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PHYSICAL LIBRATION OF A DEFORMABLE MOON

THEORY I

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I. ABSTRACT

The first part of the present semiannual report covers a discussion of the recent data on the lunar gravity field as it pertains to the mechanical ellipticities and mass distribution of the Moon. It is tentatively concluded that the Moon departs more from hydrostatic equilibrium than was believed in the pre-Orbiter era, and that its density is almost everywhere constant with a slight increase with depth.

The second part of the report contains the formulation from first principles of the problem of the physical librations of the Moon, which is assumed to be a compressible fluid of arbitrarily high viscosity. The case of solid bodies is covered by this formulation as a limiting case. A self-gravitating viscous mass is allowed to oscillate in an arbitrary time-dependent external field, which in the case of the Moon, will be represented by the gravity field emanating from the Earth. The solar attraction will be treated as is customary in the pertinent literature as a perturbation which can be inserted as one additional term in the right-hand side of the Eulerian equations. This term is available in closed form.

II. INTRODUCTION

The kinetic behavior of the Moon around its center of mass is almost solely governed by its principle moments of inertia, or more precisely, by its mechanical ellipticities which appear in the Eulerian equations of motion. Our main task in this project is to study the physical librations, assuming that the mechanical ellipticities vary in a periodic fashion, and attempt to apply the theoretical results to data that will eventually be made available by analyses of tracking data of Lunar Orbiters. The fact that tidal deformation of the lunar globe can be anticipated on the grounds that such deformations are observable on a smaller scale on the terrestrial globe, offers firsthand justification of the present effort.

It is therefore essential in this connection that a number of estimates of the mechanical ellipticities of the Moon be made at regular (if possible) intervals over a long period of time so that periodicities of various frequencies will be detected. Recent reductions of tracking data of lunar satellites indicate that pre-Orbiter estimates of these quantities are very inaccurate and, therefore, one has to depend on future estimates in order to obtain an answer regarding their time variation.

Quantities that bear on the present problem are those that give a measure of the extent to which the lunar globe departs from hydrostatic equilibrium. Large departures from a state of equilibrium indicate the existence of uncompensated differential stresses which, in turn, imply a high viscosity coefficient, or, at the extreme, a solid condition. The mass distribution is also essential here. A study of the motion of the nodal line of the Moon made on the basis of pre-Orbiter data has resulted in the paradoxical conclusion that most of the mass lies on the surface (Eckert, 1965). The hypothesis advanced to explain this result is that the higher harmonics, which have been neglected in the above analysis, are not in reality small. While this seems a very reasonable explanation, it can be added that the time variation of the various harmonics, including the one of second order, can be responsible for bringing about such a paradox. In the first part of this report we present a discussion of the results of Luna 10 from which we conclude that the Moon is almost homogeneous and that its density increases very slightly as we move toward the center. Thus, unless there are errors in the analysis by Eckert, the Moon must either have large higher harmonics, or be deformable, or both.

The differential equations which govern the motions of self-gravitating bodies about their center of gravity--whether free or forced--have been known since the early days of the history of rational mechanics; and the investigators of their solutions bearing on the precession and nutation of the Earth, or the physical librations of the Moon, included (to name only the greatest) Newton, Euler, Lagrange and Laplace. All these investigators assumed in common that the body moving about its center of gravity in an external field of force can be regarded as *rigid*; and its external form (or moments of inertia) be fixed and independent of the time. However, it was not till in the second half of the 19th century that it has been gradually realized that a self-gravitating body of the mass of the

Earth or the Moon cannot be regarded as rigid or incompressible; moreover, observations have revealed (at least in the case of the Earth) that its form responds to a fluctuating external field of force through bodily tides.

A mathematical treatment of the motion of deformable bodies about their center of mass in an external field of force was, however, slow to come (cf. Liouville, 1858; Gyldén, 1871; Oppenheimer, 1885; Darwin, 1879; Poincaré, 1910) and is still far from being adequately solved for the precession, or nutation of the Earth, while its bearing on the physical librations of the lunar globe has not yet even been considered. The aim of the present project will be to provide a more comprehensive treatment of this subject than has been done by all previous investigators, and to do so on the basis of the fundamental equations of hydrodynamics, in which the three velocity components u, v, w will be systematically expressed in terms of the independent rotations, about the three respective axes x, y, z, with angular velocities ω_x, ω_y, and ω_z. Departures from a hydrodynamical treatment; necessitated if the response of a deformable body to an external strain is that of an elastic solid rather than that of a viscous fluid, (i.e., if we deal with a "Maxwell" rather than "Kelvin-Voigt" body), will be taken up in the concluding report.

III. RECENT DATA: MECHANICAL ELLIPTICITIES

The recently published results (Akim, 1966) of the analysis of the motion of Luna 10, although in close agreement with earlier values (Goudas, 1964) concerning second zonal C_{20} and sectorial C_{22} harmonic coefficients of the lunar force function, have disclosed new facts regarding the homogeneity and departure of the Moon from a condition of hydrostatic equilibrium. In particular, as the following arguments will show, the Moon is more homogeneous than believed before on the basis of a tentative value for the third zonal harmonic (Michael *et al*, 1966, and Goudas *et al*, 1966), and it departs more from hydrostatic equilibrium than recent studies (Koziel, 1967) of the physical libration of the Moon have suggested.

Indeed, without any assumption for the mass distribution, the values of C_{20} (or c_{20}) and C_{22} (or c_{22}) given in Ref. [1], when substituted in the expressions

$$C_{20} = \frac{A + B - 2C}{2Mr_0^2}, \quad C_{22} = \frac{B-A}{4Mr_0^2} \tag{1}$$

where A, B, C are the principal moments of inertia, M the mass and r_0 the mean radius, allow us to determine the value of the ratio f of the mechanical ellipticities α [i.e., (C-B)/A] to β [i.e., (C-A)/B]. To first order terms the quantity f is given by the expression

$$f = \frac{C_{20} + 2C_{22}}{C_{20} - 2C_{22}} \tag{2}$$

and, therefore,

$$f = 0.76 \pm 0.04$$

where the uncertainty represents the maximum error. This value is three times larger than the one which corresponds to a state of hydrostatic equilibrium. The most recent studies (Koziel, 1967) of the physical libration of the Moon have resulted in the value

$$f = 0.633.$$

This discrepancy would never exist, nor would it have any consequence at all for the study of the Eulerian motion of the Moon around its center of mass, if the linearized third equation of motion did not have a singularity at the value $f = 0.66$, which makes the corrective process determining f dependent upon the initial value adopted for it. Ironically, the value determined by Hayn (1907) sixty years ago, and which is in use today by the Astronomical Ephemeris and Nautical Almanac for the computation of the ephemeris of physical libration, is $f = 0.75$, i.e., almost the exact value given by Luna 10.

The assumption of homogeneity and the values of the constants C_{20} and C_{22} permit the determination of not only the ratio f but also the moments of inertia, and therefore the mechanical ellipticities of the Moon. The pertinent formulae for the moments of inertia are

$$\begin{aligned} A &= \frac{2Mr_0^2}{5}(1 + j_{20}/2 - 3j_{22}) \\ B &= \frac{2Mr_0^2}{5}(1 + j_{20}/2 + 3j_{22}) \\ C &= \frac{2Mr_0^2}{5}(1 - j_{20}) \end{aligned} \tag{3}$$

where $j_{20} = \frac{5}{3} C_{20}$, $j_{22} = \frac{5}{3} C_{22}$ and M is the total mass of the Moon. The value of f corresponding to the above expressions (3) is the same as the one given by equation (2). However, the mechanical ellipticities α, β and γ $(\gamma = \beta - \alpha)$ are

$$\begin{aligned} \alpha &= 0.000445 \pm 0.000060 \\ \beta &= 0.000586 \pm 0.000011 \\ \gamma &= 0.000141 \pm 0.000071 \end{aligned} \tag{4}$$

for the homogeneous Moon and

$$\begin{aligned} \alpha &= 0.000478 \pm 0.000030 \\ \beta &= 0.000629 \pm 0.000006 \\ \gamma &= 0.000151 \pm 0.000036 \end{aligned} \tag{5}$$

for the true case. The uncertainties are again maximum errors and, although to the best of our knowledge the true values are definitely within the intervals indicated, we cannot yet decide the extent to which the mass distribution departs from perfect homogeneity. To say the least, the results from Luna 10 are inconclusive in regard to the hypothesis (Eckert, 1965) that the surface layers of the Moon may be more dense than the ones closer to the center. A simple calculation shows that the value of $3C/2Mr^2$ is 0.56 ± 0.18, i.e., a great deal smaller than the value 0.965 suggested by Eckert to explain the motion of the node. Keeping in mind that the value corresponding to homogeneity is 0.6 and that values less than this mean increasing density with depth, then forgetting for the time being the given uncertainty--which again is a maximum error--we find that the lunar central regions are a little denser than the outer layers. Although the uncertainty weakens the conclusion somewhat, the value given by Eckert is definitely unacceptable, since it lies outside the range indicated by the maximum error. As a result, a large portion--if not all--of the nodal motion produced by the asphericity of the Moon, should find an explanation other than a peculiar distribution of mass postulated by Eckert. The best explanation one can advance is an assertion that a large part of the motion of the node is produced by the total effect of higher-order moments which, the best estimates available indicate, are of the same order as the moments of second order.

As far as the quantity f goes, Luna 10 has placed it on the other side of the singularity, thus giving a strong confirmation of the view that the Moon is far from hydrostatic equilibrium--a fact which indicates that no liquid parts (such as a liquid core) ever existed on the Moon, or, if they did, that their relative size was or is too small.

IV. EQUATIONS OF THE PROBLEM

As is well known, the Eulerian fundamental equations of hydrodynamics governing the motion of compressible viscous fluids can be expressed in rectangular coordinates in the symmetrical form

$$\rho \frac{Du}{Dt} = \rho \frac{\partial \Omega}{\partial x} - \frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} , \tag{6}$$

$$\rho \frac{Dv}{Dt} = \rho \frac{\partial \Omega}{\partial y} - \frac{\partial P}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} , \tag{7}$$

$$\rho \frac{Dw}{Dt} = \rho \frac{\partial \Omega}{\partial z} - \frac{\partial P}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} , \tag{8}$$

where u, v, w denote the velocity components of fluid motion, at the time t, in the direction of increasing coordinates x, y, z, respectively;

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{9}$$

representing the Lagrangian time-derivative (following the motion); ρ stands for the local density of the fluid; P, for its pressure; Ω, for the total potential (internal as well as external) of all forces acting upon it; and

$$\sigma_{xx} = \frac{2}{3} \mu\{3 \frac{\partial u}{\partial x} - \Delta\}, \tag{10}$$

$$\sigma_{yy} = \frac{2}{3} \mu\{3 \frac{\partial v}{\partial y} - \Delta\}, \tag{11}$$

$$\sigma_{zz} = \frac{2}{3} \mu\{3 \frac{\partial w}{\partial z} - \Delta\}, \tag{12}$$

$$\sigma_{xy} = \mu\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\} = \sigma_{yx}, \tag{13}$$

$$\sigma_{yz} = \mu\left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\} = \sigma_{zy}, \tag{14}$$

$$\sigma_{zx} = \mu\left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\} = \sigma_{xz}, \tag{15}$$

are the respective components of the viscous stress tensor, where μ denotes the coefficient of viscosity and

$$\Delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{16}$$

the divergence of the velocity vector of the fluid.

As is well known, equations (6) - (8) safeguard the conservation of momentum of the underlying dynamical problem; and as such represent only one-half of the system necessary for a complete specification of the six dependent variables

$$u,\ v,\ w;$$

$$\rho,\ P,\ \Omega;$$

of our problem. Of the remaining three equations, two can be adjoined with relative ease: namely, the equation of continuity

$$\frac{D\rho}{Dt} + \rho\Delta = 0 \tag{17}$$

safeguarding the conservation of mass, and the Poisson equation

$$\nabla^2\Omega = -4\pi G\rho \tag{18}$$

which must be satisfied by the gravitational potential (G denoting the constant of gravitation).

The sole remaining equation required to render the solution of our system determinate (for an appropriate set of boundary conditions) must be derived from the principle of the conservation of energy, in the form of an "equation of state" relating P and ρ; but its explicit formulation will be postponed until a later stage of our analysis.

V. THE COMPONENTS OF VELOCITIES AND ACCELERATIONS

In order to apply the system of equations set up in the preceding section for the study of the motion of a self-gravitating body about its center of gravity, consider the transformation of rectangular coordinates between an *inertial* (fixed) system of *space* axes $x, y, z,$ and a *rotating* system of *body* (primed) axes x', y', z', possessing the same origin, but with the primed axes rotated with respect to the space axes by the Eulerian angles $\phi, \theta, \psi,$ in accordance with a scheme illustrated on the accompanying Figure I.

As is well known, the transformation of coordinates from the space to the body axes is governed by the following matrix equation

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \tag{19}$$

where the coefficients a_{ik}, expressed in terms of the Eulerian angles assume the explicit forms

$$\left.\begin{aligned} a_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi, \\ a_{12} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi, \\ a_{13} &= \sin\theta\sin\phi \quad ; \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} a_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, \\ a_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi, \\ a_{23} &= -\sin\theta\cos\phi \quad ; \end{aligned}\right\} \tag{21}$$

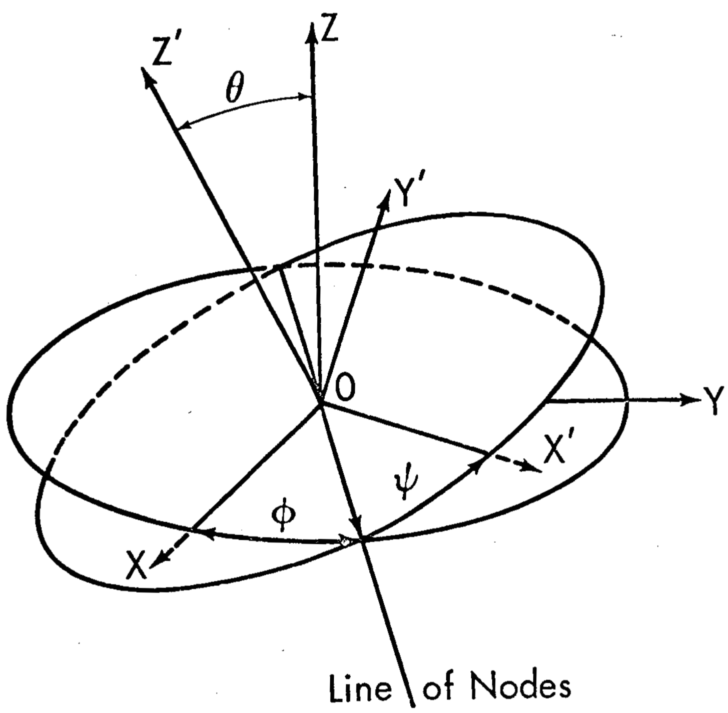

Fig. I

$$\left.\begin{aligned} a_{31} &= \sin\psi \sin\theta, \\ a_{32} &= \cos\psi \sin\theta, \\ a_{33} &= \cos\theta. \end{aligned}\right\} \tag{22}$$

In order to obtain the corresponding *space* velocity-components $u, v, w,$ let us differentiate equations (19) with respect to the time. With dots denoting hereafter ordinary (total) derivatives with respect to t, we find that

$$\dot{x} = u = \dot{a}_{11}x' + \dot{a}_{12}y' + \dot{a}_{13}z' + a_{11}\dot{x}' + a_{12}\dot{y}' + a_{13}\dot{z}', \tag{23}$$

$$\dot{y} = v = \dot{a}_{21}x' + \dot{a}_{22}y' + \dot{a}_{23}z' + a_{21}\dot{x}' + a_{22}\dot{y}' + a_{23}\dot{z}', \tag{24}$$

$$\dot{z} = w = \dot{a}_{31}x' + \dot{a}_{32}y' + \dot{a}_{33}z' + a_{31}\dot{x}' + a_{32}\dot{y}' + a_{33}\dot{z}'; \tag{25}$$

whereas the *body* velocity-components u', v', w' follow from

$$\begin{aligned} \dot{x}' = u' &= \dot{a}_{11}x + \dot{a}_{21}y + \dot{a}_{31}z \\ &\quad + a_{11}\dot{x} + a_{21}\dot{y} + a_{31}\dot{z}, \end{aligned} \tag{26}$$

$$\begin{aligned} \dot{y}' = v' &= \dot{a}_{12}x + \dot{a}_{22}y + \dot{a}_{32}z \\ &\quad + a_{12}\dot{x} + a_{22}\dot{y} + a_{32}\dot{z}, \end{aligned} \tag{27}$$

$$\begin{aligned} \dot{z}' = w' &= \dot{a}_{13}x + \dot{a}_{23}y + \dot{a}_{33}z \\ &\quad + a_{13}\dot{x} + a_{23}\dot{y} + a_{33}\dot{z}; \end{aligned} \tag{28}$$

where

$$\left.\begin{aligned} \dot{a}_{11} = a_{12}\dot{\psi} - a_{21}\dot{\phi} + a_{31}\dot{\theta}\sin\phi &= a_{31}\omega_y - a_{21}\omega_z \\ &= a_{12}\omega_{z'} - a_{13}\omega_{y'}, \end{aligned}\right\} \tag{29}$$

$$\dot{a}_{12} = -a_{11}\dot{\psi} - a_{22}\dot{\phi} + a_{32}\dot{\theta}\sin\phi = a_{32}\omega_y - a_{22}\omega_z = a_{13}\omega_{x'} - a_{11}\omega_{z'}, \tag{30}$$

$$\dot{a}_{13} = - a_{23}\dot{\phi} + a_{33}\dot{\theta}\sin\phi = a_{33}\omega_y - a_{23}\omega_z = a_{11}\omega_{y'} - a_{12}\omega_{x'}; \tag{31}$$

$$\dot{a}_{21} = a_{22}\dot{\psi} + a_{11}\dot{\phi} - a_{31}\dot{\theta}\cos\phi = a_{11}\omega_z - a_{31}\omega_x = a_{22}\omega_{z'} - a_{23}\omega_{y'}, \tag{32}$$

$$\dot{a}_{22} = -a_{21}\dot{\psi} + a_{12}\dot{\phi} - a_{32}\dot{\theta}\cos\phi = a_{12}\omega_z - a_{32}\omega_x = a_{23}\omega_{x'} - a_{21}\omega_{z'}, \tag{33}$$

$$\dot{a}_{23} = + a_{13}\dot{\phi} - a_{33}\dot{\theta}\cos\phi = a_{13}\omega_z - a_{33}\omega_x = a_{21}\omega_{y'} - a_{22}\omega_{x'}; \tag{34}$$

$$\dot{a}_{31} = a_{32}\dot{\psi} + \dot{\theta}\sin\psi\cos\theta = a_{21}\omega_x - a_{11}\omega_y = a_{32}\omega_{z'} - a_{33}\omega_{y'}, \tag{35}$$

$$\dot{a}_{32} = -a_{31}\dot{\psi} + \dot{\theta}\cos\psi\cos\theta = a_{22}\omega_x - a_{12}\omega_y = a_{33}\omega_{x'} - a_{31}\omega_{z'}, \tag{36}$$

$$\dot{a}_{33} = - \dot{\theta}\sin\theta = a_{23}\omega_x - a_{13}\omega_y = a_{31}\omega_{y'} - a_{32}\omega_{x'} \tag{37}$$

where, taking advantage of the fact that

$$\left.\begin{aligned} a_{11}\dot{a}_{11} + a_{12}\dot{a}_{12} + a_{13}\dot{a}_{13} &= 0 \\ a_{21}\dot{a}_{21} + a_{22}\dot{a}_{22} + a_{23}\dot{a}_{23} &= 0 \\ a_{31}\dot{a}_{31} + a_{32}\dot{a}_{32} + a_{33}\dot{a}_{33} &= 0 \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} a_{11}\dot{a}_{11} + a_{21}\dot{a}_{21} + a_{31}\dot{a}_{31} &= 0 \\ a_{12}\dot{a}_{12} + a_{22}\dot{a}_{22} + a_{32}\dot{a}_{32} &= 0 \\ a_{13}\dot{a}_{13} + a_{23}\dot{a}_{23} + a_{33}\dot{a}_{33} &= 0 \end{aligned}\right\} \quad (39)$$

the respective angular velocities of rotation are given by

$$\left.\begin{aligned} \omega_x &= +(a_{21}\dot{a}_{31} + a_{22}\dot{a}_{32} + a_{23}\dot{a}_{33}) \\ &= -(a_{31}\dot{a}_{21} + a_{32}\dot{a}_{22} + a_{33}\dot{a}_{23}), \end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned} \omega_y &= +(a_{31}\dot{a}_{11} + a_{32}\dot{a}_{12} + a_{33}\dot{a}_{13}) \\ &= -(a_{11}\dot{a}_{31} + a_{12}\dot{a}_{32} + a_{13}\dot{a}_{33}), \end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned} \omega_z &= +(a_{11}\dot{a}_{21} + a_{12}\dot{a}_{22} + a_{13}\dot{a}_{23}) \\ &= -(a_{21}\dot{a}_{11} + a_{22}\dot{a}_{12} + a_{23}\dot{a}_{13}), \end{aligned}\right\} \quad (42)$$

with respect to the space axes; or

$$\left.\begin{aligned} \omega_{x'} &= +(a_{13}\dot{a}_{12} + a_{23}\dot{a}_{22} + a_{33}\dot{a}_{32}) \\ &= -(a_{12}\dot{a}_{13} + a_{22}\dot{a}_{23} + a_{32}\dot{a}_{33}), \end{aligned}\right\} \quad (43)$$

$$\left.\begin{aligned} \omega_{y'} &= +(a_{11}\dot{a}_{13} + a_{21}\dot{a}_{23} + a_{31}\dot{a}_{33}) \\ &= -(a_{13}\dot{a}_{11} + a_{23}\dot{a}_{21} + a_{33}\dot{a}_{31}), \end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned} \omega_{z'} &= +(a_{12}\dot{a}_{11} + a_{22}\dot{a}_{21} + a_{32}\dot{a}_{31}) \\ &= -(a_{11}\dot{a}_{12} + a_{21}\dot{a}_{22} + a_{31}\dot{a}_{32}), \end{aligned}\right\} \quad (45)$$

with respect to the body axes; the pairs of alternative equations arising from the fact that, by a time-differentiation of the relations $a_{ij}a_{ik} = \delta_{jk}$ it follows that $a_{ij}\dot{a}_{ik} + a_{ik}\dot{a}_{ij} = 0$.

Inserting in the equations (40)-(45) from (29)-(37) it follows that, in terms of the Eulerian angles,

$$\omega_x = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi, \tag{46}$$

$$\omega_y = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi, \tag{47}$$

$$\omega_z = + \dot{\psi} \cos \theta + \dot{\phi}; \tag{48}$$

while

$$\omega_{x'} = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi, \tag{49}$$

$$\omega_{y'} = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi, \tag{50}$$

$$\omega_{z'} = \dot{\phi} \cos \theta + \dot{\psi}, \tag{51}$$

as could be also directly verified by an application of the inverse of the transformation (19), in accordance with which

$$\left.\begin{aligned} \omega_{x'} &= a_{11}\omega_x + a_{21}\omega_y + a_{31}\omega_z, \\ \omega_{y'} &= a_{12}\omega_x + a_{22}\omega_y + a_{32}\omega_z, \\ \omega_{z'} &= a_{13}\omega_x + a_{23}\omega_y + a_{33}\omega_z. \end{aligned}\right\} \tag{52}$$

With the aid of the preceding results the equations (23)-(25) or (26) - (28) for the velocity-components with respect to the space or body axes can be reduced to the forms

$$u = z\omega_y - y\omega_z + u_0' \quad (53)$$

$$v = x\omega_z - z\omega_x + v_0' \quad (54)$$

$$w = y\omega_x - x\omega_y + w_0' \quad (55)$$

or

$$u' = -z'\omega_{y'} + y'\omega_{z'} + u_0, \quad (56)$$

$$v' = -x'\omega_{z'} + z'\omega_{x'} + v_0, \quad (57)$$

$$w' = -y'\omega_{x'} + x'\omega_{y'} + w_0, \quad (58)$$

where

$$\left.\begin{aligned} u_0 &= a_{11}u + a_{21}v + a_{31}w \\ v_0 &= a_{12}u + a_{22}v + a_{32}w \\ w_0 &= a_{13}u + a_{23}v + a_{33}w \end{aligned}\right\} \quad (59)$$

are the *space* velocity components in the direction of the *rotating* axes x', y', z'; and

$$\left.\begin{aligned} u_0' &= a_{11}u' + a_{12}v' + a_{13}w' \\ v_0' &= a_{21}u' + a_{22}v' + a_{23}w' \\ w_0' &= a_{31}u' + a_{32}v' + a_{33}w' \end{aligned}\right\} \quad (60)$$

are the *body* velocity components in the direction of the *fixed* axes x, y, z.

In order to specify the appropriate forms of the components of *acceleration*, let us differentiate the foregoing expressions (53)-(58) for the velocity components with respect to the time. Doing so we find that those with respect to the *space* axes assume the forms

$$\dot{u} = w\omega_y + z\dot{\omega}_y - v\omega_z - y\dot{\omega}_z + \dot{u}_0', \tag{61}$$

$$\dot{v} = u\omega_z + x\dot{\omega}_z - w\omega_x - z\dot{\omega}_x + \dot{v}_0', \tag{62}$$

$$\dot{w} = v\omega_x + y\dot{\omega}_x - u\omega_y - x\dot{\omega}_y + \dot{w}_0', \tag{63}$$

where the velocity components u, v, w have already been given by equations (53)-(55); and where, by differentiation of (60),

$$\dot{u}_0' = a_{11}\dot{u}' + a_{12}\dot{v}' + a_{13}\dot{w}' + \dot{a}_{11}u' + \dot{a}_{12}v' + \dot{a}_{13}w', \tag{64}$$

$$\dot{v}_0' = a_{21}\dot{u}' + a_{22}\dot{v}' + a_{23}\dot{w}' + \dot{a}_{21}u' + \dot{a}_{22}v' + \dot{a}_{23}w', \tag{65}$$

$$\dot{w}_0' = a_{31}\dot{u}' + a_{32}\dot{v}' + a_{33}\dot{w}' + \dot{a}_{31}u' + \dot{a}_{32}v' + \dot{a}_{33}w'. \tag{66}$$

The first three terms in each of these expressions represent obviously the body accelerations with respect to the space axes; and we shall abbreviate them as

$$\left.\begin{aligned} a_{11}\dot{u}' + a_{12}\dot{v}' + a_{13}\dot{w}' &= (\dot{u})_0', \\ a_{21}\dot{u}' + a_{22}\dot{v}' + a_{23}\dot{w}' &= (\dot{v})_0', \\ a_{31}\dot{u}' + a_{32}\dot{v}' + a_{33}\dot{w}' &= (\dot{w})_0'. \end{aligned}\right\} \tag{67}$$

Since, moreover, by insertion from (29)-(31)

$$\dot{a}_{11}u' + \dot{a}_{12}v' + \dot{a}_{13}w' = (a_{31}\omega_y - a_{21}\omega_z)u' + (a_{32}\omega_y - a_{22}\omega_z)v' + (a_{33}\omega_y - a_{23}\omega_z)w'$$
$$= \omega_y(a_{31}u' + a_{32}v' + a_{33}w') - \omega_z(a_{21}u' + a_{22}v' + a_{23}w')$$
$$= \omega_y w_0' - \omega_z v_0'; \quad (68)$$

and, similarly,

$$\dot{a}_{21}u' + \dot{a}_{22}v' + \dot{a}_{23}w' = \omega_z u_0' - \omega_x w_0' \quad (69)$$

while

$$\dot{a}_{31}u' + \dot{a}_{32}v' + \dot{a}_{33}w' = \omega_x v_0 - \omega_y u_0', \quad (70)$$

equations (61)-(63) can be rewritten in a more explicit form

$$\dot{u} = -x(\omega_y^2 + \omega_z^2) + y(\omega_x\omega_y - \dot{\omega}_z) + z(\omega_x\omega_z + \dot{\omega}_y) + (\dot{u})_0' + 2(w_0'\omega_y - v_0'\omega_z), \quad (71)$$

$$\dot{v} = -y(\omega_z^2 + \omega_x^2) + z(\omega_y\omega_z - \dot{\omega}_x) + x(\omega_x\omega_y + \dot{\omega}_z) + (\dot{v})_0' + 2(u_0'\omega_z - w_0'\omega_x), \quad (72)$$

and

$$\dot{w} = -z(\omega_x^2 + \omega_y^2) + x(\omega_x\omega_z - \dot{\omega}_y) + y(\omega_y\omega_z + \dot{\omega}_x) + (\dot{w})_0' + 2(v_0'\omega_x - u_0'\omega_y). \quad (73)$$

The foregoing equations refer to accelerations with respect to the inertial system of space axes. Those with respect to the (rotating) *body* axes can be obtained by an analogous process from the equations

$$\dot{u}' = -w'\omega_{y'} - z'\dot{\omega}_{y'} + v'\omega_{z'} + y'\dot{\omega}_{z'} + \dot{u}_0, \tag{74}$$

$$\dot{v}' = -u'\omega_{z'} - x'\dot{\omega}_{z'} + w'\omega_{x'} + z'\dot{\omega}_{x'} + \dot{v}_0, \tag{75}$$

$$\dot{w}' = -v'\omega_{x'} - y'\dot{\omega}_{x'} + u'\omega_{y'} + x'\dot{\omega}_{y'} + \dot{w}_0, \tag{76}$$

equivalent to (61)-(63); which on being treated in the same way as the latter can eventually be reduced to the form

$$\dot{u}' = -x'(\omega^2_{y'} + \omega^2_{z'}) + y'(\omega_{x'}\omega_{y'} + \dot{\omega}_{z'}) + z'(\omega_{x'}\omega_{z'} - \dot{\omega}_{y'}) + (\dot{u})_0 - 2(w_0\omega_{y'} - v_0\omega_{z'}), \tag{77}$$

$$\dot{v}' = -y'(\omega^2_{z'} + \omega^2_{x'}) + z'(\omega_{y'}\omega_{z'} + \dot{\omega}_{x'}) + x'(\omega_{x'}\omega_{y'} - \dot{\omega}_{z'}) + (\dot{v})_0 - 2(u_0\omega_{z'} - w_0\omega_{x'}), \tag{78}$$

$$\dot{w}' = -z'(\omega^2_{x'} + \omega^2_{y'}) + x'(\omega_{x'}\omega_{z'} + \dot{\omega}_{y'}) + y'(\omega_{y'}\omega_{z'} - \dot{\omega}_{x'}) + (\dot{w})_0 - 2(v_0\omega_{x'} - u_0\omega_{y'}), \tag{79}$$

where the space velocity components u_0, v_0, w_0 in the direction of increasing x', y', z' continue to be given by equations (59); while the corresponding components of the accelerations are given by

$$\left.\begin{aligned} (\dot{u})_0 &= a_{11}\dot{u} + a_{21}\dot{v} + a_{31}\dot{w}, \\ (\dot{v})_0 &= a_{12}\dot{u} + a_{22}\dot{v} + a_{32}\dot{w}, \\ (\dot{w})_0 &= a_{13}\dot{u} + a_{23}\dot{v} + a_{33}\dot{w}. \end{aligned}\right\} \tag{80}$$

If, in particular, we consider the restricted case of a rotation about the z-axis only (so that $\omega_x = \omega_y = 0$), equations (71)-(73) will reduce to the system

$$\left.\begin{aligned} \boxed{\dot{u}} &= (\dot{u})'_0 - 2v\omega_z + x\omega_z^2 - y\dot{\omega}_z, \\ \boxed{\dot{v}} &= (\dot{v})'_0 + 2u\omega_z + y\omega_z^2 + x\dot{\omega}_z, \\ \boxed{\dot{w}} &= (\dot{w})'_0; \end{aligned}\right\} \tag{81}$$

while equations (77)-(79) will likewise reduce to

$$\left.\begin{aligned} \dot{u}' &= \boxed{(\dot{u})_0} + 2v'\omega_{z'} + x'\omega_{z'}^2 + y'\dot{\omega}_{z'}, \\ \dot{v}' &= \boxed{(\dot{v})_0} - 2u'\omega_{z'} + y'\omega_{z'}^2 - x'\dot{\omega}_{z'}, \\ \dot{w}' &= \boxed{(\dot{w})_0} \,. \end{aligned}\right\} \tag{82}$$

It is the accelerations in the cartouches of the two systems--referred as they are to the inertial space axes--which should be identified with the Lagrangian time-derivatives

$$\frac{D\vec{V}}{Dt}$$

on the left-hand sides of the equations (7) - (8) of motion if these are referred to the inertial or rotating axes of coordinates.

A closing note concerning the time differentiation of the coordinates or velocities should be added in this place. As

$$x \equiv x(t), \quad y \equiv y(t), \quad z \equiv z(t), \tag{83}$$

it follows that

$$\left.\begin{aligned} \dot{x} &\equiv u = \frac{dx}{dt} = \frac{\partial x}{\partial t}, \\ \dot{y} &\equiv v = \frac{dy}{dt} = \frac{\partial y}{\partial t}, \\ \dot{z} &\equiv w = \frac{dy}{dt} = \frac{\partial z}{\partial t}, \end{aligned}\right\} \tag{84}$$

i.e., the ordinary (total) and partial derivatives of the coordinates with respect to the time are obviously identical. This is, however, no longer true of the time-differentiation of the velocities -whether linear or angular. As

$$\left.\begin{aligned} u &\equiv u(x,y,z;t) \\ v &\equiv v(x,y,z;t) \\ w &\equiv w(x,y,z;t) \end{aligned}\right\} \tag{85}$$

or

$$\omega_{x,y,z} \equiv \omega_{x,y,z}(x,y,z;t), \tag{86}$$

where the coordinates (83) are themselves functions of the time. In consequence,

$$\begin{aligned} \dot{u} \equiv \frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \end{aligned} \tag{87}$$

by virtue of (84) ; and similarly for $\dot{v}$ and $\dot{w}$. Likewise,

$$\dot{\omega} \equiv \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z} \tag{88}$$

for $\omega \equiv \omega_{x,y,z}$.

For coordinate systems referred to the rotating body axes similar reltions hold good; care being merely taken to replace the unprimed coordinates or velocity components by the primed ones.

VI. FORMATION OF THE EULERIAN EQUATIONS FOR PRECESSION AND NUTATION

In Section II of this report we set up the general form of the equations governing the motion of compressible viscous fluids in rectangular coordinates; and in Section III we expressed its velocity components in terms of arbitrary rotations about three rectangular axes. The aim of the present section will be to combine the fundamental equations (6)-(8) rewritten in terms of the angular variables $\omega_{x,y,z}$ introduced in Section III in a form suitable for their subsequent solution.

In order to embark on this task, let us multiply equations (71)-(73) by x, y, z and form their following differences:

$$\begin{aligned} y\dot{w} - z\dot{v} = (y^2 + z^2)\dot{\omega}_x &+ (y^2 - z^2)\omega_y\omega_z \\ &- xy(\dot{\omega}_y - \omega_x\omega_z) - xz(\dot{\omega}_z + \omega_x\omega_y) \\ &- yz(\omega_y^2 - \omega_z^2) \\ &+ \{y(\dot{w})_0' - z(\dot{v})_0'\} + 2y\{v_0'\omega_x - u_0'\omega_y\} \\ &- 2z\{u_0'\omega_z - w_0'\omega_x\}, \end{aligned} \tag{89}$$

$$\begin{aligned} z\dot{u} - x\dot{w} = (z^2 + x^2)\dot{\omega}_y &+ (z^2 - x^2)\omega_x\omega_z \\ &- yz(\dot{\omega}_z - \omega_y\omega_x) - yx(\dot{\omega}_x + \omega_y\omega_z) \\ &- zx(\omega_z^2 - \omega_x^2) \\ &+ \{z(\dot{u})_0' - x(\dot{w})_0'\} + 2z\{w_0'\omega_y - v_0'\omega_z\} \\ &- 2x\{v_0'\omega_x - u_0'\omega_y\}, \end{aligned} \tag{90}$$

$$\begin{aligned}x\dot{v} - y\dot{u} = & (x^2 + y^2)\dot{\omega}_z + (x^2 - y^2)\omega_x\omega_y \\ & - zx(\dot{\omega}_x - \omega_y\omega_z) - zy(\dot{\omega}_y + \omega_x\omega_z) \\ & - xy(\omega_x^2 - \omega_y^2), \\ & + \{x(\dot{v})_0' - y(\dot{u})_0'\} + 2x\{u_0'\omega_z - w_0'\omega_x\} \\ & - 2y\{w_0'\omega_y - v_0'\omega_z\}.\end{aligned} \tag{91}$$

If so, however, equations (6) - (8) can be combined accordingly to yield

$$y\dot{w} - z\dot{v} + \frac{1}{\rho}\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\}P - \{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\}\Omega = y\mathcal{H} - z\mathcal{G}, \tag{92}$$

$$z\dot{u} - x\dot{w} + \frac{1}{\rho}\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\}P - \{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\}\Omega = z\mathcal{F} - x\mathcal{H}, \tag{93}$$

$$x\dot{v} - y\dot{u} + \frac{1}{\rho}\{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\}P - \{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\}\Omega = x\mathcal{G} - y\mathcal{F}, \tag{94}$$

where

$$\rho\mathcal{F} = \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}, \tag{95}$$

$$\rho\mathcal{G} = \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}, \tag{96}$$

$$\rho\mathcal{H} = \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}, \tag{97}$$

represent the effects of viscosity.

In order to proceed further, let us rewrite the foregoing expressions in terms of the respective velocity components. Inserting for the components σ_{ij} of the viscous stress tensor from (10) - (15) we find the expressions on the right-hand sides of equations (95) - (97) to assume the more explicit forms

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = \mu\nabla^2 u + \frac{\mu}{3}\frac{\partial\Delta}{\partial x}$$

$$+ 2\left\{\frac{\partial u}{\partial x} - \frac{\Delta}{3}\right\}\frac{\partial\mu}{\partial x}$$

$$+ \left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\}\frac{\partial\mu}{\partial y}$$

$$+ \left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\}\frac{\partial\mu}{\partial z}\,, \tag{98}$$

$$\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = \mu\nabla^2 v + \frac{\mu}{3}\frac{\partial\Delta}{\partial y}$$

$$+ 2\left\{\frac{\partial v}{\partial y} - \frac{\Delta}{3}\right\}\frac{\partial\mu}{\partial y}$$

$$+ \left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\}\frac{\partial\mu}{\partial z}$$

$$+ \left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\}\frac{\partial\mu}{\partial x}\,, \tag{99}$$

and

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = \mu\nabla^2 w + \frac{\mu}{3}\frac{\partial\Delta}{\partial z}$$

$$+ 2\left\{\frac{\partial w}{\partial z} - \frac{\Delta}{3}\right\}\frac{\partial\mu}{\partial z}$$

$$+ \left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\}\frac{\partial\mu}{\partial x}$$

$$+ \left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\}\frac{\partial\mu}{\partial y}\,, \tag{100}$$

where Δ denotes, as before, the divergence (16) of the velocity vector; and ∇^2 stands for the Laplacean operator.

Next, let us insert for the velocity components u, v, w from (53) - (55) ; by doing so we find that

$$\nabla^2 u = 2\nabla^2\omega_y - y\nabla^2\omega_z + \nabla^2 u_0' + 2\left\{\frac{\partial\omega_y}{\partial z} - \frac{\partial\omega_z}{\partial y}\right\}, \tag{101}$$

$$\nabla^2 v = x\nabla^2\omega_z - z\nabla^2\omega_x + \nabla^2 v_0' + 2\left\{\frac{\partial\omega_z}{\partial x} - \frac{\partial\omega_x}{\partial z}\right\}, \tag{102}$$

$$\nabla^2 w = y\nabla^2\omega_x - x\nabla^2\omega_y + \nabla^2 w_0' + 2\left\{\frac{\partial\omega_x}{\partial y} - \frac{\partial\omega_y}{\partial x}\right\}, \tag{103}$$

and

$$\Delta = \left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\omega_x + \left\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}\omega_y + \left\{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\}\omega_z + \frac{\partial u_0'}{\partial x} + \frac{\partial v_0'}{\partial y} + \frac{\partial w_0'}{\partial z}\ . \tag{104}$$

Before proceeding further, one feature of basic importance should be brought out which we by-passed without closer discussion at an earlier stage: namely, when by virtue of equations (53) - (57) or (58) - (60) we replaced the *three* dependent variables u, v, w or u', v', w' on their left-hand sides by *six* new variables ω_x, ω_y, ω_z and u_0', v_0', w_0' or $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ and u_0, v_0, w_0 on their right-hand sides. This deliberately created redundancy permits us to impose without the loss of generality additional constraints on these variables, not embodied in the fundamental equations of Section II; and this we propose to do at the present time. We propose, in particular, to assume that the primed axes $x'y'z'$ obtained by a rotation of the inertial system xyz, about a fixed origin, in accordance with the transformation (19) remain rectangular--an assumption to which implies, in effect, that *the Eulerian angles* θ, ϕ, ψ involved in the direction cosines a_{ik} and, therefore, in the angular velocity components $\omega_{x,y,z}$ or $\omega_{x',y',z'}$ as defined by equations (46) - (48) or (49) - (51) *are functions of* *the time* t *alone* (for should they depend, in addition, on the spatial coordinates x, y, z, a rotation as represented by equations (19) would result in a curvilinear coordinate system).

This assumption will neatly separate the physical meaning of the two groups of variables: for while the angular velocity components $\omega_{x,y,z}$ will describe a *rigid-body rotation* of our dynamical system (during which the position of each particle remains unchanged in the primed coordinates), the remaining velocity components u_0', v_0', w_0' will represent *deformation* of our body, in the primed system, in the course of time. It is, therefore, the latter which will be of particular interest for the main problem which we have in mind; and in what follows, we propose to investigate the extent to which their occurrence may modify the structure of our equations.

In order to do so we notice first that, inasmuch as the angular velocity components are hereafter to be regarded as functions of t alone it follows from (101) - (103) that

$$\left.\begin{aligned} \nabla^2 u &= \nabla^2 u_0', \\ \nabla^2 v &= \nabla^2 v_0', \\ \nabla^2 w &= \nabla^2 w_0'; \end{aligned}\right\} \tag{105}$$

and, similarly, the divergence (104) of the velocity vector will reduce to

$$\Delta_0' = \frac{\partial u_0'}{\partial x} + \frac{\partial v_0'}{\partial y} + \frac{\partial w_0'}{\partial z} . \tag{106}$$

In consequence, the corresponding expressions on the right-hand sides of equations (98) - (100) are obtained if the velocity components u, v, w present there are replaced by u_0', v_0', w_0'; and Δ by Δ_0'.

Therefore,

$$\rho\{y\mathcal{H} - z\mathcal{G}\} = \mu\{y\nabla^2 w_0' - z\nabla^2 v_0' + \frac{1}{3}D_1\Delta_0'\}$$

$$+ \frac{\partial\mu}{\partial x}\{D_1 u_0' + \frac{\partial}{\partial x}(yw_0' - zv_0')\} +$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial y}\{2D_1 v_0' + D_4 w_0'\}$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial z}\{2D_1 w_0' - D_4 v_0'\}$$

$$- \frac{2}{3}\frac{\partial u_0'}{\partial x}D_1\mu + \frac{1}{3}\xi D_4\mu, \tag{107}$$

$$\rho\{z\mathcal{F} - x\mathcal{H}\} = \mu\{z\nabla^2 u_0' - x\nabla^2 w_0' + \frac{1}{3}D_2\Delta_0'\}$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial x}\{2D_2 u_0' - D_5 w_0'\}$$

$$+ \frac{\partial\mu}{\partial y}\{D_2 v_0' + \frac{\partial}{\partial y}(zu_0' - xw_0')\}$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial z}\{2D_2 w_0' + D_5 u_0'\}$$

$$- \frac{2}{3}\frac{\partial v_0'}{\partial y}D_2\mu + \frac{1}{3}\eta D_5\mu, \tag{108}$$

and

$$\rho\{x\mathcal{G} - y\mathcal{F}\} = \mu\{x\nabla^2 v_0' - y\nabla^2 u_0' + \frac{1}{3}D_3\Delta_0'$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial x}\{2D_3 u_0' + D_6 v_0'\}$$

$$+ \frac{2}{3}\frac{\partial\mu}{\partial y}\{2D_3 v_0' - D_6 u_0'\}$$

$$+ \frac{\partial\mu}{\partial z}\{D_3 w_0' + \frac{\partial}{\partial z}(xv_0' - yu_0')\}$$

$$- \frac{2}{3}\frac{\partial w_0'}{\partial z}D_3\mu + \frac{1}{3}\zeta D_6\mu, \tag{109}$$

where the symbols D_j $(j = 1, \ldots 6)$ stand for the following operators

$$D_1 \equiv y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} , \tag{110}$$

$$D_2 \equiv z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} , \tag{111}$$

$$D_3 \equiv x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} , \tag{112}$$

$$D_4 \equiv z \frac{\partial}{\partial z} + y \frac{\partial}{\partial y} , \tag{113}$$

$$D_5 \equiv x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} , \tag{114}$$

$$D_6 \equiv x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} ; \tag{115}$$

and where

$$\xi = \frac{\partial w_0'}{\partial y} - \frac{\partial v_0'}{\partial z} , \tag{116}$$

$$\eta = \frac{\partial u_0'}{\partial z} - \frac{\partial w_0'}{\partial x} , \tag{117}$$

$$\zeta = \frac{\partial v_0'}{\partial x} - \frac{\partial u_0'}{\partial y} , \tag{118}$$

denote the components of vorticity of the deformation vector.

As the next step of our analysis, let us integrate both sides of the equations (92) - (94) over the entire mass of our configuration with respect to the mass element

$$dm = \rho dV = \rho dxdydz. \tag{119}$$

If, as usual,

$$A = \int (y^2 + z^2)\,dm, \tag{120}$$

$$B = \int (x^2 + z^2)\,dm, \tag{121}$$

$$C = \int (x^2 + y^2)\,dm \tag{122}$$

denote the *moments of inertia* of our configuration with respect to the axes x, y, z; and

$$D = \int yz\,dm, \tag{123}$$

$$E = \int xz\,dm, \tag{124}$$

$$F = \int xy\,dm \tag{125}$$

stand for the respective *products of inertia*, the mass integrals of the equations (92) - (94) combined with (89) - (91) will assume the forms

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z - D(\omega_y^2 - \omega_z^2) - E(\dot{\omega}_z + \omega_x\omega_y) \\ - F(\dot{\omega}_y - \omega_x\omega_z) \\ + 2\omega_x\int(yv_0' + zw_0')\,dm - 2\omega_y\int yu_0'\,dm - 2\omega_z\int zu_0'\,dm \\ + \int D_1 P\,dV - \int D_1\Omega\,dm = \int\{z(\dot{v})_0' - y(\dot{w})_0'\}\,dm + \int\rho\{y\mathcal{H} - z\mathcal{G}\}\,dV, \end{aligned} \tag{126}$$

$$\begin{aligned} B\dot{\omega}_y + (A - C)\omega_x\omega_z - D(\dot{\omega}_z - \omega_x\omega_y) - E(\omega_z^2 - \omega_x^2) \\ - F(\dot{\omega}_x + \omega_y\omega_z) \\ + 2\omega_y\int(xu_0' + zw_0')\,dm - 2\omega_z\int zv_0'\,dm - 2\omega_x\int xv_0'\,dm \\ + \int D_2 P\,dV - \int D_2\Omega\,dm = \int\{x(\dot{w})_0' - z(\dot{u})_0'\}\,dm + \int\rho\{z\mathcal{F} - x\mathcal{H}\}\,dV, \end{aligned} \tag{127}$$

and

$$\begin{aligned} C\dot{\omega}_z + (B - A)\omega_x\omega_y - D(\dot{\omega}_y + \omega_x\omega_z) - E(\dot{\omega}_x - \omega_y\omega_z) \\ - F(\omega_x^2 - \omega_y^2) \\ + 2\omega_z\int(yv_0' + xu_0')\,dm - 2\omega_x\int xw_0'\,dm - 2\omega_y\int yw_0'\,dm \\ + \int D_3 P\,dV - \int D_3\Omega\,dm = \int\{y(\dot{u})_0' - x(\dot{v})_0'\}\,dm + \int\rho\{x\mathcal{G} - y\mathcal{F}\}\,dV. \end{aligned} \tag{128}$$

The preceding three equations represent the exact form of the generalized Eulerian equations governing the precession and nutation of self-gravitating configurations which consist of a viscous fluid. They constitute a system of three ordinary differential equations for $\omega_{x,y,z}$ considered as functions of the time t alone. If the body in question were rigid (non-deformable)--or, if deformable, it were subject to no time-dependent deformation--all three velocity components u', v', w' relative to the rotating frame of reference (and thus, by (60), u_0', v_0', w_0') would be identically zero. In such a case, equations (126) - (128) would reduce to their more familiar form

$$A\dot{\omega}_x + (C - B)\omega_y\omega_z - D(\omega_y^2 - \omega_z^2) - E(\dot{\omega}_z + \omega_x\omega_y) - F(\dot{\omega}_y - \omega_x\omega_z) + \int D_1 P dV - \int D_1\Omega_0 dm = \int D_1\Omega_1 dm, \tag{129}$$

$$B\dot{\omega}_y + (A - C)\omega_x\omega_z - D(\dot{\omega}_z - \omega_x\omega_y) - E(\omega_z^2 - \omega_x^2) - F(\dot{\omega}_x + \omega_y\omega_z) + \int D_2 P dV - \int D_2\Omega_0 dm = \int D_2\Omega_1 dm, \tag{130}$$

and

$$C\dot{\omega}_z + (B - A)\omega_x\omega_y - D(\dot{\omega}_y + \omega_x\omega_z) - E(\dot{\omega}_x - \omega_y\omega_z) - F(\omega_x^2 - \omega_y^2) + \int D_3 P dV - \int D_3\Omega_0 dm = \int D_3\Omega_1 dm, \tag{131}$$

where we have decomposed the total gravitational potential

$$\Omega = \Omega_0 + \Omega_1 \tag{132}$$

into its part arising from the mass of the respective body (Ω_0) and that arising from external disturbing forces (Ω_1) if any.

In the case of a rigid body, the existence of hydrostatic equilibrium requires that

$$\int D_i P dV = \int D_i \Omega_0 dm \tag{133}$$

exactly for $i = 1, 2, 3$. If, moreover, we choose our system of inertial axes xyz to coincide with the principal axes of inertia of our configuration, it can be shown that all three moments of inertia (123) - (125) can be made to vanish; and for

$$D = E = F = 0 \tag{134}$$

our equations (129) - (131) will reduce further to

$$\left.\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= D_1\Omega_1 dm, \\ B\dot{\omega}_y + (A - C)\omega_x\omega_z &= D_2\Omega_1 dm, \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= D_3\Omega_1 dm, \end{aligned}\right\} \tag{135}$$

which is the familiar form of the Eulerian equations for the precession of rigid bodies.

If, however, the body in question were fluid and subject to distortion by external forces--though not necessarily (like equilibrium tides) fluctuating in time--equations (135) would cease to be exact to the extent to which equations (133) need no longer hold true. The reader may note that as long as the functions $P(r)$ and $\Omega_0(r)$ are purely radial (as they would be in the absence of any distortion) operation with D_i $(i = 1, 2, 3)$ will annihilate them completely; so that equations (133) continue to be fulfilled identically. The same argument discloses, however, that for fluid bodies, equations (133) may become inequalities to the extent brought about by distortion; and--to this extent--the Eulerian differential equations for the precession and nutation of rigid and fluid bodies may be different even if the form of the fluid does not vary with the time.

If, however, this latter condition is not fulfilled--such as, for instance, in the case when the period of axial rotation of the fluid body differs from that of the revolution of an external attracting mass producing *dynamical tides* on the rotating fluid--the velocity components u_0', v_0', w_0' will emerge to give rise to supplementary terms in the equations (126) - (128) which can be classified in two groups. Those on the left-hand sides of the respective equations are factored by the angular velocity components ω_x, ω_y, ω_z which play the role of dependent variables of our problem. However, their coefficients are not constants (like A, B, C; D, E, F), but functions of the time. The second group of new terms arising on the right-hand sides of the same equations are independent of $\omega_{x,y,z}$ and render our system non-homogeneous. The first mass-integral on the right-hand sides of equations (126) - (128) arises from the accelerations $(\dot{u})_0'$, $(\dot{v})_0'$, $(\dot{w})_0'$ experienced by the body subject to deformation--irrespective of whether the flow due to this motion is inviscid or viscous; while the second group of volume integrals (the integrands of which are given by equations (126) - (128) represent the effects of viscosity proper; and if the latter is large, these may be predominant.

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