfied. But as a consequence of choosing $\langle \underline{h}, \underline{f} \rangle$ rather than \underline{h} , $\nabla V(\underline{x})$ is unavailable and $V(\underline{x})$ must be found by solving the partial differential equation rather than from a line integration.

Example: $\dot{x}_1 = -2x_1 + 2x_2^4$

$\dot{x}_2 = -x_2$

Following Zubov's method, we might fortunately select [13] $\langle \underline{h}, \underline{f} \rangle = -24(x_1^2 + x_2^2)$. Then (17) is

$$\frac{\partial V(\underline{x})}{\partial x_1} (-2x_1 + 2x_2^4) + \frac{\partial V(\underline{x})}{\partial x_2} (-x_2) = -24(x_1^2 + x_2^2).$$

The solution to this partial differential equation can be shown to be

$$V(\underline{x}) = 6x_1^2 + 12x_2^2 + 4x_1x_2^4 + x_2^8,$$

which is positive definite and therefore a Lyapunov function.

The choice of $\dot{V}(\underline{x})$ which allows a solution to the partial differential equation is very critical. Furthermore, there is no guarantee that condition (C) is satisfied, if a suitable time derivative is found, i.e., if conditions (A) and (B) are met. Thus modifications to Zubov's procedure have been suggested.

George [14] describes the possibility of using an unknown scalar function, $u(x)$, for one component of $\nabla V (= \underline{g} + \underline{h})$ in (17). The other components of the gradient are expressed in terms of this scalar, using integral forms similar to those appearing in the development of (10). Instead of trying to solve the partial differential equation for V , he obtains an integral equation in u . For a solution of u , V follows by integration. A successive approximation technique is stated for the integral equation. Again this method can be viewed as a procedure whereby an attempt is made to select \hat{V} and calculate a gradient to satisfy conditions (A) and (B), and then check the result for condition (C).

V. Alternate Auxiliary Exact Differential Equations

Equation (2) arbitrarily defines \underline{g} as a linear combination of \underline{f} , such that $\langle \underline{g}, \underline{dx} \rangle = 0$ represents solution trajectories of $\dot{\underline{x}} = \underline{f}(\underline{x})$ in the phase space. Alternate definitions for \underline{g} could be made. The subsequent choice of \underline{h} , however, provides the freedom required to obtain a unique auxiliary exact differential equation for these various definitions. This is illustrated in the following.

$$\begin{aligned} \text{Consider} \quad \dot{x}_1 &= x_2 & &= f_1 \\ \dot{x}_2 &= -x_2 - x_1^3 & &= f_2 \end{aligned} .$$

$$\begin{aligned} \text{Choosing } g_i \text{ according to (2)} \quad g_1 &= x_2 + x_1^3 \\ g_2 &= x_2 \end{aligned}$$

$$\begin{aligned} \text{and } h_i \text{ as} \quad h_1 &= \frac{x_1}{2} - \frac{x_2}{2} \\ h_2 &= \frac{x_1}{2} \end{aligned} , \tag{18}$$

$$\dot{V} = \langle \underline{h}, \underline{f} \rangle = -\frac{1}{2}(x_2^2 + x_1^4)$$

$$\text{and} \quad V = \frac{1}{4}(x_1^2 + 2x_1x_2 + 2x_2^2 + x_1^4) . \tag{19}$$

$$\begin{aligned} \text{Alternately consider} \quad g_1 &= -f_1f_2 = x_2^2 + x_1^3x_2 \\ g_2 &= f_1^2 = x_2^2 \end{aligned} .$$

Then $\langle \underline{g}, \dot{\underline{x}} \rangle = 0$. For the same Lyapunov function given by (19), from equation (12), $\underline{h} = \nabla V - \underline{g}$:

$$h_1 = \frac{x_1}{2} + \frac{x_2}{2} + x_1^3 - x_2^2 - x_1^3 x_2$$

$$h_2 = \frac{x_1}{2} + x_2 - x_2^2.$$

It follows that V and \dot{V} are unchanged. The reader is cautioned to note that if an alternate positive definite scalar function were chosen in place of (19), an \underline{h} could be found to insure that the equation $\langle \underline{g}, \dot{\underline{x}} \rangle$ is exact. No guarantee is implied, however, that the time derivative, $\langle \underline{h}, \underline{\dot{f}} \rangle$, will be negative definite (or semi-definite).

In contrast, for a given definition of \underline{g} relative to equation (1) an alternate choice of \underline{h} will lead to an alternate Lyapunov function. For this same example and \underline{g} defined by (2), consider, instead of (18),

$$h_1 = 10x_1 = 9x_2 + 10x_1^3$$

$$h_2 = 10(x_1 + x_2).$$

Then $\dot{V} = \langle \underline{h}, \underline{\dot{f}} \rangle = -x_2^2 - 10x_1^4,$

and $V = 5x_1^2 + 10x_1x_2 + \frac{11}{2}x_2^2 + \frac{11}{4}x_1^4.$

VI. Discussion

A definitive procedure for developing Lyapunov functions has not been given. Rather the intuitive, experienced or random search methods that have characterized the subject are redirected from terms of a scalar function or gradient to components of a vector \underline{h} . The practical advantages of this alternative perspective would depend on the background of the analyst. A comparison of the seven methods that were described in Section IV, however, suggests that the methods which attempt a minimal search, i.e., those formulated around the unknown

vector \bar{h} instead of an unknown gradient or $V(x)$, would be most useful with a totally unfamiliar problem. Method (iv) shows, for example, that the well publicized variable gradient method may be less dependent on ingenuity if only the components of \bar{h} are sought. The practical difficulties experienced with Zubov's method are shown in (vii) to be the cost of circumventing what appear to be useful intermediate steps.

More important, however, the generation of Lyapunov functions from the viewpoint of auxiliary exact differential equations provides a unity to the disparate methods that have been proposed by many writers ever since Lyapunov's work was originally published. And as a consequence, they also form a conceptual base from which new procedures may more readily evolve.

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