STATUS REPORT

to

NATIONAL AERONAUTICS

and

SPACE ADMINISTRATION

Statistical Methods and Subjective Probability
Grant NGR 37-002-031

Leroy Folks: Principal Investigator

I. Introduction

Research was begun on this project in February, 1966 by the principal investigator and two part-time research assistants with the principal investigator devoting quarter time to this research. During June and July of 1966, the principal investigator devoted his full time to the project. Since August 1, 1966 the research has been carried on by two part-time research assistants.

II. Results

In the original proposal for this research grant, it was proposed "to make use of existing research in decision theory, subjective probability, Bayesian inference, and related areas to attempt to devise practical workable methods which use in a formal way the prior information." This attempt has resulted in the following:

1. Bibliography - A rather extensive review of the literature has been made and a bibliography resulting from this review is presented in appendix 1. This bibliography is being submitted to a professional

journal for publication. It is planned to supplement the bibliography as new articles are published or as other articles come to the attention of the investigators.

2. Contributions to Textbooks - The attempt to develop new methods which can be used in "cookbook" fashion has had a profound effect upon the attitude of the principal investigator toward the standard, or classical, statistical methods. Although these methods are generally presented in classrooms as decision making devices (for example, deciding whether to accept the null hypothesis or the alternative hypothesis, or deciding what number to use as an estimate for the population mean), the principal investigator is convinced that these same statistical methods are used in the majority of cases not as decision making tools but in a much more subjective way as devices for the description and presentation of data. The presentation of statistics from this point of view will appear in a forthcoming (early 1968) text book, co-authored by the principal investigator on probability and statistical inference.

3. Regression Analysis - The chief effort of the investigators has been in the area of regression analysis. The major results and / or problems which have been studied are as follows:

a. Variance of posterior marginal distribution - It was noted in a previous report that in a multiparameter model that a posterior marginal distribution may have larger variance than the corresponding prior marginal, even though the data appear to support quite strongly the prior marginal. This problem has been studied for the general linear regression model and conditions determined such that the posterior marginal will have smaller variance than the prior marginal (appendix 2).

b. Prior Distribution for Positive Slope - A problem frequently encountered in fitting straight lines to data is that the usual least squares fit does not use the prior knowledge that the slope must be non-negative (as in growth problems, for example). A natural way of dealing with such knowledge is to use a prior distribution with all mass on the positive axis. For the simple linear model: $Y_i = \beta X_i + e_i$, $e_i \sim N(0, \sigma^2)$, a number of results have been obtained by using one and two parameter gamma distributions for β (appendices 3 and 4).

For the simple linear model $Y_i = \alpha + \beta X_i + e_i$, $e_i \sim N(0, \sigma^2)$, results have been obtained by using a negative exponential prior distribution for β with a uniform distribution for α (appendix 5).

c. Prior Distribution for Slope - Known to be in an Interval - In situations where straight lines are actually fitted to data, it is usually the case that a linear model is used because of a background of knowledge which indicates not only that the linear model is appropriate but also a range of likely values for β. The investigators have studied using a prior distribution with mass concentrated in an interval as compared with the usual classical estimates (appendix 6).

4. Parameter Estimation - Rectangular - Motivated by the problem described in 3c, the investigators have used a Bayes estimator for the parameter of a uniform distribution with classical estimators subject to the restriction that the estimate lie in the same range of values where the mass of the prior distribution is concentrated (appendix 7).

5. Testing Hypotheses - Prior Distribution on the Number of Significant Effects.

a. Single Test - For a single test of size α with power p, the posterior distribution of the number of non-zero effects (0 or 1) given the results of the test is obtained when the prior probability of a non-zero effect is m (appendix 8).

b. 2 x 2 Table - Preliminary results have been obtained for the posterior distribution of the number of non-zero effects in a 2 x 2 table given the number of effects judged significant (appendix 9).

III. Further Efforts

The appendices attached to this report should be viewed as preliminary reports inasmuch as additional work is either in progress or is planned for these topics.

APPENDIX I

Bibliography of Subjective Probability and Bayesian Procedures

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APPENDIX 2

Variance of Posterior Marginal in Regression Analysis

Assume $Y = X\beta + \epsilon$, $f_N(\epsilon \mid 0,h) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}h\epsilon^2} h^{\frac{1}{2}}$, with predetermined X of rank p, sample size k. $n = X^tX$ is thus predetermined. Assume a joint prior distribution on $(\tilde{\beta},\tilde{h})$ of Normal-Gamma with parameters (b',v',n',ν'). Since n is predetermined, let $n' = n$. Then rank $n' = p' = p$ and $\nu' = \nu = k - p$.

Results

The joint posterior distribution of $(\tilde{\beta},\tilde{h})$ is Normal-Gamma with parameters

$$n'' = n' + n = 2n, \quad n''^{-1} = \tfrac{1}{2} n^{-1},$$

$$b'' = n''^{-1} (n'b' + nb) = \tfrac{1}{2}(b' + b), \text{ rank } n'' = p'' = p ,$$

$$\nu'' = \nu' + p' + \nu + p - p'' = 2k - p ,$$

$$v'' = \frac{1}{\nu''} [(\nu'v' + b'^t nb') + \nu v + b^t nb - b''^t n''b'']$$

$$= \frac{k - p}{2k - p} (v' + v) + \frac{(b - b')^t n(b - b')}{2(2k - p)} .$$

The posterior marginal of $\tilde{\beta}$ is Student with

$$E'' (\tilde{\beta}| b'',v'',\nu'',n'') = b'' = \frac{b' + b}{2} ,$$

$$V'' (\tilde{\beta}| b'',v'',\nu'',n'') = \frac{n''^{-1}v''\nu''}{\nu'' - 2} = \frac{n^{-1}v''\nu''}{2(\nu''-2)} .$$

The prior marginal of $\tilde{\beta}$ is Student with

$$E'(\tilde{\beta}| b',v',\nu'n') = b' ,$$

$$V'(\tilde{\beta}| b',v',\nu',n') = \frac{n'^{-1}v'\nu'}{\nu'-2} = \frac{n^{-1}v'\nu'}{\nu'-2} .$$

For the posterior estimate b" to be better than the prior estimate b' we would want the corresponding diagonal elements of V" less than the corresponding diagonal elements of V'. That is,

$$\frac{v''\nu''}{2(\nu''-2)} < \frac{v'\nu'}{\nu'-2} \text{ , which implies}$$

$$v < \frac{3k - p - 2}{k - p - 2} v' - \frac{(b - b')^t n(b - b')}{2(k - p)} .$$

Now $g(k) = \frac{3k - p - 2}{k - p - 2}$ is decreasing and greater than three for $k - p > 2$. Thus, if it results that $v < 3v'$, this implies $v < \frac{3k - p - 2}{k - p - 2} v'$ for $k - p > 2$. It would then seem that if the prior estimates b' and v' are based on "good" prior information $v < 3v'$ would readily follow and hence $v < \frac{3k - p - 2}{k - p - 2} v'$. Similarly, for a "good" prior estimate b' and (or) k sufficiently large $v < \frac{3k - p - 2}{k - p - 2} v' - \frac{(b - b')^t n(b - b')}{2(k - p)}$ would be expected to be satisfied. In the above sense, the posterior mean b" should be a more precise estimate of β than the prior mean b'.

Special cases: p = 2 in previous model.

(a) If in addition to the previous assumptions we take b = b' then the posterior estimate of β is b" = b' and this requires only $v < \frac{3k - 4}{k - 4} v'$ for b" to be more precise than b'.

(b) If in addition to the previous assumptions we take $b = b' = \binom{b_0}{b_1}$ and v = v' we obtain $s_{b_0}^2 = s_E^2 \left[\frac{1}{k} + \frac{\bar{X}^2}{\Sigma x^2} \right]$ and $s_{b_1}^2 = \frac{s_E^2}{\Sigma x^2}$ which are the standard estimates generally used in elementary statistical methods courses.

APPENDIX 3

Gamma Prior Distribution for Regression Through the Origin

Consider the model $y_i = X_i\beta + e_i$; with each e_i independently normally distributed with mean 0 and variance σ^2. Suppose we have the prior density on β of $f(\beta) = \frac{\beta^{\nu-1}e^{-\beta}}{\Gamma(\nu)}$; $0 < \beta < \infty$ and σ^2 is assumed to be known. Also ν is specified.

The likelihood function is given by

$$L(y \mid \beta,\sigma^2) = (2\pi)^{-\frac{1}{2}n}\sigma^{-n} \exp\left\{\frac{-(Y - X\beta)'(Y - X\beta)}{2\sigma^2}\right\} .$$

The joint posterior is

$$P(y,\beta \mid \sigma^2,\nu) = \frac{(2\pi)^{-\frac{1}{2}n}\sigma^{-n}\beta^{\nu-1}}{\Gamma(\nu)} \exp\left\{\frac{-(Y - X\hat{\beta})'(Y - X\hat{\beta})}{2\sigma^2} - (\beta - \hat{\beta})^2 \frac{X'X}{2\sigma^2} - \beta\right\},$$

where $\hat{\beta} = (X'X)^{-1}X'Y$.

It then follows that the posterior of β is

$$P(\beta \mid y,\sigma^2,\hat{\beta},\nu) = \frac{1}{K}\beta^{\nu-1}\exp \frac{-X'X}{2\sigma^2} \left[\beta - (\hat{\beta} - \frac{\sigma^2}{X'X})\right]^2$$

$$= \frac{1}{K}\beta^{\nu-1}\exp \frac{-X'X}{2\sigma^2}(\beta - \tilde{\beta})^2 \quad \text{where } \tilde{\beta} = \hat{\beta} - \frac{\sigma^2}{X'X} .$$

Comments:

1. We attempted to determine the characteristic function associated with the posterior distribution of β for the given prior and were confronted with the incomplete gamma function in the form

$$C(\beta) = \frac{1}{k} e^{-it\tilde{\beta} - \frac{\sigma^2}{2X'X}t^2} \int_0^\infty \beta^{\nu-1}\exp\left\{\frac{-X'X}{2\sigma^2}\left[\beta - (\tilde{\beta} - \frac{i\sigma^2 t}{X'X})\right]^2\right\} d\beta .$$

2. We reduced the form of the density $P(\beta|y,\sigma^2,\hat{\beta},\nu)$ to

$$\frac{1}{K}\beta^{\nu-1}e^{-a(\beta-b)^2}, \text{ where } a = \frac{X'X}{2\sigma^2}, \quad b = \hat{\beta} - \frac{\sigma^2}{X'X},$$

and attempted to determine the form of K, which was dependent upon whether b was positive or negative. We obtained

$$K = \begin{cases} \displaystyle\sum_{\alpha=0}^{\nu-1} \binom{\nu-1}{\alpha} b^{\nu-1-\alpha} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{2a^{\frac{\alpha+1}{2}}} \left[1 - F_{\frac{\alpha+1}{2}}(ab^2)\right] & \text{if } b<0, \\ \displaystyle\sum_{\alpha=0}^{\nu-1} \binom{\nu-1}{\alpha} b^{\nu-1-\alpha} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{2a^{\frac{\alpha+1}{2}}} \left[1 + (-1)^{\alpha} F_{\frac{\alpha+1}{2}}(ab^2)\right] & \text{if } b>0, \end{cases}$$

where $F_{\delta}(x) = \int_0^x \frac{u^{\delta-1}}{\Gamma(\delta)} e^{-u}\, du$.

3. We considered the posterior expectation of β and obtained

$$E''(\beta|y) = \frac{1}{K}\sum_{\alpha=0}^{\nu} \binom{\nu}{\alpha} b^{\nu-\alpha} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{2a^{\frac{\alpha+1}{2}}} \left[1 - F_{\frac{\alpha+1}{2}}(ab^2)\right].$$

4. The posterior variance obtained was

$$V''(\beta|y) = \frac{1}{2a} + \frac{(\nu-1)(\nu-2)}{(2a)^2 I}\int_{-b}^{\infty} (u+b)^{\nu-3} e^{-au^2} du - \left\{\frac{\nu-1}{2aI}\int_{-b}^{\infty} (u+b)^{\nu-2} e^{-au^2} du\right\}^2.$$

APPENDIX 4

Two Parameter Gamma Prior Distribution for Regression Through the Origin

We consider the same problem as in Appendix 3 but with a two parameter gamma prior of the form

$$f(\beta) = \frac{1}{\Gamma(\nu)} \beta^{\nu-1} c^{\nu} e^{-c\beta} .$$

Conclusions:

1. The posterior density of β is

$$f(\beta | y) = \frac{1}{I_\nu} \beta^{\nu-1} e^{-a(\beta - b)^2} d\beta ,$$

where $b = \ddot{\beta} - \frac{c\sigma^2}{X'X}$, $a = \frac{X'X}{2\sigma^2}$, and $I_\nu = \int_o^\infty \beta^{\nu-1} e^{-a(\beta - b)^2} d\beta$.

2. The posterior mean of β is given by

$$E''(\beta | y) = \frac{I_{\nu+1}}{I_\nu} \quad \text{where } I_\nu = \int_o^\infty \beta^{\nu-1} e^{-a(\beta - b)^2} d\beta .$$

3. The posterior variance of β is given by

$$V''(\beta | y) = \frac{I_\nu I_{\nu+2} - I_{\nu+1}^2}{I_\nu^2} .$$

4. We established a recurrence relation for the incomplete gamma integrals which define the posterior mean and variance in (2) and (3) above. The relation is:

$$I_{\nu+1} = \frac{\nu - 1}{2a} I_{\nu-1} + bI_\nu .$$

5. The posterior mode of β is given by

$$\beta m = \frac{b}{2} + \frac{\sqrt{a^2 b^2 - 2a(\nu - 1)}}{2a} .$$

APPENDIX 5

Simple Linear Regression with Prior Distributions on Three Parameters

Consider the model $y_i = \alpha + \beta x_i + e_i$, $e_i \sim n(0,\sigma^2)$, with prior densities

$$f(\beta) = c\ e^{-c\beta}\ ,\quad 0 < \beta < \infty\ ,$$

$$h(\alpha) = \frac{1}{\alpha_2-\alpha_1}\ ,\quad \alpha_1 \le \alpha \le \alpha_2\ ,$$

$$g(\sigma) = \frac{1}{\log(\sigma_2/\sigma_1)}\ \frac{1}{\sigma}\ ,\quad e^{\sigma_1} \le \sigma \le e^{\sigma_2}\ .$$

Conclusions:

1. The joint posterior distribution is given by

$$f''(\beta,\alpha,\sigma|\,y) = K\sigma^{-(n+1)}\exp\frac{-1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \alpha - x_i\beta)^2 + 2c\sigma^2\beta\right]$$

where $K^{-1} = \int_{o}^{\infty}\int_{\sigma_1}^{\sigma_2}\int_{\alpha_1}^{\alpha_2} f(\beta,\alpha,\sigma|\,y)\ d\alpha\ d\sigma\ d\beta$.

2. The posterior marginal density on β is

$$f''(\beta\,|y) \propto \frac{e^{-c\beta}}{[L + (\beta - \hat{\beta})^2]^{\frac{1}{2}n + 1}} \quad \text{where } L = \sum_{i=1}^{n}\frac{[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})\,]^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

APPENDIX 6

Comparison of Estimators of Slope for Regression Through the Origin

Suppose we have the model $y_i = \beta X_i + e_i$, where the e_i are independently normally distributed with mean 0 and variance 1. Also it is believed that β lies between a and b, with $a < b$.

a. We have various alternatives:

  (i) Set a prior distribution on β concentrated between a and b and obtain the Bayes estimate for β.

  (ii) Use the least squares estimator of β subject to the restriction $a \leq \tilde{\beta}_{LS} \leq b$.

b. Conjecture: method (i) results in a smaller mean square error than method (ii) regardless of the prior we pick for $\beta \epsilon [a,b]$.

c. Results:

  (i) Consider a truncated normal prior density on β,

$$D'(\beta | t, \nu^2) = k(2\pi\nu^2)^{-1/2} \exp - \frac{1}{2\nu^2}(\beta - t)^2, \quad a \leq \beta \leq b, \quad t = \frac{a+b}{2},$$

$\nu > 0$ a constant, and $k^{-1} = \int_a^b D'(\beta | t, \nu^2)\, d\beta$.

  (1) The posterior distribution for β is given by

$$D''(\beta | Y) = k_2(2\pi\gamma^2)^{-1/2} \exp - \frac{1}{2\gamma^2}(\beta - \mu'')^2, \quad a \leq \beta \leq b$$

where $k_2^{-1} = \int_a^b (2\pi\gamma^2)^{-1/2} \exp - \frac{1}{2\gamma^2}(\beta - \mu'')^2\, d\beta$

$$\mu'' = (r^2 t + \nu^2 \hat{\beta})/(r^2 + \nu^2)$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$r^2 = (X'X)^{-1}$$

$$\gamma^2 = \upsilon^2 r^2/(\upsilon^2 + r^2)$$

(2) The mean of the posterior distribution is given by

$$E''(\beta|Y) = \frac{(2\pi)^{1/2}\upsilon r \quad \exp - \frac{1}{2}(c_1 d_a - c_2)^2 - \exp - \frac{1}{2}(c_1 d_b - c_2)^2}{(r^2 + \upsilon^2)^{\frac{1}{2}} \quad N(c_1 d_b + c_2| 0,1) - N(-(c_1 d_a + c_2)| 0,1))}$$

$$+ \frac{r^2(a + b) + 2\upsilon^2\hat{\beta}}{2(\upsilon^2 + r^2)}$$

Where $c_1 = \upsilon / r \ (r^2 + \upsilon^2)^{1/2}$,

$c_2 = r(b - a)/ 2\upsilon \ (\upsilon^2 + r^2)^{1/2}$,

$d_a = \ddot{\beta} - a$,

$d_b = b - \tilde{\beta}$, and

$N(\cdot \mid 0,1)$ denotes the cumulative normal distribution with mean 0 and variance 1.

(ii) If we attack the same problem using least squares subject to the constraint $a \leq \hat{\beta}_{LS} \leq b$ we obtain:

(1)

$$\ddot{\beta}_{LS} = \begin{cases} a & \text{if } (X'X)^{-1}X'Y < a \\ (X'X)^{-1}X'Y & \text{if } a \leq (X'X)^{-1}X'Y \leq b \\ b & \text{if } (X'X)^{-1}X'Y > b \end{cases}$$

(2)

$$E(\ddot{\beta}_{LS}) = \beta + (b - \beta)[1 - N((b - \beta)/\gamma| 0,1)]$$
$$- (\beta - a) N((a - \beta)/\gamma \mid 0,1)$$
$$+ (2\pi)^{-1/2}[\exp - \frac{1}{2\gamma^2}(\beta-a)^2 - \exp - \frac{1}{2\gamma^2}$$
$$(b - \beta)^2] .$$

APPENDIX 7

Comparison of Bayes Estimator with Classical Estimator

The variable x has density $f(x|\theta) = 1/\theta$, $0 \le x \le \theta$. Suppose we use the prior density on θ of $D'(\theta)=1/(b - a)$, $a \le \theta \le b$. If we take a sample of size n from $f(x|\theta)$, $y = x_{max}$ is a complete and sufficient statistic for θ and the density of y for given θ is

$$f(y|\theta) = \frac{ny^{n-1}}{\theta^n} \quad , \; 0 \le y \le \theta \quad .$$

(A) We desire to compare the Bayes estimator for θ with estimators for θ obtained by other methods.

Results:

(1)
$$D''(\theta|y) = \begin{cases} \dfrac{(n-1)(ab)^{n-1}}{\theta^n(b^{n-1} - a^{n-1})} \, , & a \le \theta \le b, \; 0 < y \le a \\[2ex] \dfrac{(n-1)(yb)^{n-1}}{\theta^n(b^{n-1} - y^{n-1})} \, , & y \le \theta \le b, \; a < y \le b \end{cases}$$

is the posterior density of θ.

(2) The posterior mean of θ is given by

$$E''(\theta|y) = \begin{cases} \dfrac{n-1}{n-2} \; \dfrac{ab(b^{n-2} - a^{n-2})}{b^{n-1} - a^{n-1}} & \text{if } 0 \le y \le a \\[2ex] \dfrac{n-1}{n-2} \; \dfrac{by(b^{n-2} - y^{n-2})}{b^{n-1} - y^{n-1}} & \text{if } a < y \le b \, . \end{cases}$$

Note: $\lim_{n\to\infty} E''(\theta|y) = \begin{cases} a & \text{if } 0 \le y \le a \\ y & \text{if } a < y \le b \end{cases}$.

(3) $\underset{Y}{E}\,[E''(\theta|\,y)] = (a + b)/2$ where $\underset{Y}{E}$ indicates the expected value taken with respect to the marginal density of $y = x_{max}$.

(4) $M.S.E.[E''(\theta|\,y)] = \underset{Y}{E}[E''(\theta|\,y)]^2 - (a + b)\theta + \theta^2$

$$\underset{y}{E}[E''(\theta|\,y)]^2 = \frac{n-1}{(n-2)^2(b-a)\,b^{n-3}}\;\frac{(b^{n-2} - a^{n-2})^2 a^3}{b^{n-1} - a^{n-1}}$$

$$+ n\int_a^b \frac{(b^{n-2} - y^{n-2})^2 y^2}{b^{n-1} - y^{n-1}}\,dy \; .$$

(B) Now we try a "best" classical estimator $\hat{\theta}$ for θ subject to the restriction $a \le \theta \le b$.

Let $y = x_{max}$ as before and consider the estimator of θ, n fixed, defined by

$$\hat{\theta} = \begin{cases} \dfrac{n}{n+1}\,a & \text{if } y < \dfrac{n}{n+1}\,a\ , \\[2ex] \dfrac{n+1}{n}\,y & \text{if } \dfrac{n}{n+1}\,a < y < \dfrac{n}{n+1}\,b\ . \end{cases}$$

Results:

(1) $E\,[\theta] = \theta$, So θ is unbiased.

(2) $V\,(\theta) = \dfrac{1}{n(n+2)} \big[\, \theta^2 - \dfrac{n}{n+1}^{\,n+2}\; \dfrac{a}{\theta}^{\,n}\, a^2$.

APPENDIX 8

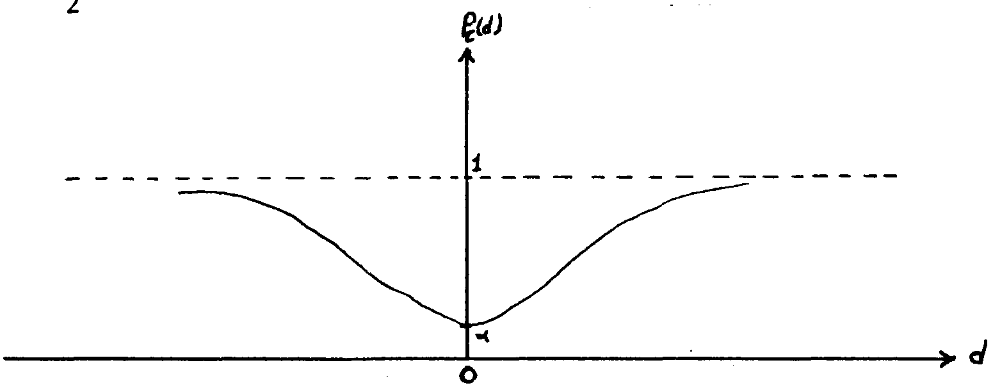
Hypothesis Testing in a Bayesian Setting

Consider the very simple situation for the model $y_{ij} = \mu + \alpha_i + e_{ij}$, $i = 1,2$; $j = 1,2, \cdots, n$; with e_{ij} distributed normally independently with mean 0 and variance σ^2. Then consider the testing of

$$\text{Hypothesis: } \alpha_1 - \alpha_2 = 0$$

$$\text{against the Alternative: } \alpha_1 - \alpha_2 \neq 0.$$

Student's t-test has certain nice properties in this setting and has a power function of the form shown in Fig. 1 below, as a function of $d = \alpha_1 - \alpha_2$.



Now suppose we have prior information of θ = the number of significant effects (i.e. $\theta = 0$ or 1) and are willing to assume the prior distribution

(1) $D'(\theta) = m^{\theta} (1 - m)^{1 - \theta}$.

If we let k = the number of effects we declare significant from our t-test, with significance level α, then the distribution of k for given θ is

(2) $f(k|\theta = 0) = \alpha^k (1 - \alpha)^{1 - k}$, $k = 0,1$

(3) $f(k|\theta = 1) = p^k (1 - p)^{1-k}$, where p is the value of the power function for each value of d. Combining (2) and (3) we may write

(4) $f(k|\theta) = [p^k(1 - p)^{1 - k}]^{\theta}[\alpha^k(1 - \alpha)^{1 - k}]^{1 - \theta}$. Then on combining (1) and (4) we obtain

(5) $D''(\theta,k) = [mp^k(1 - p)^{1 - k}]^{\theta}[(1 - m)\alpha^k(1 - \alpha)^{1 - k}]^{1 - \theta}$ which leads to

(6) $$D''(\theta|k) = \frac{[mp^k(1 - p)^{1 - k}]^{\theta}[(1 - m)\alpha^k(1 - \alpha)^{1 - k}]^{1 - \theta}}{mp^k(1 - p)^{1 - k} + (1 - m)\alpha^k(1 - \alpha)^{1 - k}}$$

Now for fixed values of θ and k various surfaces are generated by letting m and p take values $0 \le m \le 1$ and $\alpha \le p \le 1$. One such surface is indicated in figure 3. It is also mildly interesting to see the curves generated by varying m for fixed α and p, and those curves generated by varying p for fixed m and α. These curves are indicated in figs. 2a and 2b below respectively.

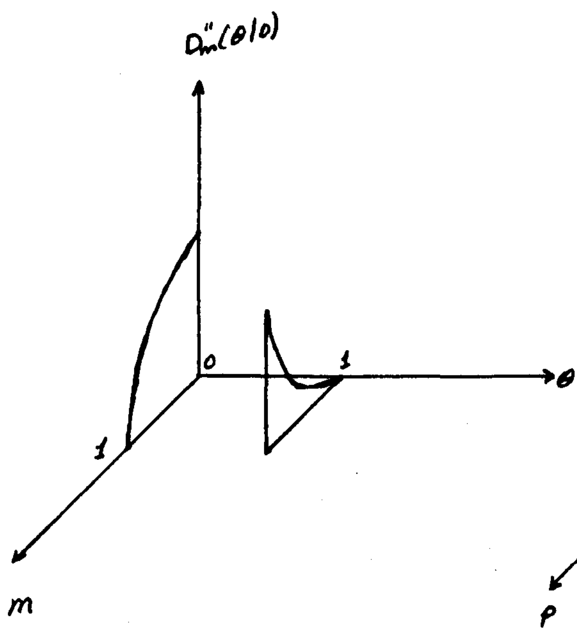


Fig. 2a

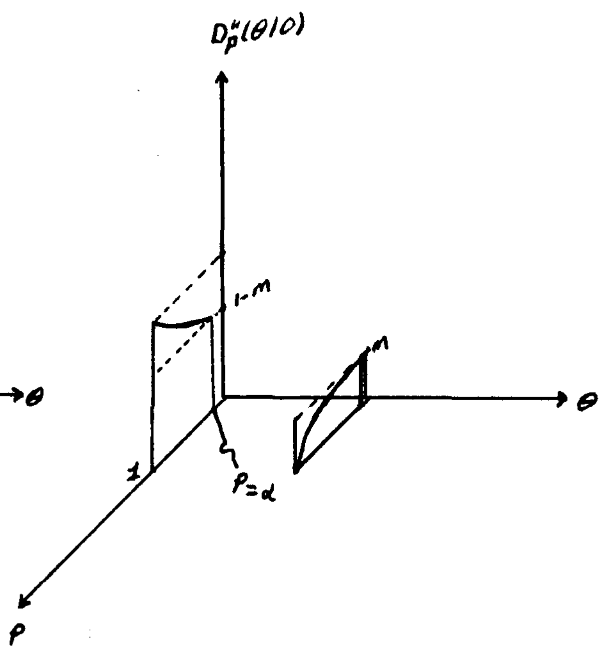


Fig. 2b

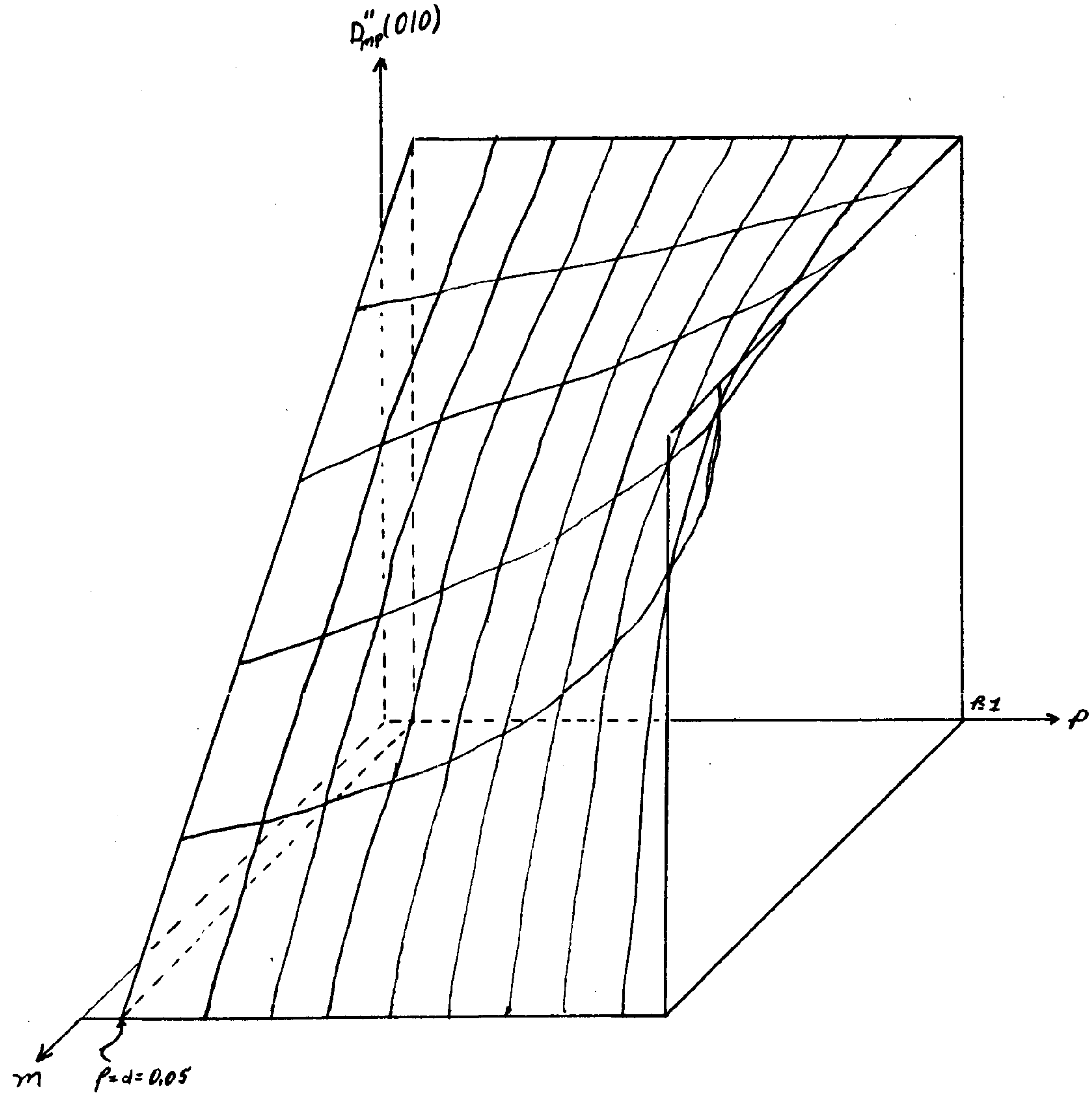


Figure 3

The surface generated by the posterior probability of "no effect" given that we accept the hypothesis at $\alpha = 0.05$ for the range of possible values of m, our prior probability of an "effect", and the possible values of the power function p.

APPENDIX 9

A Tabular Posterior Distribution

k	$\theta = 0$	$\theta = 1$	$\theta = 2$	$\theta = 3$
0	$(1-m)^3(1-\alpha)^3$	$3m(1-m)^2(1-\alpha)^2(1-p)$	$3m^2(1-m)(1-\alpha)$	$m^3(1-p)^3$
1	$3(1-m)^3\alpha(1-\alpha)^2$	$3m(1-m)^2[p(1-\alpha)^2 + 2\alpha(1-\alpha)(1-p)]$	$3m^2(1-m)[\alpha(1-p)^2 + 2p(1-p)(1-\alpha)]$	$3m^3p(1-p)^2$
2	$3(1-m)^3\alpha^2(1-\alpha)$	$3m(1-m)^2[\alpha^2(1-p) + 2\alpha(1-\alpha)p]$	$3m^2(1-m)[p^2(1-\alpha) + 2p(1-p)\alpha]$	$3m^3p^2(1-p)$
3	$(1-m)^3\alpha^3$	$3m(1-m)^2\alpha^2p$	$3m^2(1-m)\ \alpha p^2$	m^3p^3

The table above is the joint posterior distribution of ϑ = the number of significant effects and k = the number of effects we declare significant from a χ^2 test. The table was obtained from considerations similar to those for appendix 8, however the model here was $y_{ijh} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijh}$, $i,j = 1,2;$ $h = 1, \cdots, n$ with e_{ijh} distributed normally independently with mean 0 and known variance σ^2. Furthermore we tacitly assume the value of the power function for the χ^2 test is the same for each difference. Also the prior distribution becomes $D'(\theta) = \binom{3}{\theta}\ m^{\theta}\ (1 - m)^{3-\theta}$.