A MATHEMATICAL INVESTIGATION OF A HEAT TRANSFER CONFIGURATION

By

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B.S. in Mathematics and Physics,
University of North Dakota 1965

NGR-35-002-003

A Thesis
Submitted to the Faculty
of the
University of North Dakota
in partial fulfillment of the requirements
for the Degree of
Master of Science

Grand Forks, North Dakota
June
1967

This thesis submitted by Paul G. Nelson in partial fulfillment of the requirements for the Degree of Master of Science in the University of North Dakota is hereby approved by the Committee under whom the work has been done.

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ACKNOWLEDGMENTS

The writer wishes to express his sincere appreciation to Professor Edward O. Nelson for his generous assistance and constructive criticism in the writing of this thesis. A special thanks is also extended to Professors Thomas J. Robinson and Kenneth L. Hankerson for serving on the writer's committee and to Professor Milton B. Larson, whose research in heat transfer made this thesis possible.

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CHAPTER I

INTRODUCTION

One of the most powerful methods of dealing with the differential equations of applied mathematics is the Laplace transformation method. This method is especially well adapted to the solution of problems in conduction of heat.

In this introductory chapter we will define the Laplace transform, the inverse Laplace transform, and briefly discuss some of the properties of the Laplace transformation method and its use in solving systems of differential equations. The error function and Hermite polynomials appear in the body of the thesis as the result of certain inversions, so definitions of these functions will also be included.

Definition 1.1: Let T be a function of θ specified for $\theta > 0$. Then the Laplace transform of T, denoted by $L\{T(\theta)\}$, is the function defined by

$$L\{T(\theta)\} = t(s) = \int_0^\infty e^{-s\theta} T(\theta) d\theta$$

if there exists a complex number s such that the above integral converges. If no such s exists, the Laplace transform does not exist.

Note: Strictly speaking we should say that if T is a function defined by $y = T(\theta)$ then the Laplace transform of

T, denoted by $L\{T\}$, is the function defined by $L\{T\}(s) = \int_0^\infty e^{-s\theta}T(\theta)d\theta$. Instead of this notation we will use the shorter though incorrect notation $L\{T(\theta)\}$.

Definition 1.2: If the Laplace transform of a function T is t(s), i.e. if $L\{T(\theta)\} = t(s)$, then T is called an inverse Laplace transform of t(s) and we write symbolically $T = L^{-1}\{t(s)\}$ where L^{-1} is called the inverse Laplace transformation operator.

One means of obtaining the inverse of a Laplace transform is given in the following theorem.

Theorem 1.1: (Complex Inversion Theorem) If $t(s) = L\{T(\theta)\}$ then

$$T = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\theta}t(s)ds,$$

where γ is a real number so large that all the singularities of t(s) lie to the left of the line $(\gamma-i\infty,\gamma+i\infty)$.

Another useful theorem is

Theorem 1.2: Let $L\{T(\theta)\} = \sum_{n=1}^{\infty} f_n(s)$ satisfy the following conditions

1. There exists a γ such that all singularities of f and f_n lie to the left of the line $(\gamma-i\infty,\gamma+i\infty)$.

2. $L\{T_n(\theta)\} = f_n(s)$.

3. $\sum_{n=1}^{\infty} e^{s\theta}f_n(s)$ converges uniformly for s on the line $(\gamma-i\infty,\gamma+i\infty)$.

Then $T(\theta) = \sum_{n=1}^{\infty} T_n(\theta)$.

In the body of the thesis the Laplace transforms of certain functions are used. Boundary conditions are also transformed and are used as bondary conditions of the transformed differential equations. In performing these transformations, certain assumptions have been made regarding the nature of the functions involved. To help clarify what these assumptions are, we first present a set of sufficient conditions for the existence of the Laplace transform, and then justify a few of the transformations on boundary conditions that are made later in the thesis.

Definition 1.3: If real constants $M > 0$, $N > 0$ and γ exist such that for all $\theta > N$

$$e^{-\gamma\theta}|T(\theta)| < M$$

we say that $T(\theta)$ is a function of exponential order γ as $\theta \to \infty$ or, briefly, is of exponential order.

Theorem 1.3: If a function T is piecewise continuous in every finite interval $0 \leq \theta \leq N$ and of exponential order γ for $\theta > N$, then its Laplace transform t(s) exists for all $s > \gamma$.

We thus assume whenever necessary that the functions involved satisfy the conditions of Theorem 1.3.

Consider the system of partial differential equations

$$\frac{\partial^2 T_1}{\partial \psi^2} + \frac{QR^2}{k_1 S} = \frac{R^2}{\alpha_1}\frac{\partial T_1}{\partial \theta} \tag{1.1}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial \theta},$$

where Q, R, k_1, S, α_1, and α_2 are constants which are defined later, and T_1 and T_2 are functions of (ψ, θ) and (x, θ) respectively.

One of the boundary conditions used later in connection with this system is $T_1(0,\theta) = T_2(0,\theta)$, $\theta \geq 0$. The Laplace transform of this condition is written

$$t_1(0,s) = t_2(0,s).$$

To see why this is true we have only to write t_1 and t_2 in terms of their integral definitions. That is

$$t_1(0,s) = \int_0^\infty e^{-s\theta} T_1(0,\theta)\, d\theta$$

$$= \int_0^\infty e^{-s\theta} T_2(0,\theta)\, d\theta = t_2(0,s).$$

The Laplace transform of the first equation in (1.1) is given by the equation

$$\frac{d^2 t_1}{d\psi^2} + \frac{QR^2}{k_1 S}\,\frac{1}{s} = \frac{R^2}{\alpha_1}\left(s t_1 - T_1(\psi,+0)\right). \tag{1.2}$$

Let us consider what assumptions regarding the function T_1 are necessary in writing (1.2). The term we are concerned with is the term $\frac{\partial^2 T_1}{\partial \psi^2}$. In obtaining (1.2) from (1.1) we have made the assertion that

$$\int_0^\infty e^{-s\theta}\left(\frac{\partial^2 T_1}{\partial \psi^2}\right) d\theta = \frac{d^2 t_1}{d\psi^2}. \tag{1.3}$$

If T_1 is such that the order of integration and differentiation can be interchanged we have

$$\int_0^\infty e^{-s\theta}\left(\frac{\partial^2 T_1}{\partial \psi^2}\right)d\theta = \frac{d^2}{d\psi^2}\int_0^\infty e^{-s\theta}T_1 d\theta$$

$$= \frac{d^2 t_1}{d\psi^2}.$$

By Leibnitz's Rule [3, p. 322], we can interchange the order of integration and differentiation in this manner, if for each $\theta \geq 0$, T_1, $\frac{\partial T_1}{\partial \psi}$, and $\frac{\partial^2 T_1}{\partial \psi^2}$ exist and are continuous functions of ψ for $\psi > 0$. Thus the assumptions of continuity of T_1 and its first and second partial derivatives with respect to ψ, and the assumptions that T_1 and the second partial of T_1 with respect to ψ are of exponential order, allow us to write (1.3).

As a final example, we justify the following transformation of a boundary condition. Suppose we have the condition that

(1.4) $\frac{\partial T_1}{\partial \psi}$ is a nonnegative decreasing function of ψ.

We now justify the result that (1.4) transforms into the condition

(1.5) $\frac{dt_1}{d\psi}$ is a nonnegative decreasing function of ψ.

Showing that (1.4) implies (1.5) is the same as showing that (1.4) implies

$\frac{d}{d\psi}\int_0^\infty e^{-s\theta}T_1 d\theta$ is a nonnegative decreasing function of ψ.

Again utilizing Leibnitz's rule we can write

$$\frac{d}{d\psi}\int_0^\infty e^{-s\theta}T_1 d\theta = \int_0^\infty e^{-s\theta}\frac{\partial T_1}{\partial \psi}d\theta$$

if we assume that T_1 and $\frac{\partial T_1}{\partial \psi}$ are continuous functions of ψ for $\psi \geq 0$ and $\theta \geq 0$, and that they are of exponential order. Clearly

$$\int_0^\infty e^{-s\theta}\frac{\partial T_1}{\partial \psi}d\theta$$

is nonnegative since $\frac{\partial T_1}{\partial \psi}$ is nonnegative. It is also clear that

$$\int_0^\infty e^{-s\theta}\frac{\partial T_1}{\partial \psi}(\psi+\Delta\psi,\theta)d\theta \leq \int_0^\infty e^{-s\theta}\frac{\partial T_1}{\partial \psi}(\psi,\theta)d\theta$$

for any increment $\Delta\psi > 0$ since $\frac{\partial T_1}{\partial \psi}$ is a decreasing function of ψ. Therefore $\frac{dt_1}{d\psi}$ has the property that it is a nonnegative decreasing function of ψ.

Briefly, the Laplace transformation method of solving a system of differential equations involves the following steps. Application of the Laplace transformation to a partial differential equation involving only one space variable and one time variable, reduces the partial differential equation to an ordinary differential equation. The equation thus derived is referred to as the "subsidiary equation." The boundary conditions for the subsidiary equation are obtained by applying the Laplace transformation to the boundary conditions of the initial partial differential equations. By using standard methods of ordinary

differential equations, the subsidiary equations are solved using the transformed boundary conditions. The Laplace transform of the solution of the problem is then known and the problem is solved if an inversion of the transformed solution can be made.

In practice the inverse Laplace transform of a function is usually obtained from a table of inverse Laplace transforms. If the inverse of a function does not appear in a table but the function can be expressed as the product of two functions whose inverses do appear in the table, the following theorem is useful.

Theorem 1.4: (Convolution Theorem) If $L^{-1}\{t(s)\} = T(\theta)$ and $L^{-1}\{g(s)\} = G(\theta)$, then $L^{-1}\{f(s)g(s)\} = \int_0^{\theta} T(\tau)G(\theta-\tau)d\tau$.

If the transform t(s) does not appear in a table of Laplace transforms, T can often be determined from t(s) by using Theorem 1.1.

We state the following definitions of functions which will be used in the body of the thesis.

Definition 1.4: [1, p. 297] The function erf defined by

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

is called the error function.

Definition 1.5: [1, p. 297] The function erfc defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt = 1 - \operatorname{erf} z$$

is called the complementary error function.

Definition 1.6: [1, p. 299]

$$i^n \operatorname{erfc} z = -\frac{z}{n} i^{n-1} \operatorname{erfc} z + \frac{1}{2n} i^{n-2} \operatorname{erfc} z \quad (n=1,2,3,\ldots)$$

where

$$i^{-1} \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} e^{-z^2} \text{ and } i^0 \operatorname{erfc} z = \operatorname{erfc} z.$$

Definition 1.7: [1, p. 775] The functions $H_n(x)$ defined by

$$H_n(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{1}{m!(n-2m)!} (2x)^{n-2m},$$

where by $\left[\frac{n}{2}\right]$ we mean the largest integer less than or equal to $\frac{n}{2}$, are called Hermite polynomials.

CHAPTER II

STRAIGHT WIRE

We first consider the problem of obtaining the temperature distribution in a thin skin to which a straight wire has been attached. A heating rate Q is assumed to be uniform over the surface and constant with time.

The following diagram pictorially describes the physical situation.

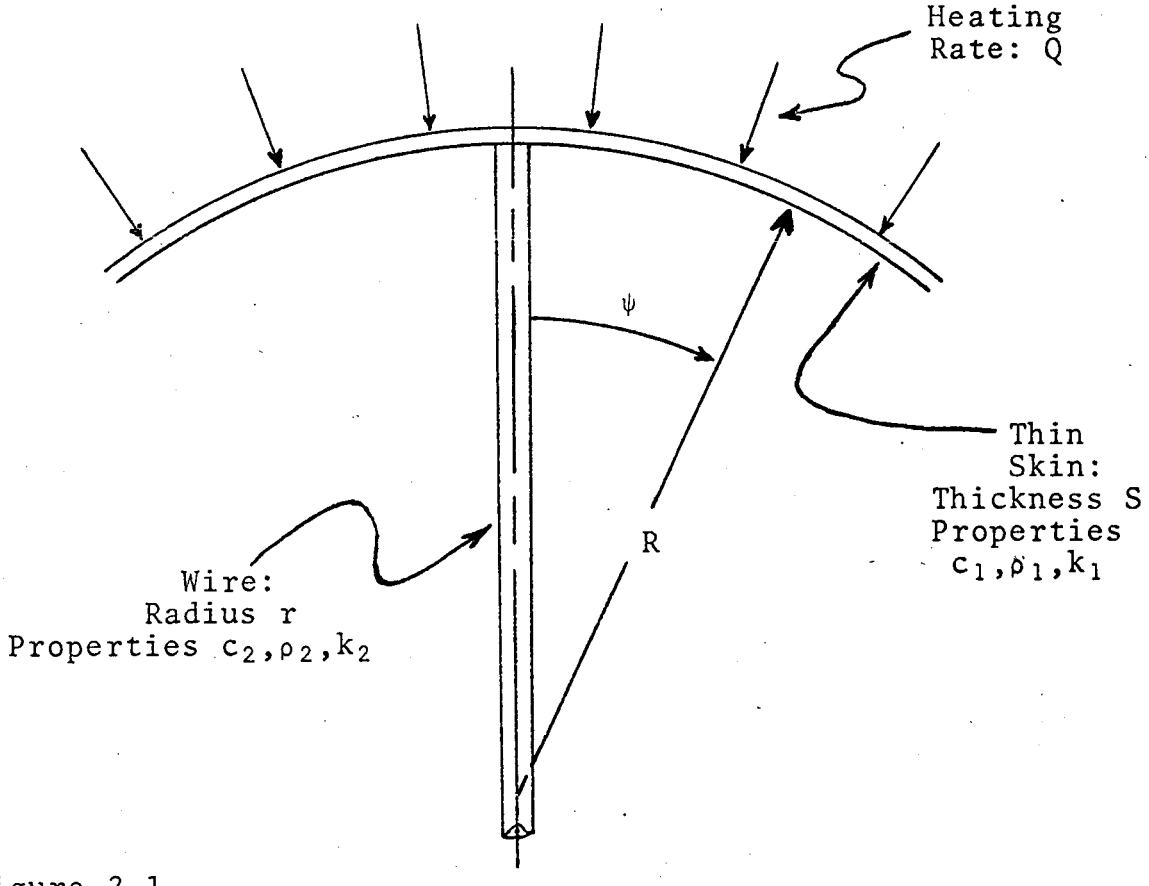

Figure 2.1.

Throughout this thesis the following notation will be used:

ψ	is the angle measured from the wire.
x	is the distance down the wire from the skin.
θ	is the time.
R	is the radius of curvature of the skin.
$T_1(\psi, \theta)$	is the temperature of the skin.
$T_2(x, \theta)$	is the temperature of the wire.
Q	is the heating rate per unit area of the skin.
S	is the skin thickness.
r	is the radius of the wire.
c_1	is the specific heat of the skin.
c_2	is the specific heat of the wire.
ρ_1	is the density of the skin.
ρ_2	is the density of the wire.
k_1	is the thermal conductivity of the skin.
k_2	is the thermal conductivity of the wire.
$\alpha_1=\frac{k_1}{c_1\rho_1}$	is the thermal diffusivity of the skin.
$\alpha_2=\frac{k_2}{c_2\rho_2}$	is the thermal diffusivity of the wire.

The following assumptions are also made:

1. The thin skin is considered to have infinite thermal conductivity in the thickness dimension.

2. There is no heat transfer from the rear side of the skin except through the wire.

3. The wire has infinite thermal conductivity in the direction perpendicular to its axis.

4. The wire is assumed to be infinitely long and heat transfer from the surface of the wire is neglected.

5. The thermal capacity of the material of the thin skin adjacent to the junction is neglected. This is the material enclosed within the radius r and of thickness S.

6. The heating rate Q is assumed to be uniform over the face and constant with time.

The differential equation that must be satisfied in the skin is

$$\frac{\partial^2 T_1}{\partial \psi^2} + \frac{QR^2}{k_1 S} = \frac{R^2}{\alpha_1}\frac{\partial T}{\partial \theta} \qquad \psi > 0, \quad \theta > 0.$$

This equation essentially describes the flow of heat in the skin. The equation describing the linear flow of heat in a wire on assumption of no radiation is

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial \theta} \qquad x > 0, \quad \theta > 0.$$

We thus have two differential equations that must be satisfied.

We next present a set of boundary conditions. Assuming that our system is initially at a zero temperature we have the condition

$$T_1(\psi,0) = T_2(x,0) = 0 \qquad \psi > 0, \; x > 0.$$

Since the temperatures $T_1(0,\theta)$ and $T_2(0,\theta)$ are temperatures at the same physical position we have $T_1(0,\theta) = T_2(0,\theta)$ $\theta \geq 0$. As we move far enough down the wire, away from the skin, the temperature approaches a constant value. This fact can be

represented by the statement

$$\frac{\partial T_2}{\partial x}(x \to \infty, \theta) = 0 \qquad \theta > 0.$$

Equating the heat lost from the skin to the heat drawn down the wire we have

$$k_1 S 2\pi r \frac{\partial T_1}{\partial (R\psi)}(0,\theta) = -k_2 \pi r^2 \frac{\partial T}{\partial x}2(0,\theta) \qquad \theta > 0.$$

The cooling effect of the wire becomes less as we move away from the wire. Thus the physical situation also dictates that $\frac{\partial T}{\partial \psi}1$ be a nonnegative decreasing function of ψ, $\psi > 0$.

The problem is thus one of solving the system of equations

$$\frac{\partial^2 T_1}{\partial \psi^2} + \frac{QR^2}{k_1 S} = \frac{R^2}{\alpha_1}\frac{\partial T_1}{\partial \theta} \tag{2.1}$$

$(\psi,\ \theta,\ x > 0)$

$$\frac{\partial^2 T_2}{\partial \psi^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial \theta} \tag{2.2}$$

with the boundary conditions

$$T_1(\psi,0) = T_2(x,0) = 0 ; \tag{2.3}$$

$$T_1(0,\theta) = T_2(0,\theta) ; \tag{2.4}$$

$$\frac{\partial T}{\partial x}2(x \to \infty, \theta) = 0 ; \qquad (\psi,\ \theta,\ x > 0) \tag{2.5}$$

$$2k_1 S \frac{\partial T_1}{\partial (R\psi)}(0,\theta) = -k_2 r \frac{\partial T_2}{\partial x}(0,\theta) ; \tag{2.6}$$

$$\frac{\partial T_1}{\partial \psi} \text{ is a nonnegative decreasing function of } \psi. \tag{2.7}$$

We now apply the Laplace transform to (2.1) and (2.2). Letting $t_1 = \int_0^\infty e^{-s\theta} T_1(\psi,\theta)\,d\theta$ and $t_2 = \int_0^\infty e^{-s\theta} T_2(x,\theta)\,d\theta$ we have

$$\frac{d^2t_1}{d\psi^2} + \frac{QR}{k_1S}\frac{1}{s} = \frac{R^2}{\alpha_1}\left(st_1 - T_1(\psi,+0)\right) \tag{2.8}$$

and

$$\frac{d^2t_1}{dx^2} = \frac{1}{\alpha_2}\left(st_2 - T_2(x,+0)\right). \tag{2.9}$$

Using boundary condition (2.3), equations (2.8) and (2.9) simplify to

$$\frac{d^2t_1}{d\psi^2} + \frac{QR^2}{k_1S}\frac{1}{s} = \frac{R^2}{\alpha_1}st_1 \tag{2.10}$$

and

$$\frac{d^2t_2}{dx^2} = \frac{1}{\alpha_2}st_2. \tag{2.11}$$

Boundary conditions (2.4)-(2.7) on T_1 and T_2 transform into the following boundary conditions on t_1 and t_2 :

$$t_1(0,s) = t_2(0,s) \ ; \tag{2.12}$$

$$\frac{dt_2}{dx} \to 0 \text{ as } x \to \infty \ ; \tag{2.13}$$

$$2k_1S\frac{dt_1}{d(R\psi)}(0,s) = -k_2r\frac{dt_2}{dx}(0,s) \ ; \tag{2.14}$$

(2.15) $\frac{dt_1}{d\psi}$ is a nonnegative decreasing function of ψ.

The general solution of (2.11) is

$$t_2 = d_1e^{x\sqrt{s/\alpha_2}} + d_2e^{-x\sqrt{s/\alpha_2}} \ . \tag{2.16}$$

Similarly the complete solution of (2.10) is

$$t_1 = d_3e^{R\psi\sqrt{s/\alpha_1}} + d_4e^{-R\psi\sqrt{s/\alpha_1}} + \frac{Q\alpha_1}{k_1S}\frac{1}{s^2}. \tag{2.17}$$

Differentiating (2.16) with respect to x and applying boundary condition (2.13) forces the result $d_1 = 0$. Thus we have

(2.18) $$t_2 = d_2 e^{-x\sqrt{s/\alpha_2}}.$$

Differentiating (2.17) with respect to ψ and applying boundary condition (2.15) yields the result $d_3 = 0$. Therefore

(2.19) $$t_1 = d_4 e^{-R\psi\sqrt{s/\alpha_1}} + \frac{Q\alpha_1}{k_1 S}\frac{1}{s^2} .$$

Using (2.18) and (2.19) in (2.12) produces the equation

(2.20) $$d_4 + \frac{Q\alpha_1}{k_1 S}\frac{1}{s^2} = d_2 .$$

Boundary condition (2.14) implies that

(2.21) $$2k_1 S \frac{1}{R} d_4 R\sqrt{s/\alpha_1} = -k_2 r d_2 \sqrt{s/\alpha_2} .$$

Solving (2.20) and (2.21) algebraically for d_4 and making this substitution in (2.19) yields the result

(2.22) $$t_1 = -\frac{a_1 a_5}{a_3+a_5}\frac{1}{s^2} e^{-R\psi\sqrt{s/\alpha_1}} + \frac{a_1}{s^2} ,$$

where

$$a_1 = \frac{Q}{c_1\rho_1 S} , \quad a_3 = \frac{2k_1 S}{(\alpha_1)^{\frac{1}{2}}} , \text{ and } a_5 = \frac{k_2 r}{(\alpha_2)^{\frac{1}{2}}} .$$

Upon inversion [1, p. 1026, Formula 29.3.86], (2.22) becomes

(2.23) $$T_1 = -\frac{A}{E} 4\theta\, i^2\text{erfc}\,\omega + a_1\theta ,$$

where

$$A = a_1 a_5, \quad E = a_3 + a_5, \text{ and } \omega = \frac{R\psi}{2(\theta\alpha_1)^{\frac{1}{2}}} .$$

CHAPTER III

BENT WIRE

In this chapter we evaluate the effects of thermal resistance and extra capacity between the wire and the skin. The extra capacity is taken into account by using a model where the wire is fastened along the skin for some length. As the thermal resistance and the length along which the wire is connected to the skin are decreased, the solution to this model should approach the solution to the straight wire model.

Extending our notation to encompass these added concepts, we let Ω represent the thermal resistance between the wire and the skin, and ℓ represent the length along which the wire is attached to the skin.

The following diagram pictorially describes the physical situation.

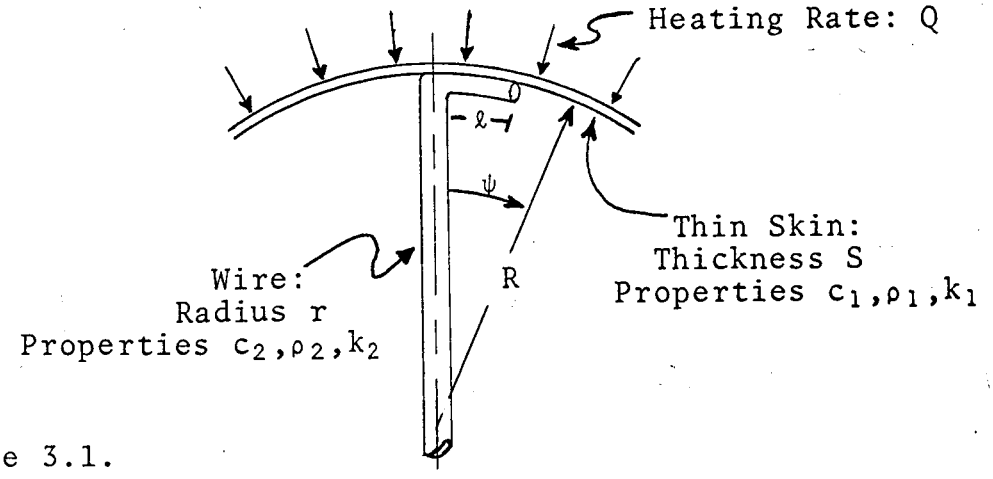

Figure 3.1.

We first consider the case where there is no resistance between the wire and the skin, but where the wire is fastened along the skin for some distance. This is the case $\Omega = 0$ and $\ell \neq 0$.

The differential equations for this problem are identical to the differential equations for the previous problem in Chapter II. The boundary conditions are also the same except for boundary condition (2.6). This condition equates the heat drawn from the skin to the heat flowing down the wire. Taking into account the "reservoir" effect of the added length of wire, boundary condition (2.6) becomes

$$k_1 2\pi rS \frac{\partial T_1}{\partial (R\psi)} (0,\theta) = c_2\rho_2\pi r^2\ell\frac{\partial T_2}{\partial \theta}(0,\theta) - k_2\pi r^2\frac{\partial T}{\partial x}^2(0,\theta).$$

Transforming, using boundary conditions, and proceeding as in Chapter II we find that

$$t_1 = \frac{a_1 a_3 e^{-R\psi\sqrt{s/\alpha_1}}}{s^2(a_4\sqrt{s}+a_3+a_5)} - \frac{a_1}{s^2} e^{-R\psi\sqrt{s/\alpha_1}} + \frac{a_1}{s^2}$$

where

$$a_1 = \frac{Q\alpha_1}{k_1 S}, \quad a_3 = \frac{2k_1 S}{\sqrt{\alpha_1}}, \quad a_4 = c_2\rho_2 r\ell, \text{ and } a_5 = \frac{k_2 r}{\sqrt{\alpha_2}}.$$

Rewriting t_1 we have

$$(3.1) \qquad t_1 = -\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^2} \frac{a_1 a_4\sqrt{s} + a_1 a_5}{a_4\sqrt{s} + a_3 + a_5} + \frac{a_1}{s^2}.$$

The problem is then one of inverting t_1. The first term of t_1 can be inverted if it is expanded using partial fractions. Rewriting part of the first term of (3.1) we

have

$$\frac{a_1 a_4 \sqrt{s} + a_1 a_5}{s^2(a_4\sqrt{s} + E)} = \frac{A}{a_4}\frac{B\sqrt{s} + 1}{s^2(\sqrt{s} + E/a_4)}$$

where

$$A = a_1 a_5 \text{ and } B = \frac{a_4}{a_5} .$$

Using the partial fraction expansion

$$\frac{Bx + 1}{x^4(x+p)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{x^4} + \frac{e}{x+p}$$

(3.1) becomes

$$t_1 = -\frac{A}{a_4}\left(\frac{a}{s^{\frac{1}{2}}} + \frac{b}{s} + \frac{c}{s^{\frac{3}{2}}} + \frac{d}{s^2} + \frac{e}{s^{\frac{1}{2}}+\frac{E}{a_4}}\right)e^{-R\psi\sqrt{s/\alpha_1}} + \frac{a_1}{s^2} ,$$

where

$$a = \frac{a_4^3(BE - a_4)}{E^4} , \quad b = \frac{-a_4^2(BE - a_4)}{E^3} ,$$

$$c = \frac{a_4(BE - a_4)}{E^2} , \quad d = \frac{a_4}{E} , \text{ and } e = -a.$$

Using the relations [1, p. 1026, Formulas 29.3.84, 29.3.86, and 29.3.88], we have upon simplification

$$T_1 = -\frac{A}{E} 4\theta(i^2\text{erfc } \omega) + \left(-\frac{A(BE - a_4)}{E^2} 2\sqrt{\theta/\pi}\right)e^{-\omega^2}$$

$$+ \left(2\sqrt{\theta}\omega \frac{A(BE - a_4)}{E^2} + \frac{Aa_4(BE - a_4)}{E^3}\right)\text{erfc } \omega$$

$$+ \left(\frac{ABE}{a_4^2} e^{\left(\frac{E}{a_4}\right)2\sqrt{\theta}\omega} e^{E^2\theta/a_4^2}\right)\text{erfc } \left(\frac{E}{a_4}\sqrt{\theta} + \omega\right) + a_1\theta ,$$

where $\omega = \frac{R\psi}{2\sqrt{\alpha_1\theta}}$.

Next we consider the case where there is no capacitance between the wire and the skin, but where there is a resistance present. This is the case $\ell = 0$ and $\Omega \neq 0$.

Again the differential equations are identical to those of the straight wire problem. The boundary condition that distinguishes this problem from the straight wire problem is boundary condition (2.4). This condition stated that $T_1(0,\theta) = T_2(0,\theta)$. The effect of a resistance between the skin and the wire is to restrict the flow of heat from the skin to the wire and cause the temperature of the wire to be less than if no resistance were present. More specifically, boundary condition (2.4) becomes in this case

$$T_1(0,\theta) - T_2(0,\theta) = \Omega k_1 2\pi r S \frac{\partial T_1}{\partial(R\psi)}(0,\theta) .$$

The solution of this case is analogous to the case $\Omega = 0$ and $\ell \neq 0$ and we present here only the result that

$$T_1 = -\frac{A}{E} 4\theta(i^2\text{erfc } \omega) + \frac{A(a_2a_5)}{E^2}(2\sqrt{\theta/\pi})\, e^{-\omega^2}$$

$$+ \left(-\frac{A(a_2a_5)}{E^2} 2\sqrt{\theta}\omega - \frac{A(a_2a_5)^2}{E^3}\right) \text{erfc } \omega$$

$$+ \left(\frac{A(a_2a_5)^2}{E^3}\, e^{\left(\frac{2E\sqrt{\theta}\omega}{a_2a_5}\right)} e^{\left(\frac{E^2\theta}{(a_2a_5)^2}\right)}\right) \text{erfc } \left(\frac{E\sqrt{\theta}}{a_2a_5} + \omega\right)$$

$$+ a_1\theta ,$$

where

$$a_2 = \frac{\Omega k_1 2\pi r S}{\sqrt{\alpha_1}} \quad \text{and} \quad a_5 = \frac{k_2 r}{\sqrt{\alpha_2}} .$$

We now attack the general problem were both Ω and ℓ are nonzero. The system of equations for this problem is

$$\frac{\partial^2 T_1}{\partial \psi^2} + \frac{QR^2}{k_1 S} = \frac{R^2}{\alpha_1} \frac{\partial T_1}{\partial \theta} \tag{3.2}$$

$$(\psi, \theta, x > 0)$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T_2}{\partial \theta} , \tag{3.3}$$

subject to the boundary conditions:

$$T_1(\psi,0) = T_2(x,0) = 0 ; \tag{3.4}$$

$$T_1(0,\theta) - T_2(0,\theta) = \Omega k_1 2\pi r S \frac{\partial T_1(0,\theta)}{\partial (R\psi)} ; \tag{3.5}$$

$$\frac{\partial T_2}{\partial x}(x \to \infty, \theta) = 0 ; \tag{3.6}$$

$$k_1 2\pi r S \frac{\partial T_1}{\partial (R\psi)} (0,\theta) = c_2 \rho_2 \pi r^2 \ell \frac{\partial T_2}{\partial \theta}(0,\theta) - k_2 \pi r^2 \frac{\partial T_2}{\partial x}(0,\theta); \tag{3.7}$$

(3.8) $\frac{\partial T_1}{\partial \psi}$ is a nonnegative decreasing function of ψ.

Upon transformation and use of boundary condition (3.4), (3.2) and (3.3) become

$$\frac{d^2 t_1}{d\psi^2} + \frac{QR^2}{k_1 S} \frac{1}{s} = \frac{R^2}{\alpha_1} s t_1 \tag{3.9}$$

and

$$\frac{d^2 t_2}{dx^2} = \frac{1}{\alpha_2} s t_2 . \tag{3.10}$$

Transforming boundary conditions (3.5)-(3.8) and again using boundary condition (3.4) we obtain the following set of transformed boundary conditions:

$$t_1(0,s) - t_2(0,s) = \frac{\Omega k_1 2\pi r S}{R} \frac{dt_1}{d\psi}(0,s) ; \tag{3.11}$$

(3.12) $$\frac{dt_2}{dx} \to 0 \text{ as } x \to \infty \ ;$$

(3.13) $$\frac{2k_1S}{R}\frac{dt_1}{d\psi}(0,s) = c_2\rho_2 r\ell s t_2(0,s) - k_2 r \frac{dt_2}{dx}(0,s) \ ;$$

(3.14) $\frac{dt_1}{d\psi}$ is a nonnegative decreasing function of ψ.

As was seen in Chapter II, solutions for (3.9) and (3.10) upon application of boundary conditions (3.12) and (3.14) are

(3.15) $$t_2 = d_2 e^{-\sqrt{s/\alpha_2}x}$$

and

(3.16) $$t_1 = d_4 e^{-R\psi\sqrt{s/\alpha_1}} + \frac{Q\alpha_1}{k_1 S}\frac{1}{s^2} \ ,$$

where d_2 and d_4 are constants that must be evaluated using conditions (3.11) and (3.13). From condition (3.11) we have

(3.17) $$d_4 + \frac{Q\alpha_1}{k_1 S}\frac{1}{s^2} - d_2 = \Omega k_1 2\pi r S \left(\frac{d_4\sqrt{s}}{\sqrt{\alpha_1}}\right) \ .$$

From condition (3.13)

(3.18) $$2k_1S\left(-d_4\sqrt{s/\alpha_1}\right) = c_2\rho_2 r\ell s d_2 - k_2 r\left(-d_2\sqrt{s/\alpha_2}\right) \ .$$

Solving (3.17) and (3.18) algebraically for d_4 we obtain

$$d_4 = -\frac{1}{s^2}\,\frac{a_1a_4\sqrt{s} + a_1a_5}{a_2a_4 s + (a_4+a_2a_5)\sqrt{s} + (a_3+a_5)} \ ,$$

where

$$a_1 = \frac{Q\alpha_1}{k_1 S} \ , \quad a_2 = \frac{\Omega k_1 2\pi r S}{\sqrt{\alpha_1}} \ , \quad a_3 = \frac{2k_1 S}{\sqrt{\alpha_1}} \ ,$$

$$a_4 = c_2\rho_2 r\ell \text{ and } \quad a_5 = \frac{k_2 r}{\sqrt{\alpha_2}} \ .$$

Thus from (3.16)

$$t_1 = -\frac{A}{s^2}\left(\frac{B\sqrt{s}+1}{Cs+D\sqrt{s}+E}\right) e^{-R\psi\sqrt{s/\alpha_1}} + \frac{a_1}{s^2} ,$$

where

$$A = a_1a_5 , \quad B = \frac{a_4}{a_5} , \quad C = a_2a_4 \neq 0 ,$$

$$D = a_4+a_2a_5 , \text{ and } E = a_3+ a_5 .$$

Inverting t_1 we obtain

$$T_1 = L^{-1}\{t_1\} = -\frac{AB}{C} L^{-1}\{P_1\} - \frac{A}{C} L^{-1}\{P_2\} + a_1\theta ,$$

where

$$P_1 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{3}{2}}(s + D/C\sqrt{s} + E/C)}$$

and

$$P_2 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^2(s + D/C\sqrt{s} + E/C)} .$$

Using the quadratic formula P_1 and P_2 may be rewritten as

(3.19) $$P_1 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{3}{2}}(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)}$$

and

(3.20) $$P_2 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^2(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)} ,$$

where

(3.21) $$G = \frac{D/C + \sqrt{D^2/C^2 - 4E/C}}{2}$$

and

(3.22) $$H = \frac{D/C - \sqrt{D^2/C^2 - 4E/C}}{2} .$$

We now consider three possible cases for values of G and H. The first case we consider will be the case where

G and H are real or complex but unequal. The second case presented assumes that G and H are complex but unequal. Finally, in the third case, we solve the problem where G and H are assumed equal.

Case I: G and H are real or complex, and unequal. Using partial fractions we obtain

$$\frac{1}{s^{\frac{3}{2}}(s^{\frac{1}{2}}+G)(s^{\frac{1}{2}}+H)} = \frac{a_6}{s^{\frac{3}{2}}} + \frac{a_7}{s} + \frac{a_8}{s^{\frac{1}{2}}} + \frac{a_9}{(s^{\frac{1}{2}}+G)} + \frac{a_{10}}{(s^{\frac{1}{2}}+H)} ,$$

where

$$a_6 = \frac{1}{GH} , \quad a_7 = \frac{-(G+H)}{(GH)^2} , \quad a_8 = \frac{-(GH) + (G+H)^2}{(GH)^3} ,$$

$$a_9 = \frac{1}{-G^3(H-G)} \quad \text{and} \quad a_{10} = \frac{1}{-H^3(G-H)} .$$

Using the above partial fraction expansion, P_1 can be inverted using the inverse transforms referred to earlier in the chapter [1, p. 1026, Formulas 29.3.84, 29.3.86, and 29.3.88]. Noting that $a_8 + a_9 + a_{10} = 0$ we obtain the result

$$L^{-1}\{P_1\} = a_6(2\sqrt{\theta/\pi}e^{-\omega^2} - 2\sqrt{\theta}\,\omega\,\text{erfc}\,\omega) + a_7\,\text{erfc}\,\omega$$
$$- a_9\,Ge^{G(2\sqrt{\theta}\omega+G\theta)}\text{erfc}\,(G\sqrt{\theta}+\omega)$$
$$- a_{10}\,He^{H(2\sqrt{\theta}\omega+H\theta)}\text{erfc}\,(H\sqrt{\theta}+\omega) ,$$

where again $\omega = \frac{R\psi}{2\sqrt{\alpha_1\theta}}$.

Again using partial fractions we obtain

$$\frac{1}{s^2(s^{\frac{1}{2}}+G)(s^{\frac{1}{2}}+H)} = \frac{a_{11}}{s^2} + \frac{a_{12}}{s^{\frac{3}{2}}} + \frac{a_{13}}{s} + \frac{a_{14}}{s^{\frac{1}{2}}} + \frac{a_{15}}{(s^{\frac{1}{2}}+G)} + \frac{a_{16}}{(s^{\frac{1}{2}}+H)} ,$$

where

$$a_{11} = \frac{1}{GH} = a_6 \ , \ a_{12} = \frac{-(G+H)}{(GH)^2} = a_7 \ ,$$

$$a_{13} = \frac{-(GH)+(G+H)^2}{(GH)^3} = a_8 \ , \ a_{14} = \frac{2(G+H)(GH) - (G+H)^3}{(GH)^4} \ ,$$

$$a_{15} = \frac{1}{G^4(H-G)} = -\frac{a_9}{G}, \text{ and } a_{16} = \frac{1}{H^4(G-H)} = -\frac{a_{10}}{H}.$$

Using the above partial fraction expansion, P_2 can be inverted using the same inverse transforms used above for inverting P_1. Noting that $a_{14} + a_{15} + a_{16} = 0$ we have

$$L^{-1}\{P_2\} = a_6 4\theta(i^2\text{erfc } \omega) + a_7(2\sqrt{\theta/\pi}e^{-\omega^2} - 2\sqrt{\theta}\omega \text{ erfc } \omega)$$

$$+ a_8\text{erfc } \omega - a_{15}Ge^{G(2\sqrt{\theta}\omega+G\theta)}\text{erfc } (G\sqrt{\theta} + \omega)$$

$$- a_{16}He^{H(2\sqrt{\theta}\omega+H\theta)}\text{erfc } (H\sqrt{\theta} + \omega).$$

Since

$$T_1 = -\frac{AB}{C} L^{-1}\{P_1\} - \frac{A}{C} L^{-1}\{P_2\} + a_1\theta$$

we have upon combining similar terms

$$(3.23) \qquad T_1 = -\frac{A}{E} 4\theta(i^2\text{erfc } \omega) + 2\frac{A}{E}\sqrt{\theta/\pi}(D/E - B)e^{-\omega^2}$$

$$+ \frac{A}{E}\left(\frac{BD}{E} + \frac{CE-D^2}{E^2} + 2\sqrt{\theta}\omega(B - D/E)\right) \text{ erfc } \omega$$

$$+ \frac{a_9A}{C}(BG - 1)e^{G(2\sqrt{\theta}\omega+G\theta)}\text{erfc } (G\sqrt{\theta} + \omega)$$

$$+ \frac{a_{10}A}{C}(BH - 1)e^{H(2\sqrt{\theta}\omega+H\theta)}\text{erfc } (H\sqrt{\theta} + \omega)$$

$$+ a_1\theta \ .$$

Because of the partial fraction expansion used, the above formula holds only when $G \neq H$.

Certain values of the original physical parameters result in complex values for G and H. If G and H are complex, the 4th and 5th terms of (3.23) are also complex. It may appear strange that an equation for a physical quantity such as temperature would involve complex terms. One might even begin to doubt the validity of the solution. This apparent irregularity is explained upon closer examination of the terms in (3.23). Using properties of the function erfc, it can be shown that the 4th and 5th terms of (3.23) are complex conjugates. Summing complex conjugates results in a real number and our worry about obtaining complex temperatures from (3.23) is now removed.

In the event that G and H are complex, equation (3.23) does not readily lend itself to numerical calculation for T_1. The following alternate solution may be of assistance when G and H are complex.

Case II: G and H are complex and different.

Our equation for T_1 is

$$T_1 = -\frac{AB}{C} L^{-1}\{P_1\} - \frac{A}{C} L^{-1}\{P_2\} + a_1\theta \,, \tag{3.24}$$

where

$$P_1 = \frac{e^{-2\sqrt{\theta}\omega\sqrt{s}}}{s^{\frac{1}{2}}(s + D/C\sqrt{s} + E/C)} \tag{3.25}$$

and

$$P_2 = \frac{e^{-2\sqrt{\theta}\omega\sqrt{s}}}{s^{2}(s + D/C\sqrt{s} + E/C)} \,. \tag{3.26}$$

Let

$$\Phi(s) = \frac{1}{s + D/C\sqrt{s} + E/C} \quad . \tag{3.27}$$

Rather than factoring the denominator of (3.27) using the quadratic formula as was done in obtaining (3.19) and (3.20), we rewrite (3.27) as follows

$$\Phi(s) = \frac{1}{s + D/C\sqrt{s} + E/C} = \frac{1}{(s + E/C)\left(1 + \frac{D/C\sqrt{s}}{s + E/C}\right)} \quad .$$

The Maclaurin expansion of $\frac{1}{1 + x}$ converges absolutely if $|x| < 1$.

Thus

$$\frac{1}{1 + \frac{D/C\sqrt{s}}{s + E/C}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{D/C\sqrt{s}}{s + E/C}\right)^n \tag{3.28}$$

will converge absolutely if $\left|\frac{D/C\sqrt{s}}{s + E/C}\right| < 1$.

Since we have assumed that G and H are complex, (3.21) and (3.22) imply that $(D/C)^2 - 4E/C < 0$. C, D, and E are nonnegative, (they are combinations of nonnegative physical parameters), so $(D/C)^2 - 4E/C < 0$ implies

$$\frac{D}{2\sqrt{EC}} < 1 \quad . \tag{3.29}$$

The maximum value of

$$\left|\frac{D/C\sqrt{s}}{s + E/C}\right| \tag{3.30}$$

is attained when

$$\frac{D/C}{2\sqrt{s}(s + E/C)} - \frac{D/C\sqrt{s}}{(s + E/C)^2} = 0 \quad . \tag{3.31}$$

Solving (3.31) algebraically for s we find that s = E/C. Therefore the maximum value of (3.30) is

$$\frac{D/C\sqrt{E/C}}{2E/C} = \frac{D}{2\sqrt{EC}} < 1 \quad \text{by (3.29)} . \tag{3.32}$$

Thus (3.28) converges absolutely

From (3.28)

$$\Phi(s) = \frac{1}{(s + E/C)\left(1 + \frac{D/C\sqrt{s}}{s + E/C}\right)} = \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}} . \tag{3.33}$$

Using partial fractions

$$\frac{1}{s^{\frac{3}{2}}(s + D/C\sqrt{s} + E/C)} = \frac{b_1}{s^{\frac{3}{2}}} + \frac{b_2}{s} + \frac{b_3}{s^{\frac{1}{2}}} + \frac{b_4 + b_5 s^{\frac{1}{2}}}{(s + D/C\sqrt{s} + E/C)} , \tag{3.34}$$

where $b_1, \ldots, b_5$ are constants. Applying (3.33) and (3.34), (3.25) becomes

$$P_1 = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left(\frac{b_1}{s^{\frac{3}{2}}} + \frac{b_2}{s} + \frac{b_3}{s^{\frac{1}{2}}} + (b_4 + b_5 s^{\frac{1}{2}}) \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}}\right) . \tag{3.35}$$

Similarly

$$P_2 = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left(\frac{c_1}{s^2} + \frac{c_2}{s^{\frac{3}{2}}} + \frac{c_3}{s} + \frac{c_4}{s^{\frac{1}{2}}} + (c_5 + c_6 s^{\frac{1}{2}}) \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}}\right) , \tag{3.36}$$

where $c_1, \ldots, c_6$ are constants obtained from the partial

fraction expansion of (3.26).

Recalling (3.24), the equation for T_1, let us consider

$$L^{-1}\{Q_1^*\} = L^{-1}\{-\frac{AB}{C} P_1^* - \frac{A}{C} P_2^*\} \tag{3.37}$$

where

$$P_1^* = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left(\frac{b_1}{s^{\frac{3}{2}}} + \frac{b_2}{s} + \frac{b_3}{s^{\frac{1}{2}}}\right) \tag{3.38}$$

and

$$P_2^* = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left(\frac{c_1}{s^2} + \frac{c_2}{s^{\frac{3}{2}}} + \frac{c_3}{s} + \frac{c_4}{s^{\frac{1}{2}}}\right) . \tag{3.39}$$

Upon simplification

$$\begin{aligned} L^{-1}\{Q_1^*\} = &- \frac{A}{E} 4\theta(i^2\operatorname{erfc}\,\omega) + 2\frac{A}{E}\sqrt{\theta/\pi}(D/E - B)e^{-\omega^2} \\ &+ \frac{A}{E}\left(\frac{BD}{E} + \frac{C}{E} - \frac{D^2}{E^2} + 2\sqrt{\theta}\omega(B - D/E)\right)\operatorname{erfc}\,\omega \\ &+ \frac{A}{E^2}\frac{1}{\sqrt{\pi\theta}}\left(BC - \frac{BD^2}{E} - \frac{2CD}{E} + \frac{D^3}{E^2}\right)e^{-\omega^2} . \end{aligned} \tag{3.40}$$

We observe that the first three terms of (3.40) correspond exactly with the first three terms of (3.23). This was to be expected because of the similarity of the partial fraction expansions used. It does however serve as a check of the previous result.

We next investigate the terms that were omitted in (3.40). These terms were

$$-\frac{AB}{C} L^{-1}\{P_1 - P_1^*\} \quad \text{and} \quad -\frac{A}{C} L^{-1}\{P_2 - P_2^*\} .$$

From (3.35), (3.36), (3.38), and (3.39)

(3.41)

$$P_1 - P_1^* = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left((b_4 + b_5\sqrt{s}) \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}}\right)$$

and

(3.42)

$$P_2 - P_2^* = e^{-2\sqrt{\theta}\omega\sqrt{s}}\left((c_5 + c_6\sqrt{s}) \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}}\right) .$$

Using the absolute convergence of (3.28) we combine (3.41) and (3.42) and rearrange terms to obtain

(3.43)

$$L^{-1}\{Q_2^*\} = L^{-1}\{-\frac{AB}{C}(P_1 - P_1^*) - \frac{A}{C}(P_2 - P_2^*)\}$$

$$= -\frac{A}{C}\left((Bb_4 + c_5)\, L^{-1}\left\{e^{-2\sqrt{\theta}\omega\sqrt{s}} \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{n/2}}{(s + E/C)^{n+1}}\right\}\right.$$

$$\left. +(Bb_5 + c_6)\, L^{-1}\left\{e^{-2\sqrt{\theta}\omega\sqrt{s}} \sum_{n=0}^{\infty}(-D/C)^n \frac{s^{(n+1)/2}}{(s + E/C)^{n+1}}\right\}\right) .$$

The conditions of Theorem 1.2 are satisfied, therefore (3.43) can be inverted termwise. The terms of Q_2^* are of the form

(3.44) $$\frac{1}{(s + E/C)^{n+1}}\, s^{n/2} e^{-2\sqrt{\theta}\omega\sqrt{s}}$$

or

(3.45) $$\frac{1}{(s + E/C)^{n+1}}\, s^{(n+1)/2} e^{-2\sqrt{\theta}\omega\sqrt{s}} .$$

Using [1, p. 1022, Formula 29.3.10] and [1, p. 1026, Formula 29.3.87] (3.44) and (3.45) can be inverted using convolution Theorem 1.4. The result for the inverse of (3.44) is

(3.46)

$$\int_0^{\theta} \frac{(\theta-\tau)^n e^{-(E/C)(\theta-\tau)}}{n!} \frac{e^{-(R^2\psi^2)/(4\alpha_1\tau)}}{2^{n+1}\sqrt{\pi}(\tau)^{(n+2)/2}} H_{n+1}\left((R\psi)/(2\sqrt{\alpha_1\tau})\right) d\tau .$$

$$(n=0,1,2,\ldots) .$$

A similar result is obtained for the inverse of (3.45). From (3.46) the first part of (3.43) becomes, neglecting a constant coefficient,

(3.47)

$$\sum_{n=0}^{\infty}\left((-D/C)^n \int_0^{\theta} \frac{(\theta-\tau)^n e^{-(E/C)(\theta-\tau)}}{n!} \frac{e^{-(R^2\psi^2)/(4\alpha_1\tau)}}{2^{n+1}\sqrt{\pi}(\tau)^{(n+2)/2}}\right.$$

$$\left. X \; H_{n+1}\left((R\psi)/(2\sqrt{\alpha_1\tau})\right) d\tau\right) \qquad (n = 0,1,2,\ldots) .$$

If the series in (3.47) can be shown to be uniformly convergent, for τ in some interval $(0,\theta_0)$, the order of integration and summation can be interchanged for $\theta \leq \theta_0$.

Using the inequality [1, p. 787, Formula 22.14.17]

(3.48) $$|H_n(x)| < e^{x^2/2} K 2^{n/2}\sqrt{n!} \qquad K \simeq 1.086435$$

the desired uniform convergence can be obtained if θ_0 is sufficiently small. For example, if $\theta_0 < (CR\psi)/(D\sqrt{2\alpha_1})$, the integrand in (3.47) converges uniformly on the interval $[0,\theta_0]$.•

In showing the above uniform convergence, a non-negative series of constants was shown to be a bound for (3.47). This convergent series and a similar one for the second part of (3.43) could be used in determining how many terms of (3.43) need be computed to obtain T_1 within a given margin of error.

Finally we consider the case where G and H are assumed equal.

Case III: G and H are equal.

Our equation for T_1 is again

$$T_1 = -\frac{AB}{C} L^{-1}\{P_1\} - \frac{A}{C} L^{-1}\{P_2\} + a_1\theta \,, \tag{3.49}$$

where

$$P_1 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{3}{2}}(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)}$$

and

$$P_2 = \frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{2}(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)} \,.$$

The equations for G and H are

$$G = \frac{D/C + \sqrt{D^2/C^2 - 4E/C}}{2}$$

and

$$H = \frac{D/C - \sqrt{D^2/C^2 - 4E/C}}{2} \,.$$

Assuming G = H we have $G = H = \frac{D}{2C}$.

Using partial fractions

$$\frac{1}{s^{\frac{3}{2}}(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)} = \frac{1}{s^{\frac{3}{2}}(s^{\frac{1}{2}} + G)^2}$$

$$= \left(\frac{1}{G}\right)^2 \frac{1}{s^{\frac{3}{2}}} - 2\left(\frac{1}{G}\right)^3 \frac{1}{s} + 3\left(\frac{1}{G}\right)^4 \frac{1}{s^{\frac{1}{2}}} - \left(\frac{1}{G}\right)^3 \frac{1}{(s^{\frac{1}{2}}+G)^2} - 3\left(\frac{1}{G}\right)^4 \frac{1}{(s^{\frac{1}{2}}+G)} \,.$$

and

$$\frac{1}{s^{2}(s^{\frac{1}{2}} + G)(s^{\frac{1}{2}} + H)} = \frac{1}{s^{2}(s^{\frac{1}{2}} + G)^2}$$

$$= \left(\frac{1}{G}\right)^2 \frac{1}{s^2} - 2\left(\frac{1}{G}\right)^3 \frac{1}{s^{\frac{3}{2}}} + 3\left(\frac{1}{G}\right)^4 \frac{1}{s} - 4\left(\frac{1}{G}\right)^5 \frac{1}{s^{\frac{1}{2}}} + \left(\frac{1}{G}\right)^4 \frac{1}{(s^{\frac{1}{2}}+G)^2} + 4\left(\frac{1}{G}\right)^5 \frac{1}{(s^{\frac{1}{2}}+G)} \,.$$

Therefore

$$L^{-1}\{P_1\} = \left(\frac{1}{G}\right)^2 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{3}{2}}}\right\} - 2\left(\frac{1}{G}\right)^3 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s}\right\}$$
$$+3\left(\frac{1}{G}\right)^4 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{1}{2}}}\right\} - \left(\frac{1}{G}\right)^3 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)^2}\right\}$$
$$-3\left(\frac{1}{G}\right)^4 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)}\right\},$$

and

$$L^{-1}\{P_2\} = \left(\frac{1}{G}\right)^2 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^2}\right\} - 2\left(\frac{1}{G}\right)^3 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{3}{2}}}\right\}$$
$$+3\left(\frac{1}{G}\right)^4 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s}\right\} - 4\left(\frac{1}{G}\right)^5 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{s^{\frac{1}{2}}}\right\}$$
$$+\left(\frac{1}{G}\right)^4 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)^2}\right\} + 4\left(\frac{1}{G}\right)^5 L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)}\right\}.$$

The only inversion above that gives any trouble is

$$L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)^2}\right\}$$

Using Theorem 1.4 we can invert this term by writing

$$L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)^2}\right\} = L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)}\ \frac{1}{(s^{\frac{1}{2}}+G)}\right\}$$

and using [1, p. 1023, Formula 29.3.37] and [1, p. 1026, Formula 29.3.88] to get

$$(3.50)\quad L^{-1}\left\{\frac{e^{-R\psi\sqrt{s/\alpha_1}}}{(s^{\frac{1}{2}}+G)^2}\right\} = \int_0^\theta \left(\frac{1}{\sqrt{\pi(\theta-\tau)}} - Ge^{G^2(\theta-\tau)}\operatorname{erfc}(G\sqrt{\theta-\tau})\right)$$

$$\times\left(\frac{1}{\sqrt{\pi\tau}}\, e^{-(R^2\psi^2)/(4\alpha_1\tau)} - Ge^{GR\psi/\sqrt{\alpha_1}}e^{G^2\tau}\operatorname{erfc}\left(G\sqrt{\tau}+\frac{R\psi}{2\sqrt{\alpha_1\tau}}\right)\right)d\tau.$$

Recalling (3.49), the equation for T_1, we have using [1, p. 1026, Formulas 29.3.84, 29.3.86, and 29.3.88] that

(3.51)

$$\begin{aligned} T_1 = {} & -\frac{A}{C}\left(\frac{1}{G}\right)^2 4\theta(i^2\operatorname{erfc}\,\omega) \\ & + \frac{A}{C}\left(\frac{1}{G}\right)^2\left(\frac{2}{G}-B\right)\left(2\sqrt{\theta/\pi}e^{-\omega^2} - 2\sqrt{\theta}\omega\,\operatorname{erfc}\,\omega\right) \\ & + \frac{A}{C}\left(\frac{1}{G}\right)^3\left(2B-\frac{3}{G}\right)\operatorname{erfc}\,\omega + \frac{A}{C}\left(\frac{1}{G}\right)^4\left(-3B+\frac{4}{G}\right)\frac{1}{\sqrt{\pi\theta}}\,e^{-\omega^2} \\ & + \frac{A}{C}\left(\frac{1}{G}\right)^3\left(B-\frac{1}{G}\right)L^{-1}\left\{\frac{e^{-2\sqrt{\theta}\omega\sqrt{s}}}{(s^{\frac{1}{2}}+G)^2}\right\} \\ & + \frac{A}{C}\left(\frac{1}{G}\right)^4\left(3B-\frac{4}{G}\right)\left(\frac{1}{\sqrt{\pi\theta}}\,e^{-\omega^2} - Ge^{G(2\sqrt{\theta}\omega+G\theta)}\operatorname{erfc}\,(G\sqrt{\theta}+\omega)\right), \end{aligned}$$

where $\omega = \dfrac{R\psi}{2\sqrt{\alpha_1\theta}}$.

We now make a comparison of some of the solutions obtained in this chapter. Our solution to the problem where $\Omega = 0$ and $\ell \neq 0$ was

$$\begin{aligned} (3.52)\quad T_1 = {} & -\frac{A}{E}\,4\theta(i^2\operatorname{erfc}\,\omega) + \left(-\frac{A(BE-a_4)}{E^2}\,2\sqrt{\theta/\pi}\right)e^{-\omega^2} \\ & + \left(2\sqrt{\theta}\omega\,\frac{A(BE-a_4)}{E^2} + \frac{Aa_4(BE-a_4)}{E^3}\right)\operatorname{erfc}\,\omega \\ & + \left(\frac{ABE}{a_4^2}\,e^{\left(\frac{E}{a_4}\right)2\sqrt{\theta}\omega}e^{E^2\theta/a_4^2}\right)\operatorname{erfc}\left(\frac{E}{a_4}\sqrt{\theta}+\omega\right) + a_1\theta\ . \end{aligned}$$

Our solution to the problem where $\ell = 0$ and $\Omega \neq 0$ was

$$(3.53)\quad T_1 = -\frac{A}{E}\,4\theta(i^2\operatorname{erfc}\,\omega) + \frac{A(a_2a_5)}{E^2}\,(2\sqrt{\theta/\pi})\,e^{-\omega^2}$$

$$+\left(-\frac{A(a_2a_5)}{E^2}\,2\sqrt{\theta}\omega - \frac{A(a_2a_5)^2}{E^3}\right)\operatorname{erfc}\,\omega$$

$$+\left(\frac{A(a_2a_5)^2}{E^3}\,e^{\left(\frac{2E\sqrt{\theta}\omega}{a_2a_5}\right)}e^{\left(\frac{E^2\theta}{(a_2a_5)^2}\right)}\right)\operatorname{erfc}\left(\frac{E\sqrt{\theta}}{a_2a_5}+\omega\right)$$

$$+ a_1\theta\ .$$

To compare these two solutions we will need to look at some of the constant terms that appear in them. For convenience we list some of these constants below.

$$a_1 = \frac{Q\alpha_1}{k_1S}\ ,\quad a_2 = \frac{\Omega k_1 2\pi rS}{\sqrt{\alpha_1}}\ ,\quad a_3 = \frac{2k_1S}{\sqrt{\alpha_1}}\ ,$$

$$a_4 = c_2\rho_2 r\ell \quad \text{and} \quad a_5 = \frac{k_2 r}{\sqrt{\alpha_2}}\ .$$

$$A = a_1a_5\ ,\ B = \frac{a_4}{a_5}\ ,\ C = a_2a_4,$$

$$D = a_4+a_2a_5, \quad \text{and} \quad E = a_3+a_5\ .$$

Observing that $\Omega = 0$ implies that $a_2 = 0$, we see that $\Omega = 0$ implies that $D = a_4$ and $C = 0$. We also observe that $\ell = 0$ implies that $a_4 = 0$ and that therefore $\ell = 0$ implies that $B = C = 0$ and $D = a_2a_5$. The above observations allow us to write one equation for T_1 for both (3.52) and (3.53). This equation is

$$(3.54)\quad T_1 = -\frac{A}{E}\,4\theta(i^2\operatorname{erfc}\,\omega) + 2\,\frac{A}{E}\sqrt{\theta/\pi}\left(\frac{D}{E} - B\right)e^{-\omega^2}$$

$$+\frac{A}{E}\left(\frac{BD}{E} + \frac{CE - D^2}{E^2} + 2\sqrt{\theta}\omega\left(B - \frac{D}{E}\right)\right)\operatorname{erfc}\,\omega$$

$$+\frac{A(D - BE)D}{E^3}e^{(E/D)(2\sqrt{\theta}\omega + \frac{E}{D}\theta)}\operatorname{erfc}\left(\frac{E}{D}\sqrt{\theta} + \omega\right) + a_1\theta\ .$$

This equation is identical to equation (3.23), the equation representing the solution to the general bent wire problem where $G \neq H$, except for the second to the last term.

CHAPTER IV

NONDIMENSIONALIZED SOLUTION

The solution of problems in conduction of heat can always be expressed in terms of a number of dimensionless quantities. It is always desirable to make this change before making numerical computations from the solutions.

The solutions obtained in previous chapters were found without nondimensionalyzing the equations. We can however obtain solutions involving dimensionless parameters by making appropriate substitutions. (A more direct method of obtaining dimensionless parameters in the solution would have been to nondimensionalize before solving the equations).

We now proceed to nondimensionalize (3.23), the general solution to the bent wire problem with $G \neq H$.

As the solution stands before nondimensionalization, we have T_1 in units of temperature and θ in units of time. We would like to introduce new dimensionless parameters to represent the magnitude of T_1 and θ. The initial differential equations will indicate substitutions leading to dimensionless parameters. Since we are interested in only T_1, the temperature on the skin, we need only look at the first equation for the bent wire problem. This equation is

$$(4.1) \qquad \frac{\partial^2 T_1}{\partial \psi^2} + \frac{QR^2}{k_1 S} = \frac{R^2}{\alpha_1} \frac{\partial T_1}{\partial \theta} .$$

We observe that multiplying both sides of (4.1) by $\frac{k_1 S}{QR^2}$ will reduce it to an expression equating two numbers without units, namely

$$(4.2) \qquad \frac{k_1 S}{QR^2} \frac{\partial^2 T_1}{\partial \psi^2} + 1 = \frac{k_1 S}{Q\alpha_1} \frac{\partial T_1}{\partial \theta} .$$

A desirable substitution for T_1 can now be seen to be $T_1 = \frac{\tau_1 QR^2}{k_1 S}$; where τ_1 is our new dimensionless parameter. Substituting τ_1 for T_1 in the right member of (4.2) we obtain

$$\frac{k_1 S}{Q\alpha_1} \frac{\partial T_1}{\partial \theta} = \frac{k_1 S}{Q\alpha_1} \frac{\partial T_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \theta}$$

$$= \frac{k_1 S}{Q\alpha_1} \frac{QR^2}{k_1 S} \frac{\partial \tau_1}{\partial \theta}$$

$$= \frac{R^2}{\alpha_1} \frac{\partial \tau_1}{\partial \theta}$$

and the desirable substitution for θ can be seen to be $\theta = \frac{R^2 \phi}{\alpha_1}$, where ϕ is our new dimensionless time parameter.

Solving for τ_1 in terms of T_1 we obtain $\tau_1 = \frac{k_1 S}{QR^2} T_1$. Thus if we multiply the solution for T_1 by the factor $\frac{k_1 S}{QR^2}$ and replace θ by $\frac{R^2 \phi}{\alpha_1}$, we will be able to obtain a nondimensionalized solution. Our solutions for T_1 was

$$(4.3)\quad T_1 = -\frac{A}{E}\,4\theta(i^2\operatorname{erfc}\,\omega) + 2\frac{A}{E}\sqrt{\theta/\pi}(D/E - B)e^{-\omega^2}$$

$$+ \frac{A}{E}\left(\frac{BD}{E} + \frac{CE-D^2}{E^2} + 2\sqrt{\theta}\omega(B-D/E)\right)\operatorname{erfc}\,\omega$$

$$+ \frac{a_9 A}{C}\,(BG - 1)e^{G(2\sqrt{\theta}\omega+G\theta)}\operatorname{erfc}\,(G\sqrt{\theta} + \omega)$$

$$+ \frac{a_{10}A}{C}\,(BH - 1)e^{H(2\sqrt{\theta}\omega+H\theta)}\operatorname{erfc}\,(H\sqrt{\theta} + \omega)$$

$$+ a_1\theta\ .$$

We first note that $\omega = \frac{R\psi}{2\sqrt{\alpha_1\theta}} = \frac{\psi}{2\sqrt{\phi}}$ is already dimensionless. Our task is thus to nondimensionalize G, H, $\frac{A}{C}$, $\frac{D}{E}$, B, $\frac{A}{E}$, and $\frac{C}{E}$. The dimensionless parameters for G and H can be obtained directly from the terms $\operatorname{erfc}\,(G\sqrt{\theta} + \omega)$ and $\operatorname{erfc}\,(H\sqrt{\theta} + \omega)$ respectively using the substitution $\theta = \frac{R^2\phi}{\alpha_1}$. Designating the corresponding dimensionless parameters with an asterisk, we find that

$$G^* = \frac{GR}{\sqrt{\alpha_1}} \quad \text{and} \quad H^* = \frac{HR}{\sqrt{\alpha_1}}\ .$$

To obtain the other dimensionless parameters we multiply T_1 by $\frac{k_1 S}{QR^2}$ and replace θ by $\frac{R^2\phi}{\alpha_1}$. Since T_1 is nondimensional, $\frac{k_1 S}{QR^2}$ times any term in the right member of (4.3) has to be nondimensional. We use this fact to find the dimensionless forms of $\frac{A}{C}$, B, $\frac{A}{E}$, $\frac{D}{E}$, and $\frac{C}{E}$.