Interim Report

CHEBYSHEV APPROXIMATIONS FOR THE STUMPFF SERIES OF ORDERS FOUR AND FIVE

Herbert E. Salzer

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Analytical Mechanics Associates, Inc.
Westbury, Long Island, N.Y.

Summary

This report gives the coefficients in the Chebyshev series for the four functions

$$f(x) = F_i(\pm x) = \frac{1}{i!} \mp \frac{x}{(i+2)!} + \frac{x^2}{(i+4)!} \mp - - - + \frac{x^{10}}{(i+20)!}$$

$$i = 4, 5 \qquad 0 \le x \le 1$$

and the two functions

$$f(x) = F_i(x) \qquad i = 4, 5 \qquad -1 \le x \le 1$$

With these coefficients, $f(x)$ can be found by a simple recurrence formula, without the need to calculate any Chebyshev polynomials. This form for $f(x)$ provides a single series which can be truncated at any term to meet varying needs in accuracy, and also avoids the considerably larger coefficients occurring in the explicit polynomial expressions for the approximations to $F_i(x)$.

We wish to approximate the Stumpff series given in [1], p. 6 and [2], p. 4, in the notation

$$F_i(\alpha) = \frac{1}{i!} - \frac{\alpha}{(i+2)!} + \frac{\alpha^2}{(i+4)!} - \frac{\alpha^3}{(i+6)!} + - - -$$

$$\equiv \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^k}{(i+2k)!} \tag{1}$$

(We here replace α by x for notational convenience.) In (1), $\alpha = x = \theta^2$ where θ may be real or purely imaginary.

The use of reduction formulas for $|\theta^2| > 1$ enables us to concentrate on $|\alpha| = |x| = |\theta^2| \leq 1$. These reduction formulas which are due to S. Pines are given for $F_4(x)$ and $F_5(x)$ in [1], p. 6, eq. (16) and for $F_6(x)$ and $F_7(x)$ in [2], p. 4, eq. (13). As an illustration, for F_4 and F_5, with which this present report is concerned, we have

$$F_4(x) = \frac{1}{8} [F_3(x/4) + F_1(x/4)\, F_3(x/4)] \tag{2}$$

and

$$F_5(x) = \frac{1}{16} [F_4(x/4) + \frac{1}{6} F_2(x/4) + F_0(x/4)\, F_5(x/4)] \tag{3}$$

The simple recursion formula

$$F_i(x) = \frac{1}{i!} - x F_{i+2}(x) \tag{4}$$

enables us to concentrate further on (1) for two conveniently located values of i, say i = 4 and 5. After discussion with some programmers, it appears that it might be helpful to have some way of approximating the series for $F_i(x)$ as far as the term $x^{10}/(i+20)!$ inclusive. Thus this present note will be concerned with

$$F_4(x) = \frac{1}{4!} - \frac{x}{6!} + \frac{x^2}{8!} - \cdots + \frac{x^{10}}{24!} \tag{5}$$

and

$$F_5(x) = \frac{1}{5!} - \frac{x}{7!} + \frac{x^2}{9!} - \cdots + \frac{x^{10}}{25!} \tag{6}$$

for $-1 \le x \le 1$. In (5) and (6) the same notation of $F_4(x)$ and $F_5(x)$ is employed for the tenth degree approximation as for the infinite series in (1). Comparison of (5) and (6) with the true values given by the infinite series shows the <u>relative</u> errors to be within approximately 0.6×10^{-25} for (5) and 1.1×10^{-26} for (6).

Three expansions will be derived. The first will be for $0 \le x \le 1$ only, corresponding to real θ, or the circular case. The second will be for $-1 \le x \le 0$ only, corresponding to imaginary θ, or the hyperbolic case. Letting $x = -x'$, $0 \le x' \le 1$, and then dropping the prime, we have $F_i(-x)$ and only + signs in the right members of (1), (4)-(6). The third case will be for $-1 \le x \le 1$, so that the identical approximation formulas for $F_i(x)$, $i = 4, 5$, will be used for positive or negative x. This third, or universal case requires less programming than the separate circular or hyperbolic cases, but in return for the doubled range in x, the series falls off less rapidly, the coefficients a_r in (8) below being around 2^r times the coefficients a_r in (7) below.

The approximations for all three cases will be left in the form of series of Chebyshev polynomials adjusted to the interval for x, without rearrangement of those series into the equivalent polynomials in x. The advantage will be threefold:

a) We avoid the larger coefficients that occur in the polynomial form.

b) We are able to see at a glance the error in stopping at any particular term of the Chebyshev series, so that a single expansion in terms of Chebyshev polynomials meets varying needs in accuracy.

c) The series itself, taken to any number of terms, is calculated directly by a simple recurrence scheme that bypasses the need for calculating the Chebyshev polynomials themselves (see [3], pp. 76-78).

For $0 \le x \le 1$ we express $f(x) = F_4(x)$, $F_5(x)$, $F_4(-x)$ or $F_5(-x)$ as a Chebyshev series in the form

$$f(x) = \frac{1}{2} a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + --- + a_n T_n^*(x) \qquad (7)$$

where $T_r^*(x) = \cos r\theta$, $\theta = \cos^{-1}(2x-1)$ and, of course, the coefficients a_r differ in each of these four cases. The index n is determined so that $|a_{n+1} T_{n+1}^*(x) + --- + a_{10} T_{10}^*(x)| \le |a_{n+1}| + --- + |a_{10}|$ (which in actual practice is just about $|a_{n+1}|$) is less than the desired truncation error.

For $-1 \le x \le 1$ we express $f(x) = F_4(x)$ or $F_5(x)$ as a Chebyshev series in the form

$$f(x) = \frac{1}{2} a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + --- + a_n T_n(x) \qquad (8)$$

where $T_r(x) = \cos r\theta$, $\theta = \cos^{-1} x$, and, as above, the a_r differ in these two cases and n is the stopping point for the desired approximation.

For (7) and (8) we let $b_{n+1} = b_{n+2} = 0$ and then find successively b_n, b_{n-1}, - - -, b_0 from

$$b_r = (4x-2)b_{r+1} - b_{r+2} + a_r \tag{9}$$

for (7), and from

$$b_r = 2xb_{r+1} - b_{r+2} + a_r \tag{10}$$

for (8). For both (7) and (8) we have

$$f(x) = \frac{1}{2}(b_0 - b_2) \tag{11}$$

To obtain (7) for $F_4(\pm x)$ and $F_5(\pm x)$ we replace x^r in (5) and (6), in terms of the Chebyshev polynomials $T_k^*(x)$, given by the formula

$$x^r = \frac{1}{2^{2r-1}} \left\{ \frac{1}{2}\binom{2r}{r} T_0^*(x) + \sum_{k=1}^{r} \binom{2r}{r-k} T_k^*(x) \right\} \tag{12}$$

after which the coefficients a_r are found by a direct calculation.

To obtain (8) for $F_4(x)$ and $F_5(x)$ we replace x^r in (5) and (6), in terms of the Chebyshev polynomials $T_k(x)$, given by the formulas

$$x^r = \frac{1}{2^{r-1}} \sum_{k=0}^{(r-1)/2} \binom{r}{k} T_{r-2k}(x) \tag{13a}$$

for r odd, and

$$x^r = \frac{1}{2^{r-1}} \left\{ \frac{1}{2} \binom{r}{r/2} T_0(x) + \sum_{k=0}^{(r/2)-1} \binom{r}{k} T_{r-2k}(x) \right\} \qquad (13b)$$

for r even, and proceed with a similar calculation for the coefficients a_r. Although for our present purposes $r \leq 10$, because of the possible further applications of (12), (13a) and (13b) to many other computational problems, we give the numerical values of the coefficients of both $T_k^*(x)$ and $T_k(x)$, up to $r = 12$, in the Appendix.

Following are the coefficients a_r, $r = 0, 1, ---, 10$, for (7) and (8). The user is reminded that the actual constant term in each series, namely $\frac{1}{2} a_0 T_0^*(x)$ and $\frac{1}{2} a_0 T_0(x)$, is half the number a_0 occurring in (9) and (10) for $r = 0$.

Table I: Coefficients for $f(x) = F_4(x)$, for (7), when $0 \leq x \leq 1$

r	a_r
0	0.08196 28745 37748 87356 66405 84
1	-0.00068 21719 17055 76698 55965 44
2	0.00000 30489 82441 27043 79448 46
3	-0.00000 00084 82185 58214 78661 10
4	0.00000 00000 16087 46174 06625 02
5	-0.00000 00000 00022 12548 05356 18
6	0.00000 00000 00000 02307 18615 75
7	-0.00000 00000 00000 00001 88667 51
8	0.00000 00000 00000 00000 00124 22
9	-0.00000 00000 00000 00000 00000 07
10	0.00000 00000 00000 00000 00000 00

Table II: Coefficients for $f(x) = F_5(x)$, for (7), when $0 \le x \le 1$

r	a_r
0	0.01647 03051 97075 80345 15031 486
1	-0.00009 78401 56475 20776 49071 272
2	0.00000 03398 04170 58306 91915 001
3	-0.00000 00007 72908 39503 69125 458
4	0.00000 00000 01239 76717 25031 307
5	-0.00000 00000 00001 47720 42685 158
6	0.00000 00000 00000 00135 88043 633
7	-0.00000 00000 00000 00000 09939 834
8	0.00000 00000 00000 00000 00005 920
9	-0.00000 00000 00000 00000 00000 003
10	0.00000 00000 00000 00000 00000 000

Table III: Coefficients for $f(x) = F_4(-x)$, for (7), when $0 \le x \le 1$

r	a_r
0	0.08474 09967 93308 59723 83476 42
1	0.00070 69753 31110 55728 26359 39
2	0.00000 31523 27765 34872 63273 09
3	0.00000 00087 43155 31301 41308 34
4	0.00000 00000 16535 55162 10400 77
5	0.00000 00000 00022 68558 95845 40
6	0.00000 00000 00000 02360 57322 66
7	0.00000 00000 00000 00001 92681 56
8	0.00000 00000 00000 00000 00126 66
9	0.00000 00000 00000 00000 00000 07
10	0.00000 00000 00000 00000 00000 00

Table IV: Coefficients for $f(x) = F_5(-x)$, for (7), when $0 \le x \le 1$

r	a_r
0	0.01686 71619 09789 45664 69213 580
1	0.00010 05960 28916 41442 62983 631
2	0.00000 03491 99069 69242 77799 145
3	0.00000 00007 92982 80386 21772 479
4	0.00000 00000 01269 63963 63803 429
5	0.00000 00000 00001 51015 17022 159
6	0.00000 00000 00000 00138 69027 193
7	0.00000 00000 00000 00000 10130 979
8	0.00000 00000 00000 00000 00006 027
9	0.00000 00000 00000 00000 00000 003
10	0.00000 00000 00000 00000 00000 000

Table V: Coefficients for $f(x) = F_4(x)$, for (8), when $-1 \le x \le 1$

r	a_r
0	0.08335 81364 86421 56668 31392 69
1	-0.00138 90955 75952 37019 96258 59
2	0.00001 24018 37511 04702 40212 23
3	-0.00000 00688 96882 71921 48319 62
4	0.00000 00002 60968 42395 82630 39
5	-0.00000 00000 00716 93868 34908 41
6	0.00000 00000 00001 49361 23559 98
7	-0.00000 00000 00000 00244 05323 67
8	0.00000 00000 00000 00000 32112 17
9	-0.00000 00000 00000 00000 00034 75
10	0.00000 00000 00000 00000 00000 03

Table VI: Coefficients for $f(x) = F_5(x)$, for (8), when $-1 \leq x \leq 1$

r	a_r
0	0.01666 94225 19033 65120 33819 817
1	-0.00019 84314 87971 93972 29051 717
2	0.00000 13779 46257 73635 77918 635
3	-0.00000 00062 63266 07292 44576 958
4	0.00000 00000 20074 33194 85438 040
5	-0.00000 00000 00047 79567 24612 936
6	0.00000 00000 00000 08785 92625 128
7	-0.00000 00000 00000 00012 84487 856
8	0.00000 00000 00000 00000 01529 149
9	-0.00000 00000 00000 00000 00001 511
10	0.00000 00000 00000 00000 00000 001

To determine at a glance the relative error in using any of these tables in connection with (9) or (10) and (11), starting with a_n and neglecting all terms beyond a_n, simply look at the ratio $a_{n+1}/(1/2\ a_o)$, which is always less than $a_{n+1}/0.04$ or $a_{n+1}/0.008$ for any of the F_4 or F_5 series respectively.

To see the improvement in the number of required terms for any desired accuracy, we also may glance at the following schedule of the upper bounds for the relative errors, say e_n, in using the uneconomized series (5) or (6) for F_4 or F_5 respectively, through the terms in x^n:

n =	0	1	2	3	4	5
e_n for F_4	3.3(-2)	6.0(-4)	6.6(-6)	5.0(-8)	2.8(-10)	1.1(-12)
e_n for F_5	2.4(-2)	3.3(-4)	3.0(-6)	1.9(-8)	9.2(-11)	3.4(-13)

$n =$	6	7	8	9	10
e_n for F_4	3.7(-15)	9.9(-18)	2.1(-20)	3.9(-23)	6.0(-26)
e_n for F_5	9.9(-16)	2.3(-18)	4.6(-21)	7.7(-24)	1.1(-26)

This schedule is based upon the worst choice of $|x| = 1$, so that whenever the largest value of $|x|$ does not exceed some $\beta < 1$, the e_n may be improved to $\beta^{n+1} e_n$. There is no such corresponding advantage in the use of the economized formulas for the smaller values of $|x|$, since they are designed primarily to minimize the maximal error over the entire range of x.

APPENDIX

Powers of x in Terms of Chebyshev Polynomials

A. x^r in Terms of $T_k^* \equiv T_k^*(x)$:

$$1 = T_0^*$$

$$x = \frac{1}{2}(T_0^* + T_1^*)$$

$$x^2 = \frac{1}{8}(3\,T_0^* + 4\,T_1^* + T_2^*)$$

$$x^3 = \frac{1}{32}(10\,T_0^* + 15\,T_1^* + 6\,T_2^* + T_3^*)$$

$$x^4 = \frac{1}{128}(35\,T_0^* + 56\,T_1^* + 28\,T_2^* + 8\,T_3^* + T_4^*)$$

$$x^5 = \frac{1}{512}(126\,T_0^* + 210\,T_1^* + 120\,T_2^* + 45\,T_3^* + 10\,T_4^* + T_5^*)$$

$$x^6 = \frac{1}{2048}(462\,T_0^* + 792\,T_1^* + 495\,T_2^* + 220\,T_3^* + 66\,T_4^* + 12\,T_5^* + T_6^*)$$

$$x^7 = \frac{1}{8192}(1716\,T_0^* + 3003\,T_1^* + 2002\,T_2^* + 1001\,T_3^* + 364\,T_4^* + 91\,T_5^* + 14\,T_6^* + T_7^*)$$

$$x^8 = \frac{1}{32768}(6435\,T_0^* + 11440\,T_1^* + 8008\,T_2^* + 4368\,T_3^* + 1820\,T_4^* + 560\,T_5^* + 120\,T_6^* + 16\,T_7^* + T_8^*)$$

$$x^9 = \frac{1}{131072}(24310\,T_0^* + 43758\,T_1^* + 31824\,T_2^* + 18564\,T_3^* + 8568\,T_4^* + 3060\,T_5^* + 816\,T_6^* + 153\,T_7^* + 18\,T_8^* + T_9^*)$$

$$x^{10} = \frac{1}{524288}(92378\,T_0^* + 167960\,T_1^* + 125970\,T_2^* + 77520\,T_3^* + 38760\,T_4^* + 15504\,T_5^* + 4845\,T_6^* + 1140\,T_7^* + 190\,T_8^* + 20\,T_9^* + T_{10}^*)$$

$$x^{11} = \frac{1}{2097152}(352716\,T_0^* + 646646\,T_1^* + 497420\,T_2^* + 319770\,T_3^* + 170544\,T_4^* + 74613\,T_5^* + 26334\,T_6^* + 7315\,T_7^* + 1540\,T_8^* + 231\,T_9^* + 22\,T_{10}^* + T_{11}^*)$$

$$x^{12} = \frac{1}{8388608}(1352078\,T_0^* + 2496144\,T_1^* + 1961256\,T_2^* + 1307504\,T_3^* + 735471\,T_4^* + 346104\,T_5^* + 134596\,T_6^* + 42504\,T_7^* + 10626\,T_8^* + 2024\,T_9^* + 276\,T_{10}^* + 24\,T_{11}^* + T_{12}^*)$$

B. x^r in Terms of $T_k \equiv T_k(x)$:

$$1 = T_0$$

$$x = T_1$$

$$x^2 = \frac{1}{2}(T_0 + T_2)$$

$$x^3 = \frac{1}{4}(3T_1 + T_3)$$

$$x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4)$$

$$x^5 = \frac{1}{16}(10T_1 + 5T_3 + T_5)$$

$$x^6 = \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6)$$

$$x^7 = \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7)$$

$$x^8 = \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8)$$

$$x^9 = \frac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9)$$

$$x^{10} = \frac{1}{512}(126T_0 + 210T_2 + 120T_4 + 45T_6 + 10T_8 + T_{10})$$

$$x^{11} = \frac{1}{1024}(462T_1 + 330T_3 + 165T_5 + 55T_7 + 11T_9 + T_{11})$$

$$x^{12} = \frac{1}{2048}(462T_0 + 792T_2 + 495T_4 + 220T_6 + 66T_8 + 12T_{10} + T_{12})$$

References

[1] Pines, S.; "Mean Conic State Transition Matrix," unpublished memorandum to Philco Corp., Western Development Lab., Palo Alto, Calif., Order No. WDL-C-2431, November 1965.

[2] Pines, S. and Fang, T.C.; "A Uniform Solution of the Euler-Lagrange Equations During Coast for the Central Force Field," unpublished report of Analytical Mechanics Associates, Inc., November 1965.

[3] Anon.; Modern Computing Methods, published by Mathematics Division of the National Physical Laboratory, 2nd ed., Philosophical Library, Inc., New York, 1961.