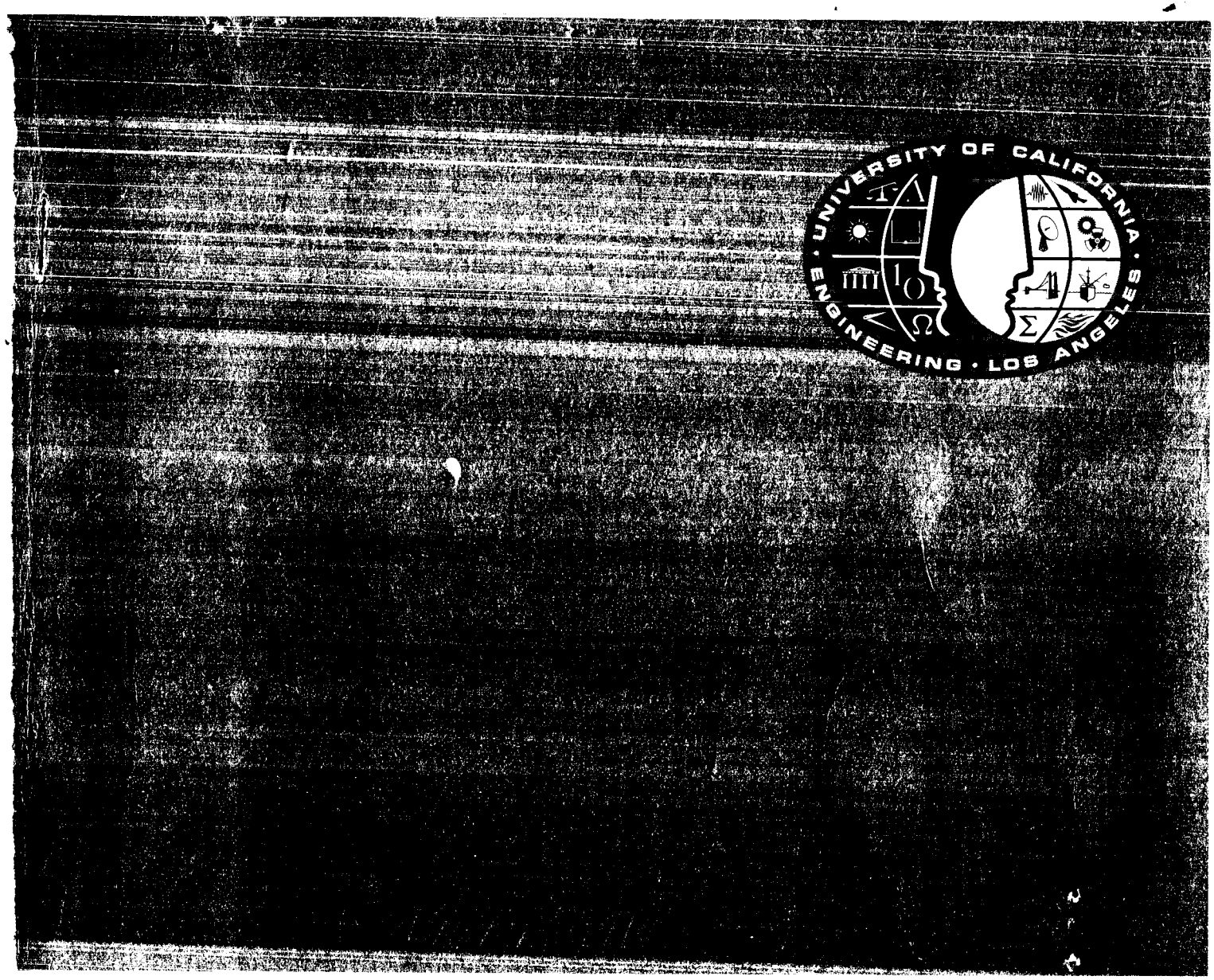

VARIABLE TIME OPTIMAL CONTROL

R. A. NIEMANN

April 1967
Report No. 66-63B

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R. A. Niemann

Department of Engineering
University of California
Los Angeles, California

FOREWORD

The research described in this report, "Variable Time Optimal Control," Number 66-63B, by R. A. Niemann, was carried out under the direction of C. T. Leondes, E. B. Stear, and A. R. Stubberud, in the Department of Engineering, University of California, Los Angeles.

This project is sponsored in part by the National Aeronautics and Space Administration under Grant NsG-237-62 to the Institute of Geophysics and Planetary Physics of the University.

This report was the basis for a dissertation submitted by the Author.

FOREWORD

The ballistic missile reference trajectory problem or the guidance and control problem for the maneuvering reentry vehicle in the case of advanced ballistic missiles all require the solution of the control problem for a time interval unknown beforehand. This is an important and basic problem in many Air Force applications yet little has been done to date on the solution of the problem of developing efficient algorithms for the solution of this problem. This report is concerned with the results of a study in this general area and contains a number of new interesting algorithms and techniques including the results of computational studies supporting all theoretical results.

The research described in this report, "Variable Time Optimal Control," No. 66-63, by Robert A. Niemann, was carried out under the direction of C. T. Leondes, A. R. Stubberud, and E. B. Stear, Co-principal Investigators in the Department of Engineering, University of California, Los Angeles.

This project is part of the continuing investigation of Advanced Ballistic Missile Retargeting sponsored by the U.S. Air Force, Ballistic Systems Division, under Contract No. AF04(694)-826.

This report was the basis for a dissertation submitted by the author.

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LIST OF SYMBOLS

$\underline{x}$	n-dimensional state variable vector
$\underline{\psi}$	n-dimensional adjoint vector
$\underline{u}$	r-dimensional control vector
$(\cdot)$	(d/dt)
$\underline{\lambda}$	initial condition on adjoint vector
$\underline{X}$	terminal condition on state variable vector
t	time
T	final time
$\left\{\frac{\partial H}{\partial \psi}\right\}, \left\{\frac{\partial H}{\partial x}\right\}$	n-dimensional vectors with components $\partial H/\partial \psi_i$ $\partial H/\partial x_i$, respectively
$\left[\frac{\partial^2 H}{\partial x \partial \psi}\right], \left[\frac{\partial^2 H}{\partial x^2}\right]$	nxn matrices with elements $\partial^2 H/\partial x_i \partial \psi_j$, $\partial^2 H/\partial x_i \partial x_j$, respectively
$\left[\frac{\partial x}{\partial \lambda}\right], \left[\frac{\partial \psi}{\partial \lambda}\right]$	nxn matrices with elements $\partial x_i/\partial \lambda_j$, $\partial \psi_i/\partial \lambda_j$, respectively
$[Y]$	nxn matrix with elements Y_{ij}
$\left[\ \right]^T, \left\{\ \right\}^T$	transpose of matrix, vector, respectively
$(\ )^k$	superscripts refer to iteration number
$(\ )_i$	subscripts refer to component number
$\lvert\ \rvert$	absolute value
ϵ	is an element of
$\subset$	is a subset of

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CHAPTER 1
INTRODUCTION

The basic function of the engineer is the design of systems which not only will perform a desired task, but will perform it in some optimum or "best" manner. In some types of engineering problems, this design relies heavily on approximate methods and engineering intuition, since many problems are not formulated in a manner so that they are amenable to rigorous analysis. Some problems, however, do have a very precise mathematical formulation, and thus very rigorous mathematical techniques may be developed for solving them. Such is the problem of optimal control.

The classical problem of optimal control involves the determination of a control which will transfer a given system described by a set of differential equations from one state to another in such a way that some performance criterion is minimized (or maximized). In such problems, the final time may be fixed or it may be variable. For each fixed time, an optimal control may be determined which minimizes the given performance criterion. The variable time problem involves finding that final time for which the minimized performance criterion is minimal over all other final times.

The mathematical theory for the formulation of necessary conditions for the solution of optimal control problems is already well developed in the literature. This theory ranges from the classical calculus of variations[6] to the more recently developed Pontryagin's maximum principle.[15] The application of this theory to complex problems is extremely difficult and often analytical solutions are impossible.

With the development of high speed digital computers, there has been considerable investigation into the development of computational algorithms for the solution of the two-point boundary value problems resulting from the application of the general optimal control theory. Such techniques as quasilinearization,[10] sequential optimization,[8] steepest descent,[4] and methods employing the second variation[3] are well reported in the literature.

However, the criteria used to determine the optimum in such techniques as Pontryagin' s maximum principle are merely necessary conditions rather than sufficient conditions. In general it is assumed that solutions obtained using such necessary conditions are indeed optimum because if one knows that a minimum (or maximum) of the performance criterion exists, and that only one solution exists which satisfies the necessary conditions, then this solution must be the desired optimum. This assumption is generally valid for the fixed time problem when one is merely looking for a solution to a set of differential equations which satisfies given boundary conditions for a fixed time. However, in the variable time problem, the additional necessary condition that the Hamiltonian $H = 0$ may be satisfied for many different final times, as will be demonstrated in the chapters to follow.

In most of the computational algorithms thus far developed for solving variable time problems, the time is varied continuously while the solutions of the differential equations are being iterated toward the desired boundary values. However, approaching the variable time problem by considering a series of fixed time problems, i.e., by first determining the optimum and solving the boundary conditions for a fixed time before varying the time to get an improved optimum, one gets a better insight into the problems

involved in using the $H = 0$ necessary condition. This paper will consider this approach to the variable time problem, and will develop computational techniques employing sufficient conditions for a local minimum for the variable time problem.

Chapter 2 concerns a brief mathematical treatment of Pontryagin' s maximum principle and a derivation of the $H = 0$ condition for variable time optimum.

In Chapter 3, two computational algorithms for solving the fixed time problem, sequential optimization and quasilinearization, are described. A numerical example is given and computational results for each algorithm are discussed.

In Chapter 4 a computational algorithm is developed which extends the two algorighms discussed in Chapter 2 to the variable time problem. This algorithm employs only the $H = 0$ necessary condition and solves the variable time problem as a series of fixed time problems.

Chapter 5 extends the method of Chapter 4 by adding sufficient conditions for a local minimum over all final times of the minimized performance criterion for various fixed final times.

In Chapter 6 the variable end point problem is considered, and a computational algorithm employing the transversality condition, the standard necessary condition for a variable end point optimum, is developed. It is shown that in using this necessary condition one encounters the same kind of problems as in using the $H = 0$ condition.

Chapter 7 presents conclusions and recommendations for future work.

CHAPTER 2

OPTIMAL CONTROL THEORY

2.1 Mathematical Formulation of the Optimal Control Problem

We are given a system whose state is described by the n-dimensional vector differential equation:

$$\dot{\underline{x}} = \underline{f}(x, u) \tag{2.1.1}$$

where

$\underline{x} = \underline{x}(t)$ is the n-dimensional phase coordinate vector

$\underline{u} = \underline{u}(t)$ is the r-dimensional control vector

$(\dot{})$ represents differentiation with respect to the independent variable time.

The state variable Equation (2.1.1) is subject to the boundary conditions:

$$\underline{x}(t)\Big|_{t=0} = \underline{x}(0)$$

$$\underline{x}(t)\Big|_{t=T} = \underline{x}(T)$$

where the final time T may be a fixed quantity in the given conditions or a variable in the optimization process.

We are also given a performance criterion or cost functional of the form:

$$J(u) = \int_0^T f_0[x(t),\ u(t)]dt \tag{2.1.2}$$

The functions $f_i(x, u)$, $i = 0, 1, 2, \ldots n$, must be continuous functions of the variables $x_1, x_2, \ldots, x_n$ and $u_1, \ldots, u_r$ and continuously differentiable with respect to $x_1, x_2, \ldots, x_n$.

The r-dimensional control vector $\underline{u}(t)$ is constrained to lie in some subset U of the r-dimensional Euclidean space E_r

(i.e., $\underline{u} \in U \subset E_r$). The space U may be equal to the space E_r, in which case the problem becomes the unbounded control problem, or may be a proper subset of E_r, in which case the problem is the bounded control problem.

The optimal control problem is to find the admissible control ($\underline{u} \in U$) which transfers the system from the given initial state to the given terminal state in such a way that the performance criterion is minimized (or maximized).

2.2 Pontryagin's Maximum Principle

The methods of solving the optimal control problem to be used in this paper will be based upon the necessary conditions for an optimum given by Pontryagin, which will now be discussed here.

In order to formulate the maximum principle, we must add to our original system a coordinate given by:

$$\dot{x}_o = f_o[x(t),\ u(t)] \tag{2.2.1}$$

thus giving a final $n + 1$ dimensional system

$$\dot{x}_i = f_i(x, u) \quad i = 0, 1, 2, \ldots n \tag{2.2.2}$$

We must also define a new $n + 1$ dimensional vector differential equation adjoint to our original system:

$$\frac{d\psi_i}{dt} = - \sum_{\alpha=1}^{n} \frac{\partial f_\alpha(x, u)}{\partial x_i} \psi_\alpha \quad i = 0, 1, \ldots n \tag{2.2.3}$$

We further define the Hamiltonian:

$$H(\psi, x, u) = \underline{f}^T \cdot \underline{\psi} = \sum_{\alpha=o}^{n} \psi_\alpha f_\alpha(x, u) \tag{2.2.4}$$

It may easily be shown from (2.2.2), (2.2.3) and (2.2.4) that:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial \psi_i} \qquad i = 0, 1, \ldots n \qquad (2.2.5)$$

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad i = 0, 1, \ldots n \qquad (2.2.6)$$

Finally let us define:

$$M(\psi, x) = \max_{u \in U} H(\psi, x, u) \qquad (2.2.7)$$

We are now ready to state the maximum principle,[15]

Theorem: Let $\underline{u}(t)$, $0 \le t \le T$, be an admissible control which transfers the state of the system (2.1.1) from $\underline{x} = \underline{x}(0)$ to $\underline{x} = \underline{x}(T)$. In order that $\underline{u}(t)$ and $\underline{x}(t)$ be optimal, it is necessary that there exist a nonzero continuous vector function $\underline{\psi}(t)$ defined by (2.2.3) such that:

1. For every t, $0 \le t \le T$, the function $H(\psi, x, u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$:

$$H(\psi, x, u) = M(\psi, x) \qquad (2.2.8)$$

2. $\psi_o(t) \le 0 \qquad M(\psi, x) = 0 \qquad (2.2.9)$
   for all t.

This is the formulation of the maximum principle for variable T. If T is fixed, the condition $H(\psi, x, u) = 0$ is removed, but H must still be a constant.

A condition on $\psi_o(t)$ may be determined. From (2.2.9) we know that $\psi_o(t) \le 0$. Since x_o does not appear explicitly in the $f_i(x, u)$, we see from (2.2.4) and (2.2.6) that $\dot{\psi}_o = 0$ and thus $\psi_o =$ constant. Furthermore, since ψ_o is the term which multiplies

$\dot{x}_o$, which is the derivative of the quantity to be minimized, ψ_o may be of arbitrary magnitude, since multiplying a function to be minimized by a constant does not affect the minimization problem. Thus for convenience we may take:

$$\psi_o(t) = -1 \tag{2.2.10}$$

2.3 Derivation of H = 0 Necessary Condition

Since we are considering the variable time problem, it will be instructive to go through the derivation of the H = 0 condition in order to determine the implications of its being a necessary rather than a sufficient condition.

Let us consider the performance criterion given by (2.2.1):

$$J(u, T, X) = \int_o^T f_o[x(t), u(t)]dt$$

where $X = \underline{x}(T)$ is the terminal state of the system. Let us define:

$$S(T, X) = J(\bar{u}, T, X) \tag{2.3.1}$$

where $\bar{u}$ is the optimal control which minimizes J for the given T and X.

Let us consider the difference δS between S for a given T and X and S evaluated for some $T + \delta T$:

$$\begin{aligned}\delta S &= S(T+\delta T, X) - S(T, X)\\ &= \int_T^{T+\delta T} f_o(x+\delta x, \bar{u}+\delta u)dt - \int_o^T f_o(x, \bar{u})dt\\ &= \int_T^{T+\delta T} f_o(x+\delta x, \bar{u}+\delta u)dt\\ &\quad + \int_o^T \left[f_o(x+\delta x, \bar{u}+\delta u) - f_o(x, \bar{u}) \right] dt \end{aligned} \tag{2.3.2}$$

It should be noted that in varying the final time T by δT, it is necessary to vary $\underline{x}(t)$ and $\underline{u}(t)$ by $\delta\underline{x}(t)$ and $\delta\underline{u}(t)$ respectively over the whole length of the trajectory in order to hold the final state X constant.

If we expand $f_o(x + \delta x, \bar{u} + \delta u)$ in a Taylor Series about $\underline{x}$ and $\underline{u}$, and neglect second and higher order terms, we obtain

$$\delta S = f_o(x, \bar{u})\Big|_{t=T} \delta T + \int_o^T \left[\sum_i \frac{\partial f_o}{\partial x_i} \delta x_i + \sum_\alpha \frac{\partial f_o}{\partial u_\alpha} \delta u_\alpha \right] dt \quad (2.3.3)$$

where the indices i and α vary over n and r, respectively.

Let us first consider the unbounded control case; we will then show how the development may easily be extended to the bounded control case.

From the maximum principle, we know that for an optimum, the Hamiltonian:

$$H = \sum_{j=0}^{n} \psi_j f_j = \sum_{j=1}^{n} \psi_j f_j - f_o \ ,$$

where $\psi_o = -1$ has been substituted, must be a maximum over u. If we assume that an optimum u exists and that it is unique, this maximum occurs at:

$$\frac{\partial H}{\partial u_\alpha} = 0 \qquad \alpha = 1, 2, \ldots . r \qquad (2.3.4)$$

but

$$\frac{\partial H}{\partial u_\alpha} = \sum_j \psi_j \frac{\partial f_j}{\partial u_\alpha} - \frac{\partial f_o}{\partial u_\alpha} = 0$$

or

$$\sum_j \psi_j \frac{\partial f_j}{\partial u_\alpha} = \frac{\partial f_o}{\partial u_\alpha} \qquad (2.3.5)$$

If we substitue (2.3.5) for $\frac{\partial f_o}{\partial u_\alpha}$ into (2.3.3), we obtain:

$$\delta S = f_o(x, \bar{u}) \Big|_{t=T} \delta T + \int_o^T \left[\sum_i \frac{\partial f_o}{\partial x_i} \delta x_i + \sum_{\alpha, j} \psi_j \frac{\partial f_j}{\partial u_\alpha} \delta u_\alpha \right] dt \qquad (2.3.6)$$

We now examine the perturbations about our given state variable equations $\dot{\underline{x}} = \underline{f}(x, u)$:

$$\frac{d}{dt}(\underline{x} + \delta \underline{x}) = \underline{f}(x + \delta x,\ u + \delta u) \qquad (2.3.7)$$

Again expanding in a Taylor Series and neglecting second and higher order terms:

$$\delta \dot{x}_j = f_j(x + \delta x,\ u + \delta u) - f_j(x, u)$$

$$= \sum_i \frac{\partial f_j}{\partial x_i} \delta x_i + \sum_\alpha \frac{\partial f_j}{\partial u_\alpha} \delta u_\alpha \qquad (2.3.8)$$

where j ranges over n.

If we solve (2.3.8) for $\sum_\alpha \frac{\partial f_i}{\partial u_\alpha} \delta u_\alpha$ and substitute it into (2.3.6) we obtain:

$$\delta S = f_o(x, \bar{u}) \Big|_{t=T} \delta T + \int_o^T \left\{ \sum_i \left[\frac{\partial f_o}{\partial x_i} - \sum_j \psi_j \frac{\partial f_j}{\partial x_i} \right] \delta x_i + \sum_j \psi_j \delta \dot{x}_j \right\} dt \qquad (2.3.9)$$

But, from (2.2.7):

$$\frac{\partial f_o}{\partial x_i} - \sum_j \psi_j \frac{\partial f_j}{\partial x_i} = -\frac{\partial H}{\partial x_i} = \dot{\psi}_i \qquad (2.3.10)$$

Substituting into (2. 2. 9)

$$\delta S = f_o(x, \bar{u}) \Big|_{t=T} \delta T + \int_o^T \sum_i \left(\dot{\psi}_i \, \delta x_i + \psi_i \, \delta \dot{x}_i\right) dt$$

$$= f_o(x, \bar{u}) \Big|_{t=T} \delta T + \int_o^T \sum_i \frac{d}{dt} \left[\psi_i \, \delta x_i\right] dt$$

$$= f_o(x, \bar{u}) \Big|_{t=T} \delta T + \sum_i \psi_i \, \delta x_i \Big|_{t=0}^{t=T} \qquad (2.3.11)$$

By our problem formulation, the $\delta x_i(0) = 0$. Let $\underline{x}^j(t)$ be the optimum trajectory for final time T and $\underline{x}^{j+1}(t)$ be the optimum trajectory for the final time $T + \delta T$. Again, by our problem formulation:

$$\underline{x}^j(T) = \underline{x}^{j+1}(T + \delta T) = X \qquad (2.3.12)$$

But, to a first order approximation:

$$\underline{x}^{j+1}(T + \delta T) = \underline{x}^{j+1}(T) + \dot{\underline{x}}^{j+1}(T) \, \delta T \qquad (2.3.13)$$

Thus:

$$\underline{x}^j(T) = \underline{x}^{j+1}(T) + \dot{\underline{x}}^{j+1}(T) \, \delta T$$

or

$$\underline{x}^{j+1}(T) - \underline{x}^j(T) = - \dot{\underline{x}}^{j+1}(T) \, \delta T = \delta \underline{x}(T) \qquad (2.3.14)$$

Therefore, substituting $\delta x_i = - f_i(x, \bar{u}) \, \delta T$ into (2. 3. 11), we obtain:

$$\delta S = \left[f_o(x, \bar{u}) - \sum_i \psi_i \, f_i(x, \bar{u}) \right] \Big|_{t=T} \delta T$$

$$= - H(\psi, x, \bar{u}) \, \delta T \qquad (2.3.15)$$

Dividing by δT and letting $\delta T \to 0$:

$$H(\psi, x, u) = - \frac{\partial S(X, T)}{\partial T} \qquad (2.3.16)$$

This development[16] applies only to unbounded control, by virtue of our use of (2. 3. 5). However, if our control is bounded, (2. 3. 5) does not apply during that portion of the optimum trajectory for which the control is bounded. Also, during this portion of the trajectory, $\delta u_\alpha = 0$. Thus, (2. 3.6) becomes:

$$\delta S = f_o(x, \bar{u}) \Big|_{t=T} \delta T + \int_o^T \left[\sum_i \frac{\partial f_o}{\partial x_i} \delta x_i \right] dt \qquad (2.3.17)$$

Also, (2. 3. 8) becomes:

$$\delta \dot{x}_i = \sum_j \frac{\partial f_i}{\partial x_j} \delta x_j$$

or

$$\delta \dot{x}_i - \sum_j \frac{\partial f_i}{\partial x_j} \delta x_j = 0 \qquad (2.3.18)$$

If we multiply (2. 3. 18) by ψ_i and sum over i:

$$\sum_i \psi_i \, \delta \dot{x}_i - \sum_{i,j} \psi_i \frac{\partial f_i}{\partial x_j} \delta x_j = 0 \qquad (2.3.19)$$

We can add (2. 3. 18) to (2. 3. 17) without changing the value of (2. 3. 17) in which case (2. 3. 17) becomes equivalent to (2. 3. 9) for the unbounded case, from whence the derivation of (2. 3. 16) for bounded control proceeds exactly as for the unbounded case.

Equation (2. 3. 16) is, of course, the Hamiltonian-Jacobi equation from the Calculus of Variations. However, its relation to Pontryagin's necessary condition for a variable time optimum has never been properly explored.

If we consider a graph of $S(T, X)$ vs. T for fixed X, that is, if we consider a graph of the values of the minimum performance

criterion for each fixed time T plotted as a function of T, the Hamiltonian H is the negative of the derivative with respect to T of this curve. Thus the condition $H = 0$ for a minimum of $S(T,X)$ is a necessary condition in the same sense that the condition that the derivative of a general function be zero is a necessary condition for the minimum of that function. However, the condition $H = 0$ will thus also hold for a local maximum of the function $S(T,X)$, and any computational algorithm employing only the $H = 0$ condition for variable time optimum can yield such a local maximum if it exists. In the chapters to follow, an example will be given in which both a local minimum and a local maximum of $S(T,X)$ do indeed exist (in fact, for which several such local maxima or minima exist).

CHAPTER 3

COMPUTATIONAL ALGORITHMS FOR SOLUTION OF FIXED TIME PROBLEMS

3.1 The Two-Point Boundary Problem

There is no particularly great mathematical sophistication required in writing out the various mathematical expressions called for in the application of the maximum principle to a given problem; even in problems of great complexity, deriving the expressions for the adjoint equations, the Hamiltonian, and the optimal control program requires only basic calculus. However, the solution of the resulting differential equations, even for very simple problems, is often impossible analytically. Furthermore, due to the fact that the boundary conditions on the differential equations are not given for all the equations at the same time instant, (i.e., the classical two point boundary value problem), computational solution on digital computers is not straight forward.

The application of the maximum principle to the given n state variable equations results in a final set of $2n$ differential equations to be solved, with the n state variable equations coupled to the n adjoint equations through the optimal control program. However, the $2n$ boundary conditions are all given on the state variable equations (n at the initial state and n at the terminal state). Thus, in order to computationally determine the complete solution to the n equations, an iterative technique proceeding from some initial guess of the solution to the final solution is needed.

Such iterative techniques for solution of the two-point boundary value problems fall into two general classes: those which start with initial estimates on the adjoint boundary values and iterate until all boundary conditions on the state variable equations are

satisfied, and those which start with initial estimates on the entire trajectories which satisfy all the boundary conditions but do not necessarily satisfy the 2n differential equations, and iterate until all the given differential equations are satisfied.

In this paper, sequential optimization will be considered as an example of the former and quasilinearization as an example of the latter. These two methods are chosen because they have not previously been extended to the variable time problem for the general case.

3.2 Sequential Optimization

This technique[8] solves the optimal control problem for a system of nonlinear differential equations by transforming the problem into a sequence of problems consisting of linear differential equations. The linear problem is an easier one to solve, and we will briefly outline a standard technique for its solution later in the paper. First we shall consider the sequential optimization technique itself.

Let us first write the first order perturbation equations for our given set of nonlinear equations $\underline{\dot{x}} = \underline{f}(x, u) = \{\partial H/\partial \psi\}$:

$$\delta\underline{\dot{x}} = \left[\frac{\partial^2 H}{\partial \psi \partial x}\right] \delta\underline{x} + \left[\frac{\partial^2 H}{\partial \psi \partial u}\right] \delta\underline{u} \qquad (3.2.1)$$

where $\left[\partial^2 H/\partial\psi\partial x\right]$ is an nxn matrix with elements $A_{ij}(t) = \partial^2 H/\partial\psi_i \partial x_j$, $\left[\partial^2 H/\partial\psi\partial u\right]$ is an nxr matrix with elements $B_{ij}(t) = \partial^2/H\partial\psi_i \partial u_j$, and $\delta\underline{\dot{x}}$, $\delta\underline{x}$ and $\delta\underline{u}$ are column vectors of order n, n, and r, respectively.

Similarly let us write first order perturbation equations for our adjoint system $\underline{\dot{\psi}} = -\{\partial H/\partial x\}$:

$$\delta\dot{\underline{\psi}} = -\left[\frac{\partial^2 H}{\partial x^2}\right]\delta\underline{x} - \left[\frac{\partial^2 H}{\partial x \partial u}\right]\delta\underline{u} - \left[\frac{\partial^2 H}{\partial x \partial \psi}\right]\delta\underline{\psi} \qquad (2.3.2)$$

where $\left[\partial^2 H/\partial x^2\right]$ is an nxn matrix with elements $\partial^2 H/\partial x_i\,\partial x_j$, $\left[\partial^2 H/\partial x \partial u\right]$ is an nxr matrix with elements $\partial^2 H/\partial x_i\,\partial u_j$, and $\left[\partial^2 H/\partial x \partial \psi\right]$ is an nxn matrix with elements $\partial^2 H/\partial x_i\,\partial \psi_i$.

Finally, let us write the second order perturbation equation for $H(\psi, x, u)$:

$$\delta H = \delta\underline{\psi}^T\left\{\frac{\partial H}{\partial \psi}\right\} + \delta\underline{x}^T\left\{\frac{\partial H}{\partial x}\right\} + \delta\underline{u}^T\left\{\frac{\partial H}{\partial u}\right\} + \frac{1}{2}\left(\delta\underline{x}^T\left[\frac{\partial^2 H}{\partial x^2}\right]\delta\underline{x} + \delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u^2}\right]\delta\underline{u}\right.$$
$$\left. + 2\,\delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u \partial \psi}\right]\delta\underline{\psi} + 2\,\delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u \partial x}\right]\delta\underline{x} + 2\,\delta\underline{\psi}^T\left[\frac{\partial^2 H}{\partial \psi \partial x}\right]\delta\underline{x}\right) \qquad (3.2.3)$$

where in writing (3.2.3) we made use of the fact that the order of taking partial derivatives is interchangeable for $H(\psi, x, u)$ continuous with continuous derivatives, e.g., $\partial^2 H/\partial\psi_i\,\partial x_j = \partial^2 H/\partial x_j\,\partial\psi_i$, and that $\partial^2 H/\partial\psi^2 = 0$ since H is linear in ψ. Also the matrix symbolism used is analogous to that used in (3.2.1) and (3.2.2).

Applying the maximum principle to our new Hamiltonian δH, i.e., setting $\partial\delta H/\partial\delta u = 0$ and solving for δu:

$$\delta\underline{u}_{opt} = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left(\left\{\frac{\partial H}{\partial u}\right\} + \left[\frac{\partial^2 H}{\partial u \partial \psi}\right]\delta\underline{\psi} + \left[\frac{\partial^2 H}{\partial u \partial x}\right]\delta\underline{x}\right) \qquad (3.2.4)$$

Note that $\{\partial H/\partial u\} = 0$ if the u employed is optimal for the nonlinear problem.

We want (3.2.1) and (3.2.2) to be adjoint to each other, and that δH be the Hamiltonian for them, i.e., that:

$$\left.\begin{aligned} \delta\dot{\underline{x}} &= \left\{\frac{\partial \delta H}{\partial \delta \psi}\right\} \\ \delta\dot{\underline{\psi}} &= -\left\{\frac{\partial \delta H}{\partial \delta x}\right\} \end{aligned}\right\} \tag{3.2.5}$$

and

$$\delta H = \delta\underline{\psi}^T \cdot \delta\dot{\underline{x}} - \delta f_o \,. \tag{3.2.6}$$

If we drop the first two terms of (3.2.3) for δH, the resulting equation:

$$\delta H = \delta\underline{u}^T\left\{\frac{\partial H}{\partial u}\right\} + \frac{1}{2}\left(\delta\underline{x}^T\left[\frac{\partial^2 H}{\partial x^2}\right]\delta\underline{x} + \delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u^2}\right]\delta\underline{u} + 2\,\delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u \partial \psi}\right]\delta\psi + 2\,\delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u \partial x}\right]\delta\underline{x} + 2\,\delta\underline{\psi}^T\left[\frac{\partial^2 H}{\partial \psi \partial x}\right]\delta\underline{x}\right) \tag{3.2.7}$$

satisfies (3.2.5). In order for (3.2.6) to hold, we must define δf_o according to the following:

$$\delta J = \int_o^T \delta f_o\, dt = -\int_o^T \left(\delta\underline{u}^T\left\{\frac{\partial H}{\partial u}\right\} + \frac{1}{2}\left(\delta\underline{x}^T\left[\frac{\partial^2 H}{\partial x^2}\right]\delta\underline{x} + 2\,\delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u \partial x}\right]\delta\underline{x} + \delta\underline{u}^T\left[\frac{\partial^2 H}{\partial u^2}\right]\delta\underline{u}\right)\right) dt \tag{3.2.8}$$

In rewriting δH according to (3.2.7), since the two terms dropped did not contain δu, (3.2.4) still holds for δu_{opt}.

Equations (3.2.1), (3.2.2), (3.2.4), (3.2.7) and (3.2.8) represent a complete formulation for an optimal control problem for the linear equations (3.2.1).

Now let us return to the general nonlinear problem. Let us assume that we have applied the maximum principle to our nonlinear $\dot{\underline{x}} = \underline{f}(x, u)$ and obtained an optimal control program $u = u(x,\psi)$. We now need only solve the two-point boundary problem.

If we knew the initial conditions on the adjoint equations $\underline{\psi}(0)$, we could integrate the equations on a digital computer to obtain the optimal trajectory. Let us make a first estimate $\underline{\lambda}^o$ for $\underline{\psi}(0)$, and integrate our state variable and adjoint equations to obtain the trajectory $\underline{x}^o(t)$. Since our $\underline{\lambda}^o$ was not exactly equal to $\underline{\psi}(0)$, $\underline{x}^o(T) \neq \underline{x}(T)$, the desired terminal state variable condition.

$$\text{Let} \qquad \delta\underline{x}^o(T) = \underline{x}(T) - \underline{x}^o(T) \tag{3.2.9}$$

Let us assume we have a method of solving the linear problem given by (3.2.1), (3.2.2), (3.2.4), (3.2.7) and (3.2.8). Let us solve this problem for boundary conditions $\delta\underline{x}(0) = 0$ and $\delta\underline{x}(T) = \delta\underline{x}^o(T)$. This solution will also produce a trajectory $\delta\underline{\psi}^o(t)$ with $\delta\underline{\psi}^o(0) = \delta\underline{\lambda}^o$. Now let a new estimate on our $\underline{\psi}(0)$ for the nonlinear problem be given by:

$$\underline{\lambda}^1 = \underline{\lambda}^o + \delta\underline{\lambda}^o \tag{3.2.10}$$

Our new iteration on the nonlinear equations will have an error in terminal conditions $\delta\underline{x}'(T)$ which is less than the original error. We can continue this process until the terminal conditions are as close as desired to the given terminal conditions.

It should be noted that according to the degree of nonlinearity of the problem and the closeness of the estimate of $\underline{\psi}(0)$ to the correct value, it may not always be possible to apply the entire error $\delta\underline{x}(T)$ to the linear problem. That is, it might be necessary to take:

$$\delta\underline{x}^n(T) = \epsilon\left(\underline{x}(T) - \underline{x}^n(T)\right) \tag{3.2.11}$$

where $0 < \epsilon \leq 1$.

The above described technique constitutes the sequential optimization method. In the course of describing the method, a

subsidiary linear problem was introduced. This problem involves solution of the linear vector equation:

$$\dot{\underline{y}} = [A(t)]\,\underline{y} + [B(t)]\,\underline{u} \tag{3.2.12}$$

subject to

$$\underline{y}(t)\,\Big|_{t=0} = \underline{y}(0)$$

$$\underline{y}(t)\,\Big|_{t=T} = \underline{y}(T)$$

$$|u_i| \le M_i(t) \qquad i = 1, 2, \ldots, m$$

The performance criterion to be optimized is a quadratic form:

$$J = \int_o^T \left(\underline{a}^T(t)\,\underline{x}(t) + \underline{b}^T(t)\,\underline{u}(t) + \underline{x}^T(t)\,[C(t)]\,\underline{x}(t) + \underline{u}^T(t)\,[D(t)]\,\underline{u}(t) \right) dt \tag{3.2.13}$$

A gradient technique for solving this subsidiary problem is described in Reference 8. In this technique, if we use the maximum principle to determine the control program, and let $\underline{\lambda}^k$ be the k^{th} estimate of the initial adjoint condition, then the $(k+1)^{th}$ is given by:

$$\underline{\lambda}^{k+1} = \underline{\lambda}^k + K[P]\,\delta\underline{y}(T) \tag{3.2.14}$$

where $\delta\underline{y}(T) = \underline{y}(T) - \underline{y}^k(T)$, $[P]$ is an $n \times m$ matrix with elements $P_{ij} = \partial\psi_j / \partial\lambda_i \,|_{t=T}$, and K is the distance along the gradient to be moved.

The $\partial\psi_j / \partial\lambda_i$ are obtained by integrating the equations:

$$\frac{d}{dt}\left[\frac{\partial y}{\partial \lambda}\right] = \left[\frac{\partial^2 H}{\partial y\,\partial \psi}\right]\left[\frac{\partial y}{\partial \lambda}\right] + \left[\frac{\partial^2 H}{\partial \psi^2}\right]\left[\frac{\partial \psi}{\partial \lambda}\right]$$

$$\frac{d}{dt}\left[\frac{\partial \psi}{\partial \lambda}\right] = -\left[\frac{\partial^2 H}{\partial y^2}\right]\left[\frac{\partial \psi}{\partial \lambda}\right] - \left[\frac{\partial^2 H}{\partial \psi\,\partial y}\right]\left[\frac{\partial \psi}{\partial \lambda}\right] \tag{3.2.15}$$

where $\left[\partial\psi/\partial\lambda\right]$ and $\left[\frac{\partial y}{\partial\lambda}\right]$ are nxn matrices with elements $\partial\psi_i/\partial\lambda_j$ and $\partial y_i/\partial\lambda_j$, respectively. The initial conditions are $\partial x_i/\partial\lambda_j = 0$ for all i, j; $\partial\psi_i/\partial\lambda_j = 0$ for $i \neq j$; and $\partial\psi_i/\partial\lambda_j = 1$ for i = j. The H and ψ are the usual Hamiltonian and adjoint variables for the state variable equation (3.2.12). It should be noted that, in order to simplify (3.2.15), $u = u(x, \psi)$ has been substituted into $H=H(\psi, x, u)$ to make $H = H(\psi, x)$, thus eliminating terms such as $\left[\frac{\partial^2 H}{\partial\psi\partial u}\right]\left[\frac{\partial u}{\partial\lambda}\right]$ in (3.2.15).

A method for choosing K, the distance along the gradient to be moved, in an optimal manner to minimize the number of steps required for convergence is given in Reference 8. Powell's acceleration method, which is also described in Reference 8, was used to obtain rapid convergence.

3.3 Quasilinearization[10]

Assume as in the previous section that we have applied the maximum principle to obtain the optimal control program and thus have only the two-point boundary value problem to solve. We thus have a 2n-dimensional system of differential equations of the form:

$$\dot{\underline{y}} = \underline{f}(\underline{y}) \tag{3.3.1}$$

where $\dot{\underline{y}}$, $\underline{y}$, and $\underline{f}$ are 2n-dimensional vectors. Again, we have substituted $u = u(x, \psi)$ in $\dot{\underline{x}} = \underline{f}(x, u)$ to give $\dot{\underline{x}} = \underline{f}(x, \psi)$ in order to allow our state variable and adjoint equations to be formulated as in (3.3.1). The 2n-boundary conditions are given by:

$$\begin{aligned} y_i(t)\big|_{t=0} &= y_i(0) \quad i = 1, 2, \ldots . n \\ y_i(t)\big|_{t=T} &= y_i(T) \quad i = 1, 2, \ldots . n \end{aligned} \tag{3.3.2}$$

This corresponds to our standard two-point boundary value problem

with $x_i(t)$ corresponding to $y_i(t)$ for $i = 1, 2, \ldots n$ and $\psi_{i-n}(t)$ corresponding to $y_i(t)$ for $i = n+1,\ n+2, \ldots, 2n$.

Let us assume that we have made an initial estimate of $\underline{y}(t)$ which satisfies the boundary conditions (3.3.2) but does not necessarily satisfy (3.3.1). We desire an iterative procedure which will finally result in $\underline{y}(t)$ which does satisfy (3.3.1).

Consider the 2n-dimensional set of linear differential equations:

$$\dot{y}_i^{n+1} = \sum_{j=1}^{2n} \frac{\partial f_i}{\partial y_j} \left(y_j^{n+1} - y_j^n \right) + f_i(\underline{y}^n) \quad i = 1, 2, \ldots 2n \tag{3.3.3}$$

where the y_j^n, $f(y_n)$ and $\frac{\partial f_i}{\partial y_j}$ are from the previous (initial) estimate and $\dot{y}_i^{n+1}$ and y_i^{n+1} are the improved estimates. If we have a method of solving the linear problem given by (3.3.3):

$$\dot{\underline{y}}^{n+1} = [A(t)]\, \underline{y}^{n+1} + \underline{B}(t) \tag{3.3.4}$$

we will eventually iterate to a solution $\underline{y}^n(t)$ which comes as close as desired to satisfying (3.3.1) at all points along the trajectory and thus as close as possible to the desired solution $\underline{y}(t)$.

The linear problem may be solved as follows. Generate by numerical integration n solutions to the homogeneous equations $\dot{\underline{y}}^{n+1} = [A(t)]\, \underline{y}^{n+1}$ using n sets of initial conditions with zeros for every $\underline{y}_i(0)$ except the y_{n+j}, $j = 1, 2, \ldots n$, which is equal to one, that is, n sets of initial conditions of the form:

$$\underline{y}^{n+1}(0) = \left\{0, 0, 0, \ldots 0, y_{n+j} = 1, 0, \ldots 0\right\} \tag{3.3.5}$$

Denote these solutions by $\underline{z}_1(t)$, $\underline{z}_2(t), \ldots \underline{z}_n(t)$. Also generate a particular solution $\underline{z}_p(t)$ by integrating (3.3.4) with initial conditions $\underline{y}_p(0) = \left\{x_1(0),\ x_2(0), \ldots x_n(0),\ k_1, k_2 \ldots k_n\right\}$ where the k_i are arbitrary. The desired solution $\underline{y}^{n+1}(t)$ of (3.3.4) is given by:

$$\underline{y}^{n+1}(t) = C_1 \underline{z}_1(t) + C_2 \underline{z}_2(t) + \text{---} + C_n \underline{z}_n(t) + \underline{z}_p(t) \tag{3.3.6}$$

where the $C_1, C_2, \ldots C_n$ are constants which need to be determined. If we set:

$$\underline{y}^{n+1}(T) = C_1 \underline{z}_1(T) + C_2 \underline{z}_2(T) + \text{---} C_n \underline{z}_n(T) + \underline{z}_p(T) \tag{3.3.7}$$

we have n equations in n unknowns $C_1, C_2 \ldots C_n$ which may be solved for the C_i. This completes the method of solving the linear problem (3.3.3).

Therefore, we have outlined a complete method of solving the two-point boundary value problem given by (3.3.1) and (3.3.2), which constitues the quasilinearization technique. Note that in this technique the boundary conditions are always satisfied while we iterate to a solution which satisfies the given differential equations, as opposed to the sequential optimization technique in which we iterate on the initial conditions until the boundary conditions are all satisfied.

3.4 Numerical Results

The problem solved using the techniques outlined in the preceding sections consisted of the system of equations given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2) x_2 - x_1 + u \end{aligned} \tag{3.4.1}$$

with boundary conditions:

$$\begin{aligned} x_1(0) &= 1.0 & x_1(T) &= -0.97 \\ x_2(0) &= 0.0 & x_2(T) &= -0.96 \end{aligned} \tag{3.4.2}$$

with fixed final time $T = 5.0$ seconds.

The performance criterion to be minimized was:

$$J(u) = \int_o^T \left(x_1^2 + x_2^2 + u^2 \right) dt \qquad (3.4.3)$$

The Hamiltonian for the system is thus given by:

$$H = \psi_1 x_2 + \psi_2 \left[\left(1 - x_1^2 \right) x_2 - x_1 + u \right] - x_1^2 - x_2^2 - u^2 \qquad (3.4.4)$$

If we set $\partial H/\partial u = 0$ and solve for u, we obtain:

$$u_{opt} = \psi_{2/2} \qquad (3.4.5)$$

If we have bounds on u of the form:

$$u_{min} \leq u \leq u_{max} \qquad (3.4.6)$$

we can see from (3.4.4) that if u_{opt} given by (3.5.6) is outside the bounds on u, then the Hamiltonian is maximized for admissible u if u is at its constraint. That is, if $u_{max} \leq \psi_{2/2}$, $u_{opt} = u_{max}$, and if $\psi_{2/2} \leq u_{min}$, $u_{opt} = u_{min}$.

The adjoint equations are given by:

$$\dot{\psi}_1 = - \frac{\partial H}{\partial x_1} = 2x_1 x_2 \psi_2 + 2x_1 + \psi_2$$
$$\dot{\psi}_2 = - \frac{\partial H}{\partial x_2} = - \psi_1 - \psi_2 \left(1 - x_1^2 \right) + 2x_2 \qquad (3.4.7)$$

Other expressions which must be derived for the application of quasilinearization and sequential optimization are given in Appendix I.

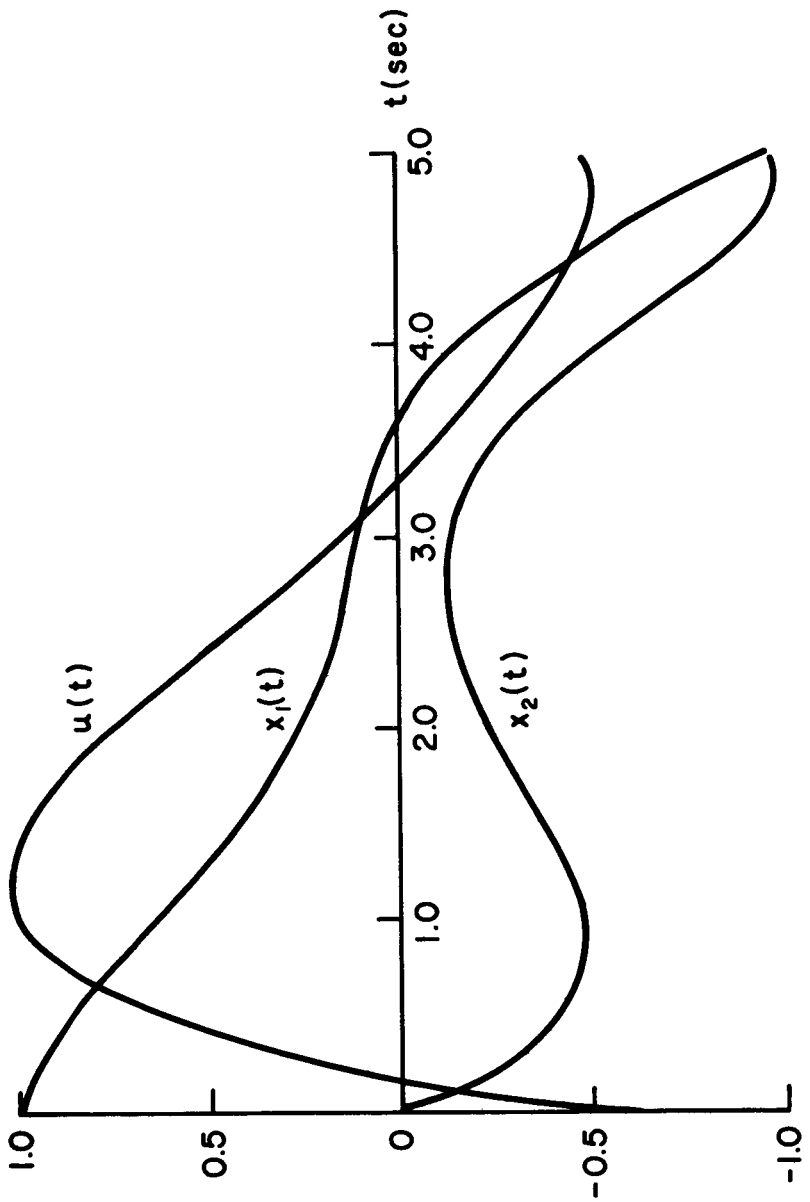
We have now reduced the problem to the two-point boundary problem and we can employ sequential optimization and quasilinearization to solve this problem.

Both these techniques were programmed on the IBM 7094 computer. The problem was solved first for the unbounded case, and the resulting optimal trajectories are shown in Figure 3.4.1. Of course, both techniques yielded the same solution.

Sequential optimization required three and a half minutes of computer time to solve the problem, compared to 32 seconds required for quasilinearization.

The quasilinearization trajectories were accurate to four decimal places all along the trajectories (that is, the last two iterations were identical to four places) and the sequential optimization trajectories were iterated to within 1% of the desired terminal conditions.

The main cause of the greater time required by sequential optimization was the difficulty in making initial estimates on the initial conditions on the adjoint variables. With no other information available, the most reasonable initial estimates to use were $\lambda_1^o = \lambda_2^o = 0$. However, using these initial estimates, the endpoints of the state variable trajectories at $T = 5$ seconds were very far from the desired values, and thus the linearization approximation did not hold and convergence could not be achieved (unless only a very small fraction of the error was applied at each iteration). This problem was solved by halting the integration at some final time less than 5 seconds when the state variables had diverged from the desired endpoints by a prescribed amount, applying sequential optimation to drive the trajectories closer to the desired endpoints, and then allowing the final time to be extended a little more until the trajectories again began to diverge too much, and proceeding in this manner until the final time reached the desired $T = 5$ seconds. The criterion used to halt the integration was:



FIXED TIME OPTIMAL TRAJECTORIES,
UNBOUNDED CONTROL

FIGURE 3.4.1

$$\left| x_i^n(t) - x_i(T) \right| \geq M \left| x_i(0) - x_i(T) \right| \qquad (3.4.7)$$

That is, if the difference between the current value of the state variables and the desired endpoint exceeded some constant M times the difference between the initial state variable condition and the endpoints, integration was halted at that time T' and T' was used as the final time until the expression on the left of (3.5.7) was less than the expression on the right, at which time integration was allowed to proceed. M must be greater than one, or else integration will halt at $t = 0$. For the given problem, it was found that an M just greater than one, say 1.05, was a good choice. Using this technique, sequential optimization would eventually converge using any arbitrary initial guess $\underline{\lambda}^o$, and thus trial and error guesses of $\underline{\lambda}^o$ to get a trajectory close enough to the desired trajectory to allow convergence were not necessary. However, even using this technique, the entire error in terminal conditions could not be applied to the linear problem, and a step size constraint $\epsilon = .7$ had to be applied as in Equation (3.2.11).

In the quasilinearization technique, initial estimates were made on the entire trajectories $\underline{x}^o(t)$ and $\underline{\psi}^o(t)$, and thus it was possible to insure that the initial guesses were not too far from the desired trajectories. $\underline{x}^o(t)$ was chosen to be a straight line between $\underline{x}(0)$ and $\underline{x}(T)$, while $\underline{\psi}^o(t)$ was chosen to be identically zero for all t. Proceeding from these initial estimates, quasilinearization converged readily to the desired solution. Also, in quasilinearization, there was no need to apply a step size constraint, i.e., the entire correction was applied at each iteration.

Sequential optimization required 6 iterations in the main program for the final time to reach the desired value of 5 seconds, and 5 more iterations to reach the desired endpoint, while quasilinearization required only 5 total iterations in the main program.

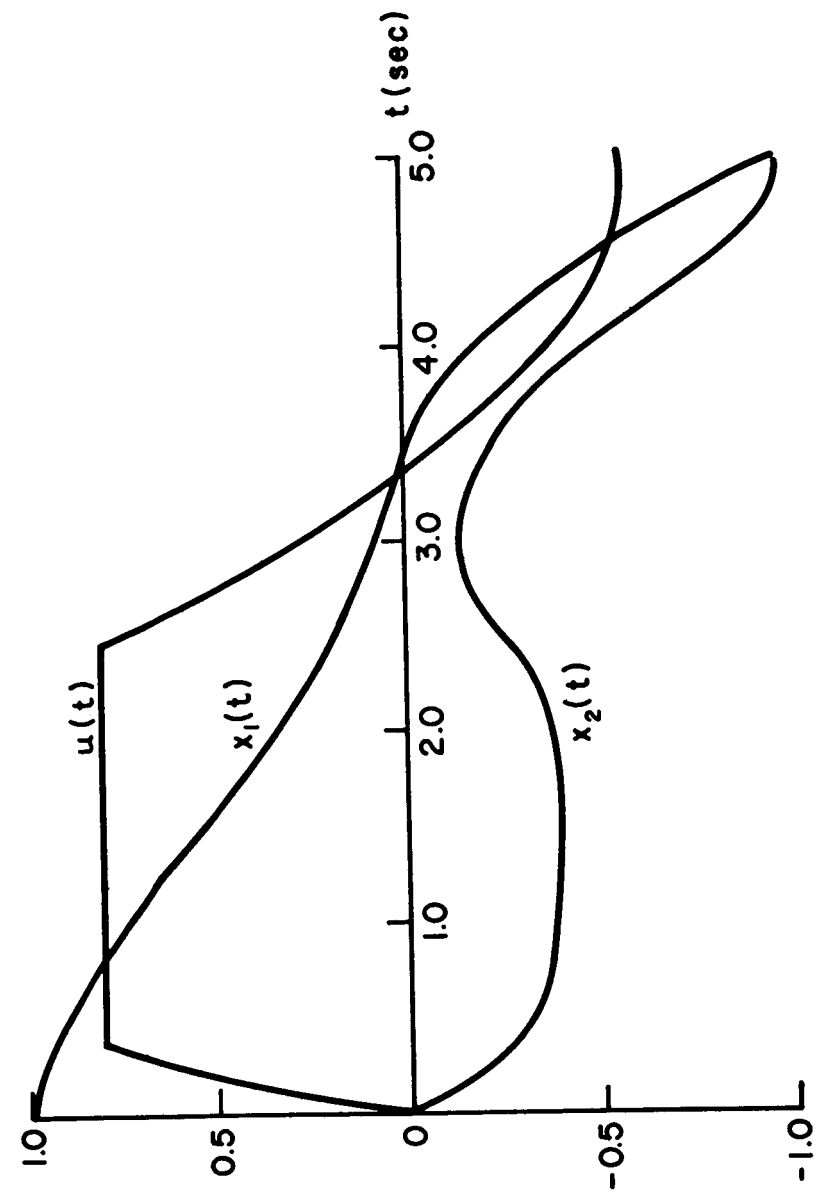
Iterations in the main programs refer to iterations on the nonlinear equations; each iteration in the main program requires several iterations in the program for solving the linear problem. The program for solving the linear problem will hereafter be referred to as the subroutine. Each entrance into the subroutine for quasilinearization requires only three integrations of the linearized equations, while each entrance into the subroutine for sequential optimization may require many more iterations, each one requiring the integration of many equations in addition to the given linearized equations.

In considering the relative efficiency of the two methods, it is difficult to decide what constitutes equivalent initial estimates and an equivalent error criterion. For example, quasilinearization requires initial estimates on the entire trajectories, while sequential optimization requires only initial estimates on the initial adjoint conditions. Also, the error criterion for halting iterations for quasilinearization requires that the entire trajectories (for all time) satisfy the given differential equations within a given error, while the error criterion for sequential optimization requires only that the terminal conditions be satisfied within a given error. In making the respective initial estimates, it is felt that for both techniques, a reasonable first guess was made which did not rely on extra information such as initial trial and error runs, and thus these guesses give reasonable indication of the relative difficulty of convergence for the general problem. In setting error criterion, it is more difficult to determine what represents an equivalent set of error values for the two methods, but that chosen for sequential optimization would appear to be looser than that chosen for quasilinearization. Thus quasilinearization appears to be clearly the more efficient technique for this problem, and probably for the general problem.

The problem given by (3. 4. 1), (3. 4. 2) and (3. 4. 3) was also solved for bounded control, with the admissible control being given by $|u| \leq 0.8$.

In extending the two techniques to bounded control, no particular difficulty was encountered. In sequential optimization, in writing linearized equations and other expressions required by the technique, terms resulting from partial derivatives with respect to u had to be set equal to zero when u reached its bounds. In quasilinearization, again, the terms in the linearized equations which resulted from u being a function of $\underline{\psi}$ and $\underline{x}$ were set equal to zero when u reached its bounds.[13]

Figure 3. 4. 2 shows the resulting optimal trajectories for the bounded control problem. Quasilinearization again solved the problem readily, in the same number of iterations as for the unbounded problem. However, considerable convergence problems were encountered in sequential optimization. Convergence could not be obtained for the bounded problem starting from arbitrary guesses on the initial adjoint conditions. Results could only be obtained by first solving the unbounded problem starting from arbitrary first guesses and then applying the control bounds to the resulting unbounded optimal trajectories, and using this as the first guess for the bounded problem, and from here convergence proceeded fairly readily. Apparently the region of convergence is smaller for the bounded problem, and because of the tendency of the state variable trajectories to diverge from the desired trajectory for arbitrary initial adjoint condition guesses, and the impossibility of reducing the constant M in (3. 5.7) below 1, the convergence problems resulted. Quasilinearization thus appears to be a much more proficient method for the bounded problem also.



FIXED TIME OPTIMAL TRAJECTORIES,
BOUNDED CONTROL

FIGURE 3.4.2

3.5 Bounded State Variable Problem

A futher variation of the optimal control problem is that in which the state variables are constrained to lie in some region. The constraints are generally given in the form:

$$\underline{C}\,(x,t) \leq 0 \tag{3.5.1}$$

where C is a vector function of dimension up to that of the state variable vector $\underline{x}$. The most simple type of constraint would be of the form:

$$\begin{aligned} &x_i(t) \leq C_i \\ &i = 1, 2, \text{---} m < n \end{aligned} \tag{3.5.2}$$

for all t, where the C_i are constants and n is the dimension of $\underline{x}$.

According to Pontryagin, for the bounded state variable problem given by (3.5.1), the optimal solution consists of segments for which $C(x,t) < 0$, during which the optimal solution satisfies the maximum principle, and segments for which $C(x,t) = 0$, for which the controller is determined so as to satisfy $C(x,t) = 0$. Since $C(x,t)$ does not necessarily contain $u(t)$ explicitly, it might be necessary to differentiate C with respect to t, and since $C(x,t) \equiv 0$, then $dC/dt \equiv 0$. Thus:

$$\left\{\frac{dC}{dt}\right\} = \left\{\frac{\partial C}{\partial t}\right\} + \left\{\frac{\partial C}{\partial x}\right\} \dot{\underline{x}} = 0 \tag{3.5.3}$$

$\dot{\underline{x}}$ will generally contain $u(t)$ explicitly, and thus we can solve (3.5.3) for $u(t)$.

For the special case given by (3.5.2), (3.5.3) becomes:

$$\dot{\underline{x}} = 0 \tag{3.5.4}$$

that is, if a given state variable $x_i(t)$ reaches its constraint, we start computing $u(t)$ so that $\dot{x}_i(t) = 0$, so that $x_i(t)$ holds at its constraint. When $\dot{x}_i(t)$ for $u(t)$ computed by the maximum

principle changes sign so that $x_i(t)$ will move off its constraint, we again use this $u(t)$.

This technique[5] presents no particular problem in applying it to sequential optimization. One need only integrate the given equations until an $x_i(t)$ reaches its constraint, and then change the optimal control program to that giving $C(x, t) = 0$, and of course changing all the other equations involved in the technique in which this change would be reflected. The integration is continued until the region in which $C(x, t) < 0$ is reached, at which point the controller and the differential equations are again changed. Also, one must continuously compute bounds on $\delta \underline{x}(t)$, which are the differences between the values of the state variables at a given time and their respective constraints, for use in the linear subroutine to insure that the next iteration $\underline{x} + \delta\underline{x}$ lies within the constraints.

Quasilinearization also presents no problem in the use of this technique. For example, for constraints of the form given by (3.5.2), one need only set the appropriate $f_i(x^n, u^n) = 0$ when $x_i^n(t)$ reaches its constraint. Furthermore, we do not need to worry about checking $\dot{x}_i^n(t)$ to determine when $x_i^n(t)$ moves off its constraint, since $x_i^n(t)$ always satisfies the given terminal conditions and thus must move off its constraint. Thus quasilinearization automatically iterates $x_i^n(t)$ until $f_i(x^n, u^n) = 0$ when $x_i^n(t) \geq C_i$, and thus until $x_i^n(t) = C_i$ in this region. In the case of sequential optimization, once $x_i(t)$ reaches its constraint, and we start computing $u(t)$ so that $\dot{x}_i(t) = 0$, $x_i(t)$ will not come off its constraint unless we use the special check on $\dot{x}_i(t) = f_i(x, u)$ with u computed by the maximum principle to tell when to change the program for $u(t)$ again.

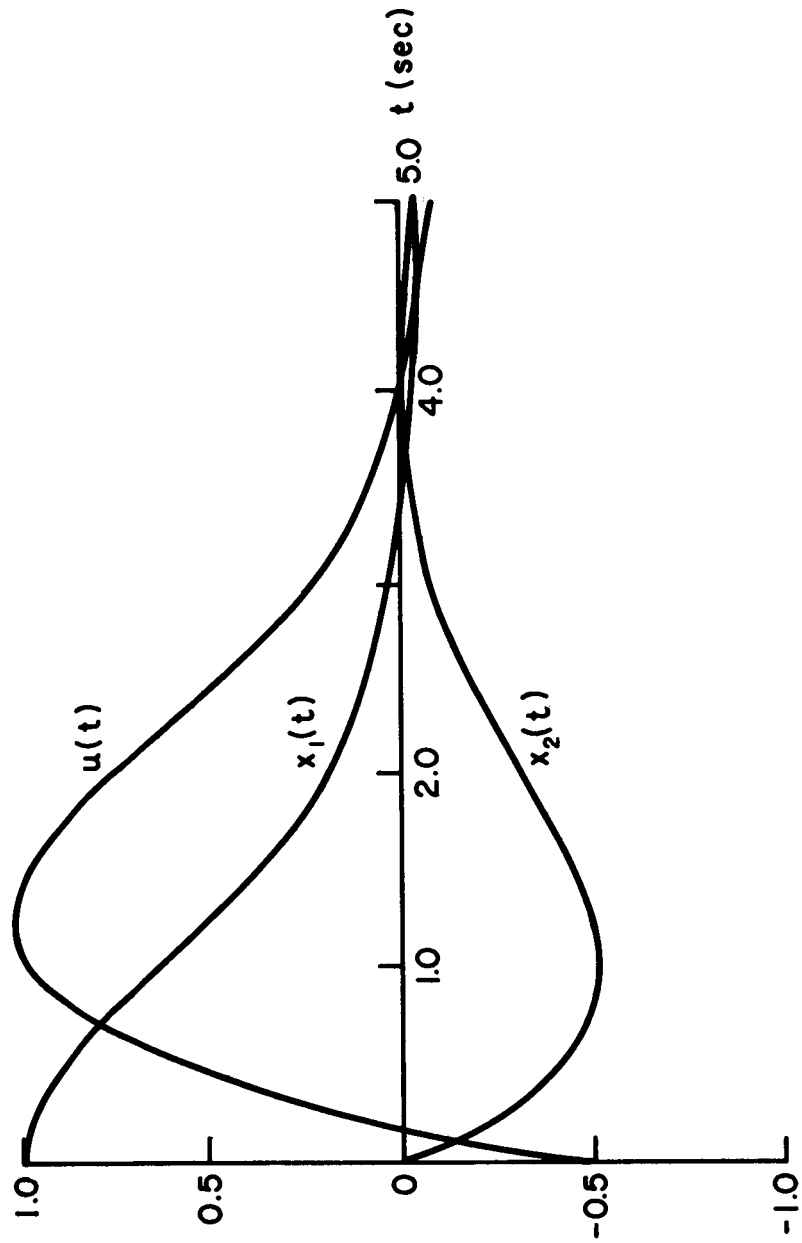
The technique described above was applied to quasilinearization so that in succeeding chapters we may demonstrate that the techniques developed for solving the variable time problem work for the bounded state variable problem. We will consider the simple case of constraints of the form (3. 5. 2). Since we cannot apply constraints of the form (3. 5. 2) to the trajectories with the terminal conditions used in Figure 3. 4. 1 (since the extreme of both curves are at the initial or final times), we will use terminal conditions of $x_1(T) = x_2(T) = -0.05$. The resulting unbounded state variable optimal trajectories are given in Figure 3. 5. 1. (It was originally intended to use these terminal conditions for all the variable time studies. However, $S(T, X)$ does not vary as much as a function of T for these conditions as the ones of Figure 3. 4. 1, and thus the maxima and minima of $S(T, X)$ are not as clearly shown. Thus it was necessary to use different terminal conditions for the variable time studies with unbounded state variables and those with bounded state variables.)

The bounds used for the given example are:

$$-0.4 \leq x_2(t) \tag{3.5.5}$$

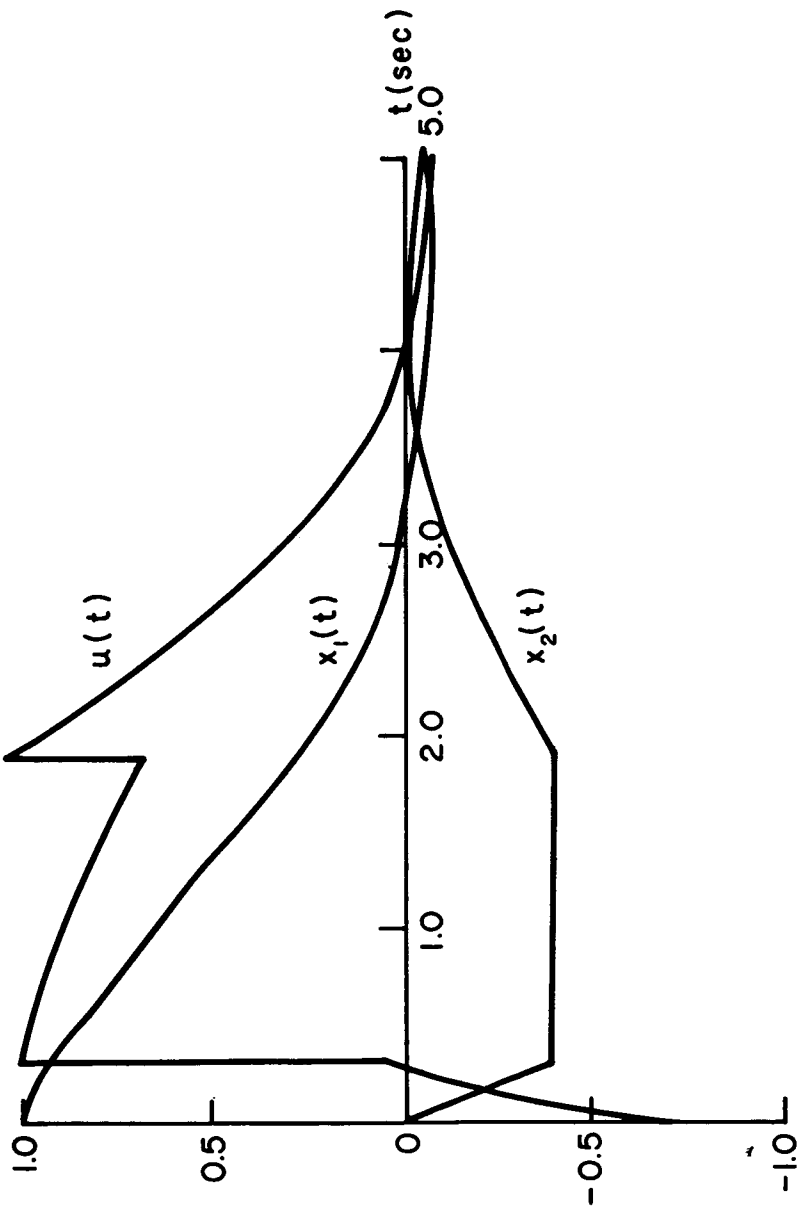
The optimal trajectories for these bounds are given in Figure 3. 5. 2. The unbounded trajectories of Figure 3. 5. 1 required 5 iterations for convergence, while those of Figure 3. 5. 2 require 11. Thus the introduction of state variable constraints slows convergence considerably.

During the period when the state variable is at its constraint, the control must be variable in order to hold the state variable at its constraint. Thus if we consider a problem with both bounded state variable and bounded control, the state variable and control cannot be at their constraints at the same time. However,



FIXED TIME OPTIMAL TRAJECTORIES,
UNBOUNDED STATE VARIABLES

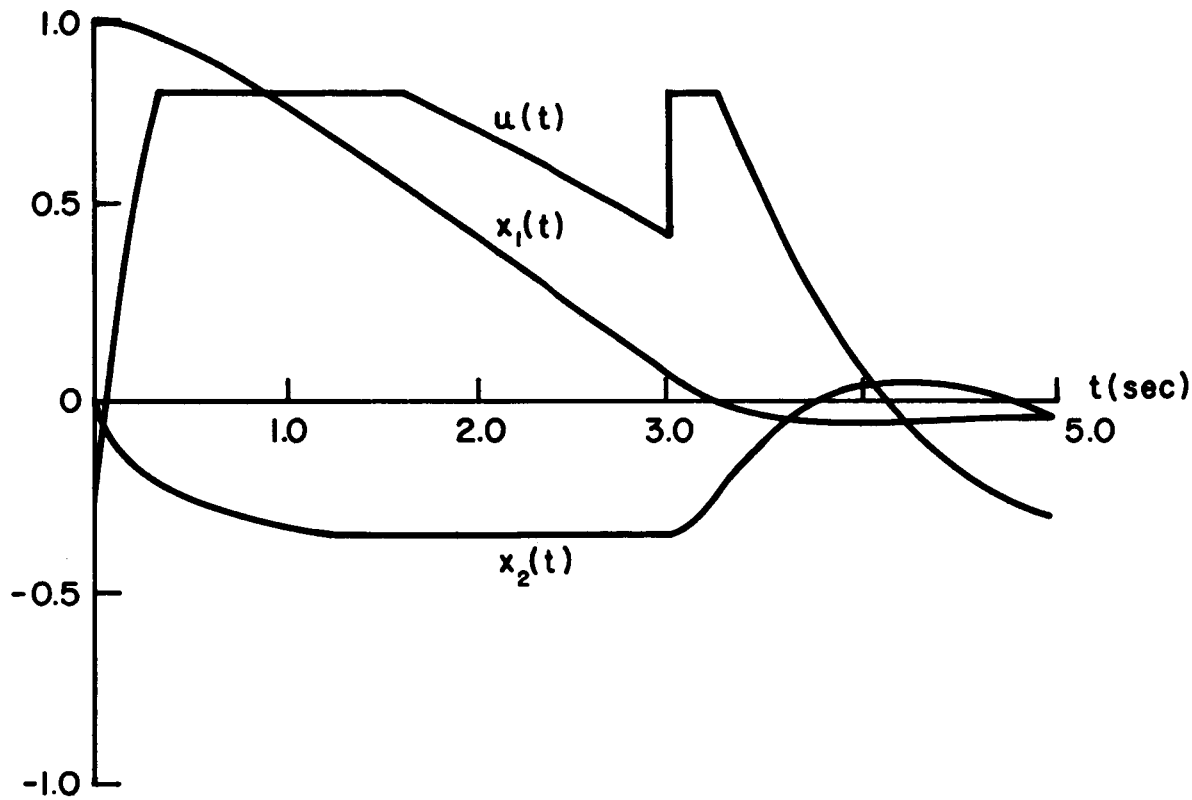
FIGURE 3.5.1



FIXED TIME OPTIMAL TRAJECTORIES,
BOUNDED STATE VARIABLES

FIGURE 3.5.2

from Figure 3.5.2 we see that the control is at its higher values during the period when the state variable is at its constraint. Therefore, bounds on the control of $|u(t)| \leq 0.8$ and bounds on the state variable of $-0.35 \leq x_2(t)$ were applied simultaneously to see if the given techniques could handle this problem. The resulting optimal trajectories are shown in Figure 3.5.3. These trajectories required 15 iterations for convergence, somewhat longer than was required for the bounded state variable alone.



OPTIMAL TRAJECTORIES BOUNDED CONTROL
AND STATE VARIABLE

FIGURE 3.5.3

CHAPTER 4

USE OF H = 0 NECESSARY CONDITION TO SOLVE THE VARIABLE TIME PROBLEM

4.1 Computational Algorithm

All methods of solving the variable time problem developed to this time use only the $H = 0$ necessary condition and assume the resulting solution is the only one and thus the desired optimum. One standard method[14] is to iterate on the final time T and $n-1$ of the n initial adjoint conditions (n total variables) until the n terminal state variable conditions are satisfied, and at each iteration use the $H = 0$ condition to determine the n^{th} initial adjoint condition. In this manner, the $H = 0$ condition is always satisfied, and when the boundary conditions are finally reached, the resulting solution is a possible solution of the variable time problem. However, in this process, the optimum solution at only one final time T is obtained.

Since we want to consider the variable time problem more or less separately from the other aspects of the optimal control problem in order to judge the validity of using only the $H = 0$ condition, we are going to derive an algorithm which solves a series of fixed time problems, iterating on the final time T until the $H = 0$ condition is satisfied. This algorithm will solve the fixed time problem for an initial guess on T, compute an increment on T, δT, such that when the fixed time problem for new final time $T + \delta T$ is solved, a Hamiltonian closer to zero will result, and through repeated iterations finally arrive at the T for which $H = 0$.

If we take our first solution for which the terminal conditions are satisfied at time T, and let the time vary by an amount δT, this will produce an error in terminal conditions which is given

to a first order of approximation by:

$$\delta \underline{x}(T) = \dot{\underline{x}}(T)\, \delta T \qquad (4.1.1)$$

If we want to resolve the two-point boundary problem for the new $T^{n+1} = T^n + \delta T^n$, we will have to vary our initial adjoint condition by some vector $\delta\underline{\lambda}$. The question is what δT will require a $\delta\underline{\lambda}$ to correct the terminal conditions which will also make the Hamiltonian equal to zero. Since the Hamiltonian is constant, we need only require that it be zero at the initial conditions.

In the previous chapter, we derived a set of differential equations for variables of the form $\partial x_i(t)/\partial \lambda_j$, which gives the effect on the value of the state variable x_i at time t of a variation in initial adjoint condition $\psi_j(0)$ of $\delta\lambda_j$. If we integrate these differential equations to time T, and let $[Y]$ represent the matrix of elements $Y_{ij} = \partial x_i/\partial \lambda_j \,|\, t = T$, then a variation in initial adjoint conditions $\delta\underline{\lambda}$ will produce, to a first order of approximation, a change in the value of the state variables at time $t = T$ given by:

$$\delta \underline{x}(T) = [Y]\, \delta\underline{\lambda} \qquad (4.1.2)$$

However, since the variation in $\underline{x}(T)$ due to a variation in final time of δT is given by (4.1.1), we want to vary $\underline{\psi}(0)$ by $\delta\underline{\lambda}$ to counteract this variation, that is, choose $\delta\underline{\lambda}$ so that:

$$\delta \underline{x}(T) = -\dot{\underline{x}}(T)\, \delta T \qquad (4.1.3)$$

Substituting (4.1.3) into (4.1.2) and solving for $\delta\underline{\lambda}$, we obtain:

$$\delta\underline{\lambda} = -[Y]^{-1}\, \dot{\underline{x}}(T)\, \delta T \qquad (4.1.4)$$

The Hamiltonian at $t = 0$ is given by:

$$H^n = \underline{f}^T(0) \bullet \underline{\psi}^n(0) - f_o(0) \qquad (4.1.5)$$

where $f_o(0) = f_o\left(\underline{x}(0), \underline{\psi}(0)\right)$, and $\underline{f}(0) = \underline{f}\left(\underline{x}(0), \psi(0)\right)$ since $\underline{u}(0) = \underline{u}\left(\underline{x}(0), \underline{\psi}(0)\right)$.

We want the new $H^{n+1} = \underline{f}^T(0) \cdot \underline{\psi}^{n+1}(0) - f_o(0) = 0$.

If we let $\underline{\psi}^{n+1}(0) = \underline{\psi}^n + \delta\underline{\lambda}^n$, and set $H^{n+1} = 0$:

$$H^{n+1} = \underline{f}\left(\underline{x}(0), \psi^n(0) + \delta\underline{\lambda}^n\right) \cdot \left(\psi^n(0) + \delta\underline{\lambda}^n\right) - f_o\left(\underline{x}(0), \psi^n(0) + \delta\underline{\lambda}^n\right) = 0 \tag{4.1.6}$$

If we substitute Equation (4.1.4) in (4.1.6), we obtain a single equation in a single unknown δT, which may be solved for δT. That is, [Y], $\dot{x}(T)$, $\underline{\psi}^n(0)$, and $\underline{x}(0)$ are all either known originally or were determined in solving the problem for the previous T.

If we then apply the $\delta\underline{\lambda}$ given by (4.1.4) for the δT determined as above and apply $\underline{\psi}^{n+1}(0) = \underline{\psi}^n(0) + \delta\underline{\lambda}^n$ to the system of equations, we will find that the Hamiltonian is exactly equal to zero. However, when we integrate the system of equations, we will find that the terminal values are slightly off. This is due to the fact that $\delta\underline{x}(T)$ is not given exactly by (4.1.1), and even if it were, the $\delta\underline{\lambda}$ given by (4.1.4) would not exactly compensate for the $\delta\underline{x}(T)$ in one step. Thus we must again apply the fixed time algorithm to exactly solve the two-point boundary problem for the new $T^{n+1} = T^n + \delta T$. This will result in a $\underline{\psi}^{n+1}(0)'$ which is not exactly equal to $\underline{\psi}^{n+1}(0) = \underline{\psi}^n(0) + \delta\underline{\lambda}$ and thus the Hamiltonian will not be exactly zero. However, the $\psi^{n+1}(0)'$ determined by the fixed time algorithm for the T^{n+1} will be close enough to $\psi^{n+1}(0)$ that the Hamiltonian will be closer to zero than it was for the T^n. Thus, this process can be repeated until the T is found which does result in $H = 0$, or at least as close to zero as we please.

It should be noted that (4.1.6) may be solved analytically only if the system of equations is linear in the control variable and the performance criterion is no more than quadratic in the control variable, in which case (4.1.6) will be quadratic in $\delta\underline{\lambda}$ and thus in δT. If this is not the case, we must write H as a second order Taylor series expansion:

$$H^{n+1} = H^n + \delta\underline{\lambda}^T\left\{\frac{\partial H}{\partial \underline{\psi}}\right\} + \delta\underline{x}^T\left\{\frac{\partial H}{\partial \underline{x}}\right\} + \delta\underline{u}^T\left\{\frac{\partial H}{\partial \underline{u}}\right\}$$

$$+ \frac{1}{2}\left(\delta\underline{x}^T\left[\frac{\partial^2 H}{\partial \underline{x}^2}\right]\delta\underline{x} + \delta\underline{u}^T\left[\frac{\partial H}{\partial \underline{u}^2}\right]\delta\underline{u} + 2\,\delta\underline{\lambda}^T\left[\frac{\partial^2 H}{\partial\underline{\psi}\partial\underline{x}}\right]\delta\underline{x}\right. \qquad (4.1.7)$$

$$\left. + 2\,\delta\underline{\lambda}^T\left[\frac{\partial^2 H}{\partial\underline{\psi}\partial\underline{u}}\right]\delta\underline{u} + 2\,\delta\underline{x}^T\left[\frac{\partial^2 H}{\partial\underline{x}\partial\underline{u}}\right]\delta\underline{u}\right) = 0$$

where $\delta\underline{x} = \underline{x}^{n+1} - \underline{x}^n$, etc. If we evaluate (4.1.7) at $t = 0$, $\delta\underline{x}(0) = 0$, $\delta\underline{u}(0) = \frac{\partial \underline{u}}{\partial \underline{\psi}}\Big|_{t=0}\delta\underline{\lambda}$, and all the matrices at $\underline{x} = \underline{x}(0)$ and $\underline{\psi}(0) = \underline{\psi}^n(0)$, (4.1.7) becomes a second order equation in $\delta\underline{\lambda}$ and thus in δT which may be solved for δT.

The resulting equation will be of the form:

$$a\,\delta T^2 + b\,\delta T + c = 0 \qquad (4.1.8)$$

If $b^2 - 4ac < 0$, (4.1.7) has no real solution, and we must take the δT which minimizes H, i.e.,:

$$2a\,\delta T + b = 0 \qquad (4.1.9)$$

or

$$\delta T = -\frac{b}{2a} \qquad (4.1.10)$$

If $b^2 - 4ac > 0$, (4.1.7) has two real solutions and we must use the smaller δT, which more closely satisfies the approximations made in deriving the algorithm.

This gives us a complete algorithm for iterating on T until the Hamiltonian is driven to zero. The following section will consider a numerical example.

4.2 Numerical Example

In this section we will discuss the application of the method developed in the previous section to the problem described in the last chapter and solved for the fixed time case using both quasilinearization and sequencial optimization. It should be noted that the variable time optimization method described in the previous chapter may be applied in exactly the same manner to both quasi-linearization and sequential optimization, and in fact to any fixed time optimization technique which uses the adjoint equations and Hamiltonian described by (2.2.3) and (2.2.4) respectively.

From Equation (4.1.4), for a two-dimensional problem:

$$\begin{Bmatrix} \delta\lambda_1 \\ \delta\lambda_2 \end{Bmatrix} = - \begin{bmatrix} \partial x_1/\partial\lambda_1 & \partial x_1/\partial\lambda_2 \\ \partial x_2/\partial\lambda_1 & \partial x_2/\partial\lambda_2 \end{bmatrix}^{-1}_{t=T} \begin{Bmatrix} \dot{x}_1(T) \\ \dot{x}_2(T) \end{Bmatrix} \delta T \qquad (4.2.1)$$

$$= \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \delta T$$

where the $\dot{x}_i(T)$ and $\partial x_i/\partial\lambda_j$ are available in computer storage after solving the fixed time problem by either sequential optimization or quasilinearization. The differential equations used for determining $\partial x_i/\partial\lambda_j$ are given in Appendix I. R_1 and R_2 are given by:

$$R_1 = -\left[\partial x_2/\partial\lambda_2\Big|_{t=T}\dot{x}_1(T) - \partial x_2/\partial\lambda_1\Big|_{t=T}\dot{x}_2(T)\right]/\det\left|\partial x/\partial\lambda\right|$$

$$R_2 = -\left[\partial x_1/\partial\lambda_1\Big|_{t=T}\dot{x}_2(T) - \partial x_1/\partial\lambda_2\Big|_{t=T}\dot{x}_1(T)\right]/\det\left|\partial x/\partial\lambda\right| \quad (4.2.2)$$

$$\det\left|\partial x/\partial\lambda\right| = \left(\partial x_1/\partial\lambda_1 \cdot \partial x_2/\partial\lambda_2 - \partial x_2/\partial\lambda_1 \cdot \partial x_1/\partial\lambda_2\right)\Big|_{t=T}$$

The Hamiltonian for our example, from (3.4.4), with $u = \psi_2/2$ substituted for unbounded control, is given by:

$$H(\psi, x) = \psi_1 x_2 + \psi_2(1 - x_1^2)\, x_2 - \psi_2 x_1 + \psi_2^2/2 - x_1^2 - x_2^2 - \psi_2^2/4 \quad (4.2.3)$$

Since $H(\psi, x)$ is constant for all t, we need only insure that our $H(\psi + \delta\psi, x) = 0$ at $t = 0$. Substituting the initial conditions $x_1(0) = 1.0$, $x_2(0) = 0$, and $\psi_2(0) = \lambda_2$.

$$H(\lambda) = \lambda_2^2/4 - \lambda_2 - 1 \quad (4.2.3)$$

Therefore

$$\begin{aligned} H(\lambda + \delta\lambda) &= (\lambda_2 + \delta\lambda_2)^2/4 - (\lambda_2 + \delta\lambda_2) - 1 \\ &= \lambda_2^2/4 - \lambda_2 - 1 + \delta\lambda_2^2/4 + \frac{(\lambda_2 - 2)}{2}\delta\lambda_2 \\ &= H(\lambda) + \delta\lambda_2^2/4 + \frac{(\lambda_2 - 2)}{2}\delta\lambda_2 \end{aligned} \quad (4.2.4)$$

where $H(\lambda)$ and λ_2 are also available in computer storage after solution of the fixed time problem. Thus, substituting $\delta\lambda_2 = R_2\,\delta T$ from (4.2.1); and setting $H(\lambda + \delta\lambda) = H(\delta T) = 0$:

$$\frac{R_2^2}{4}\,\delta T^2 + \frac{(\lambda_2 - 2)}{2}\,R_2\,\delta T + H(\lambda) = 0 \qquad (4.2.5)$$

This equation may be solved for δT, and thus $T + \delta T$ is our new estimate for the optimal final time. If (4.2.5) has no real solution, we use the δT that minimizes (4.2.5) according to (4.1.10).

In applying sequential optimization for the new fixed final time $T + \delta T$, we use as initial estimates on $\underline{\psi}(0)$:

$$\begin{aligned} \lambda_1^{n+1} &= \lambda_1^n + R_1\,\delta T \\ \lambda_2^{n+1} &= \lambda_2^n + R_2\,\delta T \end{aligned} \qquad (4.2.6)$$

Using these estimates, the new fixed time problem will be solved more rapidly than the original fixed time problem, for which arbitrary initial estimates of $\underline{\lambda}^o = 0$ were used.

In quasilinearization, if $\delta T > 0$, the optimal $\underline{x}(t)$ and $\underline{\psi}(t)$ for the previous T are used as initial estimates for the new $T + \delta T$ for $0 \le t \le T$, and $\underline{x}(t) = \underline{x}(T)$ and $\psi(t) = \psi(T)$ are used as initial estimates for $T \le t \le T + \delta T$. If $T < 0$, the same initial estimates on $\underline{x}(t)$ and $\underline{\psi}(t)$ given in section (3.4) for the first fixed time are used.

For bounded control, if the control for the fixed time case is at its bound at $t = 0$, $H(\lambda + \delta\lambda)$ of (4.2.4) becomes:

$$\begin{aligned} H(\lambda + \delta\lambda) &= H(\lambda) - \delta\lambda_2(1 - u) \\ &= H(\lambda) - R_2(1-u)\,\delta T \end{aligned} \qquad (4.2.7)$$

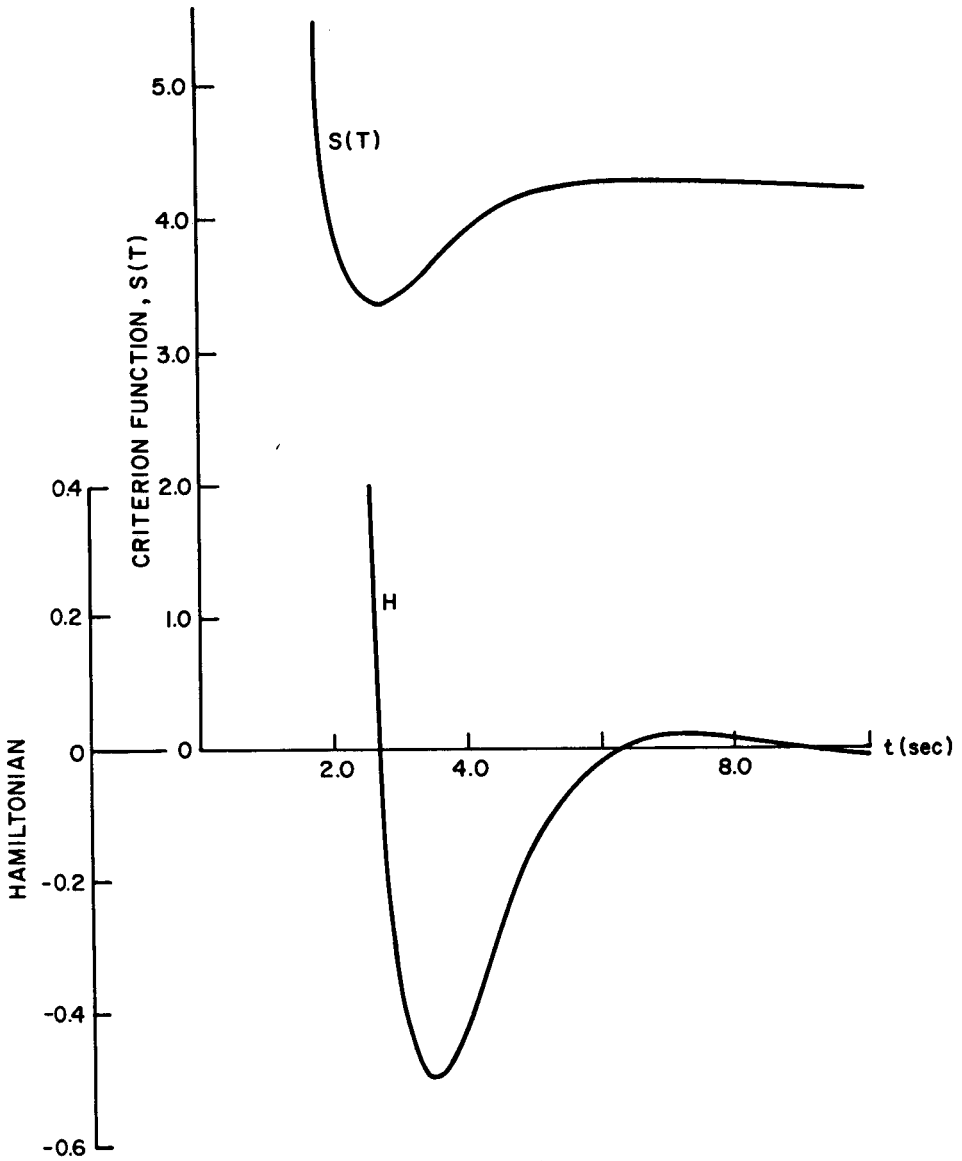
and we use

$$\delta T = \frac{H(\lambda)}{R_2(1-u)} \qquad (4.2.8)$$

4.3 Discussion of Numerical Results

Figure 4.3.1 and Table 4.3.1 give the values of the Hamiltonian and the minimized performance criterion, S(T), for various values of fixed final time for unbounded control. As can be seen from the figure and table, the Hamiltonian goes through zero at $T \cong 2.6$ seconds, at which time S(T) goes through a minimum. This point is the desired variable time optimum. However, the Hamiltonian also goes through zero at $T \cong 6.1$ seconds, at which time S(T) attains a local maximum. Again, at $T = 9.3$ seconds, $H = 0$. According to the table, S(T) is not a local minimum at this point, but this is due to accumulated integration errors for large T's and the fact that S(T) is only varying in the fourth decimal place. S(T) should be a local minimum here (although a greater minimum than that attained at $T = 2.6$ seconds). For higher T's, H would probably continue to oscillate with decreasingly small amplitudes about $H = 0$, and S(T) would continue to achieve relative maxima and minima, but with increasingly small difference between them.

Any technique for solving the variable time problem which uses only the $H = 0$ criterion could arrive at any of the various $H = 0$ points, some being local maxima. For example, using the technique outlined in preceding sections, if the initial guess on T was less than 3.4 seconds (the point where H achieves its first extremum), the variable time solution obtained will be the correct one of $T = 2.6$ seconds. However, if the initial guess on T is greater than 3.4 seconds (but less than seven seconds) the solution obtained will be the local maximum at $T \cong 6.1$ seconds. This demonstrates the contention made earlier in this paper that all those methods given in the literature for solving the variable time problem



MINIMIZED CRITERION FUNCTION AND HAMILTONIAN VERSUS FINAL TIME

FIGURE 4.3.1

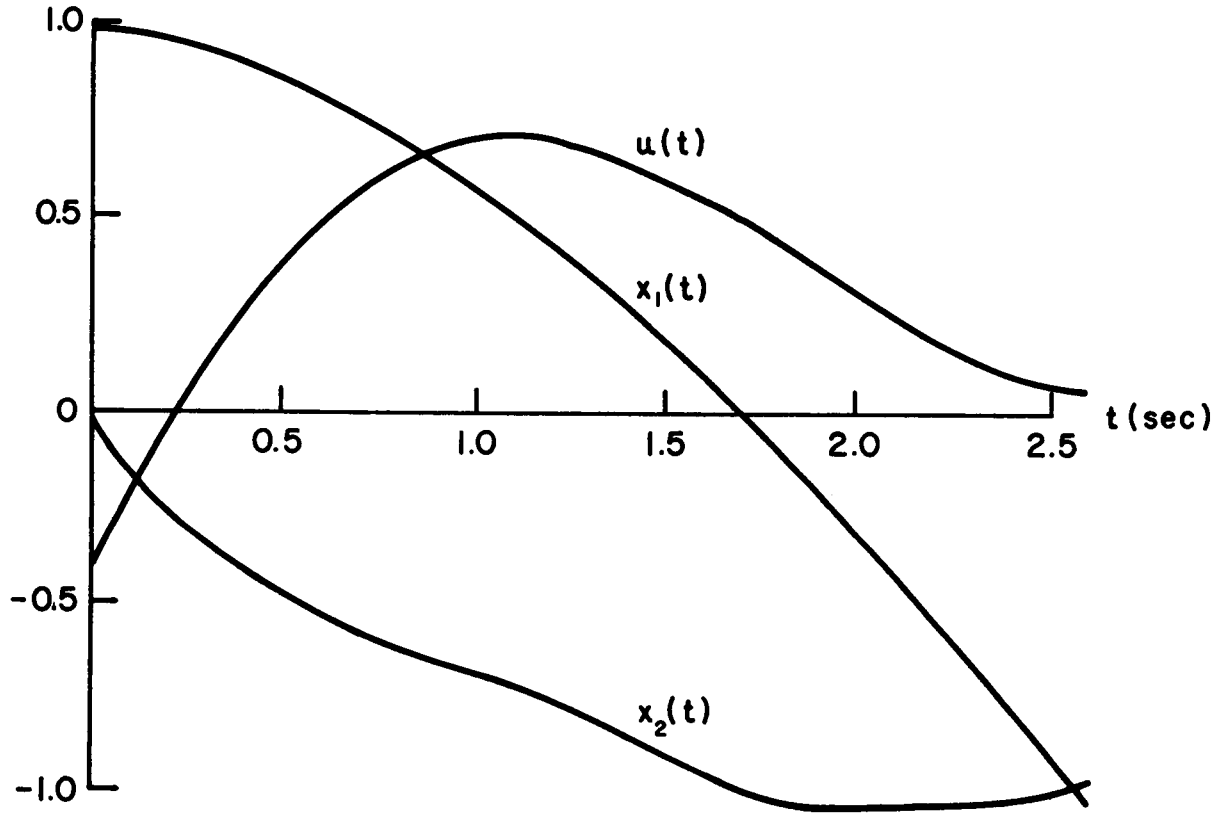
TABLE 4.3.1

T (sec)	H	S(T)
2.0	2.8466	4.0320
2.2	1.3928	3.6217
2.4	0.5522	3.4347
2.6	0.0569	3.3780
2.8	-0.2338	3.3981
3.0	-0.3974	3.4627
3.2	-0.4792	3.5512
3.4	-0.5067	3.6502
3.6	-0.4981	3.7510
3.8	-0.4659	3.8475
4.0	-0.4194	3.9360
4.2	-0.3654	4.0144
4.4	-0.3088	4.0816
4.6	-0.2533	4.1376
4.8	-0.2013	4.1828
5.0	-0.1544	4.2181
5.2	-0.1134	4.2447
5.4	-0.0786	4.2636
5.6	-0.0498	4.2762
5.8	-0.0269	4.2837
6.0	-0.0090	4.2871
6.2	0.0041	4.2874
6.4	0.0135	4.2855
6.6	0.0195	4.2821
6.8	0.0229	4.2777
7.0	0.0243	4.2729
7.2	0.0240	4.2679
7.4	0.0227	4.2631
7.6	0.0207	4.2587
7.8	0.0184	4.2547
8.0	0.0156	4.2512
8.2	0.0131	4.2482
8.4	0.0104	4.2458
8.6	0.0080	4.2439
8.8	0.0059	4.2423
9.0	0.0025	4.2358
9.2	0.0010	4.2350
9.4	-0.0004	4.2345
9.6	-0.0013	4.2343
9.8	-0.0019	4.2343
10.0	-0.0033	4.2339

which use only the $H = 0$ condition are inadequate for the general problem.

Using sequential optimization, and starting at $T = 3.2$ seconds, the variable time algorithm arrived at the solution $T = 2.6$ seconds after iterating through $T = 2.0$ and 2.4 seconds (or three total iterations on T). The variable time optimal trajectories for $T = 2.6$ seconds are shown in Figure 4.3.2. The first iteration overshot the desired T because the initial guess of $T = 3.2$ was near the extremum of H where H was varying slowly with respect to T, and thus the method computed a δT to drive H to zero which was too large. Thus, in order to make the technique work in regions where $\partial H/\partial T \cong 0$, an upper bound should be placed on the allowed δT (say, $\delta T \max = T/3$).

The total computer time required by sequential optimization to solve the problem starting from $T = 3.2$ seconds was four minutes. Starting from the same T, quasilinearization solved the problem in one minute. Note that the relative efficiency of quasilinearization over sequential optimization has decreased somewhat from what it was for the fixed time problem (from about 7 to 1 to about 4 to 1). This is due to the fact that the main inefficiency in sequential optimization is in making initial guesses on $\underline{\psi}(0)$, and such guesses only have to be arbitrary for the first iteration on T; thereafter, the technique provides a reasonable good first guess on $\psi(0)$ for each succeeding iteration on T.



VARIABLE TIME OPTIMAL TRAJECTORIES,
UNBOUNDED CONTROL

FIGURE 4.3.2

CHAPTER 5

SUFFICIENT CONDITIONS FOR A VARIABLE TIME LOCAL MINIMUM

5.1 Development of Sufficient Conditions

In the preceding chapter we showed that the $H = 0$ condition for a variable time optimum can sometimes lead to an incorrect solution. That is, computational algorighms based only upon the $H = 0$ condition can result in a solution which is a local maximum of $S(T)$ with respect to T as well as a local minimum, since $H = -\frac{\partial S}{\partial T}$. In the standard minimization problem, a sufficient condition for a local minimum is that the second derivative of the function to be minimized be greater than zero. Since $H = -\frac{\partial S}{\partial T}$, for our problem this sufficient condition is given by:

$$\partial^2 S/\partial T^2 = -\frac{\partial H}{\partial T} > 0$$

or

$$\frac{\partial H}{\partial T} < 0 \qquad (5.1.1)$$

Since the Hamiltonian is constant with time for the solution for a given final time T, $\partial H/\partial T$ will be the same regardless of at what time it is evaluated. For convenience we will evaluate it at time $t = 0$.

Since $H = H(\psi, x, u)$:

$$\frac{\partial H}{\partial T} = \left\{\frac{\partial H}{\partial \psi}\right\}^T \left\{\frac{\partial \psi}{\partial T}\right\} + \left\{\frac{\partial H}{\partial x}\right\}^T \left\{\frac{\partial x}{\partial T}\right\} + \left\{\frac{\partial H}{\partial u}\right\}^T \left\{\frac{\partial u}{\partial T}\right\} \qquad (5.1.2)$$

If we evaluate $\partial H/\partial T$ at $t = 0$, $\partial x/\partial T = 0$ since the initial condition is constant, and $\left\{\frac{\partial u}{\partial T}\right\} = \left\{\frac{\partial u(x, \psi)}{\partial T}\right\} = \left[\frac{\partial u}{\partial \psi}\right]\left\{\frac{\partial \psi}{\partial T}\right\}$ (where the term $\left[\frac{\partial u}{\partial x}\right]\left\{\frac{\partial x}{\partial T}\right\} = 0$ since $\left\{\frac{\partial x}{\partial T}\right\} = 0$).

Thus:

$$\frac{\partial H}{\partial T} = \left\{\frac{\partial H}{\partial \lambda}\right\}^T \left\{\frac{\partial \lambda}{\partial T}\right\} + \left\{\frac{\partial H}{\partial u}\right\}^T \left[\frac{\partial u}{\partial \lambda}\right] \left\{\frac{\partial \lambda}{\partial T}\right\} \qquad (5.1.3)$$

where $\psi = \lambda$ has been substituted since we are evaluating at $t = 0$.

From (4.1.4)

$$\delta\underline{\lambda} = -[Y]^{-1}\, \dot{\underline{x}}(T)\, \delta T$$

or

$$\frac{\delta\lambda}{\delta T} = -[Y]^{-1}\, \dot{\underline{x}}(T) \qquad (5.1.4)$$

But

$$\lim_{\delta T \to 0} \frac{\delta\lambda}{\delta T} = \left\{\frac{\partial \lambda}{\partial T}\right\} \qquad (5.1.5)$$

Therefore

$$\left\{\frac{\partial \lambda}{\partial T}\right\} = -[Y]^{-1}\, \dot{\underline{x}}(T) \qquad (5.1.6)$$

Substituting (5.1.6) into (5.1.3)

$$\frac{\partial H}{\partial T} = -\left(\left\{\frac{\partial H}{\partial \lambda}\right\}^T + \left\{\frac{\partial H}{\partial u}\right\}^T \left[\frac{\partial u}{\partial \lambda}\right]\right) \cdot [Y]^{-1}\, \dot{\underline{x}}(T) \qquad (5.1.7)$$

In (5.1.7) all the variables on the right side of the equation may easily be evaluated for an optimal solution for variable final time obtained using the method of the previous chapter which uses only the $H = 0$ necessary condition. If $\frac{\partial H}{\partial T} < 0$, we have a local minimum; if $\frac{\partial H}{\partial T} > 0$ we have a local maximum.

In the next section, we will develop a computational algorithm which will drive the solution to a point where $\frac{\partial H}{\partial T} < 0$ if the first variable time solution obtained has $\frac{\partial H}{\partial T} > 0$. It should be noted that if a problem is such that $S(T)$ has more than one local maximum and local minimum, such a method will not insure that the solution arrived at will be the local minimum with the least magnitude, only that it will be a local minimum and not a local maximum.

5.2 Computational Algorithm

Through the method of Chapter 4 we arrived at an approximation for H as a function of δT which contains up to second order terms in δT, and thus has two solutions for which $H = 0$. We chose the δT which had the smaller magnitude, since the approximations are more accurate for this δT, and if H is zero at only one T in the actual problem, the method will converge to that T. However, if for the given problem there is more than one T for which $H = 0$, the method will converge to the closest T according to the approximations in the method. (The method might converge to a T which is slightly farther from the first estimate on T than is some other T for which $H = 0$. For example, in the numerical example of the last chapter, when the initial estimate of T was chosen to be four seconds, the method converged to the solution at $T = 6.1$ seconds, instead of the closer one at $T = 2.6$ seconds. Apparently, in the approximation for H as a function of δT, the solution on the right looked closer than the one on the left, that is, the positive δT which solved $H = 0$ was smaller than the negative one, even though $T = 2.6$ was actually closer than $T = 6.1$ was to $T = 4$).

If the first estimate on T is such that $\frac{\partial H}{\partial T} < 0$, the solution finally arrived at by the method of the previous chapter might still be such that $\frac{\partial H}{\partial T} > 0$, due to the facts mentioned in the previous paragraph and the fact that the extremum of H vs. T might not be symmetrically located between the two points where $H = 0$ on each side. However, if our first estimate on T is such that $\frac{\partial H}{\partial T} > 0$, and we develop a method for driving T far enough into the region where $\frac{\partial H}{\partial T} < 0$, the method will converge to a point where $\frac{\partial H}{\partial T} < 0$, i.e., a relative minimum.

If we can compute $\partial^2 H/\partial T^2$ in some way, we then know which direction to vary T to make $\partial H/\partial T < 0$ if $\partial H/\partial T > 0$ for our first estimate. That is, we can take:

$$\delta T = - K_T \frac{\partial H/\partial T}{\partial^2 H/\partial T^2} \tag{5.2.1}$$

where the value of K_T depends upon how far into the $\partial H/\partial T < 0$ region it is desired to drive T to insure convergence to a point where $\partial H/\partial T < 0$ (K_T should be at least two).

If, after applying the δT given by (5.2.1), the $\partial H/\partial T$ is still greater than zero due to inaccuracies in the approximations, or if the T is not driven far enough into the region where $\partial H/\partial T < 0$ to insure convergence in that region and the solution returns to the $\partial H/\partial T > 0$ region, K_T may be increased and (5.2.1) reapplied.

We now must derive a method of computing $\partial^2 H/\partial T^2$. Equation (5.1.3) gives $\partial H/\partial T$ as a function of $\underline{x}$, $\underline{u}$, $\underline{\lambda}$, and $\{\partial\lambda/\partial T\}$. Let:

$$\frac{\partial H}{\partial T} = G(\underline{x}, \underline{u}, \underline{\lambda}, \underline{\lambda}') \tag{5.2.2}$$

where $\lambda' = \{\partial\lambda/\partial T\}$.

Then

$$\frac{\partial^2 H}{\partial T^2} = \left\{\frac{\partial G}{\partial u}\right\}^T \left[\frac{\partial u}{\partial \lambda}\right] \left\{\frac{\partial \lambda}{\partial T}\right\} + \left\{\frac{\partial G}{\partial \lambda}\right\}^T \left\{\frac{\partial \lambda}{\partial T}\right\} + \left\{\frac{\partial G}{\partial \lambda'}\right\} \left\{\frac{\partial^2 \lambda}{\partial T^2}\right\} \tag{5.2.3}$$

where again terms involving $\partial x/\partial T$ are zero and $\frac{\partial \lambda'}{\partial T} = \frac{\partial^2 \lambda}{\partial T^2}$.

If we can compute $\left\{\frac{\partial^2 \lambda}{\partial T^2}\right\}$, we have a complete method of computing $\partial^2 H/\partial T^2$. First, we must rewrite (4.1.3) including second order terms

$$\delta \underline{x}(T) = \dot{\underline{x}}(T)\, \delta T + \frac{1}{2} \ddot{\underline{x}}(T)\, \delta T^2 \tag{5.2.4}$$

where $\ddot{\underline{x}}(T)$ may easily be computed from the given equations. Also, we must expand the elements $Y_{ij} = \partial x_i/\partial \lambda_j$ of $[Y]$ to include first order effects of δT, that is, let $[Y + \dot{Y}\,\delta T]$ be the nxn matrix of elements $Y_{ij} + \dot{Y}_{ij}\,\delta T = \partial x_i/\partial \lambda_j + \frac{d}{dt}\frac{\partial x_i}{\partial \lambda_j} \times \delta T$. The $\frac{d}{dt}\frac{\partial x_i}{\partial \lambda_j}$ are used in the computer program to determine the $\partial x_i/\partial \lambda_j$ and thus are available in computer storage; expressions for the $\frac{d}{dt}\,\partial x_i/\partial \lambda_j$ are given in (3.2.15).

Thus, we may rewrite (4.1.4) with second order terms in δT included:

$$\delta\underline{\lambda} = -\left[Y + \dot{Y}\,\delta T\right]^{-1}\left(\dot{\underline{x}}(T)\,\delta T + \frac{1}{2}\,\ddot{\underline{x}}(T)\,\delta T^2\right) \qquad (5.2.5)$$

The inverse matrix $[Y + \dot{Y}\,\delta T]^{-1}$ is equal to the transposed matrix of cofactors of $[Y + \dot{Y}\,\delta T]$ divided by its determinant. The cofactors of $[Y + \dot{Y}\,\delta T]$ will have terms involving δT to the power n-1 if the matrix is of order n. Since we are only interested in expanding $\delta\underline{\lambda}$ to second order terms in δT, we can neglect those terms in $[Y + \dot{Y}\,\delta T]^{-1}$ of order higher than one (since all terms in $[Y + \dot{Y}\,\delta T]^{-1}$ are multiplied by δT in (5.2.5). Let $[Z + \dot{Z}\,\delta T]$ be the transposed matrix of cofactors of $[Y + \dot{Y}\,\delta T]$ with terms of higher order than δT truncated. In evaluating the determinant of $[Y + \dot{Y}\,\delta T]$, we arrive at terms of order as high as δT^n. Again, we can truncate terms of order higher than one. Let $d_1 + d_2\,\delta T$ be the truncated expression for the determinant. Thus

$$[Y + \dot{Y}\,\delta T]^{-1} \cong \frac{[Z + \dot{Z}\,\delta T]}{d_1 + d_2\,\delta T} \qquad (5.2.6)$$

to a first order approximation. Let us expand $1/(d_1 + d_2\,\delta T)$ and again neglect terms of higher order than one

$$\frac{1}{d_1 + d_2\,\delta T} \cong \frac{1}{d_1} - \frac{d_2}{d_1^2}\,\delta T \qquad (5.2.7)$$

Therefore

$$[Y+\dot{Y}\,\delta T]^{-1} \cong \left(\frac{1}{d_1} - \frac{d_2}{d_1^2}\,\delta T\right)[Z+\dot{Z}\,\delta T]$$

$$\cong \frac{[Z]}{d_1} + \left[\frac{\dot{Z}}{d_1} - \frac{d_2}{d_1^2}\,Z\right]\delta T \qquad (5.2.8)$$

where again terms involving δT^2 were truncated. Substituting (5.2.8) into (5.2.5) and truncating terms of higher order than δT^2

$$\delta\underline{\lambda} = -\frac{[Z]}{d_1}\,\underline{x}(T)\,\delta T - \left(\frac{1}{2d_1}\,[Z]\,\ddot{\underline{x}}(T) + \left[\frac{\dot{Z}}{d_1} - \frac{d_2}{d_1^2}\,Z\right]\dot{\underline{x}}(T)\right)\delta T \qquad (5.2.9)$$

However, we also have, to second order approximation

$$\delta\underline{\lambda} = \left\{\frac{\partial\lambda}{\partial T}\right\}\delta T + \frac{1}{2}\left\{\frac{\partial^2\lambda}{\partial T^2}\right\}\delta T^2 \qquad (5.2.10)$$

Therefore

$$\left\{\frac{\partial^2\lambda}{\partial T^2}\right\} = -\frac{1}{d_1}\,[Z]\,\ddot{\underline{x}}(T) - 2\left[\frac{\dot{Z}}{d_1} - \frac{d_2}{d_1^2}\,Z\right]\dot{\underline{x}}(T) \qquad (5.2.11)$$

Also, $\left\{\frac{\partial\lambda}{\partial T}\right\} = -\frac{[Z]}{d_1}\,\dot{\underline{x}}(T)$, which is the same as the expression in (5.1.6).

Therefore, (5.2.11) with (5.1.6) and (5.2.3) give us a method of computing $\partial^2 H/\partial T^2$. While many approximations were made in deriving this method, we do not need a really exact value for $\partial^2 H/\partial T^2$, but just one which is accurate enough to use (5.2.1) to drive T into a region where $\partial H/\partial T < 0$, for which this method should be sufficiently accurate. The following section will consider the application of this method to the same numerical example studied in previous chapters.

5.3 Numerical Example

From (4.2.3), we have that, at $t = 0$

$$H(\lambda) = \frac{\lambda_2^2}{4} - \lambda_2 - 1 \tag{5.3.1}$$

Thus

$$\frac{\partial H}{\partial T} = \frac{\lambda_2}{2}\frac{\partial \lambda_2}{\partial T} - \frac{\partial \lambda_2}{\partial T}$$

$$= \frac{\partial \lambda_2}{\partial T}\left(\frac{\lambda_2}{2} - 1\right) \tag{5.3.2}$$

From (4.2.1) and (5.1.6),

$$\frac{\partial \lambda_2}{\partial T} = R_2 \tag{5.3.3}$$

where R_2 is defined in (4.2.2).

Differentiating (5.3.2) again with respect to T

$$\frac{\partial^2 H}{\partial T^2} = \frac{\partial^2 \lambda_2}{\partial T^2}\left(\frac{\lambda_2}{2} - 1\right) + \frac{1}{2}\left(\frac{\partial \lambda_2}{\partial T}\right)^2 \tag{5.3.4}$$

In computing $\partial^2 \lambda_2 / \partial T^2$, we first write $[Y + \dot{Y}\delta T]^{-1}$ for a two-dimensional system

$$[Y + \dot{Y}\,\delta T]^{-1} = \begin{bmatrix} \frac{\partial x_1}{\partial \lambda_1} + \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_1}\right)\delta T & \frac{\partial x_1}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_2}\right)\delta T \\ \frac{\partial x_2}{\partial \lambda_1} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_1}\right)\delta T & \frac{\partial x_2}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_2}\right)\delta T \end{bmatrix}^{-1} \tag{5.3.5}$$

The transpose of the cofactors of $[Y + \dot{Y}\delta T]$ is given by

$$[Z + \dot{Z}\,\delta T] = \begin{bmatrix} \frac{\partial x_2}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_2}\right)\delta T & -\frac{\partial x_2}{\partial \lambda_1} - \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_1}\right)\delta T \\ -\frac{\partial x_1}{\partial \lambda_2} - \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_2}\right)\delta T & \frac{\partial x_2}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_2}\right)\delta T \end{bmatrix} \tag{5.3.6}$$

Thus

$$\begin{aligned} Z_{ij} &= -Y_{ij} \quad \dot{Z}_{ij} = -\dot{Y}_{ij} \qquad \text{for } i \neq j \\ Z_{11} &= Y_{22} \quad \dot{Z}_{11} = \dot{Y}_{22} \\ Z_{22} &= Y_{11} \quad \dot{Z}_{22} = \dot{Y}_{11} \end{aligned} \tag{5.3.7}$$

Also

$$\begin{aligned} \det[Y + \dot{Y}\,\delta T] = &\left(\frac{\partial x_1}{\partial \lambda_1} + \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_1}\right)\delta T\right)\left(\frac{\partial x_2}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_2}\right)\delta T\right) \\ &- \left(\frac{\partial x_2}{\partial \lambda_1} + \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_1}\right)\delta T\right)\left(\frac{\partial x_1}{\partial \lambda_2} + \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_2}\right)\delta T\right) \end{aligned} \tag{5.3.8}$$

Therefore, multiplying out (5.3.8) and identifying terms involving δT^0 and δT^1, we obtain d_1 and d_2 of (5.2.7), (5.2.8) and (5.2.9):

$$d_1 = \frac{\partial x_1}{\partial \lambda_1} \cdot \frac{\partial x_2}{\partial \lambda_2} - \frac{\partial x_2}{\partial \lambda_1} \cdot \frac{\partial x_1}{\partial \lambda_2}$$

$$\begin{aligned} d_2 = &\frac{\partial x_1}{\partial \lambda_1} \cdot \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_2}\right) + \frac{\partial x_2}{\partial \lambda_2} \cdot \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_1}\right) \\ &- \frac{\partial x_2}{\partial \lambda_1} \cdot \frac{d}{dt}\left(\frac{\partial x_1}{\partial \lambda_2}\right) - \frac{\partial x_1}{\partial \lambda_2} \cdot \frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_1}\right) \end{aligned} \tag{5.3.9}$$

Equations (5.3.7) and (5.3.9) give us the expressions necessary to evaluate $\partial^2\lambda_2/\partial T^2$ by (5.2.11). Thus we have derived all the expressions necessary to calculate $\partial H/\partial T$ and $\partial^2 H/\partial T^2$ to apply the method developed in 5.1 and 5.2.

If the control is at its bound at $t = 0$, $H(\lambda)$ at $t = 0$ becomes:

$$H(\lambda) = -\lambda_2(1 - u) - 1 \tag{5.3.10}$$

Thus

$$\frac{\partial H}{\partial T} = (u - 1) \frac{\partial \lambda_2}{\partial T} \tag{5.3.11}$$

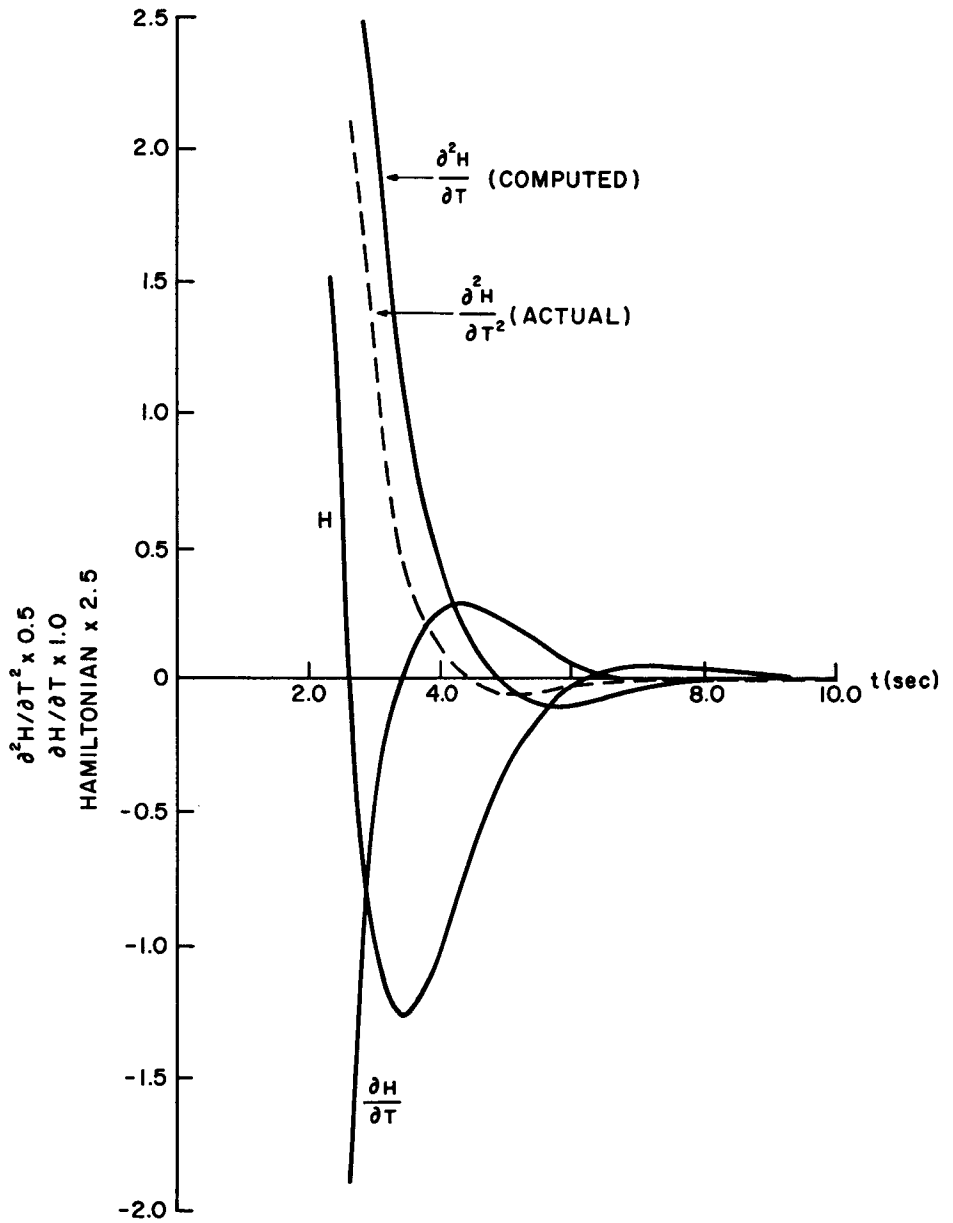
$$\frac{\partial^2 H}{\partial T^2} = (u - 1) \frac{\partial^2 \lambda_2}{\partial T^2} \tag{5.3.12}$$

where the $\partial\lambda_2/\partial T$ and $\partial^2\lambda_2/\partial T^2$ are given by the same expressions as for the unbounded case.

5.4 Discussion of Results

Figure 5.4.1 and Table 5.4.1 show H, $\partial H/\partial T$ and $\partial^2 H/\partial T^2$ computed from the methods outlined earlier in the chapter as a function of T. As can be seen from these, the values of $\partial H/\partial T$ are fairly accurate, i.e., they are the same as those that would be obtained by differentiating the H vs. T curve. This was to be expected, since the approximations made in making first order approximations for $\delta\underline{x}$ and $\partial\lambda/\partial T$ become exact as δT approaches zero.

However, the computed values for $\partial^2 H/\partial T^2$ are not exact, as can be seen by comparing the computed values with the correct values, shown by the dashed curve in Figure 5.4.1, obtained by differentiating the $\partial H/\partial T$ vs. T curve. This error is due to the method used in Equation (5.2.5) in representing the first order variation in the $[\partial x/\partial\lambda]^{-1}$ matrix as a function of δT. This method used a first order approximation for the value of $[\partial x/\partial\lambda]^{-1}$ at the final time $T + \delta T$. However, in actuality, the effective value of $[\partial x/\partial\lambda]^{-1}$ is somewhere between its value at final time T and time $T + \delta T$. If its value at final time T had been used, the first order variation in $[\partial x/\partial\lambda]^{-1}$ with δT would have been completely lost, thus making second order terms in δT in (5.2.9) inaccurate. A compromise would be to use a value of $[\partial x/\partial\lambda]^{-1}$ evaluated at some



H, $\partial H/\partial T$, and $\partial^2 H/\partial T^2$ vs. FINAL TIME

FIGURE 5.4.1

TABLE 5.4.1

T sec.	H	$\partial H/\partial T$	$\partial^2 H/\partial T^2$
2.0	2.847	- 9.53	32.0
2.2	1.393	- 5.44	18.5
2.4	0.552	- 3.19	11.5
2.6	0.057	- 1.88	7.57
2.8	- 0.234	- 1.09	5.29
3.0	- 0.397	- 0.585	3.87
3.2	- 0.479	- 0.255	2.91
3.4	- 0.507	- 0.036	2.22
3.6	- 0.4981	0.111	1.69
3.8	- 0.466	0.203	1.27
4.0	- 0.419	0.256	0.917
4.2	- 0.365	0.280	0.629
4.4	- 0.309	0.283	0.385
4.6	- 0.253	0.270	0.208
4.8	- 0.201	0.248	0.063
5.0	- 0.154	0.220	- 0.045
5.2	- 0.113	0.190	- 0.121
5.4	- 0.079	0.159	- 0.171
5.6	- 0.050	0.129	- 0.200
5.8	- 0.027	0.102	- 0.211
6.0	- 0.009	0.077	- 0.210
6.2	0.004	0.056	- 0.200
6.4	0.014	0.038	- 0.183
6.6	0.020	0.023	- 0.162
6.8	0.023	0.012	- 0.140
7.0	0.024	0.0025	- 0.117
7.2	0.024	- 0.0041	- 0.095
7.4	0.023	- 0.0086	- 0.075
7.6	0.021	- 0.0115	- 0.057
7.8	0.018	- 0.0130	- 0.041
8.0	0.016	- 0.0134	- 0.028
8.2	0.013	- 0.0131	- 0.017
8.4	0.010	- 0.0123	- 0.0087
8.6	0.008	- 0.0111	- 0.0022
8.8	0.0059	- 0.0097	0.0026
9.0	0.0025	- 0.0083	0.0059
9.2	0.0010	- 0.0069	0.0080
9.4	- 0.0004	- 0.0055	0.0091
9.6	- 0.0013	- 0.0043	0.0094
9.8	- 0.0019	- 0.0032	0.0092
10.0	- 0.0033	- 0.0023	0.0086

final time between T and $T + \delta T$, say $T + \frac{\delta T}{2}$, in which case $[\partial x/\partial\lambda + \frac{1}{2}\frac{d}{dt}(\partial x/\partial\lambda)]$ would be used. However, this proved to be no more accurate than using the value at $T + \delta T$. In fact, for some values of T, $\partial^2 H/\partial T^2$ was more accurate when $[\partial x/\partial\lambda]$ was evaluated at a final time fairly close to $T + \delta T$, and sometimes the value closer to T gave a more accurate $\partial^2 H/\partial T^2$. However, for the most part, the original method used evaluating $[\partial x/\partial\lambda]^{-1}$ at $T + \delta T$ gave a $\partial^2 H/\partial T^2$ more consistently of the same general shape as the actual curve, and thus was used in the final computer results. For any initial guess on T, the method did indeed converge to a point at which $\partial^2 H/\partial T^2 < 0$, i.e., a local minimum of $S(T)$.

With an initial estimate of $T = 4.6$ seconds, the method converged to $T = 2.6$ seconds in steps of $T = 4.6$, 3.05, 2.2, 2.45, and 2.6 seconds, with $K_T = 2.0$ used for this run. The first iteration, as expected, drove T into the region where $\partial H/\partial T < 0$, and from there the method converged to the local minimum at $T = 2.6$ seconds. In the method of Chapter 4, with only the $H = 0$ criterion used, any initial estimate of T greater than 3.4 seconds would converge to the local maximum of $S(T)$ at $T = 6.1$ seconds.

When an initial estimate of $T = 5.2$ seconds was used, (after $\partial^2 H/\partial T^2$ changes sign) the method converged to $T = 9.3$ seconds in steps of 5.2, 7.8, 9.0, and 9.3 seconds. When only the $H = 0$ condition was used, the solution at $T = 6.1$ seconds was arrived at starting from $T = 5.2$ seconds. The solution at $T = 9.3$ seconds is a local minimum of $S(T)$ (although Table 5.4.1 does not show this, due to integration errors), but not the least local minimum, which is at $T = 2.6$ seconds.

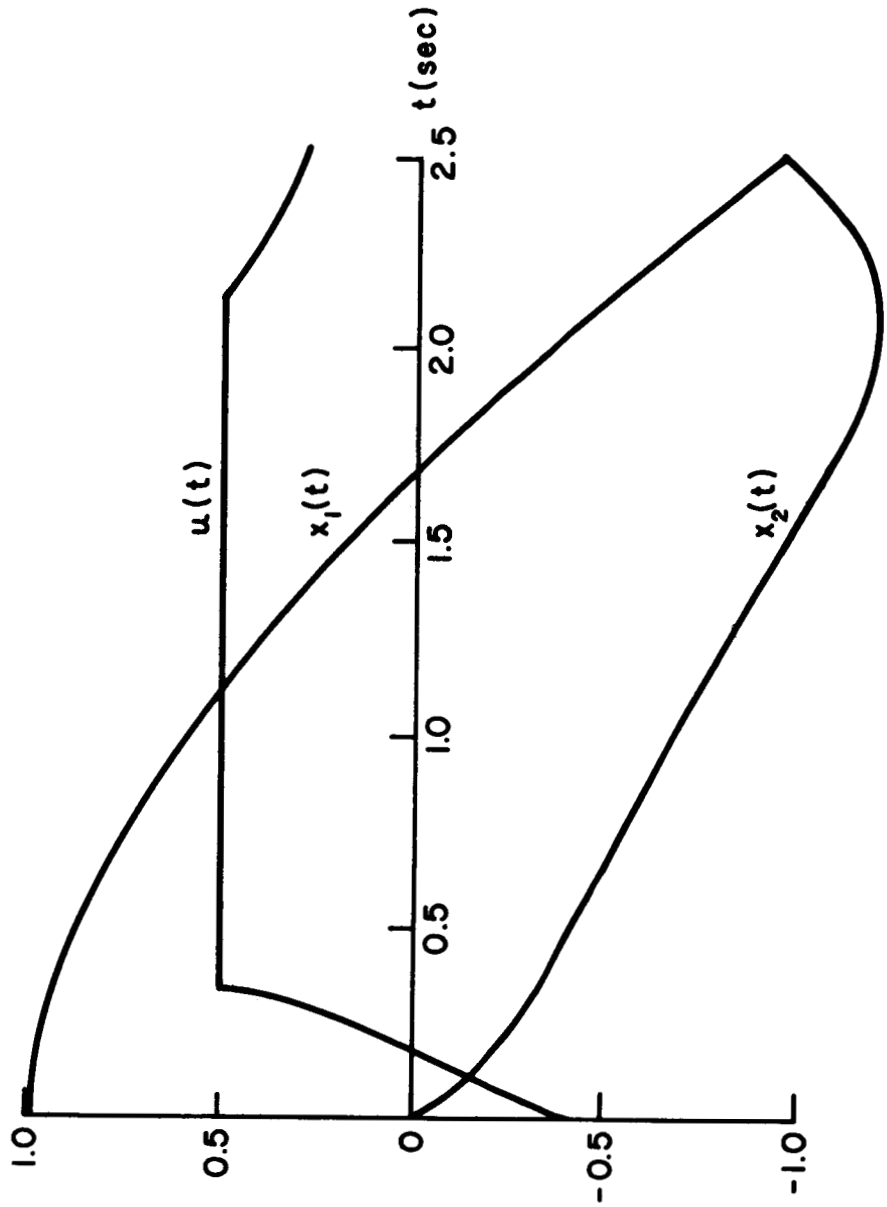
For the bounded control case, it was found that the most efficient way of solving the variable time problem was first to solve

it for unbounded control and use the optimal final time T for unbounded control for the first estimate on T for the variable time optimal bounded problem. As can be seen from Figure 4.3.2, the maximum value the control attains for the variable time optimal T = 2.6 seconds is about 0.7. Thus, if we were to start with an initial estimate of T = 5 seconds for the bounded case, and applied bounds of ± 0.8 as was the case in Figure 3.4.2, although the optimal control for the initial estimate of T is bounded, the final variable time solution at T = 2.6 seconds is unbounded. If we take the bounds as ± 0.5 to obtain a bounded control for T = 2.6 seconds, but apply these bounds to an initial estimate of T = 5 seconds, the control is almost totally bounded, and convergence is difficult. However, if we first solve the unbounded problem, we will avoid having to solve the bounded problem in areas of T where the optimal solution is almost totally bounded, where convergence is less efficient.

If the bounded problem is solved in this manner, with bounds of ± 0.5, we obtain the variable time optimal solution at T = 2.525 seconds shown in Figure 5.4.2. As can be seen, even though the control is well bounded, the variable time optimal T is not too far from the value of T = 2.6 seconds for the unbounded case.

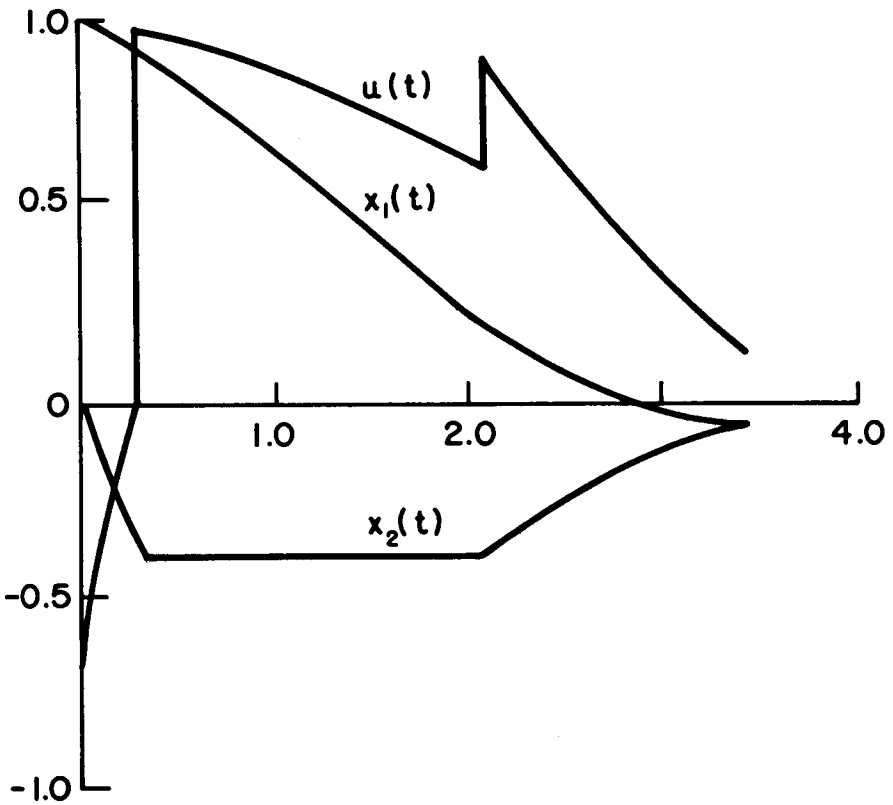
For the bounded state variable problem, the extension to variable time was straightforward, requiring merely the addition of the technique of section 3.5 to the methods of this and the preceeding chapters. For the bound $-0.4 \leq x_2(t)$, the variable time optimum occurred at T = 3.45 seconds, and the resulting optimal trajectories are shown in Figure 5.4.2.

In making the extension of the bounded state variable and bounded control problems to variable time, the only possible source of difficulty in using the variable time techniques of Chapters 4 and



VARIABLE TIME OPTIMAL TRAJECTORIES,
BOUNDED CONTROL

FIGURE 5.4.2



VARIABLE TIME OPTIMAL TRAJECTORIES,
BOUNDED STATE VARIABLES

FIGURE 5.4.3

5 was that in using Equation (4.1.4) to obtain the change in initial adjoint condition $\delta\underline{\lambda}$ as a function of δT, no allowance was made for the fact that this $\delta\underline{\lambda}$ might produce a $\delta\underline{x}(t)$ or $\delta\underline{u}(t)$ which exceeds the constraints. However, (4.1.4) is only a first order approximation, anyway, and neglecting this effect did not produce enough additional inaccuracy to prevent convergence.

CHAPTER 6

THE VARIABLE END POINT PROBLEM

6.1 Theoretical Background

In the first five chapters we considered optimization problems for which the terminal condition $\underline{x}(T)$ was some fixed constant vector $\{x_1(t), x_2(T), \ldots x_n(T)\}$ in the n-dimensional Euclidean space E_n. We will now consider problems in which the terminal vector $\underline{x}(T)$ is merely constrained to lie in some (n-k)-dimensional manifold M of E_n. Such a manifold is defined as the set of all points $\underline{x} = \{x_1, x_2, \ldots, x_n\}$ which satisfy simultaneously the k equations

$$\begin{aligned} &h_1(x_1, x_2, \ldots, x_n) = 0 \\ &h_2(x_1, x_2, \ldots, x_n) = 0 \\ &\text{----------} \\ &h_k(x_1, x_2, \ldots, x_n) = 0 \end{aligned} \tag{6.1.1}$$

and where the vectors

$$\text{grad } h_1(x), \text{ grad } h_2(x), \ldots \text{grad } h_k(x) \tag{6.1.2}$$

are linearly independent, where

$$\text{grad } h(x) = \left\{ \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \ldots \frac{\partial h}{\partial x_n} \right\} \tag{6.1.3}$$

Pontryagin[15] gives a necessary condition for the optimal control for the problem given by (2.1.1) and (2.1.2), with fixed initial conditions and terminal condition constrained by $\underline{x}(T) \in M$. Let V be the tangent plane to the manifold M at a given point $\underline{x}$. This tangent plane has dimension (n-k). The desired additional necessary condition for this problem, known as the transversality condition, is that for $\underline{x}(t)$ to be optimal, the terminal adjoint vector $\underline{\psi}(T)$ must be orthogonal to every vector $\underline{\theta} \in V$, i.e.,

$$\underline{\psi}^T(T) \cdot \underline{\theta} = 0 \tag{6.1.4}$$

for all $\underline{\theta} \in V$.

Since V is $(n-k)$-dimensional, (6.1.4) yields $(n-k)$ independent relationships, which, along with the k equations (6.1.1), give sufficient relationships to solve the problem.

We will now show the equivalence between this condition and the condition that the first variational of $S(T, X)$ given by (2.3.1) with respect to $\underline{X} = \underline{x}(T)$, is zero, i.e.,

$$\delta S = \left\{\frac{\partial S}{\partial \underline{X}}\right\}^T \cdot \delta \underline{X} = 0 \tag{6.1.5}$$

where the $\delta \underline{X}$ must be consistent with the constraints (6.1.1). First let us write

$$\begin{aligned} \delta S &= S(T, X + \delta X) - S(T, X) \\ &= \int_0^T f_o(x + \delta x, \bar{u} + \delta u)dt - \int_0^T f_o(x, \bar{u})dt \end{aligned} \tag{6.1.6}$$

Note that (6.1.6) is equivalent to the second integral in (2.3.2). Thus, if we proceed as we did in going from (2.3.2) to (2.3.11), we arrive at a result similar to (2.3.11) (minus the first term with δT)

$$\delta S = \sum_{i=1}^{n} \psi_i \, \delta x_i \Big|_{t=0}^{T} \tag{6.1.7}$$

However, in (2.3.11) the δx_i arose from the variation δT, while here it is due to the $\delta \underline{X}$, and, in fact, here $\delta \underline{x} = \delta \underline{X}$. Thus, letting $\delta \underline{x}(0) = 0$ and $\delta \underline{x}(T) = \delta \underline{X}$

$$\delta S = \underline{\psi}^T(T) \cdot \delta \underline{X} \tag{6.1.8}$$

However, we have stated that $\delta \underline{X}$ must be consistent with (6.1.1), i.e., that, to a first order of approximation

$$\left\{\frac{\partial h_1}{\partial X}\right\}^T \cdot \delta\underline{X} = 0$$

$$\left\{\frac{\partial h_2}{\partial X}\right\}^T \cdot \delta\underline{X} = 0 \qquad (6.1.9)$$

$$\text{----------}$$

$$\left\{\frac{\partial h_k}{\partial X}\right\}^T \cdot \delta\underline{X} = 0$$

where $\left\{\frac{\partial h_i}{\partial X}\right\} = \left\{\frac{\partial h_i}{\partial X_1}, \frac{\partial h_i}{\partial X_2}, \cdots \frac{\partial h_i}{\partial X_n}\right\}$.

But the vectors $\left\{\partial h_i / \partial X\right\}$ are directed along the normals to the respective curves $h_i(x) = 0$, and thus the $\delta\underline{X}$ satisfying (6.1.9) are tangent to the respective curves $h_i(x) = 0$. Thus the $\delta\underline{X}$ satisfying all of (6.1.9) are tangent to all the curves $h_i(x) = 0$, and thus to the manifold M. Thus the $\delta\underline{X}$ in (6.1.8) are equivalent to the $\underline{\theta}$ in (6.1.4) (when $\delta\underline{X}$ goes to zero so that (6.1.9) become exact), demonstrating the equivalence between the transversality condition (6.1.4) and the condition $\delta S = 0$ of (6.1.5). Thus, the transversality condition for variable end point optimum, like the $H = 0$ condition for variable time optimum, is a necessary condition which holds for a variable end point maximum of $S(T, X)$ as well as a minimum of $S(T, X)$. Thus, in using this necessary condition, we run into the same problems as we did in using the $H = 0$ condition. Before commenting further on this, let us develop a computational algorithm using the transversality condition.

6.2 Computational Algorithm

We want to start with an initial estimate of $\underline{X}$ satisfying (6.1.1), solve the fixed end point problem using the methods of Chapter 3, and then obtain a new estimate $\underline{X} + \delta\underline{X}$ so as to drive the condition (6.1.4) to zero.

We know that (6.1.4) gives us $r = n-k$ independent equations in the $2n$ variables $x_1, x_2 \ldots x_n$, $\psi_1(T)$, $\psi_2(T), \ldots \psi_n(T)$. Let the r equations be given by

$$\begin{aligned} M_1\big(X,\ \psi(T)\big) &= 0 \\ M_2\big(X,\ \psi(T)\big) &= 0 \\ &\text{----------} \\ M_r\big(X, \psi(T)\big) &= 0 \end{aligned} \tag{6.2.1}$$

Let us assume that our j^{th} estimate yields

$$M_i^j\left(X^j,\ \psi^j(T)\right) = \ell_i^j \tag{6.2.2}$$

We want to compute $\delta\underline{X}^{j+1}$ so that

$$M_i^{j+1}\left(X^j + \delta X^j,\ \psi^j(T) + \delta\psi^j(T)\right) = 0 \tag{6.2.3}$$

at least to a first order of approximation. Since $\underline{X}^j$ and $\underline{\psi}^j(T)$ are known from the previous trajectory, we have $2n$ unknowns $\delta X_1,\ \delta X_2, \ldots, \delta X_n, \delta\psi_1(T),\ \delta\psi_2(T), \ldots\ldots \delta\psi_n(T)$.

We know from Chapter 4 that a change in the terminal point $\delta\underline{X}$ requires a change in initial adjoint vector $\delta\underline{\lambda}$ given approximately by (4.1.2)

$$\delta\underline{X} = [Y]\ \delta\underline{\lambda}$$

where $Y_{ij} = \partial x_i/\partial\lambda_j\big|_{t=T}$. Thus

$$\delta\underline{\lambda} = [Y]^{-1}\ \delta\underline{X} \tag{6.2.4}$$

In a manner similar to (4.1.2), we can write that a given $\delta\underline{\lambda}$ will produce a change $\delta\underline{\psi}(T)$ in $\underline{\psi}(T)$ given approximately by

$$\delta\underline{\psi}(T) = [W]\ \delta\underline{\lambda} \tag{6.2.5}$$

where $W_{ij} = \partial\psi_i/\partial\lambda_j\big|_{t=T}$, where the $\partial\psi_i/\partial\lambda_j$ are obtained, like the $\partial x_i/\partial\lambda_j$, from the sensitivity function Equations (3.2.15).

Therefore, substituting (6.2.4) in (6.2.5)

$$\delta\underline{\psi}(T) = [W]\,[Y]^{-1}\,\delta\underline{X} \qquad (6.2.6)$$

Substituting (6.2.6) into (6.2.3) for $\delta\underline{\psi}(T)$, we reduce the number of unknowns in (6.2.3) from 2n to n. Also, the k Equations (6.1.9) can be used to eliminate k of the δX_i's, thus reducing the number of unknowns in (6.2.3) to r = n-k, which is equal to the number of equations. Thus (6.2.3) can be solved for (n-k) of the n δX_i's, giving (n-k) of the desired $X_i^{j+1} = X_i^j + \delta X_i$. The k Equations (6.1.1) can then be solved for the remaining k X_i^{j+1}'s, thus insuring that $\underline{X}^{j+1} \epsilon M$. Although, due to the first order approximations made, the $m_i^{j+1}\left(X, \psi(T)\right)$ will not exactly equal zero, they will be closer to zero than the previous $m_i^j\left(X, \psi(T)\right)$, and thus eventually the method will converge so that the $m_i\left(X, \psi(T)\right)$ are as close to zero as we please. This technique will be illustrated in the next section by a numerical example.

6.3 Numerical Example

We will use the same numerical example as in previous chapters, except that here we will consider the problem in which the terminal condition is constrained by

$$x_1^2 + x_2^2 = 0.5 \qquad (6.3.1)$$

That is, among all controls which transfer the system of Equations (3.4.1) from the initial point $(x_1, x_2) = (1, 0)$ to some point on the circle given by (6.3.1), we want to find the one which minimizes the performance criterion given by (3.4.3). Thus, (6.3.1) corresponds to (6.1.1), with k = 1. We will first solve the problem for fixed final time T, and then proceed on to the variable time, variable end point problem.

For the circle given by (6.3.1), the slope of the line drawn from the origin to a given point (x_1, x_2) is equal to x_2/x_1, and the slope of the perpendicular to this line, which is also tangent to the circle, is equal to $-x_1/x_2$. Thus the vector tangent to the circle at the given point is given by $(1, -x_1/x_2)$, or alternately, by $(-x_2, x_1)$.

From the transversality condition, we know that for u(t) to be optimal $\underline{\psi} = (\psi_1, \psi_2)$ must be perpendicular to this tangent at the final time t = T, that is,

$$\left(\psi_1(T), \psi_2(T)\right) \cdot (-X_2, X_1) = -\psi_1(T) \cdot X_2 + \psi_2(T) \cdot X_1 = 0 \tag{6.3.2}$$

Thus, we have $n - k = 2 - 1 = 1$ additional equation, which, along with (6.3.1), is sufficient to solve the problem.

If we start with some initial estimate of the terminal conditions (X_1, X_2) consistent with (6.3.1) and solve the fixed end point problem, (6.3.2) will be equal to some value probably not equal to zero. We want to perturb $\underline{X}^n$ by some $\delta\underline{X}$ to get some $\underline{X}^{n+1} = \underline{X}^n + \delta\underline{X}$ so that (6.3.2) is equal to zero to a first order of approximation. Thus, we want

$$-(\psi_1 + \delta\psi_2)(X_2 + \delta X_2) + (\psi_2 + \delta\psi_2)(X_1 + \delta X_1) = 0 \tag{6.3.3}$$

From (6.2.6) we know that

$$\begin{bmatrix} \delta\psi_1(T) \\ \delta\psi_2(T) \end{bmatrix} = \begin{bmatrix} \partial\psi_1/\partial\lambda_1 & \partial\psi_1/\partial\lambda_2 \\ \partial\psi_2/\partial\lambda_1 & \partial\psi_2/\partial\lambda_2 \end{bmatrix}_{t=T} \begin{bmatrix} \partial x_1/\partial\lambda_1 & \partial x_1/\partial\lambda_2 \\ \partial x_2/\partial\lambda_1 & \partial x_2/\partial\lambda_2 \end{bmatrix}^{-1}_{t=T} \begin{bmatrix} \delta X_1 \\ \delta X_2 \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \delta X_1 \\ \delta X_2 \end{bmatrix} \tag{6.3.4}$$

where, if we multiply out the matrices

$$K_{11} = \left(\frac{\partial x_2}{\partial \lambda_2} \cdot \frac{\partial \psi_1}{\partial \lambda_1} - \frac{\partial x_2}{\partial \lambda_1} \cdot \frac{\partial \psi_1}{\partial \lambda_2} \right) \Big/ \det \left| \frac{\partial x}{\partial \lambda} \right|$$

$$K_{21} = \left(\frac{\partial x_2}{\partial \lambda_2} \cdot \frac{\partial \psi_2}{\partial \lambda_1} - \frac{\partial x_2}{\partial \lambda_1} \cdot \frac{\partial \psi_2}{\partial \lambda_2} \right) \Big/ \det \left| \frac{\partial x}{\partial \lambda} \right| \qquad (6.3.5)$$

$$K_{12} = \left(- \frac{\partial x_1}{\partial \lambda_2} \cdot \frac{\partial \psi_1}{\partial \lambda_1} + \frac{\partial x_1}{\partial \lambda_1} \frac{\partial \psi_1}{\partial \lambda_2} \right) \Big/ \det \left| \frac{\partial x}{\partial \lambda} \right|$$

$$K_{22} = \left(- \frac{\partial x_1}{\partial \lambda_2} \cdot \frac{\partial \psi_2}{\partial \lambda_1} + \frac{\partial x_1}{\partial \lambda_1} \cdot \frac{\partial \psi_2}{\partial \lambda_2} \right) \Big/ \det \left| \frac{\partial x}{\partial \lambda} \right|$$

where

$$\det \left| \frac{\partial x}{\partial \lambda} \right| = \frac{\partial x_1}{\partial \lambda_1} \cdot \frac{\partial x_2}{\partial \lambda_2} - \frac{\partial x_2}{\partial \lambda_1} \cdot \frac{\partial x_1}{\partial \lambda_2}$$

Also, from (6.3.1) and (6.1.9), to a first order of approximation, we have

$$X_1 \, \delta X_1 + X_2 \, \delta X_2 = 0 \qquad (6.3.5)$$

Solving (6.3.5) for δX_1, substituting into (6.3.3) and (6.3.4), and then substituting (6.3.4) into (6.3.3) for $\delta \psi_1$ and $\delta \psi_2$ we obtain

$$\begin{aligned} &- \left(\psi_1 + K_{11} \, \delta X_1 - K_{12} \frac{X_1}{X_2} \, \delta X_1 \right) \left(X_2 - \frac{X_1}{X_2} \, \delta X_1 \right) \\ &+ \left(\psi_2 + K_{21} \, \delta X_1 - \frac{X_1}{X_2} \, K_{22} \, \delta X_1 \right) \left(X_1 + \delta X_1 \right) = 0 \end{aligned} \qquad (6.3.6)$$

Collecting terms involving δX_1 and constants, and neglecting terms involving δX_1^2 since they are inaccurate anyway, we have an equation of the form

$$A \, \delta X_1 + B = 0 \qquad (6.3.7)$$

where

$$A = \psi_1 \frac{X_1}{X_2} - X_2 \left(K_{11} - K_{12} \frac{X_1}{X_2}\right) + \psi_2 + X_1 \left(K_{21} - \frac{X_1}{X_2} K_{22}\right) \tag{6.3.8}$$

$$B = - X_2 \psi_1 + X_1 \psi_2$$

Thus

$$\delta X_1 = - B/A \tag{6.3.9}$$

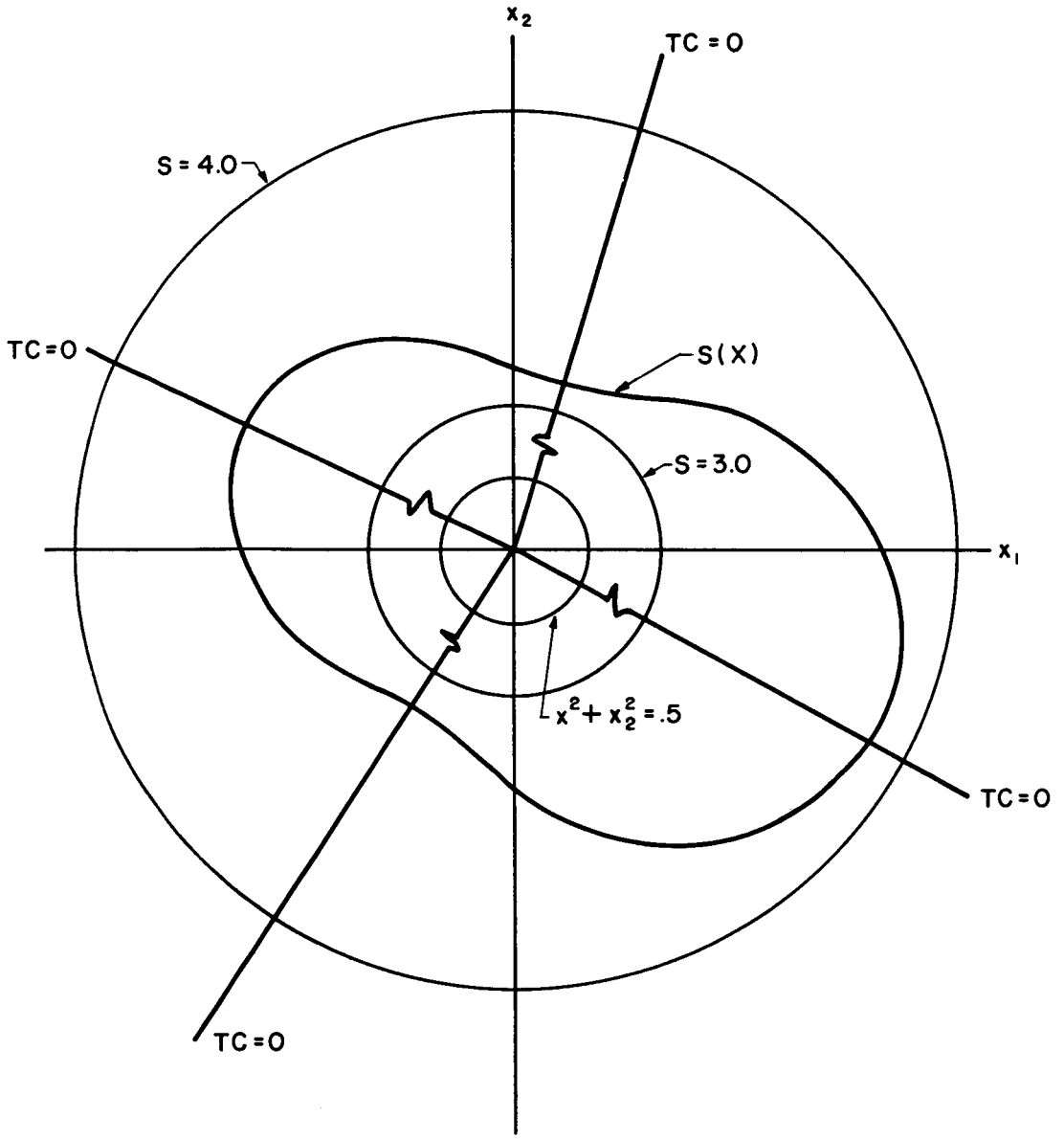
Also, we have

$$\delta X_2 = - \frac{X_1}{X_2} \delta X_1 \tag{6.3.10}$$

However, we cannot use both of these δX_i, since then (6.3.1) would not be satisfied exactly for $\underline{X}^{n+1} = \underline{X}^n + \delta \underline{X}$. Thus, if we take the smaller of the δX_i, and let $X_i^{n+1} = X_i^n + \delta X_i$ for that i, and then solve (6.3.1) for the other X_i^{n+1}, we will have our new estimate of $\underline{X}^{n+1}$ satisfying (6.3.1). We continue with this procedure until the transversality condition is as close to zero as we desire.

6.4 Numerical Results

Figure 6.4.1 gives a polar graph of the minimized criterion function $S = S(T, X)$ as a function of the end point $\underline{X}$ for a fixed final time $T = 5$ seconds. On this graph, S is plotted on lines radially outward from the origin with the S for a given $\underline{X}$ being plotted on the radial line through the given point on the $x_1^2 + x_2^2 = 0.5$ circle in the $x_1 - x_2$ plane. The points on the circle where the transversality condition is zero are indicated by the radial lines marked TC = 0. As can be seen from this graph, there is a point where the transversality condition is zero in each of the four quadrants, with two points corresponding to local maxima and two to local minima. The local maxima occur in the second and fourth



MINIMIZED CRITERION FUNCTION vs. END POINT

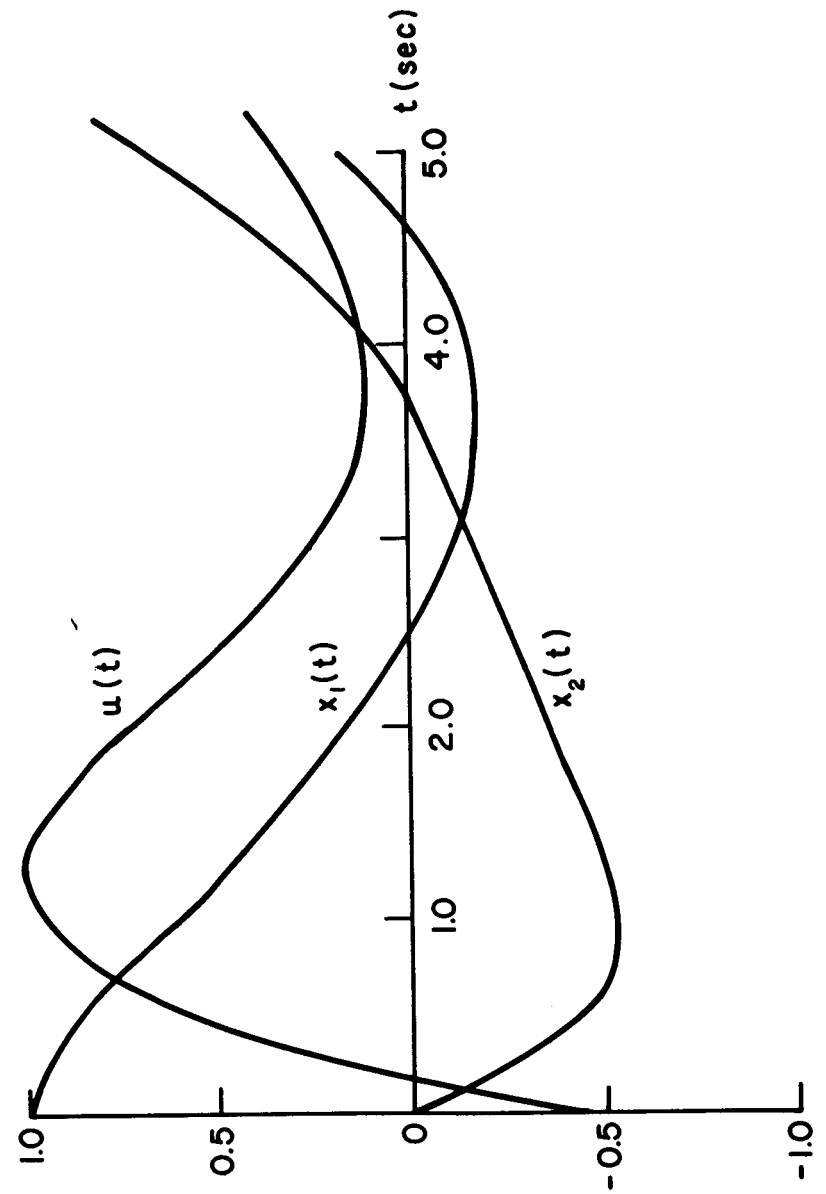
FIGURE 6.4.1

quadrants at $\underline{X} = (-0.64, 0.30)$ and $(0.62, -0.35)$ with $S = 3.51$ and 3.85, respectively. The local minima occur in the first and third quadrants at $\underline{X} = (0.19, 0.68)$ and $(-0.38, -0.60)$ with $S = 3.08$ and 3.14, respectively. Which of these extrema is obtained depends upon the first estimate of $\underline{X}$ used. Initial estimates were used at the intersection of the circle with the 45° line in each of the four quadrants, i.e., at $\underline{X} = (\pm 0.50, \pm 0.50)$, and in each case the method converged to the extremum in the corresponding quadrant, e.g., an initial estimate of $(+0.5, +0.5)$ converged to the extremum at (0.19, 0.68). Figure 6.4.2 shows the optimal trajectory at $\underline{X} = (0.19, 0.68)$.

To solve the variable time, variable end point problem, one can either first solve the variable end point problem for a given T, then solve the variable time problem for the resulting $\underline{X}$, and then resolve the variable end point problem for the new T, and continue until both the transversality condition and Hamiltonian are as close as desired to zero, or solve the variable time problem first for a given $\underline{X}$, then solve the variable end point problem, etc. The former procedure was followed here. Starting at $T = 5$ seconds and $\underline{X} = (0.5, 0.5)$, the method converged to a local minimum at $T = 5.25$ seconds and $\underline{X} = (0.21, 0.68)$, with $S = 3.08$. Starting at $T = 4$ seconds and $\underline{X} = (-0.5, -0.5)$, the method converged to a local minimum at $T = 1.15$ seconds and $\underline{X} = (0.56, -0.44)$, with $S = 1.64$. Thus, the problem of multiple extrema is compounded when we have both a variable time and variable end point problem, and the solution we obtain is highly dependent on the choice of initial guesses.

6.5 Sufficient Conditions for a Local Minimum

If $S(T, X)$ is at an extremum with respect to X, a sufficient condition for the extremum to be a minimum is that the second



VARIABLE END POINT OPTIMAL TRAJECTORIES

FIGURE 6.4.2

variation $\delta^2 S$ of S with respect to X be greater than zero. If $\delta S = \{\partial S/\partial X\}^T \cdot \delta \underline{X} = 0$, then the condition $\delta^2 S > 0$ is given by

$$\delta^2 S = \delta \underline{X}^T [\partial^2 S/\partial X^2] \, \delta \underline{X} > 0 \qquad (6.5.1)$$

where $[\partial^2 S/\partial X^2]$ is an nXn matrix with elements $\partial^2 S/\partial X_i \, \partial X_j$. In (6.5.1), all the perturbations δX_i must be consistent with the constraints on $\underline{X}$, i.e., must satisfy Equations (6.1.9). Since (6.1.9) give us k relations between the δX_i, we may use (6.1.9) to eliminate k of the δX_i in (6.5.1) and thus we can reduce (6.5.1) to a relation consisting of (n-k)-dimensional vectors and a (n-k)X(n-k) matrix. Condition (6.5.1) is thus equivalent to the condition that the resulting (n-k)X(n-k) matrix is positive definite.

We now need a method of computing the $\partial S/\partial X_i \, \partial X_j$. Comparing (6.1.5) and (6.1.8), we see that

$$\left\{\frac{\partial S}{\partial X}\right\} = \underline{\psi}(T) \qquad (6.5.2)$$

Therefore

$$\frac{\partial^2 S}{\partial X_i \partial X_j} = \frac{\partial \psi_i(T)}{\partial X_j} \qquad (6.5.3)$$

We also know that

$$\delta \underline{\psi}(T) = \left[\frac{\partial \psi(T)}{\partial X}\right] \delta \underline{X} \qquad (6.5.4)$$

Comparing (6.5.4) with (6.2.6), we see that

$$\left[\frac{\partial \psi(T)}{\partial X}\right] = [W]\,[Y]^{-1} \qquad (6.5.5)$$

where the [W] and [Y] are defined as in (6.2.6). Thus this gives us all the information we need to compute $\delta^2 S$ and thus to apply the sufficient condition (6.5.1).

Let us illustrate this technique by applying it to our numerical example. From (6.5.5)

$$\frac{\partial \psi_i(T)}{\partial X_j} = K_{ij} \qquad i,j = 1,2 \qquad (6.5.6)$$

where the K_{ij} are defined in (6.3.5). Thus for our two-dimensional problem, (6.5.1) becomes

$$\begin{Bmatrix} \delta X_1 \\ \delta X_2 \end{Bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \delta X_1 \\ \delta X_2 \end{Bmatrix} > 0 \qquad (6.5.7)$$

or

$$K_{11}\,\delta X_1^2 + \delta X_1\,\delta X_2\,(K_{12}+K_{21}) + K_{22}\,\delta X_2^2 > 0 \qquad (6.5.8)$$

But, from the constraint Equation (6.3.5)

$$\delta X_1 = -\frac{X_2}{X_1}\,\delta X_2 \qquad (6.5.9)$$

Thus (6.5.8) becomes

$$\left[\left(\frac{X_2}{X_1}\right)^2 K_{11} - \frac{X_2}{X_1}\left(K_{21}+K_{12}\right) + K_{22}\right]\delta X_2^2 > 0 \qquad (6.5.10)$$

Thus the sufficient condition for a local minimum is that

$$\left(\frac{X_2}{X_1}\right)^2 K_{11} - \frac{X_2}{X_1}\left(K_{21}+K_{12}\right) + K_{22} > 0 \qquad (6.5.11)$$

However, as was the case for the similar sufficient condition derived in Chapter 5 for variable time minimum, this condition will not insure that we have an absolute minimum, only that we have a local minimum. Thus this condition will not distinguish between the minima of the first and third quadrants of Figure (6.4.1).

6.6 Moving End Point Problem

In this section we will consider the class of problems in which end point is moving as a function of time. An example of this

problem is that of reaching a point fixed on the surface of the rotating earth. This problem is neither a simple variable time problem, since the end point is varying, nor is it a variable time, variable end point problem similar to that considered earlier in this chapter in which the variations in final time and end point were independent and the H = 0 condition and the transversality condition could both be satisfied simultaneously. For the moving end point problem, we must derive a single necessary condition containing both the variations due final time and end point. The first order variation in $S(T, X)$ for a variation in both T and $\underline{X}$ is given by

$$\delta S = - H \delta T + \left\{\frac{\partial S}{\partial X}\right\}^T \delta \underline{X} \tag{6.6.1}$$

since $H = - \partial S / \partial T$. The end point $\underline{X}$ depends explicitly on the final time T. Let us represent this dependence by $\underline{X} = \underline{X}(T)$. (Note that $\underline{X}(T)$ is a vector which is a function of T rather than a vector evaluated at $t = T$). Therefore, the variation $\delta \underline{X}$ is given by

$$\delta \underline{X} = \left\{\frac{dX}{dT}\right\} \delta T \tag{6.6.2}$$

where $\{dX/dT\}$ is an n-dimensional vector with components $dX_i(T)/dT$. Substituting (6.6.2) into (6.6.1)

$$\begin{aligned} \delta S &= - H \delta T + \left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{dX}{dT}\right\} \delta T \\ &= \left(- H + \left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{dX}{dT}\right\} \right) \delta T \end{aligned} \tag{6.6.3}$$

Dividing by δT and letting $\delta T \to 0$

$$\frac{dS}{dT} = - H + \left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{dX}{dT}\right\} \tag{6.6.4}$$

The desired necessary condition for the moving end point problem is thus given by

$$-H + \left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{dX}{dT}\right\} = 0 \qquad (6.6.5)$$

To derive a computational algorithm employing this condition, we follow a procedure similar to that used in section 4.1. We calculate an estimate of H at final time $T + \delta T$ the same way as we did in Equation (4.5.6), except that $\delta\underline{\lambda} = -[Y]^{-1}(\dot{\underline{x}}(T) - \{dX/dT\})\delta T$ is used in place of (4.1.4), since the desired change in terminal conditions is given by $-(\dot{\underline{x}}(T) - \{dX/dT\})\delta T$ instead of $-\dot{\underline{x}}(T)\,\delta T$ as in (4.1.3). The relationship for the second term of (6.6.5) evaluated at $T + \delta T$ is given by

$$\left(\left\{\frac{\partial S}{\partial X}\right\}^T + \left\{\frac{dX}{dT}\right\}^T \left[\frac{\partial^2 S}{\partial X^2}\right] \delta T\right)\left(\left\{\frac{dX}{dT}\right\} + \left\{\frac{d^2X}{dT^2}\right\} \delta T\right) =$$

$$\left(\left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{d^2X}{dT^2}\right\} + \left\{\frac{dX}{dT}\right\}^T \left[\frac{\partial^2 S}{\partial X^2}\right] \left\{\frac{dX}{dT}\right\}\right) \delta T \qquad (6.6.6)$$

$$+ \left\{\frac{dX}{dT}\right\}^T \left[\frac{\partial^2 S}{\partial X^2}\right] \left\{\frac{d^2X}{dT^2}\right\} \delta T^2$$

where $[\partial^2 S/\partial X^2]$ and $\{\partial S/\partial X\}$ are the same as given in (6.5.2) and (6.5.3), respectively. Combining this relationship with that for the estimate of H at $T + \delta T$, and setting it equal to zero as in (6.6.5), we get an equation which can be solved for the single unknown δT, which will drive (6.6.5) closer to zero. Repeated application of this process will result in driving (6.6.5) as close to zero as we please.

We will apply this technique to the same problem considered throughout this paper, with the initial condition $\underline{x}(0) = (1, 0)$, and the terminal condition moving around the circle $x_1^2 + x_2^2 = 0.5$

at a constant counterclockwise angular velocity starting at the point (0.5, 0). Thus the terminal condition $\underline{X}(T)$ is given by

$$\begin{Bmatrix} X_1(T) \\ X_2(T) \end{Bmatrix} = \begin{Bmatrix} 0.5 \cos \omega t \\ 0.5 \sin \omega t \end{Bmatrix} \tag{6.6.7}$$

where ω is the constant angular velocity.

The various derivatives of $\underline{X}(T)$ with respect to T needed in (6.6.6) are given by

$$\begin{Bmatrix} \frac{dX_1}{dT} \\ \frac{dX_2}{dT} \end{Bmatrix} = \begin{Bmatrix} -0.5\,\omega\, X_2(T) \\ 0.5\,\omega\, X_1(T) \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{d^2X_1}{dT^2} \\ \frac{d^2X_2}{dT^2} \end{Bmatrix} = \begin{Bmatrix} -0.5\,\omega^2 X_1(T) \\ -0.5\,\omega^2 X_2(T) \end{Bmatrix} \tag{6.6.8}$$

The $\{\partial S/\partial X\}$ and $[\partial^2 S/\partial X^2]$ are the same as those given in section 6.5

$$\begin{Bmatrix} \frac{\partial S}{\partial X} \end{Bmatrix} = \underline{\psi}(T)$$

$$\begin{bmatrix} \frac{\partial^2 S}{\partial X^2} \end{bmatrix} = [K] \tag{6.6.9}$$

where the K_{ij} are defined in (6.3.5).

If we let the equation for (6.6.5) evaluated at $T + \delta T$ be given by $A\,\delta T^2 + B\,\delta T + C = 0$, the values of A, B, and C for our numerical example are given by

$$A = -\frac{\left(R_2'\right)^2}{4} + \begin{Bmatrix} -.5\omega X_2(T) \\ .5\omega X_1(T) \end{Bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} -.5\omega^2 X_1(T) \\ -.5\omega^2 X_2(T) \end{Bmatrix}$$

$$B = -\frac{\left(\lambda_2 - 2\right)}{2} R_2' + \begin{Bmatrix} \psi_1(T) \\ \psi_2(T) \end{Bmatrix}^T \begin{Bmatrix} -.5\omega X_2(T) \\ .5\omega X_1(T) \end{Bmatrix} \qquad (6.6.10)$$

$$+ \begin{Bmatrix} -.5\omega X_2(T) \\ .5\omega X_1(T) \end{Bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} -.5\omega X_2(T) \\ .5\omega X_1(T) \end{Bmatrix}$$

$$C = -H + \begin{Bmatrix} \psi_1(T) \\ \psi_2(T) \end{Bmatrix}^T \begin{Bmatrix} -.5\omega X_2(T) \\ .5\omega X_1(T) \end{Bmatrix}$$

where R_2' is the same as that given in (4.2.2) with $\dot{x}_1(T)$ and $\dot{x}_2(T)$ being replaced by $\dot{x}_1(T) + 0.5\omega x_2(T)$ and $\dot{x}_2(T) - 0.5\omega x_1(T)$, respectively. All the other terms in (6.6.10) are evaluated at final time T for the previous iteration and are available in computer storage. Thus the equation $A\,\delta T^2 + B\,\delta T + C = 0$ can be solved for δT.

The value of ω used was such as to require 20 seconds for the end point to move completely around the circle, i.e., $\omega = 2\pi/20$. Starting from an initial estimate of T = three seconds, the method converged in three iterations to a local minimum at $T = 4.75$ seconds, $\underline{X} = (0.036,\ 0.499)$, with $S = 2.975$. Starting from $T = 6.5$ seconds, the method converged in two iterations to a local maximum at $T = 8.4$ seconds, $\underline{X} = (-0.438,\ 0.241)$, with $S = 3.271$. Starting from $T = 12$ seconds, the method converged in 3 iterations to a local minimum at $T = 13.5$ seconds, $\underline{X} = (-0.216,\ -0.451)$, with $S = 2.992$. Note that in this case, there is very little difference between the two local minima. It is likely that S would

continue to go through local maxima and minima as T continues to increase.

Sufficient conditions for a local minimum may be developed by differentiating (6.6.4) with respect to T

$$\frac{d^2S}{dT^2} = -\frac{dH}{dT} + \left\{\frac{\partial S}{\partial X}\right\}^T \left\{\frac{d^2X}{dT^2}\right\} + \left\{\frac{dX}{dT}\right\}^T \left[\frac{\partial^2 S}{\partial X^2}\right] \left\{\frac{dX}{dT}\right\} \qquad (6.6.11)$$

A relationship for dH/dt may be obtained from (5.1.7) by replacing $\underline{\dot{x}}(T)$ by $\underline{\dot{x}}(T) - \{dX/dT\}$

$$\frac{dH}{dT} = -\left(\left\{\frac{\partial H}{\partial \lambda}\right\}^T + \left\{\frac{\partial H}{\partial u}\right\}^T \left[\frac{\partial u}{\partial \lambda}\right]\right) \cdot [Y]^{-1} \left(\underline{\dot{x}}(T) - \left\{\frac{dX}{dT}\right\}\right) \qquad (6.6.12)$$

All other expressions in (6.6.11) have already been evaluated. The sufficient condition for a local minimum then is that $d^2S/dT^2 > 0$.

CHAPTER 7

SUMMARY AND CONCLUSIONS

Two techniques have been given for solving the fixed time optimal control problem, sequential optimization and quasilinearization. Quasilinearization proved to be the superior method in all respects for the problem considered, both from the point of view of efficiency of computer time and the methods' convergence properties. Convergence problems are encountered in sequential optimization for the bounded control case, while in quasilinearization such problems do not occur, except when the solution approaches the totally bounded case (bang-bang control). For this case, the time at which the control switches from one bound to the other becomes critical, but this problem can be overcome by using a method which decreases the integration interval in the region of the switching time.

A second variation of the sequential optimization technique exists (Reference 8) but was not considered here. This variation starts with an initial estimate on the entire control program $u(t)$ and uses the $\delta u(t)$ obtained in solving the linear problem for an improved estimate. In this variation, an initial estimate on $u(t)$ of $u(t) \equiv 0$ can be used, which does not allow the trajectories to diverge from the desired terminal conditions quite as much as when arbitrary estimates on initial adjoint conditions are used. Thus, the method iterating $u(t)$ is more efficient from the point of view of computer time than the one iterationing on $\psi(0)$. The former solves the fixed time problem with $T = 5.0$ seconds in approximately two minutes of computer time, compared to 3.5 minutes for the latter method and a half minute for quasilinearization. However, when estimates on $u(t)$ are used, adjoint differential equations are not used, and thus one cannot evaluate the Hamiltonian and solution of

the variable time problem becomes difficult. Furthermore, quasi-linearization is preferable to either variation of sequential optimization, anyway.

This paper has clearly demonstrated that methods for solution of the variable time problem which employ only the $H = 0$ condition are inadequate for the general problem. Such methods can yield a solution which is a local maximum of $S(T)$, if one exists, as well as a local minimum. A method of generating $\partial H/\partial T$ and $\partial^2 H/\partial T^2$ (equal to $-\partial^2 S/\partial T^2$ and $-\partial^3 S/\partial T^3$, respectively) has been developed which will insure that the solution arrived at will at least be a local minimum, although not necessarily the least local minimum, if more than one exist. It is questionable whether it is possible to develop a technique which, when in the neighborhood of a local minimum, can tell whether another local minimum exists.

In any problem in which both a local minimum and a local maximum exist, there must exist a second local minimum (which might occur at $T = \infty$). Thus, the techniques developed in this paper will not completely solve any problem which was not capable of being solved by some technique in the literature. That is, if there is only a local minimum, the $H = 0$ condition is sufficient; if there is a local minimum and a local maximum, there are two local minima, and we still have no way of insuring we obtain the least local minimum. However, we have developed a technique which widens the region of initial estimate on T from which we will converge to the desired absolute minimum and we have insured the attainment of at least a local minimum and not a local maximum. Also, by starting at a couple of widely differing initial estimates on T, we can quickly determine the rough shape of the $S(T)$, H, $\partial H/\partial T$, and $\partial^2 H/\partial T^2$ curves, and thus we can see if the possibility of more than one local minimum exists. If more than one does exist, we must by trial and

error test enough regions of initial estimate on T to insure that we do finally have a solution which is an absolute minimum.

Similar problems were also shown to exist in using the transversality condition, which is also merely a necessary condition, to solve the variable end point problem.

There are still many areas related to the subjects covered in this paper which require further study and development. First of all, of course, is the need for a technique which will automatically find the absolute minimum of S(T, X) with respect to both T and X, when several local minima exist, but as was stated above, this possibility doesn't look too promising. Also, a really thorough study into the relative efficiency of all the various optimization techniques, i.e., quasilinearization, sequential optimization, steepest descent, dynamic programming, etc., applied to a wide range of problems is greatly needed. Generally, an investigator will apply a technique to a given problem, and state the results for that technique and for that problem, and thus we really have no feel for which of the many techniques is most appropriate for a given class of problems. (This investigator did make such a comparison for two techniques applied to the same problem, but a comparison of more techniques for a greater variety of problems is needed). Similarly, a study is needed into the relative efficiency of the various methods for solving the variable time problem, for example techniques which vary the final time T every iteration and thus only arrive at an optimal trajectory for the optimal T, compared to the technique developed in this paper, which solves the fixed time problem for each T before proceeding on to a new estimate of T. Although intuitively one might guess that the former is more efficient, it is likely that in varying T continuously each iteration, convergence is slowed considerably in solving the two point boundary problem, while

when solving the problem first for the initial estimate of T and then obtaining an improved estimate of T, the solution of the first two-point boundary problem is probably much more rapid, and after obtaining the solution to the two-point boundary problem for the first T, solution for the succeeding values of T proceed much more rapidly than for the first T, due to the fact that we have a better first estimate for the new solution. Thus it is the really not clear which approach would be more efficient.

While a wealth of techniques already exist for solving the optimal control problem, new, more efficient techniques are constantly being developed, and more efforts in this direction, are, of course, needed. Also, the development of more efficient computer programming techniques would be helpful. Finally, a constant attempt to broaden the scope of application of these techniques to actual problems in the real world, i.e., problems relating to economic and social systems as well as to physical systems, rather than abstract mathematical problems, is also needed, an area which is too often neglected.

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APPENDIX I

DERIVATION OF EQUATIONS FOR NUMERICAL EXAMPLE

In this section of the Appendix, some of the more complex equations which were not developed in the body of the paper which are needed to apply the techniques developed in this paper to the given numerical example will be developed.

For the sequential optimization technique, application of (3.2.1) and (3.2.2) to (3.4.1) and (3.4.7) respectively yields the linearized perturbation equations

$$\begin{aligned}
\delta\dot{x}_1 &= \delta x_2 \\
\delta\dot{x}_2 &= \left(1 - x_1^2\right) \delta x_2 - \left(2x_1 x_2 + 1\right) \delta x_1 + \delta\psi_2/2 \\
\delta\dot{\psi}_1 &= 2\left(x_2 \psi_2 + 1\right) \delta x_1 + 2x_1 \psi_2 \, \delta x_2 + \left(1 + 2x_1 x_2\right) \delta\psi_2 \qquad \text{(I.1)} \\
\delta\dot{\psi}_2 &= 2x_1 \psi_2 \, \delta x_1 + 2 \, \delta x_2 - \delta\psi_1 - \left(1 - x_1^2\right) \delta\psi_2
\end{aligned}$$

where $\delta u = \delta\psi_2/2$ has been substituted since $u = \psi_2/2$. When the control u is at its bounds, the last term on the right of the second of Equations (I.1) is set equal to zero.

We must also derive equations equivalent to (3.2.15) for the linearized Equations (I.1) with δx and $\delta\psi$ of (I.1) equivalent to y and ψ in (3.2.15) respectively and δH given by (3.2.7) equivalent to H in (3.2.15). To simplify this derivation, we observe that in (3.2.15) we have assumed that the optimal control program $u = u(x, \psi)$ has been substituted into H making H a function only of x and ψ. If we do the same in (3.2.7), and observe that $\frac{\partial H}{\partial u} = 0$ when u is not at its bounds and $\delta u = 0$ if u is at its bounds, making the first term in (3.2.7) always zero, we may rewrite (3.2.7) as

$$\delta H = \frac{1}{2}\left(\delta \underline{x}^T \left[\frac{\partial^2 H}{\partial x^2}\right] \delta \underline{x} + 2\,\delta \underline{\psi}^T \left[\frac{\partial^2 H}{\partial \psi \partial x}\right] \delta \underline{x} + \delta \underline{\psi}^T \left[\frac{\partial^2 H}{\partial \psi^2}\right] \delta \underline{\psi}\right) \qquad \text{(I. 2)}$$

If we apply (3. 2. 15) to (I. 2), with the equivalence noted in the preceding paragraph, we obtain

$$\frac{d}{dt}\left[\frac{\partial \delta x}{\partial \lambda}\right] = \left[\frac{\partial^2 H}{\partial x \partial \psi}\right]\left[\frac{\partial \delta x}{\partial \lambda}\right] + \left[\frac{\partial^2 H}{\partial \psi^2}\right]\left[\frac{\partial \delta \psi}{\partial \lambda}\right]$$

$$\frac{d}{dt}\left[\frac{\partial \delta \psi}{\partial \lambda}\right] = -\left[\frac{\partial^2 H}{\partial x^2}\right]\left[\frac{\partial \delta x}{\partial \lambda}\right] - \left[\frac{\partial^2 H}{\partial \psi \partial x}\right]\left[\frac{\partial \delta \psi}{\partial \lambda}\right] \qquad \text{(I. 3)}$$

Thus, we see that the coefficients in the Equations (3. 2. 15) for the linearized equations are the same as they would be for the original nonlinear equations. That is, $\frac{\partial \delta x}{\partial \lambda} = \frac{\partial x}{\partial \lambda}$ and $\frac{\partial \delta \psi}{\partial \lambda} = \frac{\partial \psi}{\partial \lambda}$. This is an expected result, since (3. 2. 15) are simply sensitivity function equations for changes in ψ and x due to changes in λ, and an incremental change in λ would have an incremental change in ψ and x which is equivalent to an incremental change in $\delta\psi$ and δx. Applying (I. 3) to (I. 1), we obtain

$$\frac{d}{dt}\left(\frac{\partial \delta x_1}{\partial \lambda_i}\right) = \frac{\partial \delta x_2}{\partial \lambda_i}$$

$$\frac{d}{dt}\left(\frac{\partial \delta x_2}{\partial \lambda_i}\right) = \left(1 - x_1^2\right)\frac{\partial \delta x_2}{\partial \lambda_i} - \left(2x_1 x_2 + 1\right)\frac{\partial \delta x_1}{\partial \lambda_i} + \frac{1}{2}\frac{\partial \delta \psi_2}{\partial \lambda_i} \qquad \text{(I. 4)}$$

$$\frac{d}{dt}\left(\frac{\partial \delta \psi_1}{\partial \lambda_i}\right) = 2\left(x_2 \psi_2 + 1\right)\frac{\partial \delta x_1}{\partial \lambda_i} + 2x_1 \psi_2 \frac{\partial \delta x_2}{\partial \lambda_i} + \left(1 + 2x_1 x_2\right)\frac{\partial \delta \psi_2}{\partial \lambda_i}$$

$$\frac{d}{dt}\left(\frac{\partial \delta\psi_2}{\partial \lambda_i}\right) = 2x_1\psi_2 \frac{\partial \delta x_1}{\partial \lambda_i} + 2\frac{\partial \delta x_2}{\partial \lambda_i} \qquad i = 1, 2$$

$$- \frac{\partial \delta\psi_1}{\partial \lambda_i} - \left(1 - x_1^2\right)\frac{\partial \delta\psi_2}{\partial \lambda_i}$$

It should be noted that the coefficients of terms in (I. 4) are also the same as those in (I. 1). This is obvious from the similarity of coefficients in Equations (I. 3) and (3. 2. 1) and (3. 2. 2), after we have made the necessary substitution of $u = u(x, \psi)$ in (3. 2. 1) and (3. 2. 2) and made the necessary changes in partial differentation. That is, if we make unit changes in initial conditions on the linearized perturbation equations, they are equivalent to sensitivity function equations.

For the quasilinearization technique, we must determine the partial derivatives $\partial f_i / \partial y_j$ in (3. 3. 3). However, since $y_i = x_i$ and $f_i = \partial H / \partial x_i$, $i = 1, \ldots, n$; and $y_i = \psi_{i-n}$ and $f_i = -\partial H / \partial \psi_{i-n}$, $i = n+1, \ldots 2n_j$ then

$$\frac{\partial f_i}{\partial y_j} = \frac{\partial^2 H}{\partial x_i \partial x_j} \qquad j, i = 1, 2, \ldots n \qquad \text{(I. 5)}$$

$$\frac{\partial f_i}{\partial y_j} = \frac{\partial^2 H}{\partial x_i \partial \psi_{j-n}} \qquad i = 1, 2, \ldots n;\ j = n+1,\ n+2, \ldots 2n$$

$$\frac{\partial f_i}{\partial y_j} = -\frac{\partial^2 H}{\partial \psi_{i-n} x_j} \qquad i = n+1,\ n+2, \ldots 2n;\ j = 1, 2, \ldots n$$

$$\frac{\partial f_i}{\partial y_j} = -\frac{\partial^2 H}{\partial \psi_{i-n} \partial x_{j-n}} \qquad j, i = n+1,\ n+2, \ldots 2n$$

However, again, these coefficients are equivalent to those in (3.2.15) and (3.2.1) and (3.2.2), (with appropriate changes for the substitution $u = u(\psi, x)$ made). Thus, for the given numerical example, (3.3.4) is given by (I.1) with $y_i^{n+1} - y_i^n = \delta x_i$ and $\dot{y}_i^{n+1} - f_i(y^n) = \delta \dot{x}_i$ for $i = 1, 2, \ldots n$, and $y_i^{n+1} - y_i^n = \delta \psi_{i-n}$ and $\dot{y}_i^{n+1} - f_i(y^n) = \delta \dot{\psi}_{i-n}$ for $i = n+1, n+2, \ldots 2n$. However, although the equations for sequential optimization and quasilinearization are the same, they are used in an entirely different manner, the former iterating on the initial adjoint conditions and the latter iterating on the entire trajectory while holding boundary conditions constant.

In employing the variable time techniques of Chapters 4 and 5, we need to compute $\frac{d}{dt}\frac{\partial x}{\partial \lambda}$, $\frac{d}{dt}\frac{\partial \psi}{\partial x}$, $\frac{\partial x}{\partial \lambda}$ and $\frac{\partial \psi}{\partial \lambda}$. However, as mentioned earlier, these are identical to $\frac{d}{dt}\frac{\partial \delta x}{\partial \lambda}$, $\frac{d}{dt}\frac{\partial \delta \psi}{\partial \lambda}$, $\frac{\partial \delta x}{\partial \lambda}$, and $\frac{\partial \delta \psi}{\partial \lambda}$ in (I.3). Also for the variable time algorithm we need $\ddot{x}(T)$. These may be determined by straightforward differentation of (3.4.1)

$$\begin{aligned} \ddot{x}_1 &= \dot{x}_2 \\ \ddot{x}_2 &= \left(1 - x_1^2\right) \dot{x}_2 - \left(2x_1x_2 + 1\right) \dot{x}_1 + \frac{\dot{\psi}_2}{2} \end{aligned} \tag{I.6}$$

where the last term in the second equation is zero when u is at its bounds.

For the bounded state variable problem, when $x_2(t)$ is at its constraint

$$\begin{aligned} u(t) &= -\left(1 - x_1^2\right) x_2 + x_1 \\ \dot{\psi}_1 &= 2x_1 + 2u\left(1 + 2x_1x_2\right) \\ \dot{\psi}_2 &= -x_1 + 2x_2 + 2u\left(x_1^2 - 1\right) \end{aligned} \tag{I.7}$$

$$\frac{d}{dt}\left(\frac{\partial x_2}{\partial \lambda_i}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial \psi_1}{\partial \lambda_i}\right) = \left(2 + 4ux_2\right)\frac{\partial x_1}{\partial \lambda_i} + 4ux_1\frac{\partial x_2}{\partial \lambda_i}$$

$$\frac{d}{dt}\left(\frac{\partial \psi_2}{\partial \lambda_i}\right) = 4ux_1\frac{\partial x_1}{\partial \lambda_i} + 2\frac{\partial x_2}{\partial \lambda_i} - \frac{\partial \psi_1}{\partial \lambda_i} \qquad i = 1, 2$$

Other equations are the same as for the unbounded state variable case.

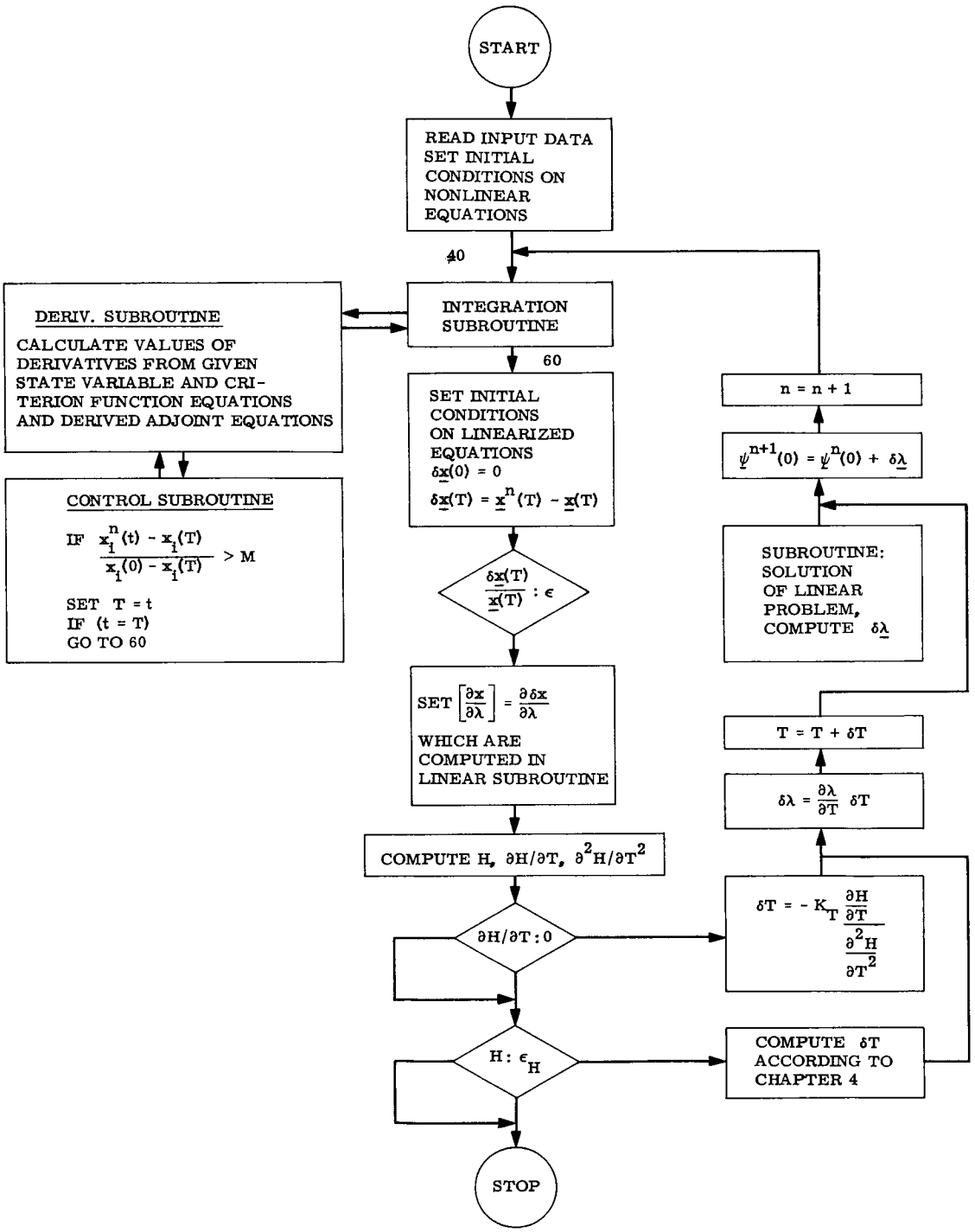
APPENDIX II

On the following two pages are given flow charts for the computer programs for the sequential optimization and quasilinearization techniques, respectively, for the variable time problem. The techniques were programmed in FORTRAN IV computer language for the IBM 7094 computer at UCLA. The integration subroutine was developed in Reference 8 and made use of Gill's method.[7]

The differential equations were all programmed into a separate subroutine called DERIV which was called from the integration subroutine. Also a subroutine CNTRL was called from the integration subroutine to determine at what time T to halt the integration. (For sequential optimization CNTRL thus contains the special provision for halting the integration when the trajectories diverge too far from the desired end points to allow convergence of the method from arbitrary first guesses on initial adjoint conditions). Thus the integration subroutine was general and could be used without change for any given problem.

For sequential optimization the general subroutine developed in Reference 8 for solving the linearized problem was used.

COMPUTER PROGRAM BLOCK DIAGRAM
SEQUENTIAL OPTIMIZATION



COMPUTER PROGRAM BLOCK DIAGRAM
QUASILINEARIZATION