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THE SOLUTION OF THE EULERIAN GYROSCOPE EQUATIONS BY MEANS OF LIE SERIES MAKING USE OF RECURRENCE FORMULAS.

by

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2) The Equations to be Studied.

The equations to be studied have the following form:

$$\ddot{\varphi} = -\dot{\varphi}\dot{\chi}\,\bar{c}c(B_1 + B_2) - \dot{\varphi}\dot{\vartheta}\,\frac{(\bar{b} + B_2\bar{c}^2\bar{b} - c^2B_1\bar{b})}{b} -$$

$$- \frac{\dot{\vartheta}\dot{\chi}}{b}(-1 + B_2\bar{c}^2 - c^2B_1) - \dot{\varphi}^2 c\bar{c}\bar{b}\,(B_1 + B_2) + \frac{N_1}{I_1}\,\frac{c}{b} +$$

$$+ \frac{N_2}{I_2}\,\frac{\bar{c}}{b} \qquad (1a)$$

$$\ddot{\vartheta} = \dot{\varphi}^2 b\bar{b}\,(-B_1\bar{c}^2 + B_2c^2) + \dot{\varphi}\dot{\vartheta}c\bar{c}\,(B_1 + B_2) -$$

$$- b\dot{\varphi}\dot{\chi}\left[1 + B_1\bar{c}^2 - B_2c^2\right] + \dot{\vartheta}\dot{\chi}\,(B_1 + B_2)\,c\bar{c} + \frac{N_1}{I_1}\,\bar{c} -$$

$$- \frac{N_2}{I_2}\,c \qquad (1b)$$

and

$$\ddot{\chi} = \dot{\vartheta}^2B_3c\bar{c} + \dot{\varphi}^2c\bar{c}(-B_3b^2 + \bar{b}^2(B_1 + B_2)) + \frac{\dot{\varphi}\dot{\vartheta}}{b}(1 + b^2B_3(c^2 - \bar{c}^2)) +$$

$$+ \bar{b}^2(B_2\bar{c}^2 - c^2B_1) + \dot{\varphi}\dot{\chi}b\bar{c}c\,(B_1 + B_2) + \dot{\vartheta}\dot{\chi}(B_2\bar{c}^2 - B_1c^2 - 1)\,\cdot$$

$$\cdot\,\frac{\bar{b}}{b} + \frac{N_3}{I_3} - \frac{N_1}{I_1}\,\frac{c\bar{b}}{b} - \frac{N_2}{I_2}\,\frac{\bar{c}}{b}\,\bar{b} \qquad (1c)$$

where φ, χ, ϑ are the Eulerian angles, N_i the external torques, and the B_i abbreviations for:

$$B_1 = \frac{T_{23}}{I_1}, \quad B_2 = \frac{T_{31}}{I_2}, \quad B_3 = \frac{T_{12}}{I_3} \qquad (2)$$

The I_i are the main moments of inertia, and the T_{ij} the components of the inertia tensor; obviously, the B_i, I_i, T_{ij} are constants.

1) Introduction

The differential equations derived in Report 13 describe the motion of a satellite about its center of mass; essentially, they are the Eulerian equations of motion for an asymmetric heavy gyroscope; their general solution in terms of Lie series was given already in Report 13.

As far as the numerical evaluation is concerned three possible ways offer themselves.

1. Repeated application of the D-operator ($Z = \sum_{0}^{\infty} \frac{t^i}{i!} D^i z$) yields very complex results for $i > 3$, such that we have to restrict ourselves to few terms; thus, we have to choose a small step length $\Delta t = t_2 - t_1$ from which, on the other hand, an increase of the calculation time arises. Furthermore, the truncation error might be considerable.
2. Derivation of recurrence formulas for $D^i z$; in this case, we could choose a relatively large step length /1-4/.
3. The third approach starts from a representation of the solution in terms of a main part and a "perturbation integral" to be evaluated by iteration:

$$Z = e^{tD_1} z + \sum_{0}^{\infty} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[D_2 D_{\bar{z}}^{\alpha} \right]_{\bar{a}} d\tau$$

In the present report, the method of recurrence formulas is treated in extenso.

As shown in Report 13, the a, b, c, $\bar{a}$, $\bar{b}$, $\bar{c}$ are transcendental functions of the Eulerian angles forming the components of the matrix A, which transforms from the body-fixed (X, Y, Z) to the rigid rotating (ξ, η, ζ)-system, i.e.:

$$A = \begin{pmatrix} \bar{c}\bar{a} - ca\bar{b}, & -\bar{c}a\bar{b} - c\bar{a}, & ab \\ \bar{c}a + c\bar{a}\bar{b}, & \bar{c}\bar{a}\bar{b} - ca, & -\bar{a}b \\ cb, & \bar{c}b, & \bar{b} \end{pmatrix} \tag{3}$$

with

$$\begin{array}{lll} a = \sin\varphi & b = \sin\vartheta & c = \sin\chi \\ \bar{a} = \cos\varphi & \bar{b} = \cos\vartheta & \bar{c} = \cos\chi \end{array} \tag{4}$$

The torques N_i are given by:

$$\begin{aligned} N_1 &= \omega^2(3a_{32}a_{33} - a_{22}a_{23})\,T_{23} + 2\omega(T_3\dot{\theta}_2a_{23} - T_2a_{22}\dot{\theta}_3) \\ N_2 &= \omega^2(3a_{31}a_{33} - a_{21}a_{23})\,T_{31} + 2\omega(T_1a_{21}\dot{\theta}_3 - T_3a_{23}\dot{\theta}_1) \\ N_3 &= \omega^2(3a_{31}a_{32} - a_{21}a_{22})\,T_{12} + 2\omega(-T_1a_{21}\dot{\theta}_2 + T_2a_{22}\dot{\theta}_1) \end{aligned} \tag{5}$$

We now replace the system (1a, b, c) of three second-order differential equations by the following system of six first-order differential equations:

$$\begin{aligned} \dot{z}_j &\equiv z_{j+3} \\ \dot{z}_j^{\bullet} &\equiv \dot{z}_{j+3} = f_j = \sum_{i=1}^{5} n_i d_{ji} + \sum_{i=6}^{8} \bar{n}_i d_{ji} \end{aligned} \tag{6}$$

$$(j = 1, 2, 3)$$

where use has been made of the following designations and abbreviations:

$$\varphi = Z_1, \quad \vartheta = Z_2, \quad \chi = Z_3$$
$$\dot{\varphi} = Z_4, \quad \dot{\vartheta} = Z_5, \quad \dot{\chi} = Z_6 \tag{7}$$

as well as

$$n_1 = \dot{\varphi}\dot{\chi}, \; n_2 = \dot{\varphi}\dot{\vartheta}, \; n_3 = \dot{\vartheta}\dot{\chi}, \; n_4 = \dot{\varphi}^2,$$
$$n_5 = \dot{\vartheta}^2, \; \bar{n}_6 = N_1, \; \bar{n}_7 = N_2, \; \bar{n}_8 = N_3 \tag{8}$$

The d_{ij} are given by:

$$d_{11} = - c\bar{c} \; (B_1 + B_2)$$
$$d_{12} = - \frac{\bar{b}}{b} \; (1 + B_2\bar{c}^2 - c^2B_1)$$
$$d_{13} = - \frac{1}{b} \; (B_2\bar{c}^2 - c^2B_1 - 1) \tag{9}$$
$$d_{14} = - c\bar{c}\bar{b} \; (B_1 + B_2)$$
$$d_{16} = \frac{1}{I_1} \quad \frac{c}{b}$$
$$d_{17} = \frac{1}{I_2} \quad \frac{c}{b}$$
$$d_{21} = -b \left[1 + B_1\bar{c}^2 - B_2c^2 \right]$$
$$d_{22} = c\bar{c} \; (B_1 + B_2) = -d_{11}$$
$$d_{23} = -d_{11} \tag{10}$$
$$d_{24} = b\bar{b} \; (-B_1\bar{c}^2 + B_2c^2)$$
$$d_{26} = \frac{\bar{c}}{I_1}$$
$$d_{27} = - \frac{c}{I_2}$$
$$d_{31} = - d_{14}$$
$$d_{32} = \frac{1}{b} \; (1 + b^2B_3 \; (c^2 - \bar{c}^2))$$
$$d_{33} = \frac{\bar{b}}{b} \; (B_2\bar{c}^2 - B_1c^2 - 1)$$

$$d_{34} = c\bar{c}(-B_3 b^2 + \bar{b}^2(B_1 + B_2))$$

$$d_{35} = B_3 c\bar{c} \qquad (11)$$

$$d_{36} = \frac{1}{I_3},\ d_{37} = \frac{c\bar{b}}{I_1 b},\ d_{38} = -\frac{\bar{c}\bar{b}}{I_2 b}$$

The formal solution of the system is given by:

$$Z_\sigma = e^{tD} z_\sigma = \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} D^\varrho z_\sigma \ (\sigma = 1, 2, 3)$$

and

$$\dot{Z}_\sigma = \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} D^{\varrho+1} z_\sigma \qquad (12)$$

Now we attempt to derive recurrence formulas connecting higher-order powers of the D-operator with lower ones; for this purpose we write $D^{\varrho+2} z_\sigma$ in the following form:

$$D^{\varrho+2} z_i = D^\varrho (D^2 z_i) = D^\varrho f_i = D^\varrho \Big(\sum_{i=1}^{5} n_i d_{ji} + \sum_{i=6}^{8} \bar{n}_i d_{ji} \Big) =$$

$$= \sum_{j_1=0}^{\varrho} \binom{\varrho}{j_1} \Big\{ D^{j_1} n_i D^{\varrho-j_1} d_{ji} + D^{j_1} \bar{n}_i D^{\varrho-j_1} d_{ji} \Big\} =$$

$$= \sum_{j_1=0}^{\varrho} \binom{\varrho}{j_1} \Big\{ \Big[\sum_{j_2=0}^{j_1} \binom{j_1}{j_2} (D^{j_2} z_{\lambda_1} D^{j_1-j_2} z_{\lambda_2}) \Big] \cdot$$

$$\cdot D^{\varrho-j_1} d_{ji} + D^{j_1} N_\sigma D^{\varrho-j_1} d_{ji} \Big\} \qquad (13)$$

where $j_1 \leq \varrho$, $j_2 \leq \varrho$

$\underline{D^{\varrho-j_1} d_{ji}}$:

$$\underline{D^{\varrho-j_1} d_{11}} = -(B_1 + B_2) D^{\varrho-j_1} c\bar{c} \qquad (14)$$

$$D^{\varrho-j_1} c\bar{c} = D^{j_4} c\bar{c} = \sum_{j_5}^{j_4} \binom{j_4}{j_5} D^{j_5} c D^{j_4-j_5}\bar{c} \qquad (15)$$

$$D^{j_5} c = +D^{j_5-1}(\bar{c} z_6) = \sum_{j_6}^{j_5-1} \binom{j_5-1}{j_6} D^{j_6}\bar{c} D^{j_5-1-j_6} z_6 \qquad (16a)$$

$$D^{j_4-j_5}\bar{c} = D^{j_4-j_5-1}(c z_6) = \sum_{j_6}^{j_4-j_5-1} \binom{j_4-j_5-1}{j_6} D^{j_6} c D^{j_4-j_5-1-j_6} z_6 \qquad (16b)$$

where $j_6 \leq j_5 - 1, \quad j_4 - j_5 - 1 - j_6 \leq \varrho, \quad j_5 - j_1 - j_6 \leq \varrho \qquad (17)$

<u>$D^{\varrho-j_1} d_{12}$:</u>

$$D^{\varrho-j_1} d_{12} = - D^{\varrho-j_1} \frac{\bar{b}}{b} - B_2 D^{\varrho-j_1}\left(\frac{\bar{b}\bar{c}}{b}\right)^2 + B_1 D^{\varrho-j_1}\left(\frac{\bar{b}c^2}{b}\right) \qquad (18)$$

$$D^{\varrho-j_1} \frac{\bar{b}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} \frac{1}{b} D^{\varrho-j_1-j_3}\bar{b} \qquad (19)$$

$$D^{j_3} \frac{1}{b} = - D^{j_3-1}\left(\frac{\bar{b}}{b^2} z_5\right) = - D^{j_3-1}(\bar{b} b^{-1} b^{-1} z_5) =$$

$$= - \sum_{j_4}^{j_3-1} \binom{j_3-1}{j_4} D^{j_4}(\bar{b} b^{-1}) D^{j_3-1-j_4}(b^{-1} z_5) \qquad (20)$$

$$D^{j_4}(\bar{b} b^{-1}) = \sum_{j_5}^{j_4} \binom{j_4}{j_5} D^{j_5}\bar{b} D^{j_4-j_5} b^{-1} \qquad (20a)$$

$$D^{j_3-1-j_4}(b^{-1} z_5) = \sum_{j_6}^{j_3-1-j_4} \binom{j_3-1-j_4}{j_6} D^{j_6} b^{-1} D^{j_3-1-j_4-j_6} z_5 \qquad (20b)$$

$$D^{\varrho-j_1-j_3}\bar{b} = D^{l_1}\bar{b} = - D^{l_1-1}(b z_5) =$$

$$= - \sum_{l_2}^{l_1-1} \binom{l_1-1}{l_2} D^{l_2} b D^{l_1-1-l_2} z_5 \qquad (21a)$$

$$D^{l_1} b = D^{l_1 - 1}(\bar{b} z_5) = \sum_{l_2}^{l_1 - 1} \binom{l_1 - 1}{l_2} D^{l_2} \bar{b} D^{l_1 - 1 - l_2} z_5 \tag{21b}$$

$$l_1 - 1 - l_2 \leq \varrho, \quad j_4 \leq \varrho, \quad j_3 - 1 - j_4 \leq \varrho \tag{22}$$

$$D^{\varrho - j_1}\left(\frac{\bar{b}}{b}\bar{c}^2\right) = \sum_{j_3}^{\varrho - j_1} \binom{\varrho - j_1}{j_3} D^{j_3} \frac{\bar{b}}{b} D^{\varrho - j_1 - j_3} \bar{c}^2 \tag{23}$$

$D^{j_3} \frac{\bar{b}}{b}$: see (19)

$$D^{\varrho - j_1 - j_3} \bar{c}^2 = D^{l_1}(\bar{c}\bar{c}) = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_2} \bar{c} D^{l_1 - l_2} \bar{c} \tag{24}$$

$D^{l_2} \bar{c}$: see (16)

$$D^{\varrho - j_1}\left(\frac{\bar{b} c^2}{b}\right) = \sum_{j_3}^{\varrho - j_1} \binom{\varrho - j_1}{j_3} D^{j_3} \frac{\bar{b}}{b} D^{\varrho - j_1 - j_3} c^2 \tag{25}$$

$D^{j_3} \frac{\bar{b}}{b}$: see (19)

$$D^{\varrho - j_1 - j_3} c^2 = D^{l_1}(cc) = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_1} c D^{l_1 - l_2} c \tag{26}$$

D^{l_1} c: see (16)

$\underline{D^{\varrho - j_1} d_{13}}$: $$- B_2 D^{\varrho - j_1} \frac{\bar{c}^2}{b} + B_1 D^{\varrho - j_1} \frac{c^2}{b} + D^{\varrho - j_1} \frac{1}{b} \tag{27}$$

$$D^{\varrho - j_1} \frac{\bar{c}^2}{b} = \sum_{j_3}^{\varrho - j_1} \binom{\varrho - j_1}{j_3} D^{j_3} \frac{1}{b} D^{\varrho - j_1 - j_3} \bar{c}^2 \tag{28}$$

$D^{j_3} \frac{1}{b}$: see (20)

$D^{\varrho-j_1-j_3}\,\bar{c}^2$: see (24)

$$D^{\varrho-j_1}\,\frac{c^2}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\,\frac{1}{b}\,D^{\varrho-j_1-j_3}\,c^2 \tag{29}$$

$D^{j_3}\,\frac{1}{b}$: see (20)

$D^{\varrho-j_1-j_3}\,c^2$: see (26)

$D^{\varrho-j_1}\,\frac{1}{b}$: see (20)

$D^{\varrho-j_1}\,d_{14}$: $-(B_1 + B_2)\,D^{\varrho-j_1}(\bar{b}c\bar{c}) = D^{\varrho-j_1}\,d_{14}$ (30)

$$D^{\varrho-j_1}(\bar{b}c\bar{c}) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\,\bar{b}D^{\varrho-j_1}(c\bar{c}) \tag{31}$$

$D^{j_3}\,\bar{b}$: see (21)

$D^{\varrho-j_1}\,c\bar{c}$: see (15)

$D^{\varrho-j_1}\,d_{15}$:

$$D^{\varrho-j_1}\,d_{16} = \frac{1}{I_1}\,D^{\varrho-j_1}\,\frac{c}{b} \tag{32}$$

$$D^{\varrho-j_1}\,\frac{c}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\,c\,D^{\varrho-j_1}\,\frac{1}{b} \tag{33}$$

$D^{j_3}\,c$: see (16)

$D^{\varrho-j_1}\,\frac{1}{b}$: see (20)

$D^{\varrho-j_1} d_{17}$:

$$D^{\varrho-j_1} d_{17} = \frac{1}{I_2} D^{\varrho-j_1} \frac{\bar{c}}{b} \tag{34}$$

$$D^{\varrho-j_1} \frac{\bar{c}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} \bar{c} \, D^{\varrho-j_1-j_3} \frac{1}{b} \tag{35}$$

$D^{j_3} \bar{c}$: see (16)

$D^{\varrho-j_1-j_3} \frac{1}{b}$: see (20)

$D^{\varrho-j_1} d_{21}$:

$$D^{\varrho-j_1} d_{21} = - D^{\varrho-j_1} b - B_1 D^{\varrho-j_1} b\bar{c}^2 + B_2 D^{\varrho-j_1} bc^2 \tag{36}$$

$D^{\varrho-j_1} b$: see (21)

$$D^{\varrho-j_1} b\bar{c}^2 = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} b \, D^{\varrho-j_1-j_3} \bar{c}^2 \tag{37}$$

$D^{j_3} b$: see (21)

$D^{\varrho-j_1-j_3} \bar{c}^2$: see (24)

$$D^{\varrho-j_1} (bc^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} b D^{\varrho-j_1-j_3} c^2 \tag{38}$$

$D^{j_3} b$: see (21)

$D^{\varrho-j_1-j_3} c^2$: see (26)

$D^{\varrho-j_1} d_{22}$:

$$D^{\varrho-j_1} d_{22} = - D^{\varrho-j_1} d_{11} \tag{39}$$

$\underline{D^{\varrho-j_1} d_{23}}$: $D^{\varrho-j_1} d_{23} = - D^{\varrho-j_1} d_{11}$ (40)

$\underline{D^{\varrho-j_1} d_{24}}$: $D^{\varrho-j_1} d_{24} = - B_1 D^{\varrho-j_1}(b\bar{b}\bar{c}^2) + B_2 D^{\varrho-j_1}(b\bar{b}c^2)$ (41)

$$D^{\varrho-j_1}(b\bar{b}\bar{c}^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}(b\bar{b}) D^{\varrho-j_1-j_3}\,\bar{c}^2 \qquad (42)$$

$D^{l_1}\,\bar{c}^2$: see (24)

$$D^{j_3}(b\bar{b}) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} b\, D^{j_3-j_4}\,\bar{b} \qquad (43)$$

$D^{j_4}\, b$: see (21)

$D^{j_3-j_4}\,\bar{b}$: see (21)

$\underline{D^{\varrho-j_1} d_{26}}$:

$$D^{\varrho-j_1} d_{26} = \frac{1}{I_1} D^{\varrho-j_1}\,\bar{c} \quad \text{see (16)} \qquad (44)$$

$\underline{D^{\varrho-j_1} d_{27}}$:

$$- \frac{1}{I_2} D^{\varrho-j_1}\, c\text{: see (16)} \qquad (45)$$

$\underline{D^{\varrho-j_1} d_{31}}$:

$$D^{\varrho-j_1} d_{31} = - D^{\varrho-j_1} d_{14} \qquad (46)$$

$\underline{D^{\varrho-j_1} d_{32}}$:

$$D^{\varrho-j_1} \frac{1}{b} + B_3 D^{\varrho-j_1}(bc^2) - B_3 D^{\varrho-j_1}(b\bar{c}^2) \qquad (47)$$

$D^{\varrho-j_1} \frac{1}{b}$: see (20)

$$D^{\varrho-j_1}(bc^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} b D^{\varrho-j_1-j_3}\, c^2 \qquad (48)$$

$D^{j_3}\, b$: see (21)

$D^{\varrho-j_1-j_3}\, c^2$: see (26)

$D^{\varrho-j_1}\, d_{33}$:

$$B_2 D^{\varrho-j_1} \frac{\bar{b}\bar{c}^2}{b} - B_1 D^{\varrho-j_1} \frac{\bar{b}c^2}{b} - D^{\varrho-j_1} \frac{\bar{b}}{b} = D^{\varrho-j_1}\, d_{33} \qquad (49)$$

$D^{\varrho-j_1} \frac{\bar{b}\bar{c}^2}{b}$: see (23)

$D^{\varrho-j_1} \frac{\bar{b}c^2}{b}$: see (25)

$D^{\varrho-j_1} \frac{\bar{b}}{b}$: see (19)

$D^{\varrho-j_1}\, d_{34}$:

$$D^{\varrho-j_1}\, d_{34} = -B_3 D^{\varrho-j_1}(c\bar{c}b^2) + B_1 D^{\varrho-j_1}(c\bar{c}\bar{b}^2) + $$
$$+ B_2 D^{\varrho-j_1}(c\bar{c}\bar{b}^2) \qquad (50)$$

$$D^{\varrho-j_1}(c\bar{c}b^2) = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}(c\bar{c}) D^{\varrho-j_1-j_3}\, b^2 \qquad (51)$$

$D^{j_3}(c\bar{c})$: see (16)

$$D^{l_1} b^2 = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_2} b D^{l_1-l_2}\, b \qquad (52)$$

$D^{l_2}\, b$: see (21)

$$D^{\varrho-j_1}(c\bar{c}b^2) := \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3}\, c\bar{c}\, D^{\varrho-j_1-j_3}\, b^2 \qquad (53)$$

$$D^{\varrho-j_1-j_3} b^2 = \sum_{j_4}^{\varrho-j_1-j_3} \binom{\varrho-j_1-j_3}{j_4} D^{j_4}\, b\, D^{\varrho-j_1-j_3}\, b \qquad (54)$$

$D^{j_4}\, b$: see (21)

$D^{j_3}\, c\bar{c}$: see (15)

$D^{\varrho-j_1} d_{35}$:

$$D^{\varrho-j_1} d_{35} = B_3 D^{\varrho-j_1}(c\bar{c}) \text{ see (15)} \tag{55}$$

$D^{\varrho-j_1} d_{36}$:

$$D^{\varrho-j_1} d_{36} = 0 \tag{56}$$

$D^{\varrho-j_1} d_{37}$:

$$D^{\varrho-j_1} d_{37} = -\frac{1}{I_1} D^{\varrho-j_1} \frac{c\bar{b}}{b} \tag{57}$$

$$D^{\varrho-j_1} \frac{c\bar{b}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{j_3} c \, D^{\varrho-j_1-j_3} \frac{\bar{b}}{b} \tag{58}$$

$D^{j_3} c$: see (16)

$D^{\varrho-j_1-j_3} \frac{\bar{b}}{b}$: see (19)

$$D^{\varrho-j_1} d_{38}: -\frac{1}{I_2} D^{\varrho-j_1} \frac{\bar{c}\bar{b}}{b} \tag{59}$$

$$D^{\varrho-j_1} \frac{\bar{c}\bar{b}}{b} = \sum_{j_3}^{\varrho-j_1} \binom{\varrho-j_1}{j_3} D^{\varrho-j_1-j_3} \bar{c} \, D^{j_3} \frac{\bar{b}}{b} \tag{60}$$

$D^{j_3} \frac{\bar{b}}{b}$: see (19)

$D^{l_1} \bar{c}$: see (16)

The recurrence formulas for $D^{j_1} N_i$ $(i=1,2,3)$ can be derived in analogy to the expressions presented above.

$D^{j_1} N_i$:

$D^{j_1} N_1$:

$$D^{j_1} N_1 = \omega^2 T_{23} D^{j_1}(3a_{32}a_{33} - a_{22}a_{23}) - 2\omega T_2 D^{j_1}(a_{22}\dot{\theta}_3) +$$
$$+ 2\omega T_3 D^{j_1}(a_{23}\dot{\theta}_2) \tag{61}$$

$$D^{j_1}(a_{32}a_{33}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3}a_{32}\, D^{j_1-j_3}\, a_{33} \qquad (62)$$

$$D^{j_3}a_{32} = D^{j_3}(\bar{c}b) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4}\,\bar{c}\, D^{j_3-j_4}\, b \qquad (63)$$

$D^{j_4}\,\bar{c}$: see (16), $D^{j_3-j_4}\, b$: see (21)

$$D^{j_1-j_3}\, a_{33} = D^{j_1-j_3}\,\bar{b}\text{: see (21)} \qquad (64)$$

$$D^{j_1}(a_{22}a_{23}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3}a_{22}\, D^{j_1-j_3}\, a_{23} \qquad (65)$$

$$D^{j_3}\, a_{22} = D^{j_3}(\bar{c}a + c\bar{a}\bar{b}) = D^{j_3}(\bar{c}a) + D^{j_3}(c\bar{a}\bar{b}) \qquad (66)$$

$$D^{j_3}(\bar{c}a) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4}\,\bar{c}\, D^{j_3-j_4}\, a \qquad (67)$$

$D^{j_4}\,\bar{c}$: see (16), $D^{j_3-j_4}\, a = D^{l_1}\, a =$

$$= D^{l_1-1}(\bar{a}z_4) = \sum_{l_2}^{l_1-1} \binom{l_1-1}{l_2} D^{l_2}\,\bar{a}D^{l_1-1-l_2}\, z_4 \qquad (68)$$

$$D^{l_1}\,\bar{a} = -\,D^{l_1-1}(az_4) = -\sum_{l_2}^{l_1-1} \binom{l_1-1}{l_2} D^{l_2} aD^{l_1-1-l_2} z_4 \qquad (69)$$

$$D^{j_3}(\bar{a}\bar{b}c) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4}(\bar{a}\bar{b}) D^{j_3-j_4}\, c \qquad (70)$$

$D^{j_3-j_4}\, c$: see (16); $D^{j_4}(\bar{a}\bar{b}) = \sum_{j_5}^{j_4} \binom{j_4}{j_5} D^{j_5}\,\bar{a}D^{j_4-j_5}\bar{b}$ (71)

$D^{j_5}\,\bar{a}$: see (69,70), $D^{j_4-j_5}\,\bar{b}$: see (21) (72)

$$D^{j_2-j_3} a_{23} = - D^{j_2-j_3} \bar{a}b = - \sum_{j_4}^{j_2-j_3} \binom{j_2-j_3}{j_4} D^{j_4}\bar{a} D^{j_2-j_3-j_4} b \qquad (73)$$

$D^{j_4}\bar{a}$: see (69,70); $D^{j_2-j_3-j_4} b$: see (21) (74)

$$D^{j_1}(a_{22}\theta_3) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22} D^{j_1-j_3} \theta_3 \qquad (75)$$

$D^{j_3} a_{22}$: see (65) (76)

$$D^{j_1-j_3} \theta_3 = D^{j_1-j_3}(z_4\bar{b} + z_6) =$$

$$= \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4} z_4 D^{j_1-j_3-j_4} \bar{b} + D^{j_1-j_3} z_6 \qquad (77)$$

$D^{j_1-j_3-j_4}\bar{b}$: see (21); $j_4 \leq \varrho$, $j_1-j_3 \leq \varrho$ (78)

$$D^{j_1}(a_{23}\theta) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{23} D^{j_1-j_3} \theta_2 \qquad (79)$$

$D^{j_3} a_{23}$: see (13) (80)

$$D^{j_1-j_3} \theta_2 = D^{j_1-j_3}(z_4\bar{c}b - z_5 c) =$$

$$= \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} \left\{ D^{j_1-j_3}(z_4\bar{c}b) + D^{j_1-j_3}(z_5 c) \right\} \qquad (81)$$

$$D^{j_1-j_3}(z_4\bar{c}b) = \sum_{j_5}^{j_1-j_3} \binom{j_1-j_3}{j_5} D^{j_5}(\bar{c}b) D^{j_1-j_3-j_5} z_4 \qquad (82)$$

$$j_1-j_3-j_5 \leq \varrho$$

$$D^{j_5}(\bar{c}b) = \sum_{j_6}^{j_5} \binom{j_5}{j_6} D^{j_6} \bar{c} D^{j_5-j_6} b \qquad (83)$$

$$D^{j_6}\bar{c}\text{: see (16); } D^{j_5-j_6}\, b\text{: see (21)} \tag{84}$$

$$D^{j_1-j_3}(z_5 c) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4} z_5\, D^{j_1-j_3-j_4}\, c \tag{85}$$

$$D^{j_4}\, c\text{: see (16)} \qquad j_4 \leq \varrho \tag{86}$$

<u>$D^{j_1} N_2$:</u>

$$D^{j_1} N_2 = D^{j_1}\Big\{(3a_{31}a_{33} - a_{21}a_{23})\,\omega^2 T_{31} +$$

$$+ 2\omega(T_1 a_{21}\dot{\theta}_1 - T_3 a_{23}\dot{\theta}_1)\Big\} \tag{87}$$

$$D^{j_1}(a_{31}a_{33}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{31}\, D^{j_1-j_3}\, a_{33} \tag{88}$$

$$D^{j_3} a_{31} = D^{j_3}(cb) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4}\, c\, D^{j_3-j_4}\, b \tag{89}$$

$$D^{j_4}\, c\text{: see (16); } D^{j_3-j_4}\, b\text{: see (21)} \tag{90}$$

$$D^{j_1-j_3}\, a_{33}\text{: see (64)} \tag{91}$$

$$D^{j_1}(a_{22}a_{23}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22}\, D^{j_1-j_3}\, a_{23} \tag{92}$$

$$D^{j_1-j_3}\, a_{23}\text{: see (73)} \tag{93}$$

$$D^{j_3}\, a_{22}\text{: see (66)} \tag{94}$$

$$D^{j_3}(a_{21}\dot{\theta}_3) = \sum_{j_4}^{j_3} \binom{j_3}{j_4} D^{j_4} a_{21} D^{j_3-j_4}\, \dot{\theta}_3 \tag{95}$$

$$D^{j_3-j_4}\, \dot{\theta}_3\text{: see (77)} \tag{96}$$

$$D^{j_4} a_{21} = D^{j_4}(\bar{c}a + c\bar{a}\bar{b}) =$$

$$= \sum_{j_5}^{j_4} \binom{j_4}{j_5} \left\{ D^{j_4} (\bar{c}a) + D^{j_4} (c\bar{a}\bar{b}) \right\} \tag{97}$$

$$D^{j_4}(\bar{c}a)\text{: see (67)}, \quad D^{j_4}(c\bar{a}\bar{b})\text{: see (68)} \tag{98}$$

$$D^{j_1}(a_{23}\dot{\theta}_1) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{23} D^{j_1-j_3} \dot{\theta}_1 \tag{99}$$

$$D^{j_3} a_{23}\text{: see (73)} \tag{100}$$

$$D^{j_1-j_3} \dot{\theta}_1 = D^{j_1-j_3}(z_4 cb + z_5 \bar{c}) =$$

$$= D^{j_1-j_3}(z_4 cb) + D^{j_1-j_3}(z_5 \bar{c}) \tag{101}$$

$$D^{j_1-j_3}(z_4 cb) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_1-j_3} z_4 D^{j_1-j_3-j_4}(cb) \tag{102}$$

$$j_1 - j_3 \leq \varrho$$

$$D^{j_1-j_3-j_4}(cb) = D^{l_1}(cb) = \sum_{l_2}^{l_1} \binom{l_1}{l_2} D^{l_2} c D^{l_1-l_2} b \tag{103}$$

$$D^{l_2} c\text{: see (16)}; \quad D^{l_1-l_2} b\text{: see (21)} \tag{104}$$

$$D^{j_1-j_3}(z_5 \bar{c}) = \sum_{j_4}^{j_1-j_3} \binom{j_1-j_3}{j_4} D^{j_4} z_5 D^{j_1-j_3-j_4} \bar{c} \tag{105}$$

$$j_4 \leq \varrho$$

$$D^{j_1-j_3-j_4} \bar{c}\text{: see (16)} \tag{106}$$

$D^{j_1} N_3$:

$$D^{j_1} N_3 = D^{j_1} \Big\{ (3a_{31}a_{32} - a_{21}a_{22}) \omega^2 T_{12} +$$

$$+ 2\omega(-T_1 a_{21} \dot{\theta}_2 + T_2 a_{22} \dot{\theta}_1) \Big\} \tag{107}$$

$$D^{j_1}(a_{31}a_{32}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{31} D^{j_1-j_3} a_{32} \tag{108}$$

$$D^{j_3} a_{31}\text{: see (89); } D^{j_1-j_3} a_{32}\text{: see (63)} \tag{109}$$

$$D^{j_1}(a_{21}a_{22}) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{21} D^{j_1-j_2} a_{22} \tag{110}$$

$$D^{j_3} a_{21}\text{: see (97); } D^{j_1-j_2} a_{22}\text{: see (65)} \tag{111}$$

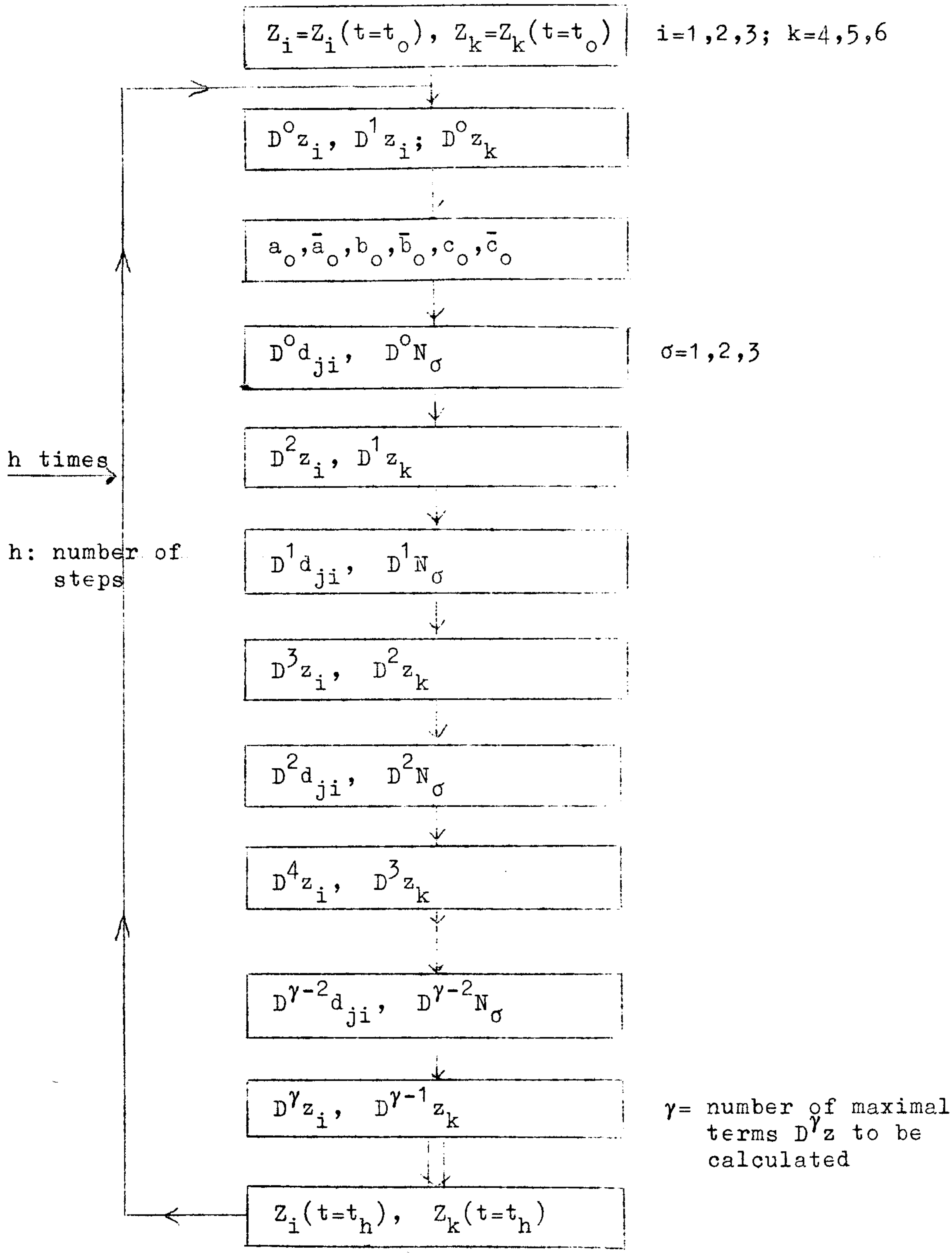
$$D^{j_1}(a_{21}\theta_2) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{21} D^{j_1-j_3} \theta_2 \tag{112}$$

$$D^{j_3} a_{21}\text{: see (97); } D^{j_1-j_3} \theta_2\text{: see (81)} \tag{113}$$

$$D^{j_1} (a_{22}\ \theta_1) = \sum_{j_3}^{j_1} \binom{j_1}{j_3} D^{j_3} a_{22} D^{j_1-j_3} \theta_1 \tag{114}$$

$$D^{j_3} a_{22}\text{: see (66); } D^{j_1-j_3} \theta_1\text{: see (101)} \tag{115}$$

F L O W   D I A G R A M



Conclusion.

The recurrence formulas have shown their effectiveness in the solution of several other types of differential equations /1 - 4/, e.g., the general second-order differential equations (homogeneous and inhomogeneous); the formulas derived in this report are, however, very complex and no statements can be made as to their usefulness in numerical calculations since investigations concerning these problems are impeded by their short duration of our contract. Numerical calculations are carried out for the cases D^0z, D^1z, D^2z, D^3z - even the application of D^3 is a very complicated procedure; obviously, these terms do not suffice for more exact computations (about 20 terms D^iz would be appropriate /4/); it is for this reason that recurrence formulas are derived for the D^iz.

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