NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-816

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John D. Anderson

JET PROPULSION LABORATORY

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Abstract

Doppler tracking data from the *Mariner II* spacecraft, which came within 41,000 km of Venus in December 1962, are used to obtain the mass of Venus and the astronomical unit. Also, a measure of the lunar inequality by means of the monthly periodic variation in the Doppler curve permits a determination of the Earth-Moon mass ratio.

The method of data reduction is a least-squares differential correction of the spacecraft's orbit along with the three constants and several other parameters necessary to describe important non-gravitational forces. The geocentric location of the tracking stations and the heliocentric position of Venus are subject to correction also. The differential coefficients, which relate variations in the constants and parameters to variations in the Doppler data, are obtained by numerically integrating a set of variational equations along with the equations of motion. Residuals are formed by directly subtracting the computed Doppler data from the observed values. Corrections for light-time, atmospheric refraction, and station timing are applied to the computed data.

The results indicate that the Sun-Venus mass ratio is 408505 ±6, the number of light seconds in one astronomical unit is 499.0036 ±0.0017 sec and the Earth-Moon mass ratio is 81.3001 ±0.0013. Information on the locations of the tracking stations and the direction and distance of Venus at the time of the encounter of *Mariner II* with the planet is also given.

Determination of the Masses of the Moon and Venus and the Astronomical Unit from Radio Tracking Data of the *Mariner II* Spacecraft

I. Introduction

The primary purpose of this study is to show how tracking data from what are generally termed deep-space probes can be used to provide fundamental information on the system of astronomical constants and on the ephemerides of the Earth and planets. In particular, data from the *Mariner II* spacecraft, which was launched from the Earth on August 27, 1962 and came within 41,000 km of Venus on December 14, 1962, are used to obtain a determination of the mass ratio μ of the Moon to the Earth, the mass M_v^s of Venus in units of the solar mass, the number of kilometers A in one astronomical unit, the three-dimensional geocentric position of Venus at the time of the spacecraft's closest approach to Venus, and, finally, the geocentric coordinates of the tracking stations at Goldstone, California.

The effect of new values of the constants on the entire system of astronomical constants is explored in detail but no attempt is made to investigate the long-term effect on the Venus ephemeris of an improved mass ratio and measurement of the position. To do this properly would require a combination of the *Mariner II* result with the radar Venus bounce measurements and optical observations taken over a period of many years. Also, a second *Mariner* spacecraft to Venus is planned in 1967 and the likelihood of combining a 1962 and 1967 position determination for an ephemeris improvement offers far greater possibilities than using the single 1962 measurement. One of the results of the investigation is that it is not possible to improve any of the Earth's orbital elements because of the relatively short duration of the accurate *Mariner II* data from September 5 to December 20, 1962. However, other space probes of longer duration, for example the current *Pioneer* series, can provide improvements in some of the Earth's elements and suggest an area for future study.

The geodetic implications of the station location determination are not discussed here, although G. Veis* is currently comparing such determinations from *Mariner* and *Ranger* space probes with station locations as obtained from several thousand optical observations of satellites.

In describing the methods and results of the reduction of the *Mariner II* data, the following organization has been adopted. In Section II a summary of the study is given. The system of astronomical constants is discussed

*Private communication, G. V. Veis, Smithsonian Astrophysical Observatory.

in Section III from the viewpoint of the history of the subject, and the problems that give rise to a system of constants are considered. This material is largely tutorial in nature and provides some degree of motivation for the detailed analyses of later sections. In Section III-E formulae are given which permit the evaluation of results from the *Mariner II* data in terms of the complete system of constants. Arguments are presented in Section III-F to explain how it is possible to obtain certain constants from Doppler data of the type for *Mariner II*.

Detailed methods for obtaining the constants from the real data are described in Section IV, starting with a discussion of the method of least squares as used to obtain differential corrections to a preliminary orbit and set of constants. Also included in this section are descriptions and justifications for formulae relating to the computation of the orbit, the Doppler residuals, and the differential coefficients for the least-squares method. The orbit computation is complicated by the introduction of solar radiation pressure and certain low-thrust forces on the spacecraft. Also it is found that in order to compute sufficiently accurate residuals, it is necessary to compute the Doppler shift to terms in $1/c^2$, where c is the velocity of light, and, additionally, to consider light-time and refraction corrections. The interpretation of the time assigned to an individual Doppler measurement is discussed in Section IV-C-3.

Numerical results are presented in Section V. In Sections V-A and V-B the numerical accuracy of the methods described in Section IV is investigated, and in Section V-C the nature and degree of information about the constants are explored before the introduction of the *Mariner II* data. The remainder of the section is concerned with a number of least-squares solutions for the constants, and in Section V-E the results of the investigations are summarized.

Mathematical derivations, which contribute little to a basic understanding of the methods, are relegated to the appendices, although Appendix E contains a listing of all the data used in the solutions of Section V-D along with two sets of residuals from the best of these solutions.

II. Summary

Preliminary determinations of certain astronomical constants from Doppler tracking data of the *Mariner II* spacecraft have been reported before in 1963 (Ref. 1) and 1965 (Ref. 2). However, both of the earlier determinations suffer from serious defects in the methods used to obtain the constants; and, in fact, the preliminary nature of those determinations was founded on the presence of systematic errors which have been removed only recently.

In Ref. 1 the least-squares solution for the constants and the six orbital parameters of the spacecraft's orbit give a mass ratio μ^{-1} of the Earth to Moon of 81.3012 ±0.0034 and a gravitational constant GM_v for Venus of 324857 ± 24 km^3 sec^{-2}; but the values included systematic errors caused by a neglect of low-thrust forces from the spacecraft's attitude-control system and by an inability to adjust the coordinates of Venus during encounter. The solution of Ref. 2 removed these deficiencies by including parameters for the low-thrust forces and the orbit of Venus in the normal equations of the differential correction, but it did not apply the corrected parameters to a recomputation of residuals in the data. Until the corrections to the orbital parameters of Venus could be applied to the computation of a new ephemeris, along with the necessary recomputation of the spacecraft's orbit, it was impossible to verify numerically the computation of the differential coefficients. Subsequent to the publication of results in Ref. 2, it was found that an error in the computer program used to compute the differential coefficients had indeed introduced erroneous corrections to the orbit of Venus. Therefore, the value of GM_v (324806 ± 20 km^3 sec^{-2}) given in Ref. 2 was similarly erroneous.

The question of the effect of errors in the Earth's ephemeris on the *Mariner II* solutions was considered in Ref. 2 by including orbital elements of the Earth in the normal equations. The conclusion that the determinations of the constants are not sensitive to reasonable errors in the Earth's ephemeris is still valid, as is the other important conclusion of Ref. 2 that the determinations are significantly sensitive to expected variations in the position of Venus at encounter but not to its velocity.

With the removal of systematic effects that influenced the previous solutions, although some errors remain as a result of computing the Doppler data in single precision on the IBM 7094 computer, the preliminary nature of the values of the constants no longer holds and future modifications should not be significant with respect to the stated uncertainties on the constants. The Earth-Moon mass ratio is now μ^{-1} = 81.3001 ± 0.0013, the gravitational constant is GM_v=324871.5±2.5 km^3 sec^{-2}, and the number of light seconds in one astronomical unit (a.u.) is τ_A = 499.0036 ± 0.0017 sec. A reliable value of the a.u. from *Mariner II* was not available before. It is expressed by the constant τ_A because this is the measured quantity, but it can be converted to A in km by multiplying by the velocity of light c. With c given by 299792.5 km/sec, the corresponding value of A from the *Mariner* data is $A = 149597550 \pm 500$ km.

A recent prepublication result by Ash, Shapiro and Smith at the Lincoln Laboratory gives $499.004786 \pm 5 \times 10^{-6}$ sec for τ_A as obtained from a combination of post 1950 meridian-circle observations of Mercury, Venus and the Sun with radar measurements of Mercury and Venus. They also obtain a value for the Sun-Venus mass ratio of 408250, using general relativity theory and a value of 408450 using Newtonian theory with an uncertainty of $\pm$ 120 in both cases. The *Mariner II* data provide a more direct determination of GM_v than of the mass ratio, but the latter can be computed by the formula k^2A^3/GM_v where k is the Gaussian gravitational constant. The result is 408505 ± 6 which, strangely enough, is consistent with the Lincoln Laboratory determination using Newtonian theory but differs by a little more than two times the uncertainty for the value obtained with general relativity theory. The resolution of this inconsistency will have to await further determinations of the mass by the 1967 *Mariner* probe to Venus and by additional analyses with radar bounce data. Also, Rabe and Francis are obtaining new values of the constants, including the mass of Venus, from observations of the minor planet Eros. A further refinement of the *Mariner II* results will not occur until the computations are done in double precision, probably with a more sophisticated parameter estimation scheme to handle the low-thrust forces.

A summary of how three astronomical constants can be determined from the *Mariner II* data and how they are correlated with other parameters in the least-squares solution is given in the following table.

Constant	Source of determination	Significant correlations
Earth-Moon mass ratio	Monthly periodic variation in Doppler curve	None
Mass of Venus	Encounter data	Spacecraft orbit Ephermeris of Venus Astronomical unit
Astronomical unit	Combination of cruise and encounter data	Spacecraft orbit Ephemeris of Venus Mass of Venus

For a more quantitative evaluation of the correlations, see Section V-D-5 and Table 17.

A. Data Reduction

The useful *Mariner II* tracking data consist of Doppler measurements made at the Goldstone station of the Deep Space Instrumentation Facility (DSIF). Other stations in this facility also tracked *Mariner II*, but the accuracy of the data was not controlled by an atomic frequency standard at that time and, for this determination of the constants, only the Goldstone data are used.

The method of solution is that of weighted least squares with a modification to allow the introduction of *a-priori* information into the process. As in any least-squares solution it is necessary to compute residuals in the data, and by convention the sense of the residuals is the observed minus the computed (*O-C*) values. The adopted procedure is to simply represent the Doppler measurement as accurately as necessary by a mathematical formula and then to form the *O-C* subtraction. The actual measurement *O* is stored on magnetic tape. An accurate representation of the data involves considerations of light-time, refraction corrections, and an interpretation of the station procedure used to record the time of an observation.

Practically all least-squares data reductions in celestial mechanics are accomplished by differentially correcting nominal or standard values of the orbital and astronomical constants to obtain the least-squares solution for both the orbit and constants. Fortunately, a preliminary orbit for *Mariner II* is available (Ref. 3) and it is possible to proceed directly to the differential correction. Therefore, the formation of differential coefficients is the major concern of the current solution for the constants. The parameters included in the differential correction are the following:

Set I: Orbital elements of the spacecraft expressed as six cartesian coordinates of position and velocity at an arbitrary epoch.

Set II: The astronomical constants μ, τ_A and the mass of Venus.

Set III: Non-gravitational parameters representing forces from solar radiation pressure and attitude control gas jetting.

Set IV: Orbital elements of the Earth and Venus. Only the three heliocentric cartesian position coordinates of Venus at planetary encounter are actually corrected.

Set V: Two coordinates for each tracking station — the distance of the station from the Earth's axis of rotation and the longitude. The component parallel to the Earth's axis cannot be determined.

The many formulae used to compute the differential coefficients associated with these five parameter sets are derived in Section IV, and, also, one can find there a description of the mechanization of the least-squares process itself. The computer program used to obtain the results is a version of the JPL single precision orbit determination program (Ref. 4 and 5), although major modifications were carried out in order to extend the differential correction to the low-thrust and ephemeris parameters.

B. Significance of Low-Thrust Forces

Although there is nothing particularly interesting in the nature of the small forces on the spacecraft from the attitude-control system, at least from the viewpoint of the astronomical constant determination, it is necessary to give as careful attention to the establishment of a proper representation of these forces as to the constants themselves. Such attention is necessary because any misrepresentation of the forces will introduce systematic errors into the solution for the constants of interest.

Ideally, the estimation scheme should take account of the fact that the magnitude and possibly the direction of the forces are random variables which take on different values at different points in time. However, the least-squares program used to obtain the current results does not adapt easily to this sort of estimation scheme, and the procedure adopted for the consideration of the small forces is to represent them by a vector whose magnitude is a quadratic function in time. Ultimately the *Mariner II* data could be subjected to a more sophisticated estimator, and the resulting values for the constants would be slightly more credible because of a greater assurance that systematic errors have been eliminated; but for now, the forces are assumed to obey the quadratic model and the coefficients in the expression for the force are obtained by least squares simultaneously with the constants. The results of this approach indicate that it is quite reasonable, and for comparison, another representation which assumes that the forces vary as the inverse square of the spacecraft's distance from the Sun gives very unreasonable results as shown in Section V-D-1.

The quadratic representation of the low-thrust forces is written as

$$\ddot{\mathbf{r}}_T = (f_1\,\mathbf{U} + f_2\,\mathbf{T} + f_3\,\mathbf{N})\,(1 - \alpha_1\,\tau - \alpha_2\,\tau^2) \qquad (1)$$

where τ is the time from some arbitrary epoch and **U**, **T** and **N** are unit vectors along the spacecraft's principal axes. The least-squares fit to the coefficients in Eq. (1) from the cruise data, defined by data in the time interval from September 5 to December 7, 1962 before Venus dominated the spacecraft's motion, is

$$f_1 = (-0.03 \pm 0.35) \times 10^{-10}\ \text{km/sec}^2$$

$$f_2 = (-0.36 \pm 0.02) \times 10^{-10}\ \text{km/sec}^2$$

$$f_3 = (-0.15 \pm 0.15) \times 10^{-10}\ \text{km/sec}^2$$

$$\alpha_1 = (0.07 \pm 0.29) \times 10^{-7}\ \text{sec}^{-1}$$

$$\alpha_2 = (0.81 \pm 0.73) \times 10^{-14}\ \text{sec}^{-2}$$

epoch = 1962, September 5, 0ʰ0 E. T.

The solution indicates a decreasing force whose magnitude behaves as shown in Fig. 1 with the error bounds computed from the formal uncertainties on the parameters as determined from diagonal elements in the inverted matrix of the least-squares normal equations. Accelerations are converted to forces by taking 1.9822 $\times\ 10^5$ gms as the mass of spacecraft. The curve as extrapolated into the encounter region of time should not be considered reliable. In Sections V-D-3 and V-D-4 when the constants are obtained from encounter data, the five f and α parameters are included also for correction. However, the cruise data provide preliminary values which can be used in more complete solutions. This is significant when one realizes that without any information on the values for f_1, f_2 and f_3, the behavior of the solution to variations in α_1 and α_2 is quite nonlinear with respect to other parameters in the solution. For example, if f_1, f_2 and f_3 are all zero, then any correction to α_1 and α_2 will satisfy the least-squares fit to the data.

C. The Earth-Moon Mass Ratio

The Earth-Moon mass ratio μ^{-1} is obtained by measuring the amplitude of the periodic component in the Doppler data resulting from the motion of the Earth about the Earth-Moon barycenter, or in effect by obtaining a dynamical measurement of the lunar inequality. For this purpose, the cruise data alone are used so that systematic effects in the encounter data from the mass and position of Venus and the a.u. are neglected. Actually, in the differential correction program it is not the mass ratio μ

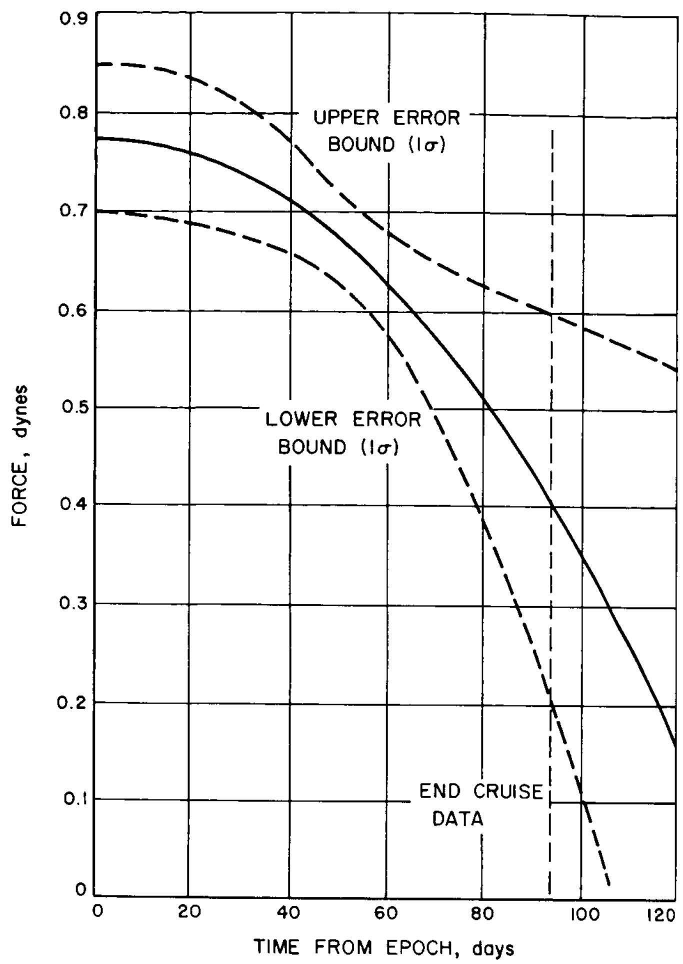

Fig. 1. Low-thrust force from f and α parameters of solution I

but the gravitational constants *GM* and *GE* for the Moon and Earth that are available for correction. Then the mass ratio is given by

$$\mu^{-1} = \frac{GE}{GM} \tag{2}$$

and with the formal uncertainties on *GE* and *GM* from the normal equations given by σ_{GE} and σ_{GM} and the correlation between them by $\rho_{GE\ GM}$, the uncertainty on μ^{-1} can be computed by

$$\left(\frac{\sigma_{\mu^{-1}}}{\mu^{-1}}\right)^2 = \left(\frac{\sigma_{GE}}{GE}\right)^2 - 2\rho_{GE,\,GM}\frac{\sigma_{GE}}{GE}\frac{\sigma_{GM}}{GM} + \left(\frac{\sigma_{GM}}{GM}\right)^2 \tag{3}$$

Two basic solutions are made: the first, Solution A, holds *GE* fixed at its value as determined by the *Ranger* series of probes (Ref. 6), and the second, Solution B includes *GE* in the least-squares solution as a parameter for correction. A solution for both *GE* and *GM* is not possible because the spacecraft is too far from Earth at the beginning of the cruise data. Thus, the constant *GE* is given an *a-priori* uncertainty of ±10 km³/sec² which is more than 10 times larger than the uncertainty for the *Ranger* determination. This *a-priori* uncertainty conditions the normal equations and permits a solution for both *GE* and *GM* within the limits set by the *a-priori* uncertainty.

The results of the two solutions are given in the following table (computation of mass ratio μ^{-1}):

Parameters	Solution A	Solution B
GE (km³ sec⁻²)	398601.27	398598.23
GM (km³ sec⁻²)	4902.8442	4902.8096
σ_{GE}/GE	0	23.882×10^{-6}
σ_{GM}/GM	13.756×10^{-6}	21.634×10^{-6}
$\rho_{GE,\,GM}$	0	0.76235
σ_μ/μ	13.756×10^{-6}	15.831×10^{-6}
μ^{-1}	81.3000 ± 0.0011	81.3000 ± 0.0013

The fact, that the correlation $\rho_{GE,GM}$ in Solution B is of the right sign and magnitude to cancel the large errors in *GE* and *GM* (see Eq. 3) and thus give a result comparable to Solution A, is evidence that the *Mariner* data yield μ and not the gravitational constant itself. Also the equivalence of the value of μ in the two solutions indicates that the determination is not sensitive to reasonable errors in the geocentric gravitational constant *GE*.

D. The Remaining Constants

A number of least-squares solutions, whose purpose is to provide values for the astronomical unit (a.u.) and the mass of Venus with as little systematic error as currently possible, are displayed in Section V-D along with a listing of the data and the residuals associated with the determined constants in Appendix E. All the parameters in the five sets of Section II-A are differentially corrected with the exception of the elements of the Earth's orbit. It was shown in Ref. 2 that a correction of the Earth's ephemeris is not necessary in the case of the *Mariner II* data. A brief description of the various solutions can be given here, but for a more detailed discussion of the motivation and results of each, it is necessary to examine Section V-D.

As a first solution for the mass of Venus, only the encounter data are used and the Lincoln Laboratory value of the a.u. is included in the determination with an *a-priori* uncertainty of ±100 km. Effectively, a mass is determined based on this value of the a.u. Also the position of Venus at encounter is included in this solution.

The remaining solutions for the mass and position of Venus use all the *Mariner II* data. The numerical stability of the solutions is investigated by statistically combining individual solutions from different batches of data and by performing another solution with all the data collected in a single batch. Also results are obtained based on different fixed values of the a.u. as well as with the a.u. in the solution for correction. Solutions with a fixed a.u. are investigated because the a.u. from *Mariner II* can be determined to about ±500 km which is considerably larger than that claimed by Lincoln Laboratories. Thus, although it is important to obtain an independent solution for the a.u. from the *Mariner II* data, it is also important to investigate the solution for the mass and position of Venus when the a.u. from optical and radar bounce data of the planet is held fixed. In this way the parameters that can be determined best from the *Mariner II* data can be given with less uncertainty than when the a.u. is included for correction.

The results of the various solutions demonstrate that the solutions are quite stable for the *Mariner* peculiar constants and that when *a-priori* information on the value of the a.u. is ignored, the resulting solution for it is consistent with the Lincoln Laboratory value, although much more uncertain.

The values for the astronomical constants given at the beginning of this section are a composite of the various solutions of Section V-D. The important conclusion of the solutions is that the values are reasonably free of systematic errors. In addition to these constants other determined parameters from the five sets of Section II-A are of interest. For example, the coordinates of the two Goldstone stations, stations 11 and 12, that tracked *Mariner II* are available as the distance $R \cos \phi'$ of the station from the Earth's axis of rotation and the station longitude east of Greenwich. Because no account is taken of the wandering of the pole during the three months of the *Mariner* data, the coordinates are referred to some mean pole during the time interval of the data. Of course, the coordinates could be referred to a reference pole, but for the purposes of this work it is satisfactory to consider them as referred to the pole of October 1962, since the uncertainty in the coordinates is on the order of a few meters anyhow. The results are:

$$\textit{Station 11: } R\cos\phi' = (5206333.6 \pm 4.2)\ \text{m}$$

$$\lambda = 243^\circ\ 09'\ 02''\!.05 \pm 0''\!.32$$

$$\textit{Station 12: } R\cos\phi' = (5212037.6 \pm 3.8)\ \text{m}$$

$$\lambda = 243^\circ\ 11'\ 39''\!.98 \pm 0''\!.32$$

The determination of the position of Venus at planetary encounter can be converted to a geocentric direction and distance. The direction can be determined to better than ½ sec of arc, which is accurate enough for serious consideration in any attempt to improve the ephemeris of Venus. Also, the distance can be obtained only by radar, and the value from *Mariner II* provides a value which is independent of radar bounce determinations. The coordinates are given as geocentric right ascension α and declination δ in true equatorial coordinates for 1962, December 14, $20^{h}\!.0$ E.T. The distance r is in units of the a.u. based on a velocity of light of 299792.5 km/sec. Again, the fundamental length is light seconds. The coordinates are:

$$\alpha = 14^{h}\ 51^{m}\ 58^{s}\!.282 \pm 0^{s}\!.015$$

$$\delta = -13^\circ\ 39'\ 28''\!.03 \pm 0''\!.4$$

$$r = 0.38640514 \pm 0.97 \times 10^{-6}\ \text{a.u.}$$

III. Astronomical Constants

In any physical theory which purports to represent observed phenomena, certain constants are introduced to assure the compatibility of theoretical predictions with the actual observations. Physical constants, for example the velocity of light c or the gravitational constant G, are determined by constructing experiments which are particularly sensitive to a single constant of interest. On the other hand astronomical constants have, until the advent of space technology, been determined by making observations of natural bodies over a period of many years. Only within the past few years has it become possible to accomplish experiments with artificial satellites and space probes which can determine some astronomical constants more accurately than was possible previously.

Because of the importance of a continual comparison of the theories of celestial mechanics with observation, it is important that a self-consistent set of astronomical constants and ephemerides exist. Thus in astronomy, more than in other physical sciences, there is a natural reluctance to introduce new determinations of the constants into the theories. Of course from the viewpoint of space technology, where the constants and ephemerides are required to achieve precise space navigation, consistency is not as important as making certain that the most current constants and ephemerides are used. In recognizing the validity of both viewpoints, Herrick (Ref. 7) has made the distinction between astronomical and "astrodynamical" values of the constants and ephemeral data. He clarifies this distinction as follows:

> "In the construction of astronomical almanacs and ephemerides, we are interested primarily not in what we may know to be currently the best theories, the best values of constants, or the best ephemeral data, but in a consistent set of theories, constants, and ephemerides that will make possible the use of many decades of observation in the ultimate determination of improved ephemerides. In astrodynamics, on the other hand, we are concerned with both constants and ephemerides that agree with the most recent observational data. For the constants, this concern implies a continual updating; for the ephemerides, it implies the use—anathema to the astronomer—of empirical corrective terms. These should be designed to introduce recent observational data, however, without destroying the dependence of the mean motion, for example, upon long-term astronomical theories and observations."

This quotation summarizes the motivation for undertaking a study of constants as done here. However, before discussing the current set of constants and its implication, we include a brief history of the subject.

Clemence (Ref. 8) suggests that Simon Newcomb, sometime before 1877, was probably the first to recognize the need to systematize the astronomical constants. In 1896 the directors of the principal national ephemerides met in Paris and adopted uniform values of some of the constants. In 1911 they met again and established a cooperative effort in the construction of ephemerides. However, a well-defined system of constants was not yet available. The first definitive work on the entire system of constants was that of de Sitter (Ref. 9) as edited and completed by Brouwer after de Sitter's death in 1934. He chose eight fundamental constants which were mutually independent and gave enough theoretical relationships to allow the evaluation of 23 derived constants from the eight fundamental ones. In addition, linear differential correction formulae were given so that any future correction to a fundamental constant could be propagated easily throughout the whole system of constants. Current discussions of constants still use this method of presentation (see Ref. 7, 8, and 10). De Sitter's fundamental constants were the following:

R_1—the mean radius of the Earth at latitude $\phi = \sin^{-1}\sqrt{1/3}$

g_1—the acceleration of gravity at mean latitude $\phi = \sin^{-1}\sqrt{1/3}$

H—the dynamical flattening, $(C - A)/C$

χ, λ—constants depending on the inner constitution of the Earth

$\pi_\odot$—solar parallax

c—velocity of light

μ^{-1}—reciprocal of Moon's mass in units of the mass of the Earth

The third international meeting on constants occurred in 1950, again in Paris, and among other things recommended the introduction of ephemeris time as the basis of time measurement. With the approval of this recommendation by the International Astronomical Union (IAU) in 1952, the orbital motion of the Earth-Moon system about the Sun replaced the Earth's rotation as the natural clock for the calibration of all time standards.

The most recent meeting to discuss the system of astronomical constants was held in 1963 as IAU Symposium No. XXI and the papers presented there were published in a single volume (Ref. 11). For the first time, representatives from the field of space technology were present and in fact Clemence (Ref. 8, p. 97) states that the immediate incentive for the meeting was the application of space technology to the general subject of astronomical constants. This can be appreciated by reviewing some of the events which had occurred before 1963 and which had contributed to an improvement in the system of constants. First of all, observations of artificial Earth satellites had directly yielded geodetic parameters which determine the external gravitational potential of the Earth. Kaula (Ref. 11, p. 21) prepared a review of these parameters for the 1963 symposium and listed current values of the geocentric gravitational constant GE and a set of harmonic coefficients in the Earth's potential function. The new field of Radar Astronomy had succeeded in recording signals bounced off the Moon and Venus and the observations had been reduced to obtain the mean distance to the Moon (Ref. 11, p. 81) and the a.u. (Ref. 11, pp. 153, 177, and 217). Also, Hamilton informally reported to the 1963 symposium preliminary results from the tracking of *Mariner II* with respect to a determination of the masses of the Moon and Venus.

Because of these recent improvements, a series of eight resolutions was passed at the 1963 symposium which effectively urged the recognition of the importance of radar bounce and space probe observations to the field of astronomical constants. In Resolution 4, a major revision in the selection of fundamental constants was recommended which would reflect the recent advances in the field, and in August-September, 1964 the General Assembly of the IAU approved the resulting system of constants as established by a Working Group appointed by the Executive committee of the IAU. This group consisted of W. Fricke (Chairman), D. Brouwer, J. Kovalevsky, A. A. Mikhailov, and G. A. Wilkins (Secretary). Their entire report is given in Ref. 11, pp. 101–107.

The selection of fundamental constants in 1964 was primarily made on the basis of the direct nature of their determination. Because there remains some degree of arbitrariness in the selection, the Working Group chose to refer to the new set as primary instead of fundamental. Also, they designated two of the constants as "defining constants" in the sense that they were recognized as necessary to define the units used in the theories of Celestial Mechanics. The list follows as a direct quote from the report of the Working Group. Hereafter, these constants will be referred to as the IAU list of constants.

Defining constants

1	Number of ephemeris seconds in 1 tropical year	$s = 31556925.9747$
2	Gaussian gravitational constant defining the a.u.	$k = 0.01720209895$

Primary constants

3	Measure of 1 a.u. in metres	$A = 149600 \times 10^6$
4	Velocity of light in metres per second	$c = 299792.5 \times 10^3$
5	Equatorial radius for Earth in metres	$a_e = 6378160$
6	Dynamical form-factor for Earth	$J_2 = 0.0010827$
7	Geocentric gravitational constant (units: m^3s^{-2})	$GE = 398603 \times 10^9$
8	Ratio of the masses of the Moon and Earth	$\mu = 1/81.30$
9	Sidereal mean motion of Moon in radians per second (1900)	$n^*_{☾} = 2.661699489 \times 10^{-6}$
10	General precession in longitude per tropical century (1900)	$p = 5025''.64$
11	Obliquity of the ecliptic (1900)	$\epsilon = 23° \, 27' \, 08''.26$
12	Constant of nutation (1900)	$N = 9''.210$

In addition to listing the constants, the Working Group also included a set of notes to go along with them. These are not given here because a detailed discussion of the constants follows. However, it is recommended that they be consulted by anyone who is seriously interested in the subject. Likewise, the auxiliary constants and factors and the derived constants are not given here verbatim, but any which are required for the reduction of the *Mariner II* data or which are necessary for a general discussion of constants will be introduced in context as needed.

On the basis of the preceding discussion of the history of the astronomical constants, one receives a definite impression that we are in a period of rapid change with respect to both concepts and numerical values. This is emphasized by the fact that the 1964 system of constants is already out of date from the viewpoint of astrodynamics, where accurate constants are needed for precise space navigation. Of particular significance is a recent measurement* of the light time τ_A associated with one astronomical unit by means of a combination of post-1950 meridian-circle observations of Mercury, Venus, and the Sun with radar measurements of Mercury and Venus. The result is claimed accurate to 5×10^{-6} light-sec or to a relative accuracy of 0.01×10^{-6}, which makes τ_A one of the best determined constants in the solar system. Certainly, it is known more accurately than the velocity of light c whose relative uncertainty is about 2×10^{-6}. Thus, it is clear that τ_A should be listed as a primary constant in place of the constant A which, as a derived constant, would be given by $c \, \tau_A$. Other improvements in the IAU constants have occurred through a determination of GE from the *Ranger* series of space probes (Ref. 6) and through the direct determination of μ from the present analysis of the *Mariner II* data. Also

*Shapiro, Irwin I., Lincoln Laboratories, private communication.

the planetary masses have been improved by the determination of the mass of Venus here and by a preliminary determination of the mass of Mars from *Mariner IV* and the mass of Mercury from observations of the minor planet, Eros. In Table 1 the "best" set of constants available at this writing (January 1967) is listed along with its sources, but it is quite likely that further modifications will be required within a few months—a reasonable time interval considering the progress being made.

The masses of the outer planets are based on recommendations of Clemence from his study of the subject.

Before beginning a discussion of the constants, it is important to realize that there are three basic problems in celestial mechanics which necessitate a system of constants in the first place. These are:

1. The problem of describing the heliocentric motions of the planets (planetary theory).
2. The description of the geocentric motion of the Moon (lunar theory).
3. The description of the size, shape and orientation of the Earth in space.

The third problem is necessary because astronomical observations are made from the environment of the Earth, and the location of the observer in space is important for an accurate representation of observations. Also a description of the Earth's gravity field is important to the study of the motions of the Moon, artificial Earth satellites, and space probes, and so certain principles from the field of Geodesy must be introduced in the discussion of this third problem. However, only concepts needed to explain the IAU constants and their astronomical implications will be introduced here. This is because we are interested in the Earth only as an observational reference system and make a distinction between constants necessary to describe this reference and constants necessary for a geodetic study of the Earth.

In the next three sections, each of the basic problems given in the foregoing will be taken up, in turn, as they pertain to the system of constants. In this way it is hoped that some motivation for the selection of constants will be provided and consequently that the entire system will be more easily understood.

A. Planetary Motions

Let $\mathbf{r}_i$ be the heliocentric position vector of a planet of mass m_i. Then the differential equations of motion

Table 1. Values for astronomical constants as of January, 1967

Constant	Value	Source
τ_A	$499.004785 \pm 5 \times 10^{-6}$ sec	Lincoln Lab.
c	299792.5 ± 0.5 km/sec	IAU (1964)
A	149597892 ± 250 km	$c\,\tau_A$
a_e	6378.160 ± 0.080 km	IAU
J_2	$0.0010827 \pm 0.3 \times 10^{-6}$	IAU
GE	398601.3 ± 0.8 km^3/sec^2	*Ranger* probes (Ref. 6)
μ^{-1}	81.3001 ± 0.0013	*Mariner II*
$n^*_{☾}$	$2.661699489 \times 10^{-6} \pm 5 \times 10^{-16}$ sec^{-1}	IAU
p	$5026''.39 \pm 0''.2$	Morgan and Oort (Ref. 12)
ϵ	$23° \, 27' \, 08''.26 \pm 0''.1$	IAU
N	$9''.210 \pm 0''.01$	IAU
Auxiliary Constants		
Constant	**Value**	**Source**
Solar parallax, $\pi_\odot$	$8''.794174 \pm 0''.00011$	arc sin (a_e/A)
Gravitational constant, G	$(6.673 \pm 0.003) \times 10^{-23}$ km^3 sec^{-2} gm^{-1}	Heyl (Ref. 13)
Heliocentric gravitational constant, GS	$(132712.50 \pm 0.66) \times 10^{-6}$ km^3 sec^{-2}	$k^2 A^3$
Mass of Sun, S	$(1.9888 \pm 0.0009) \times 10^{33}$ gm	GS/G
Ratio of masses of Sun and Earth	332945.5 ± 1.8	GS/GE
Ratio of masses of Sun and Earth + Moon	328900.0 ± 1.8	$GS/GE \, (1 + \mu)$
Mass Ratios of Sun to Planets		
Planet	**Value**	**Source**
Mercury	6005000 ± 18000	Rabe and Francis*
Venus	408505 ± 6	*Mariner II*
Mars	3098600 ± 600	Null**
Jupiter	1047.44 ± 0.02	Clemence (Ref. 8)
Saturn	3499.1 ± 0.4	Clemence (Ref. 8)
Uranus	22930 ± 6	Clemence (Ref. 8)
Neptune	19070 ± 21	Clemence (Ref. 8)
Pluto	400000 ± 40000	Clemence (Ref. 8)

*Private communication; value is preliminary and is from 1966 reduction of normal places for minor planet Eros.

**Private communication; value is from reduction of Doppler tracking data from *Mariner IV*.

that describe the motions of the system of 9 planets are

$$\frac{d^2\mathbf{r}_i}{dt^2} = -k^2(1+m_i)\frac{\mathbf{r}_i}{r_i^3} + k^2\sum_{j=1}^{9*} m_j\left(\frac{\mathbf{r}_{ij}}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_j^3}\right)$$

$$i = 1,2,\cdots,9 \qquad (4)$$

where the unit of mass is the mass of the Sun, the vector $\mathbf{r}_{ij}$ is equal to $\mathbf{r}_j - \mathbf{r}_i$ and the asterisk on the summation sign means that the case where $j = i$ is excluded from the sum.

These equations will represent the motions of the planets, within the accuracy of current observations, with the exception of two small effects. The first is a small correction to the perihelion of Mercury because of relativistic perturbations, and the second is a perturbation in the heliocentric motion of the Earth–Moon system because of the motions of both the Earth and Moon about their center of mass or barycenter. However, for a consideration of constants required to represent the planetary motions, Eq. (4) is quite satisfactory, especially if $\mathbf{r}_i$ for the Earth is taken as the position of the Earth–Moon barycenter. As for the actual solutions of the equations, they are exceedingly complicated and will not be discussed here. For the inner planets, solutions are obtained by forming approximate analytical solutions by the methods of general perturbations. The solution for the five outer planets has been obtained by numerically differentiating the equations directly (Ref. 14).

The important point with respect to Eq. (4) is that the only constants required to specify planetary motions are the masses m_i and the constant of proportionality k. However, in compiling a system of constants the standard procedure is to list the planetary masses in a separate table and not to include them in the list of primary constants. Therefore, the discussion of heliocentric constants is limited to a specification of k. Of course its value will depend on the units chosen for length and time and so it is important to define these units precisely. Note that the unit of mass has already been specified as the mass of the Sun. As the result of a precise definition of units, the analysis given in this section will also involve the constants s, c and A. Thus, the two defining constants in the IAU list and two of the ten primary constants are specified by a consideration of planetary motions.

*1. **Unit of length.*** The basis for the value assigned to k is the determination of Gauss (Ref. 15) in 1809. He chose as a unit of length the mean heliocentric distance of the Earth from the Sun and the unit of time was selected as the mean solar day. Then he evaluated k by means of Kepler's third law which would follow directly from the integration of Eq. (4) for the Earth-Moon system if the perturbation by the other planets were neglected. Even with these perturbations, Kepler's third law still holds on the average, and today certain systematic effects of the other planets could be included in it by using the results of the analytical theory of the Earth's motion. However, Gauss used the third law in its two-body form as follows:

$$P_\oplus^2 = \frac{(2\pi)^2 a_\oplus^3}{k^2(1 + m_\oplus)} \qquad (5)$$

Actually it is better to use the mass of the Earth-Moon system in Eq. (5), rather than just the mass of the Earth, because it is the Earth-Moon barycenter that most nearly follows two-body motion. However, Herrick states (Ref. 7, p. 28) that it is not clear whether Gauss made the distinction. At any rate we are interested only in the numerical values assigned to the period $P_\oplus$ of the Earth's revolution and to the mass $m_\oplus$. Since Gauss took $a_\oplus$ as his unit of length, it is unity by definition. His values for $P_\oplus$ and $m_\oplus$ are

$$P_\oplus = 365.256,3835 \text{ days}$$

$$m_\oplus = 1/354,710$$

and the resulting value of k, the Gaussian gravitational constant, is

$$k = 0.017,202,09895$$

which agrees with the value in the IAU list.

Now the values for both $P_\oplus$ and $m_\oplus$ have been improved since Gauss evaluated k in 1809, and if his units were retained, it would be necessary to compute an improved value of k every time new information was obtained on the period and mass of the Earth-Moon system. Instead of this procedure, the IAU in 1938 decided to adopt k as a fixed constant at the value given by Gauss and in the process abandon the Earth's mean distance as the unit of length. In effect, the fixing of k defined the unit of length basic to computations in celestial mechanics. Thus the IAU now calls k a defining constant in that it serves to define the astronomical unit of length (a.u.). Note that it has units of a.u.$^{3/2}$ day^{-1} (solar mass)$^{-1/2}$.

*2. **The unit of time.*** Now the units of mass and length have been completely specified and all that remains is to select some unit of time. The use of the day has not been carefully defined up to this point. It is clear that any unit of time must be based on some natural time interval which can be observed either directly or more realistically, through a highly accurate theory with observations spread over many decades. The first natural unit of time selected for astronomical work was the period of rotation of the Earth. However, in recent times the non-uniform rate of the Earth's rotation has been recognized, and so a simple statement about the period of rotation as the unit of time is insufficient without also giving the epoch associated with that period. However, an accurate theory for the rotation of the Earth does not exist, and even if one chose a rotation rate at some epoch, it would be impossible to relate this natural interval of time to observations made at times separated from that epoch. The calibration of a clock in terms of the Earth's rotation rate at one time cannot be related through theory to some defining rotation rate an another time.

Therefore, the natural standard of time used in astronomy since 1960 is not the rotation of the Earth but the revolution of the Earth-Moon system about the Sun. In this way, time calibration is based on the theories of celestial mechanics rather than on theories of the rotation of the Earth. More precisely, the standard for time measurements is the tropical year which is defined as the interval between successive crossings of the equator by the Sun as it goes from the southern to northern hemisphere. By observing the Sun, either directly or indirectly through other bodies in the solar system, it is possible to empirically determine the length of the tropical year in terms of some arbitrary clock units used for timekeeping (e.g., the resonance frequency of the cesium atom). However, this length is not a constant for two reasons. The first is that the period of the Earth's orbit is not constant because of perturbations by the other planets. The second reason is that the point of crossing, the vernal equinox, is not fixed with respect to the stars because the Earth is undergoing precession and nutation as it rotates in space. Therefore, the reference for time is arbitrarily chosen as the instantaneous tropical year at the beginning of 1900. The instantaneous tropical year is derived from the angular rate of the mean Sun on Jan. 0, 1900, 12^h0. Once the tropical year (1900) has been adopted as the basic unit of time, as the a.u. was the basic unit of length, then other more useful time intervals, for example the ephemeris second or ephemeris day of precisely 86400 sec, can be defined as some fraction of the tropical year (1900). This is exactly what the IAU has done in adopting a value for s, the number of ephemeris seconds in 1 tropical year. Actually, this number of 31,556,925.9747 sec had been adopted earlier by the Comité International des Poids et Mesures in 1957 (Ref. 16). Again, it is because the theories of celestial mechanics are so accurate that it is possible to determine the length of the tropical year (1900) by making observations several decades later. At present this interval is determined in units of one cycle of oscillation of cesium for zero magnetic field. With the definition of the ephemeris second given by the constant s, the number of cycles in one ephemeris second is 9,192,631,770 $\pm$ 20 (Ref. 17) which represents the standard for all timekeeping. In Section IV-C-3 we will delve into the observational aspects of timekeeping in more detail. For the present we are concerned only in the implications with respect to the IAU constants.

Commission 4 of the IAU is currently considering the fundamental unit of time. It seems likely that at a future date the basic unit will be defined in terms of an atomic frequency instead of as an astronomical frequency, the revolution of the Earth-Moon system about the Sun.

3. The constants A and c. If we were concerned only with theory, there would be no need for the velocity of light c. However the primary purpose in constructing theories of planetary and lunar motions is to compare the implications of the theories with actual observations. Thus the speed of propagation of the electromagnetic signal used by the observer is essential for an accurate representation of his data. The importance of c to the representation of Doppler tracking data is clear from the derivation of Section IV-B. However, as far as the astronomer is concerned, since the units of length and time are clearly defined, he is really only interested in c in units of a.u. per ephemeris sec. Therefore, there is a strong argument to include c (a.u./sec) or its inverse τ_A (sec/a.u.) as a primary constant. The recent direct measurement of τ_A by Ash and Shapiro, to almost 10 significant figures emphasizes this argument. However, the IAU has chosen to do otherwise and their alternative of making τ_A a derived constant will be clarified in a moment.

Still, suppose that τ_A were selected as a primary constant. Then, is anything else required to represent optical and radio observations of solar system objects? The answer is in the negative, unless one insists on using some laboratory unit as an alternative unit of length to the a.u. In fact, an inspection of the IAU constants reveals that the meter is used as the common unit of length. Therefore, it is necessary to have a clear definition of this admittedly extraneous unit of length. The simplest approach would be to do as in the case of the definition of the second and adopt some defining constant as the number of meters in one a.u. Clearly, this is analogous to adopting s, the number of ephemeris seconds in one tropical year, and of course it is perfectly satisfactory within the framework of astronomy; but such a definition of the meter would come into conflict with that used by the physicist. Before 1960 the meter was defined as the distance between two lines engraved on a platinum-iridium bar, the International Prototype Meter; but since 1960 the meter has been defined as exactly 1,650,763.73 wavelengths of the orange-red line in the spectrum of Krypton 86, the unperturbed transition between the levels $2p_{10}$ and $5d_5$. It is necessary to be consistent with this definition whenever the meter is used as the unit length, and for this reason the IAU has selected c (meters/sec) as a primary constant. There is significance in the fact that the velocity of light in laboratory units is the only primary constant that also qualifies as a physical constant measured entirely by laboratory equipment. In effect, c (meters/sec) allows an expression of astronomical results in units of the meter as defined by the orange-red Krypton 86 line.

As pointed out before, the constant τ_A, the number of light-seconds in one a.u., is basic to the representation of astronomical observations. Now with the addition of the constant c (meters/sec) the number of standard meters A in one a.u. can be computed by the formula

$$A = c\,\tau_A \tag{6}$$

Clearly, any two of the constants in Eq. (6) can be called primary and the third will automatically be classified as a derived constant. The IAU has chosen A and c as the two primary constants, even though τ_A is more fundamental to astronomical observations.

The determinations of the constant c have been summarized by Herrick (Ref. 7, p. 110–114), and it appears that Froome's value as obtained with a microwave interferometer (Ref. 18) is the most reliable. The IAU has followed the 1963 recommendation of the International Union of Pure and Applied Physics in adopting a value for c which is essentially Froome's value.

B. Lunar Motion

The constants introduced in the previous section are sufficient to describe planetary motions in units of the meter and the ephemeris second if the masses of the planets are given in solar mass units. Now the additional constants (GE, μ and $n^*_{\text{☾}}$) necessary to represent the motion of the Moon are explained in terms of that motion.

Although theories of the Moon's motion rely on complicated transformations of coordinates and approximate solutions to the equations of motion through the methods of general perturbations, it is still valid to discuss the problem in terms of a much simpler formulation in inertial cartesian coordinates. Then, an insight into the parameters which influence the lunar ephemeris can be gained by considering this more basic form of the equations of motion.

With respect to the nature of the motion, it is sufficient for this discussion to restrict the equations to the solar terms only. Because the lunar ephemeris is expressed in geocentric coordinates, all that is involved is the relative motion form of the three-body equations of motion with the Sun as the perturbing body. The solution of this three body system is referred to as the main problem in lunar theory and additional perturbations caused by planetary and oblateness effects are treated separately in the theories.

The three-body equations of motion are given in units of meters and seconds by

$$\frac{d^2\mathbf{r}}{dt^2} = -\,G(E+M)\frac{\mathbf{r}}{r^3} + GS\left(\frac{\mathbf{r}_{☾\oplus}}{r^3_{☾\oplus}} - \frac{\mathbf{r}'_1}{r'^3_1}\right) \tag{7}$$

where G is the universal gravitational constant (m^3 sec^{-2} gm^{-1}) and E, M and S represent the masses of the Earth, Moon and Sun, respectively in grams. The position vectors $\mathbf{r}_{☾\oplus}$ and $\mathbf{r}'_1$, represent the selenocentric and geocentric coodinates of the Sun, and $\mathbf{r}$ is the geocentric position of the Moon. As in planetary theory, the masses always occur with G and are combined with it into a single constant. Thus, the proportionality constant in the two body term of Eq. (7) can be expressed in terms of the IAU constants GE and $\mu = M/E$.

$$G(E + M) = GE(1 + \mu) \tag{8}$$

To obtain GS, all that is required is to convert k^2(a.u.^{3} day^{-2}) to units of m^3 sec^{-2} with the constant A and the number of seconds in a day (86400 sec day^{-1}).

$$GS = \frac{k^2 A^3}{(86400)^2} \tag{9}$$

Also the vectors $\mathbf{r}_{☾\oplus}$ and $\mathbf{r}'_1$ can be expressed in terms of $\mathbf{r}$ and the barycentric coordinates $\mathbf{r}'$ of the Sun which are available from the theory for the solar ephemeris.

$$\mathbf{r}_{☾⊕} = \mathbf{r}' - \frac{1}{1+\mu}\mathbf{r} \tag{10}$$

$$\mathbf{r}'_1 = \mathbf{r}' + \frac{\mu}{1+\mu}\mathbf{r} \tag{11}$$

The purpose of writing the expressions of Eq. (7) through (11) is to show that the equations of motion for the Moon depend only on GE, μ, A, k^2 and the solar ephemeris $\mathbf{r}'$. Thus, the solution of the equations will also depend on these parameters and in addition will require six arbitrary constants of the motion. The introduction of planetary perturbations will not add any other constants besides the masses of the planets in solar mass units, and the small oblateness effects can be handled with the parameters J_2 and a_e discussed in the next section.

The sidereal mean motion $n^*_☾$ of the Moon has not been required to specify the lunar motion by the preceding arguments. Therefore, there remains a question as to why it is included as a primary constant in the IAU list. The reason is that it can be measured to ten significant figures and is at least two orders of magnitude more accurate than the lunar ephemeris itself. Thus, the standard procedure is to remove one degree of freedom in the selection of arbitrary constants for the motion, and instead to apply a constraint to the ephemeris such that the mean motion (1900) is a constant given by $n^*_☾$.

In practice the invariability of $n^*_☾$ in the lunar ephemeris can be assured by choosing the Moon's geocentric mean distance $a_☾$, derived from Kepler's third law, as a parameter in the ephemeris. The third law for lunar motion is given by

$$a_☾ = F_2 \left[\frac{GE(1+\mu)}{n^{*2}_☾}\right]^{1/3} \tag{12}$$

where F_2 is a factor which comes from lunar theory and which accounts for solar perturbations on the lunar motion. The IAU designates $a_☾$ as a derived constant and gives F_2, the value 0.999093142 (Ref. 8, p. 102).

C. Earth Constants

In the preceding two sections, the IAU constants required to construct planetary and lunar ephemerides have been introduced, and it has been shown that no additional constants are required for this purpose. Now the remaining constants which are needed to describe the location of the observer in space are taken up in two groups. The first group consists of the parameters a_e and J_2 which are intended to roughly describe the shape and size of the Earth and to serve as a basis for a detailed description of its gravity field. The second group consists of the last three primary constants in the IAU list and can be interpreted as specifying the orientation of the Earth with respect to the coordinate systems of the lunar, solar, and planetary ephemerides.

1. The constants a_e and J_2. For use in Celestial Mechanics, the constants a_e and J_2 are best understood in terms of the Earth's potential function U which is usually expressed as an infinite series in spherical harmonic functions. With the notation recommended by the IAU Commission No. 7 on Celestial Mechanics (Ref. 19), the function U is written

$$U = \frac{GE}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_n^m(\sin\beta)\,(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\right] \tag{13}$$

where r is the geocentric distance, P_n^m is the associated Legendre polynomial, β is the latitude and λ is the longitude. The coefficient J_2 is representative of an alternative notation where $J_n = -C_{n,o}$ and for the case where the expansion is in terms of only the zonal harmonics ($m=0$), the potential is usually written in terms of J_n as follows:

$$U = \frac{GE}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_e}{r}\right)^n P_n(\sin\beta)\right] \tag{14}$$

The coefficient J_1 is eliminated by taking the origin of coordinates at the center of mass of the Earth.

Both expansions for U reveal that the Earth's mean equatorial radius a_e is nothing more than a convenient scale factor in the description of the gravity field, and if one chose to define a new harmonic coefficient equal to $J_n a_e^n$, then the constant a_e would not occur at all. Thus, if we were only interested in defining U, there would be no real concern that the constant a_e be an accurate representation of the actual mean equatorial radius. In fact the only significant usefulness in a reasonably accurate value for a_e, at least for astronomical work, is to provide a scale for a reference ellipsoid which will permit a first approximation to the geocentric coordinates of observatories and tracking stations. Thus, the IAU defines a_e in terms of an actual mean radius, but as the equatorial radius of an ellipsoid of revolution that approximates the geoid.

The other geodetic constant J_2 is called the dynamical form factor for the Earth and is simply the coefficient in Eq. (11) applied to the Earth. It is called dynamical because it occurs in the exterior potential and is determined from the motions of artificial Earth satellites. Also, it is possible to derive the shape of an ellipsoid of revolution by assuming an equal potential surface with parameters GE, a_e and J_2. Thus, J_2 is a form factor in a sense, although the derived flattening of such an ellipsoid will not necessarily agree with a geometrically determined flattening for the reference ellipsoid approximating the geoid. Of course even for the dynamically determined ellipsoid, the associated potential involving GE, a_e and J_2 is insufficient to represent the motions of artificial Earth satellites. What is needed is enough terms beyond J_2 to adequately represent the actual exterior potential of the Earth. In this sense, the additional harmonic coefficients are much like the planetary masses needed to define the potential field of the solar system and in a similar fashion are listed separately from the primary and derived constants. Although the IAU does not include a listing of the coefficients in their system of constants, such lists have been compiled. For example Anderle (Ref. 20) has determined all the coefficients through the sixth order from Doppler observations of satellites and includes the seventh zonal coefficient as well.

2. Constants of precession and nutation. The three constants which describe the orientation of the Earth in space are the precessional constant p, the obliquity of the ecliptic ϵ and the constant of nutation N, all three given at the epoch (1900). It is not immediately obvious why only three constants are needed to describe the orientation and motion of a coordinate system fixed in the Earth when the dynamics of rigid body motion generally results in six arbitrary constants, for example, three Eulerian angles and their rates at some epoch t_o. The simplification for the Earth occurs because two of the principal moments of inertia are assumed equal. This removes one degree of freedom. A second is removed because the fixed inertial coordinate system is taken coincident with the actual Earth-fixed coordinate system at the epoch. The third degree of freedom is the rotation rate of the Earth, which is included in an observational sense in the procedures of timekeeping (see Section IV-C-3).

The three constants determined from observation are p, the constant of precession, which is the speed of the general precession in celestial longitude, the obliquity of the ecliptic ϵ which is the angle between the poles of the ecliptic and equator, and the constant of nutation N which is the amplitude of the principal term in the nutation in celestial longitude. This term is produced by the periodic motion of the Moon's node on the ecliptic and has the same period, approximately 18⅔ years. The incorporation of these three observationally determined constants into a description of the orientation of the Earth in space is given by formulae in Ref. 21, pp. 28–31. The details are not reproduced here.

It is interesting to note that in the IAU list of constants, only the constants p, ϵ and N are not changed from the values used by Newcomb, even though Morgan and Oort (Ref. 12) have determined p to be about $0''.8$ larger than the IAU value. Clemence (Ref. 8, p. 100) discusses the reasons for not changing p. They have to do with the reduction of star positions to a common epoch for the measurement of proper motions. Comparison of recent astrometric observations of a star with those made some 50 years ago is greatly simplified if both observations were reduced with a common value of p, even if it is not the best available. Of course this sort of argument is just an extreme case of the general situation for the constants; for astronomical purposes, they should not be changed at frequent intervals.

D. Effect of the Constants on Computational Procedures

The differential correction of the astronomical constants is formulated within the framework approved by the International Astronomical Union (IAU) in 1964 (Ref. 8) as described in Section III. There are, therefore, two constants which are absolutely not subject to correction. These are s, the number of ephemeris seconds in 1 tropical year (1900), and k, the Gaussian gravitational constant. Their values from Section III are

$$s = 31556925.9747 \text{ sec}$$

$$k = 0.01720209895 \text{ a.u.}^{3/2} \text{ (day)}^{-1} \text{ (solar mass)}^{-1}$$

The constants s and k define the fundamental units used in the solutions for the constants with the conversion between units of ephemeris days and seconds given by

$$1 \text{ day} = 86400 \text{ sec}$$

The unit of mass is the solar mass and the unit of length is the astronomical unit (a.u.). The IAU list of primary constants includes the velocity of light c in meters per sec. For the purposes of determining constants from tracking data, it is proper to consider c as a fixed constant which defines the meter as a secondary unit of length. Therefore, the IAU value is treated here as a defining constant along with s and k.

$$c = 299792.5 \text{ km/sec}$$

Whenever laboratory units are used in this and other sections, they are always kilometers, grams, and ephemeris seconds. However, when a value A of the a.u. in kilometers is determined from the data, it should be understood that this is only a convenient way of expressing the radar measurement of τ_A, the number of light seconds in one a.u. The value of A is directly proportional to the adopted value of c through the relation $A = c\ \tau_A$ (Eq. 6) where τ_A is the constant actually determinable from the radar data.

Within the computer program constructed for the reduction of the data, it is possible to correct the following IAU constants to satisfy a least-squares fit to the observations. The values of the constants are those adopted by the IAU.

$A = 149.600 \times 10^6$ km (astronomical unit, a.u.)

$J_2 = 0.0010827$ (dynamical form-factor for Earth)

$GE = 398603$ km^3/sec^2 (geocentric gravitational constant)

$\mu = 1/81.30$ (Moon-to-Earth mass ratio)

The *Mariner II* data are insensitive to reasonable corrections to the constant J_2. The use of the other three constants in the differential correction procedure is described in Section V. The other primary constants are held fixed at the values given below.

$a_e = 6378.166$ km (Earth's mean equatorial radius)

$n^*_{☾} = 2.661699489 \times 10^{-6}$ sec^{-1} (sidereal mean motion of Moon)

$p = 5025''.64$ (precessional constant, 1900)

$\epsilon = 23° 27' 08''.26$ (obliquity of ecliptic, 1900)

$N = 9''.210$ (constant of nutation, 1900)

The value of a_e differs from that of 6378.160 km given by the IAU, but it has been adopted by NASA for their trajectory calculations (Ref. 22). The difference of 6 m is insignificant in comparison with the IAU limits (6378.080 to 6378.240 km) on the true value of a_e (Ref. 8). The value $n^*_{☾}$, as given, is assured by the use of Brown's lunar theory as a basis for the JPL lunar ephemeris (Ref. 23) which has been used in this work. That the three constants p, ϵ and N take on the IAU values can be verified by comparing the formulas for precession and nutation in the JPL trajectory program (Ref. 24) with those in the explanatory supplement to the ephemeris (Ref. 21). All computations of the *Mariner II* orbit are accomplished with this trajectory program.

Not all of the derived constants in the IAU list are of interest in the reduction of the *Mariner II* data. The fundamental importance of τ_A has already been mentioned. Similarly, the Doppler data from a probe gravitationally dominated by the Sun are capable of yielding a value for the constant of the lunar inequality L. This results from a measurement of the mean linear velocity V_L of the Earth about the center of mass or barycenter of the Earth-Moon system. In terms of other astronomical constants, V_L is given by

$$V_L = L\, n^*_{☾} \text{ a.u./sec} \quad (15)$$

where

$$L = \frac{\mu}{1+\mu}\,\frac{a_{☾}}{A} \quad (16)$$

Thus because $n^*_{☾}$ is so accurate, on the order of 10 significant figures, the constant L is measured directly and μ is derived from Eq. (16). However, the mean lunar distance $a_{☾}$ is itself a derived constant and another relationship for $a_{☾}$ is required before L can be expressed as a function of μ and other primary constants. This is (Ref. 8, p. 102)

$$a_{☾} = F_2 \left[\frac{GE(1+\mu)}{n^{*2}_{☾}} \right]^{1/3} \quad (17)$$

where

$$F_2 = 0.999093142$$

As far as the actual computations of the *Mariner II* orbit are concerned, an adjustment of the derived constant $a_☾$ is accomplished by scaling the JPL lunar ephemeris by a constant R_{em} rather than by the equatorial radius a_e. This is done for computational convenience only. The JPL ephemeris is constructed by evaluating the Brown Improved Lunar Theory (Ref. 25) and then by converting the results to rectangular coordinates (Ref. 23). Therefore, by applying a conversion factor R_{em} to the rectangular coordinates, the entire ephemeris is converted to kilometers. An alternative procedure would be to multiply the terms in the sine parallax by a constant to produce the derived value of $\sin \pi = a_e/a_☾$ and then to apply the constant a_e to convert the ephemeris to kilometers. Instead the value of the mean sine parallax ($\sin \pi_☾ = 3422''.54$) adopted in the Brown Improved Lunar Theory is left unchanged and R_{em} is computed for a given mean distance $a_☾$ by the formula

$$R_{em} = \left(\frac{3422.54}{206264.806}\right) a_☾ \qquad (18)$$

For the IAU constants, the derived value of R_{em} is

$$R_{em} = \left(\frac{3422.54}{206264.806}\right)(384400) = 6378.327 \text{ km}$$

which is also the value recommended by JPL (Ref. 23, p. 2).

The constant of the parallactic inequality $P_☾$, which is also given as a derived constant by the IAU, is not applied to the JPL lunar ephemeris. Thus the longitude of the Moon does not differ from the given in the Improved Lunar Ephemeris (ILE) because it is the ILE tables (Ref. 25, Table 3) which define the JPL ephemeris. The constant $P_☾$ (term 21 in the ILE listing) has been changed from $-124''.785$ as given by Brown (Ref. 26, Section 266) to $-125''.154$ in the ILE of 1954. The IAU value of $-124''.986$ is probably accurate to at least 5 significant figures, so the longitude of the Moon in the JPL ephemeris is in error by about $0''.15$, or, equivalently, about 280 meters. Of course this figure represents the amplitude of a periodic term with a period of about 29.53 days, or, more precisely, the synodic month. A 280 meter periodic error in the Moon's longitude does not affect the *Mariner II* data except for a negligible contribution to the determination of the lunar inequality. The determination of μ is based on an average linear velocity of the Earth about the Earth-Moon barycenter, where the average occurs over the period of a month. Therefore, a small periodic error of the same period should essentially be averaged out. Also any gravitational attraction of the Earth and Moon as separate bodies is extremely small because all useful *Mariner II* data occur after eight days from injection into the Earth-Venus transfer orbit. At the time of the first useful Doppler observation, the solar attraction is already 85 times the attraction of the Earth-Moon system.

The recent work of W. J. Eckert on a further improvement in the lunar ephemeris would eliminate many errors present in the ephemeris used here. However, the effect of using Eckert's improved ephemeris should be negligible as far as the *Mariner II* data are concerned.

Another derived constant of importance to the *Mariner II* reduction is the heliocentric gravitational constant $GS(\text{km}^3/\text{sec}^2)$ which enters into the calculation of the *Mariner II* trajectory. In Section III-D, the use of laboratory units in the formulation of the equations of motion is explored in detail. For the purposes of this discussion, it is sufficient to simply recognize that GS is related to the Gaussian gravitational constant k and the astronomical unit A as follows:

$$GS = (86400)^{-2} k^2 A^3 \qquad (19)$$

Again for the IAU value of A, the derived value of GS is

$$GS = 1.327{,}181{,}07 \times 10^{11} \text{ km}^3/\text{s}^2$$

Values for the planetary masses are also required in the computation of the *Mariner II* orbit. The JPL trajectory program used here assumes the following ratios of the Sun's mass to that of the planet.

Mercury	6110000	Saturn	3499.1
Venus	408539	Uranus	22930
Mars	3098600	Neptune	19070
Jupiter	1047.44	Pluto	400000

The values are in agreement with recommendations of Clemence (Ref. 8) except for Mars where preliminary reductions of the *Mariner IV* data have been taken into account.*

E. Basis for Corrections to the Constants

The importance of the *Mariner II* data to the system of astronomical constants lies in an independent determination of A, μ and the mass M_v^s of Venus in units of the Sun's mass. For purposes of relating this determination to other work in the field, the framework for astrodynamic constants analysis established by Herrick (Ref. 7) is used.

In this system, for example, the conversion of a.u. length units to kilometers is

$$\tilde{A} = A(1 + \hat{A}) \text{ km/a.u.} \qquad (20)$$

where $\tilde{A}$ represents some standard value, in particular the IAU 1964 value, and $\hat{A}$ is a relative correction term determined from recent observational data. Thus, $\hat{A}$ will be subject to error. Herrick gives for $\hat{A}$ the value $(-7 \pm 13) \times 10^{-6}$ although he has recently raised this number to reflect the Ash and Shapiro value of the astronomical unit. One of the advantages of this method of handling the constants is that corrections like $\hat{A}$ are dimensionless. All corrections and uncertainties are in this way expressed on a relative or percentage basis, and various constants can be compared immediately as to their relative accuracies. The value of $\pm 13 \times 10^{-6}$ associated with $\hat{A}$ indicates that A is good to about 4.8 significant figures. For a further discussion of $\hat{A}$, see Section V-D.

Similarly, the other two constants determinable from the *Mariner II* data can be expressed in the forms

$$\mu = \tilde{\mu}(1 + \hat{\mu}) \qquad (21)$$

$$M_V^S = \tilde{M}_V^S(1 + \hat{M}_V^S) \qquad (22)$$

Because of the way in which the computer program is organized to solve for the lunar inequality, the constant obtained from the least squares solution is not μ but the selenocentric gravitational constant $k_{gm}^2 = GM(\text{km}^3/\text{sec}^2)$.

$$k_{gm}^2 = \tilde{k}_{gm}^2 (1 + 2\hat{k}_{gm}) \qquad (23)$$

The numerical values associated with Eq. (20), (21), (22), and (23), before the solutions of Section V are introduced, are given by

$$\tilde{A} = 149.6 \times 10^6 \text{ km} \qquad \hat{A} = (-7 \pm 13) \times 10^{-6}$$

$$\tilde{M}_V^S = 1/408539 \qquad \hat{M}_V^S = (0 \pm 250) \times 10^{-6}$$

$$\tilde{\mu} = 1/81.30 \qquad \hat{\mu} = (0 \pm 120) \times 10^{-6}$$

$$\tilde{k}_{gm}^2 = 4902.87 \qquad 2\hat{k}_{gm} = (-5.3 \pm 120) \times 10^{-6}$$

The value of $\tilde{k}_{gm}^2$ is obtained from μ and the geocentric gravitational constant ($k_{ge}^2 = GE$) according to the formulas

$$k_{gm}^2 = \mu k_{ge}^2 \qquad (24)$$

and

$$2\hat{k}_{gm} = \hat{\mu} + 2\hat{k}_{ge} \qquad (25)$$

The formula for k_{ge}^2 is

$$k_{ge}^2 = \tilde{k}_{ge}^2 (1 + 2\hat{k}_{ge}) \qquad (26)$$

The adopted $\tilde{k}_{ge}$ is the IAU 1964 value and $\hat{k}_{ge}$ is based on the determinations of k_{ge} from *Ranger* space probes to the Moon (Ref. 6).

$$\tilde{k}_{ge}^2 = 398603 \text{ km}^3/\text{sec}^2$$

$$2\hat{k}_{ge} = (-5.3 \pm 2.0) \times 10^{-6}$$

We now collect the necessary formulas required for an analysis of the effect of the determinations of $\hat{A}$, $\hat{M}_V^S$ and $\tilde{k}_{gm}^2$ on the system of astronomical constants. In all of this, it is important to remember that a value for the speed of propagation c was adopted ($c = 299792.5$ km/s) and when results are given in units of the km, the basic unit is the light-second instead. The constant c is simply an agreed upon conversion factor.

*Private communication with G. W. Null, Jet Propulsion Laboratory, Pasadena, California.

A summary of this section, in the form of formulae for future reference, is given in the following list. (Cf. Ref. 7)

$$\tau_A = \hat{A} - 0.037 \times 10^{-6} \tag{27}$$

$$2\hat{k}_{gs} = 3\hat{A} + 0.81 \times 10^{-6} \tag{28}$$

$$\hat{\mu} = 2\hat{k}_{gm} - 2\hat{k}_{ge} \tag{29}$$

$$a_{☾} = \frac{2}{3}\hat{k}_{ge} - \frac{2}{3}\hat{n}^*_{☾} + \frac{1}{3}\frac{\mu}{1+\mu}\hat{\mu} \tag{30}$$

$$\hat{L} = \frac{1}{1+\mu}\hat{\mu} + \hat{a}_{☾} - \hat{A} \tag{31}$$

$$\hat{P}_{☾} = -\frac{2\mu}{1-\mu^2}\hat{\mu} + \hat{a}_{☾} - \hat{A} \tag{32}$$

$$\hat{S} = 2\hat{k}_{gs} - \hat{G} - 0.01 \times 10^{-3} \tag{33}$$

Numerical values in these formulas are also from Ref. 7 and are consistent with the IAU 1964 list of constants. For completeness, other relevant numerical values and uncertainties follow.

$\tilde{A} = 149.6 \times 10^6$

$\hat{A} = (-7 \pm 13) \times 10^{-6}$

$\tilde{G} = 6.673 \times 10^{-23}$

$\hat{G} = (0.0 \pm 0.4) \times 10^{-3}$

$\tilde{k}^2_{gs} = 1.327{,}180 \times 10^{11}$

$2\hat{k}_{gs} = (-20 \pm 39) \times 10^{-6}$

$\tilde{k}^2_{ge} = 398{,}603$

$2\hat{k}_{ge} = (-5.3 \pm 2.0) \times 10^{-6}$

$\tilde{S} = 1.9889 \times 10^{-33}$

$\hat{S} = (-0.03 \pm 0.4) \times 10^{-3}$

$\tilde{\tau}_A = 499.012$

$\hat{\tau}_A = (-7 \pm 14) \times 10^{-6}$

$\tilde{P}_{☾} = -124''\!.986$

$\hat{P}_{☾} = (0.0 \pm 20) \times 10^{-6}$

$\tilde{a}_{☾} = 384{,}400$

$\hat{a}_{☾} = (-1.8 \pm 0.8) \times 10^{-6}$

$\tilde{L} = 6''\!.43987$

$\hat{L} = (0.0 \pm 120) \times 10^{-6}$

$n^*_{☾} = 2.661699489 \times 10^{-6}$

$\hat{n}^*_{☾} = (0.0000 \pm 0.0002) \times 10^{-6}$

$\tilde{\mu} = 1/81.30$

$\hat{\mu} = (0.0 \pm 120) \times 10^{-6}$

$\tilde{k}^2_{gm} = 4902.87$

$2\hat{k}_{gm} = (0.0 \pm 120) \times 10^{-6}$

$\tilde{R}_{em} = 6378.327$

$\hat{R}_{em} = (-1.8 \pm 0.8) \times 10^{-6}$

F. Determination of the Constants

Before beginning a discussion of the detailed methods used to obtain values of astronomical constants from the *Mariner II* data, it is advisable to consider in general the relationship of the observed Doppler curve to the system of constants and to appreciate, in a descriptive sense, the nature of the determination of the masses and the astronomical unit from that curve. Clearly, since the orbit of the *Mariner II* probe depends to some degree on all the constants, and because the Doppler data can be interpreted as measurements of range rate, which in turn depend on the orbit, there is the possibility of being able to determine any of the constants. However, as in all observations of natural and artificial bodies, the orbit of the body in question is particularly sensitive to some of the constants, and is, moreover, practically insensitive to reasonable corrections to other constants. The purpose, therefore, of this discussion is to show that an adequate representation of the *Mariner II* Doppler data, where an adequate representation is defined in terms of removing all measurable systematic effects in the Doppler residuals, can be accomplished only by using relatively accurate values of the masses of the Moon and Venus and the astronomical unit. Then selecting values for the three constants such that their respective systematic effects are removed from the Doppler residuals, constitutes a determination of the constants. In all the intricacies of the least-squares procedure used in the reductions of Section V, it is important not to lose sight of the fact that the final goal of the analysis is simply this selection of the constants and that the procedure is used primarily in order to systematize the determination and reduce it to numerical operations.

In the following, each of the three constants will be considered in turn and the nature of their determination will be discussed. It will be shown that the mass of the

Moon is derived from the monthly motion of the Earth about the Earth-Moon barycenter and thus is dependent on the Doppler curve throughout the cruise portion of the flight where the Sun dominates the orbital motion. Also, the mass of Venus, as expected, is determined from the Doppler curve obtained during planetary encounter, while the astronomical unit depends on forcing consistency between the cruise and encounter data.

1. Mass of the Moon. In order to analyze the motion of the Earth about the Earth-Moon barycenter and to derive an approximate expression for the component of this motion in the Doppler curve, consider the probe in the cruise portion of the flight and assume that it and the Earth are motionless with respect to the heliocentric frame of reference. This removes the geocentric motion of the probe as a contribution to the Doppler curve. Also neglect the contribution from the geocentric motion of the station about the Earth's polar axis. Then with the angular velocity of the Earth-Moon barycenter given by the sidereal mean motion of the Moon $n^*_{☾}$, the remaining component $\dot{\rho}_B$ in the Doppler curve caused by the barycentric motion is approximately

$$\dot{\rho}_B = R_B \, n^*_{☾} \cos\beta \sin(\lambda - \lambda_{☾}) \tag{34}$$

where β and λ are the geocentric celestial latitude and longitude of *Mariner II*, $\lambda_{☾}$ is the longitude of the Moon and R_B is the mean geocentric position of the barycenter given as a fraction of the mean distance of the Moon $a_{☾}$ by

$$R_B = \frac{\mu}{1 + \mu} a_{☾} \tag{35}$$

If R_B is expressed in units of the a.u., then it is called the constant of the lunar inequality $L = R_B/A$. Again the expression for $\dot{\rho}_B$ represents nothing besides the barycentric motion, and if all other contributions to the motion are filtered out of the actual Doppler curve, then what remains is capable of yielding a determination of R_B and hence μ through a measurement of the amplitude of the $\dot{\rho}_B$ curve. In practice the filtering of all components is accomplished simultaneously with the $\dot{\rho}_B$ component by means of the least-squares procedure. However, conceptually it is proper to think in terms of the determination of the amplitude of a periodic component in the Doppler curve with a period approximately equal to the Moon's orbital period. Note that the mean motion $n^*_{☾}$ of the Moon can be considered perfectly known with respect to the uncertainty in R_B. Also the latitude β is not constant and fortunately its variation over the duration of the *Mariner II* data allows its separation from the constant R_B. If it remained constant, only the product $R_B \cos\beta$ could be determined.

With respect to the potential accuracy of the determination of R_B, from which μ can be obtained, it is a well known statistical result that the error in the determination of the amplitude of a sine function from N independent samples of that function is given by $\sqrt{2}\,\sigma/\sqrt{N}$ where σ is the measurement error. The cruise solutions of Section V use about 1000 points with an assumed measurement error of about 0.003 m/sec in range rate. Therefore, the most optimistic error estimate of the amplitude $R_B \, n^*_{☾} \cos\beta$ would be about 0.0001 m/sec based on the preceding formula. The function $\cos\beta$ is near unity and thus the amplitude itself of the $\dot{\rho}_B$ curve is about 10 m/sec. It can therefore be determined to about 0.001%, and from Eq. (35) the mass ratio can be determined to about the same percentage error. Actually the least-squares solution of Section V gives an uncertainty about twice this large which is nevertheless in excellent agreement with the rough calculation performed in this section. Note that before *Mariner II*, the uncertainty in μ from optical observations was in the region of 0.04%, so the tracking of space probes has improved its accuracy by at least an order of magnitude.

2. Mass of Venus. The mass of Venus is determined by the shape of the Doppler curve during the encounter of *Mariner II* with the planet. This curve can be approximated quite accurately in the vicinity of encounter by means of the velocity curve used for spectroscopic binaries (Ref. 27, p. 359).

$$\dot{\rho} = V + K\left[\cos(v + \omega) + e\cos\omega\right] \tag{36}$$

where V is the geocentric radial velocity of the planet, v is the true anomaly in the planet-centered-orbit of the probe, ω is the argument of the periapsis, e is the eccentricity, and K is a constant given by

$$K = \frac{n\,a \sin i}{(1 - e^2)^{1/2}} \tag{37}$$

The inclination of the orbit i and the angle ω are given with respect to a plane, "the plane of the sky," oriented such that the Earth-planet line is normal to the plane. Because the orbit of the probe about the planet is represented by a hyperbola, it is more meaningful to express K as a function of the hyperbolic elements $b = -a\sqrt{e^2 - 1}$ and $V_\infty = [GM_V/(-a)]^{1/2}$. The semi-minor axis b in the hyperbola is the distance of closest approach of the asymptotes to the mass-center at the focus. It is the distance of closest approach of the probe to the planet if the planet is massless and exerts no bending on the encounter trajectory. The hyperbolic excess velocity V_∞ is the planet-centered-velocity of the probe at infinity, and again, for a massless planet represents the constant velocity of the probe along the asymptote. In terms of these parameters K is given by

$$K = \frac{GM_V}{b\,V_\infty} \sin i \qquad (38)$$

where GM_V is the gravitational constant for Venus in units such as m^3/sec^2. Figure 2 shows a comparison of the actual *Mariner II* geocentric range-rate curve and the curve computed from Eq. (36) with the orbital elements and V held constant at the values associated with closest approach.

Now an efficient utilization of the encounter Doppler curve is best achieved by adding as much information as possible from the Doppler data outside the region of planetary encounter. A determination of the heliocentric orbit of the probe before and after encounter allows a measurement of the bending of the trajectory and the hyperbolic excess velocity V_∞. Also the orientation of the hyperbolic orbit in space, and thus the orientation with respect to the "plane of the sky," is determined. That this information is easily obtained can be seen (Fig. 3) by forming the planet-centered-velocity of the probe before and after encounter by means of a subtraction of the heliocentric velocity of Venus from the heliocentric velocities of the probe. The angle between these two planet-centered-velocity vectors is just the supplement of the total bending angle, the magnitude of both vectors is V_∞ and the plane which they define is the plane of the orbit. Of course, all of this is actually a two-body idealization of the more accurate three body system, but it is sufficiently realistic to show that the Doppler data outside the encounter region can provide values for the parameters V_∞, i, ω and e with the eccentricity given as the inverse of the cosine of one-half the bending angle. The constant K in Eq. (36) can also be determined without the encounter data by using a modified form of Eq. (38).

$$K = \frac{V_\infty}{\sqrt{e^2 - 1}} \sin i \qquad (39)$$

Thus, the only unknowns remaining in the encounter representation of $\dot{\rho}$ (Eq. 36) are the planet's radial velocity V and the constants GM_V and the time of periapsis passage T, which enter in specifying the true anomaly v as a function of time through the hyperbolic form of Kepler's equation. To show that these three remaining constants can be obtained from an observed $\dot{\rho}$ curve of the form of Eq. (36) and Fig. 2, a graphical method of

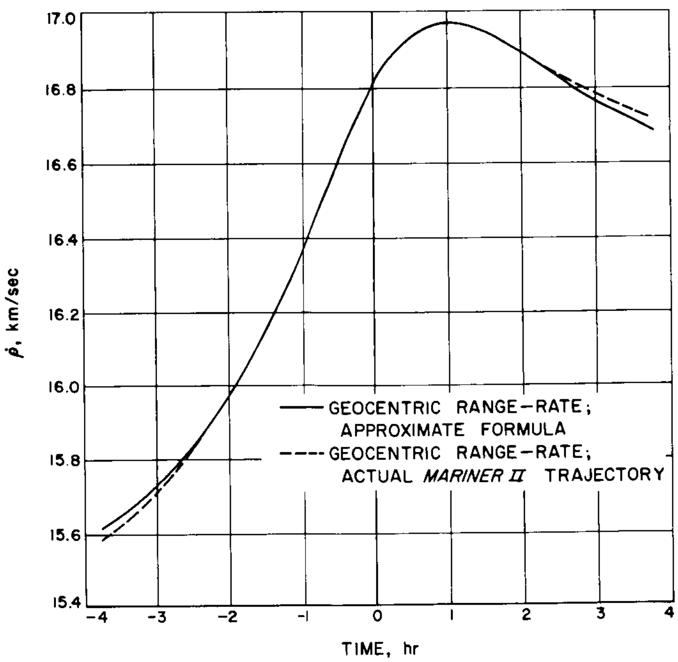

Fig. 2. Doppler curve during planetary encounter

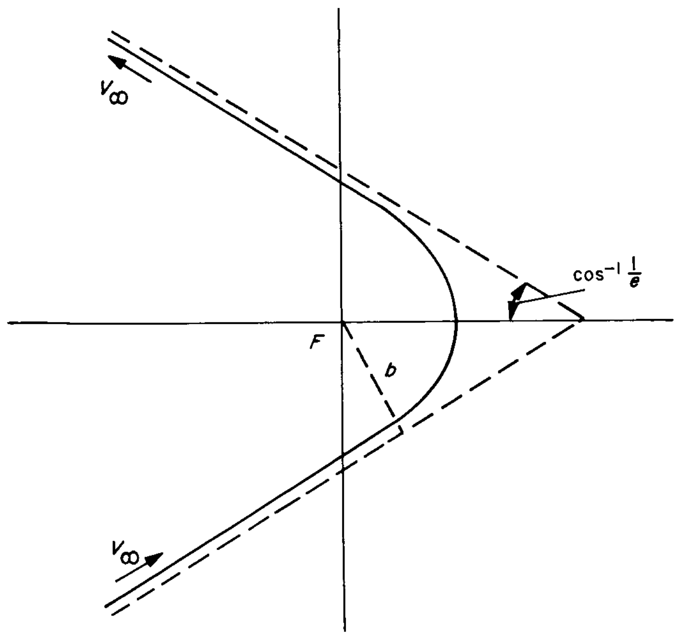

Fig. 3. Geometry of encounter hyperbola

solution is presented in the following, although the actual determination of Section V uses the more rigorous least-squares procedure.

The observed Doppler curve can be plotted as range-rate $\dot{\rho}$ versus the time t. Also because K, ω and e are known, a plot of $\dot{\rho} - V$ vs the true anomaly can be constructed from Eq. (36). Now at some point in the trajectory, the function cos $(v + \omega)$ will take on either a maximum value $(+1)$, a minimum (-1) or both values depending on the value of ω and the degree of bending. In Fig. 2, the maximum value for *Mariner II* occurs about one hour after encounter. In any case the two graphs $\dot{\rho} - V$ vs v and $\dot{\rho}$ vs t can be superimposed and the maximum and/or minimum points on the two curves can be made coincident by shifting the graphs in both the ordinate and abscissa with respect to each other. Then by reading the abscissas of both graphs, a one-to-one correspondence between the true anomaly and time can be established. The difference in the ordinates is simply the radial velocity of the planet V. The point at which $v = 0$ in the just established v vs t curve defines the time of periapsis passage T, and thus the only remaining constant to be determined is the gravitational constant GM_V. One method of doing this is to numerically or graphically differentiate the v vs t curve to obtain the rate of the true anomaly at the periapsis point $(v = 0)$. Call this rate $\dot{v}_P$ and use the angular momentum integral of the two body problem to obtain a formula for GM_V in terms of known quantities.

$$GM_V = \frac{\sqrt{e^2 - 1}}{(e - 1)^2} \frac{V_\infty^3}{\dot{v}_P} \qquad (40)$$

Of course to express the mass of Venus in units of mass of the Sun it is necessary to divide GM_V by $GS = k^2 A^3$ to obtain

$$M_V^S = \frac{1}{k^2 \dot{v}_P} \frac{\sqrt{e^2 - 1}}{(e - 1)^2} \left(\frac{V_\infty}{A} \right)^3 \qquad (41)$$

An interesting result of approaching the solution for the mass in the foregoing qualitative manner is that it becomes apparent that the encounter doppler data yield a value for GM_V, not M_V^S, and that the latter quantity is dependent on the assumed value of the astronomical unit A in meters. This conclusion is supported by the actual least-squares solutions for M_V^S in Section V. Only when all the data, both cruise and encounter, are reduced for values of M_V^S and A does an independent determination of A occur with a corresponding separation of M_V^S and A. However, the nature of the determination of A is discussed in the next section.

3. The astronomical unit. As shown in the previous section, it is not possible to determine the number of meters in the astronomical unit from an encounter Doppler curve alone. In addition, the sensitivity of range-rate to reasonable variations in A is not great enough during the three months of the cruise data to provide a solution. However, the least-squares solutions of Section V show that the combined data can determine A to the order of 500 km and separate it from the solution for the mass of Venus. The purpose of this section is to offer some reasonable justification for such a separation. Roughly, it has to do with the importance of obtaining accurate heliocentric conditions of Venus to assure consistency between the heliocentric orbit of the probe as determined from the cruise data, and the planet-centered-orbit as determined from the encounter data. One method of varying the heliocentric conditions is to vary A, which in effect scales the ephemeris of Venus as given in astronomical units. The probe's heliocentric orbit, as determined by Doppler data taken during the cruise portion of the flight, is in units of meters, say, and thus it is necessary to obtain the correct conditions for Venus in the same units by selecting a proper value of A. Also, in Section V it is concluded that even more degrees of freedom in the coordinates of Venus are necessary to force consistency in the data, and adjustments are made to the ephemeris itself as well as to A. However the present discussion is restricted to A and the multidimensional correction of the ephemeris is left for the rigorous reduction. However, the arguments presented for the validity of a solution for A can be extended, at least conceptually, to the general situation of a full correction to the coordinates of Venus.

The necessity for an accurate value of A is already evident in the reduction of the last section, although in determining the mass the value of A was assumed known and its role in the reduction was neglected. Now, however, it is important to recognize that an inconsistency in the reduction for V, GM_V and the orbital elements can arise in the following way. When the heliocentric velocity vectors (m/sec) were converted to Venus centered vectors (also m/sec) it was necessary to use the heliocentric velocity vector of Venus in the transformation of coordinates. However this is available in a.u./sec from the ephemeris of Venus, and thus the ephemeris velocity must be multiplied by A. Now after the hypothetical graphical solution for V is obtained we have a value for the geocentric radial velocity of Venus (m/sec) which can also be computed in a.u.'s from the Earth and Venus ephemerides. Therefore by forming the ratio V (m/sec)/V (a.u./sec) a value for A is obtained after the solution for V, GM_V and so forth. Unless the value of A chosen for the heliocentric to Venus centered transformation is correct, there will be an inconsistency between it and the value obtained after the solution. Of course, this discrepancy can be removed by forming the difference in the two values of A for various trial values of the A used in the coordinate transformation, and by plotting this difference vs A. Then, the A for which the difference is zero corresponds to a determination from the combined cruise and encounter data. Again, the procedure will not actually be performed here, it serves only as a basis for the analysis of the feasibility of a solution; but the mass M_V^S, A and the coordinates of Venus will all be obtained simultaneously in the least-squares solutions.

IV. Methods of Data Reduction

The *Mariner II* tracking data consist of Doppler measurements made at the Goldstone station of the Deep Space Instrumentation Facility (DSIF) under the direction of JPL. Other stations in this facility, in particular the one at Johannesburg, South Africa, also tracked *Mariner II*, but the data were not of the same quality as the Goldstone data. Only the Goldstone station was equipped with an atomic frequency standard during the *Mariner II* tracking period. Therefore, for the purposes of this determination of the constants, only the Goldstone data are considered.

The method of solution is that of weighted least squares with a modification to allow the introduction of *a-priori* information into the process. As in any least squares solution it is necessary to compute residuals in the data, and by convention the sense of the residuals is the observed minus the computed (*O-C*) values. The adopted procedure is to simply represent the Doppler measurement as accurately as necessary by a mathematical formula and then to form the *O-C* subtraction. The actual measurement *O* is stored on magnetic tape. An accurate representation of the data will involve considerations of light-time, atmospheric refraction corrections and an interpretation of the station procedure used to record the time of an observation.

Practically all least squares data reductions in celestial mechanics are accomplished by differentially correcting nominal or standard values of the orbital and astronomical constants to obtain the least squares solution for both the orbit and constants. This procedure is unavoidable because of the non-linear nature of orbital problems. In some cases the orbit is unknown and approximate solutions for a preliminary orbit must be developed which use some part of the total collection of observations. Fortunately, preliminary orbital elements for *Mariner II* are available (Ref. 3) and we can proceed directly to the differential correction. Therefore, the formation of differential coefficients, which relate incremental variations in the data to variations in the orbital elements and constants, is a primary concern of this analysis.

A. The Method of Least Squares

The general problem of fitting a set of observations by the method of least squares can be stated as follows. Suppose that some variable z, in our case the *Mariner II* Doppler data, is a function $f(t)$ of the time t. For example, the Doppler curve during the planetary encounter period looks something like Fig. 4. Now measurements of the function $f(t)$ are made at discrete times, a few points are shown in the figure, and the unavoidable situation is that the measured function $f(t)$ will not be smooth. However, we are interested in fitting to the observations a smooth function $g(x,t)$ of a multidimensional parameter set x. This function is chosen to represent the theory associated with the laws of celestial mechanics and the Doppler representation. Thus, the parameters x will include the six orbital elements for the space probe and also various constants such as the mass of Venus. The only possibility of obtaining a meaningful determination of the relevant astronomical constants is to choose the function $g(x, t)$ to account for all physical phenomena inherent in the Doppler data. Of course, there is a family of curves $g(x, t)$ for various values of x, and the problem is to find a particular parameter set x^* which will approximate the sampled function $z = f(t)$ by the particular function $g(x^*, t)$. The method of least squares chooses the parameters x^* which minimize the sum of squares of

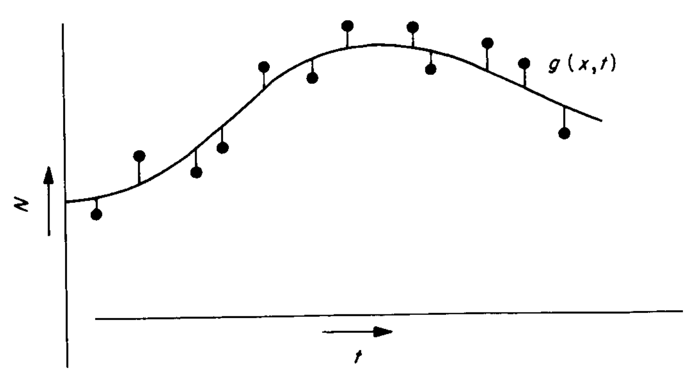

Fig. 4. Curve g (x, t) fitted to a set of data

the vertical deviations of the measured points from the curve $g(x^*, t)$. The deviations from the curve are called residuals and for a set of n discrete measurements $\hat{z}_i = f(t_i)$ $(i = 1, 2, \cdots, n)$ at times t_i, they can be computed by the formula $\hat{z}_i - g(x, t_i)$. Then the least squares method is given in analytical form as the minimization of the function S where

$$S = \sum_{i=1}^{n} [\hat{z}_i - g(x, t_i)]^2 \tag{42}$$

The minimization itself is accomplished by setting the partial derivatives of S with respect to each parameter in the set x equal to zero and by solving the resulting system of equations for x. In Section IV-A-1, the mathematical details of the solution are explored further.

1. Weighted least squares and* a-priori *information. A description of the use of the method of least squares can be found in many standard texts in the field of statistics. In addition Brouwer and Clemence (Ref. 28) have specifically discussed the method from the viewpoint of its application in celestial mechanics. The purpose of this section, therefore, is not to explore the method in detail but to take a very general approach and show (a) that the least squares solution can be conditioned by the introduction of *a-priori* information on the parameters, and (b) that a repeated application of the differential correction process can result in convergence to the least squares estimate even when the initial correction is outside of a linear region. These two aspects of the method are not widely understood.

The approach of this section relies heavily on matrix notation because it lends itself to general discussions of multidimensional systems. In particular, we will reduce the least squares problem to the numerical solution of a system of n nonlinear algebraic equations in n unknowns. The details of how the resulting equations expressed in matrix form can be converted to an actual computational procedure will not be included, but Ref. 28 contains an excellent discussion of the computational steps involved in obtaining the solution. One point should be made. Simply because matrix notation is used in this strictly theoretical approach, it should not be assumed that "standard" matrix manipulation computer routines will be adequate, either with respect to accuracy or efficiency, in solving the particular problem of least squares. It is necessary to investigate the accuracy and efficiency of the computations at each stage of the solution rather than reduce the numerical analysis to a series of matrix multiplications and inversions.

As a first step in the matrix development, consider an expression for the sum of squares of the residuals. Let the set of actual data be represented by a column matrix $\hat{z}$ and designate the set of parameters by x. Then the computed values of the data are given by the matrix function $z(x)$. The notational convention is that lower case letters represent column matrices and capitalized letters represent rectangular and square matrices. Now if the superscript T is used to indicate the transpose of a matrix, then the sum of squares of the residuals can be written

$$[\hat{z} - z(x)]^T [\hat{z} - z(x)]$$

where the transpose of a column matrix is a row matrix, and, by the definition of matrix multiplication, the indicated product results in a scalar quantity. In weighted least squares each observation is multiplied by a scalar weighting factor w before the sum of squares is formed. This can be represented in matrix notation by the use of a diagonal weighting matrix W with the square of the weight situated on the diagonal element corresponding to the observation which is to receive that particular weight. Then, if the weighted sum of squares is the scalar function $S(x)$, we have

$$S(x) = [\hat{z} - z(x)]^T W [\hat{z} - z(x)] \tag{43}$$

At this point the concept of *a-priori* information on the parameters can be introduced by recognizing that at least some parameters, for example, the astronomical unit, are not completely unknown, but that *a-priori* values $\tilde{x}$, sometimes called standard values, are available with an associated estimate of their respective uncertainties. If these uncertainties are arranged in the form of an *a-priori* covariance matrix $\tilde{\Gamma}_x$, and if the *a-priori* parameters themselves are treated as additional observations, then it is sensible to add a function

$$(\tilde{x} - x)^T \tilde{\Gamma}_x^{-1} (\tilde{x} - x)$$

to S. The matrix $\tilde{\Gamma}_x^{-1}$ is the weighting matrix for the additional observations $\tilde{x}$ and it may have off-diagonal elements to account for any *a-priori* statistical correlation between the parameters.

From a statistical viewpoint the addition of the *a-priori* term produces a combined estimate of the parameters, where the combination occurs between previous determinations of x and the one which results from the data $\hat{z}$ alone. Of course prior determinations of some or all parameters can still be ignored by inserting zeros in the matrix Γ_x^{-1}. Let the new least squares function with the *a-priori* information be $Q(x)$. Then

$$Q(x) = S(x) + (\tilde{x} - x)^T \tilde{\Gamma}_x^{-1} (\tilde{x} - x) \tag{44}$$

With this definition of $Q(x)$ the weighted least squares method can now be reduced to the following statement. Given a set of data $\hat{z}$ and a mathematical representation $z(x)$ of those data in terms of a set of parameters x, find particular values x^* of the parameters which will make $Q(x)$ an absolute minimum.

Certainly a necessary condition for this minimization is that at x^*, arbitrary infinitesimal variations in x will result in no variation in $Q(x)$. Thus, the first variation $dQ(x)$ in $Q(x)$ must be zero at x^* for all variations dx. Differentiate Eq. (44) with respect to x.

$$dQ(x) = -2\, dx^T A^T W [\hat{z} - z(x)] - 2\, dx^T \tilde{\Gamma}_x^{-1} (\tilde{x} - x) \tag{45}$$

where the matrix A is an array of the differential coefficients relating variations in the parameters to variations in the data. It is defined by

$$dz = A\, dx \tag{46}$$

The only way that $dQ(x)$ can be zero for arbitrary dx is for the matrix multiplying dx^T in Eq. (45) to be null. Thus $Q(x)$ is minimized if the following system of equations is satisfied

$$A^T W[\hat{z} - z(x)] + \tilde{\Gamma}_x^{-1} (\tilde{x} - x) = 0 \tag{47}$$

The least squares problem has, as anticipated, been reduced to the solution of a system of non-linear algebraic equations in x. Because the second variation $d[dQ(x)]$ of $Q(x)$ is always positive, the solution cannot yield a maximum value of $Q(x)$. If the variation in A is neglected, this second variation can be written as

$$d[dQ(x)] = 2\, dx^T (A^T W A + \Gamma_x^{-1}) dx \tag{48}$$

and since the matrix $A^T W A + \Gamma_x^{-1}$ is positive definite, the quadratic form of Eq. (48) is positive.

Any numerical technique that will yield the solution x^* to Eq. (47) is satisfactory in arriving at the least squares solution. However, if the Newton-Raphson method is used in its multidimensional form, the resulting system of linear differential correction formulae is precisely the set of normal equations which are solved in the classical least squares procedure. The advantage of approaching the problem from the viewpoint of the solution of a system of equations given by Eq. (47) is that now the repeated application of the classical differential correction process is meaningful. If the Newton-Raphson method converges, it will converge to the solution of Eq. (47) and hence to the value of x that minimizes the function $Q(x)$.

Designate the estimate of the solution x^* at the nth iteration by $x^{(n)}$. Then the improved estimate $x^{(n+1)}$ is given by

$$(A^T W A + \tilde{\Gamma}_x^{-1}) (x^{(n+1)} - x^{(n)}) = A^T(x^{(n)}) W[\hat{z} - z(x^{(n)})] + \tilde{\Gamma}_x^{-1} (\tilde{x} - x^{(n)}) \tag{49}$$

By neglecting the dependence of the matrix A on the parameters x, what is really generated is a modified Newton-Raphson procedure, where the first derivative of the function in Eq. (47) is only approximated by Eq. (48). Of course, if the matrix A is independent of x, then no iteration is required since the solution is obtained by a linear correction to the initial estimate of x^*. However, this is not the case for practically all problems in celestial mechanics, and successive iterations by the modified Newton-Raphson scheme are often required.

For comparison purposes the general result Eq. (49) can be reduced to the system of normal equations usually handled in least squares. Remove the effect of the *a-priori* information by setting $\tilde{\Gamma}_x^{-1} = 0$, and let $\Delta x = x^{(1)} - x^{(0)}$, the differential correction to the preliminary orbit and constants given by $x^{(0)}$. Also, let $\Delta z = \hat{z} - z(x^{(0)})$, the residuals based on the preliminary orbit. Then

$$(A^T W A)\, \Delta x = A^T W\, \Delta z \tag{50}$$

which is nothing more than a matrix expression for the familiar normal equations. The matrix A represents the coefficients in the equations of condition, Eq. (46).

All solutions for the *Mariner II* orbit and constants carried out in Section V are accomplished by means of Eq. (49), and the uncertainties and correlations in the resulting parameters $\underset{\sim}{x}^*$ are computed under the assumption that $(A^TWA + \underset{x}{\widetilde{\Gamma}}^{-1})^{-1}$ is the covariance matrix for the parameters. This is valid if W and $\underset{x}{\widetilde{\Gamma}}^{-1}$ are the inverse covariance matrices for the data and *a-priori* parameters, respectively, and if there are no sources of uncertainty outside of the data and *a-priori* parameters. Further statistical interpretations of the matrices in Eq. (49) are best made in conjunction with the actual solutions of Section V.

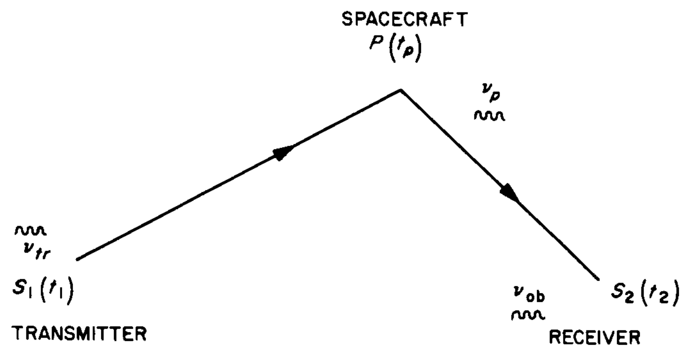

Fig. 5. Transmission of radio signal

B. Representation of the Radio Tracking Data

Before residuals can be computed in a least-squares procedure, it is necessary to construct a sufficiently accurate mathematical representation of the radio observation. The first step is to write an expression for the output of the electronic equipment itself in terms of transmitted and received frequencies. Then these frequencies can be related through theoretical considerations of the Doppler effect. Finally, methods for including effects of atmospheric refraction, light-time and the alignment of the station time with ephemeris time (ET) and universal time (UT1) will have to be described.

The observational equipment consists of a transmitter and 85-foot parabolic antenna at site S_1 (Fig. 5) which transmits a signal at frequency ν_{tr} to the spacecraft P where the signal is multiplied by a constant k and then transmitted to a second radar site S_2 equipped with a receiver and again an 85-foot parabolic antenna. The received signal at frequency ν_{ob} is compared electronically with the multiplied transmitted frequency and the difference in frequencies is accumulated in a cycle count device over some period of time τ. Thus a single observation consists of the following three numbers

- N_ν — An integer number of cycles accumulated by a cycle count device which records the number of positive zero crossings of the differenced electromagnetic signals.
- τ — The count-time or the interval of time over which the cycle count is accumulated.
- t_{ob} — The station time associated with the end of the count-time interval.

Actually to make the scale of the observation independent of the count interval, the cycle count is divided by τ when the observation is recorded. Appendix E contains a listing of the actual *Mariner II* data used in the solutions of Section V.

Call the normalized Doppler observation f and designate the number of cycles counted in the infinitesimal time interval t to $t + dt$ by $F(t)$. Then the total count is obtained by integrating over the interval $t_{ob} - \tau$ to t_{ob}.

$$f = \frac{N_\nu}{\tau} = \frac{1}{\tau}\int_{t_{ob}-\tau}^{t_{ob}} F(t)dt \tag{51}$$

The function $F(t)$ is written

$$F(t) = \nu_o + K'(k'\nu_{tr} - \nu_{ob}) \tag{52}$$

where $k'\nu_{tr} - \nu_{ob}$ represents a Doppler shift in the observed frequency with respect to the frequency $k'\nu_{tr}$ transmitted at the spacecraft. The constant frequency ν_o is added to establish a reference for no relative motion and is taken large enough to assure that $F(t)$ is never negative. It would not be possible to interpret a negative frequency with the cycle count device. The constant K' is also a part of the electronics at the receiver.

Rather than use standard quadrature formulas for the evaluation of the integral of $F(t)$, the function is expanded in a power series about the mid-point t_m of the count interval $(t_{ob} - \tau \leqq t \leqq t_{ob})$ and the result is integrated term by term. The purpose of this expansion is to avoid an evaluation of $F(t)$ at several points within the interval and instead to evaluate all quantities at the single time t_m. Thus

$$F(t) = F(t_m) + \dot{F}(t_m)\,(t-t_m) + \frac{1}{2}\,\ddot{F}(t_m)\,(t-t_m)^2 + \cdots \tag{53}$$

and the integral of $F(t)$ is

$$\int_{t_{ob-T}}^{t_{ob}} F(t)dt = \int_{t_{m-1/2T}}^{t_{m+1/2T}} [F(t_m) + \dot{F}(t_m)(t-t_m) + 1/2\ddot{F}(t_m)(t-t_m)^2 + \cdots]dt \quad (54)$$

or upon performing the integral and substituting the result into Eq. (51), the expression for f becomes

$$f = F(t_m) + \frac{\tau^2}{24}\ddot{F}(t_m) + \cdots, \quad (55)$$

The representation of the observation f is now reduced to a formation of the function $F(t)$ and its even ordered derivatives. We will return to a consideration of the truncation of the series for F after the function $F(t)$ has been specified.

1. Doppler frequency shift. The Doppler formula used, for example, in the determination of radial velocities of celestial objects from the shift in optical spectral lines is simply $\Delta\nu/\nu = \dot{\rho}/c$; where $\dot{\rho}$ is the relative radial velocity or range-rate of the source of the radiation with respect to the observer. For the *Mariner II* radar data, the combination of the range-rate of the spacecraft with respect to both the receiver and transmitter on the Earth results in a doubling of the frequency shift to order $1/c$. Therefore, the frequency term in Eq. (52) is of the form

$$k'\nu_{tr} - \nu_{ob} = k'\nu_{tr}\left[\frac{2\dot{\rho}}{c} + O\left(\frac{1}{c^2}\right)\right] \quad (56)$$

Because of the nature of the radar data, it is necessary to include some of the $1/c^2$ terms in the expression for the Doppler shift if the inherent accuracy of the data is to be fully exploited.

Consider first the transmission from the spacecraft at P to the station at S_2 and denote an inertial origin or frame of reference by 0 (Fig. 6). From the theory of relativity the proper time $d\tau_p$ associated with the spacecraft is given by Ref. 29

$$d\tau_p^2 = \left(1 - \frac{s_p^2}{c^2} + \frac{2\Phi_p}{c^2}\right) dt_p^2 \quad (57)$$

wheres s_p is the speed of P with respect to the origin of O, Φp is the gravitational potential energy at P and c is the constant speed of propagation of the signal. This formula states that if an observer at rest with respect to O measures a time increment dt_p, then the corresponding increment measured in the frame of reference of P is $d\tau_p$. Actually the gravitational term is only approximate, although for the Schwarzschild metric the next term is of the order of $1/c^4$. It should be noted that by including relativistic terms in the derivation of the frequency shift, it is unnecessary to consider separately the effects of the aberration of electromagnetic waves. These terms in the Doppler shift are analogous to the aberration effect in optical angles where the correction can be considered as a relativistic transformation of the angles to the observer's frame of reference.

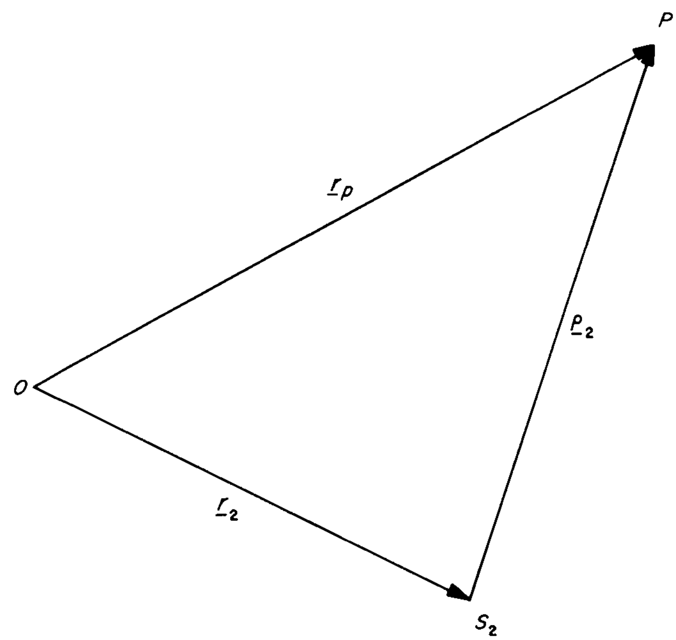

Fig. 6. Position of station S_2 and spacecraft P in inertial space

Similarly to P, the proper time for the station S_2 is

$$d\tau_2^2 = \left(1 - \frac{s_2^2}{c^2} + \frac{2\Phi_2}{c^2}\right) dt_2^2 \quad (58)$$

Now the number of cycles n_p transmitted by the spacecraft in the interval τ_p to $\tau_p + d\tau_p$ is $\nu_p(\tau_p)d\tau_p$ while the number n_2 received at S_2 in the interval τ_2 to $\tau_2 + d\tau_2$ is $\nu_2(\tau_2)d\tau_2$. In order that the number of cycles transmitted and received be the same, it follows that $n_p = n_2$ and

$$\frac{\nu_2(\tau_2)}{\nu_p(\tau_p)} = \frac{d\tau_p}{d\tau_2}$$

or to terms in $1/c^2$

$$\frac{\nu_2(\tau_2)}{\nu_p(\tau_p)} = \left[1 + \frac{1}{2c^2}\left(\dot{s}_2^2 - \dot{s}_p^2\right) - \frac{1}{c^2}\left(\Phi_2 - \Phi_p\right)\right]\frac{dt_p}{dt_2} \tag{59}$$

The derivative dt_p/dt_2 can be obtained by considering the finite propagation of the signal over the distance from P to S_2. Thus

$$t_p = t_2 - \frac{\rho_2}{c} \tag{60}$$

where ρ_2 is the magnitude of the vector $\boldsymbol{\rho}_2$ defined by

$$\boldsymbol{\rho}_2 = \mathbf{r}_p(t_p) - \mathbf{r}_2(t_2) \tag{61}$$

The vector $\mathbf{r}_p(t_p)$ is the inertial position of the *Mariner II* spacecraft when it transmits the signal at t_p and $\mathbf{r}_2(t_2)$ is the inertial position of the station S_2 at the time of reception t_2. Strictly in accordance with the theory of relativity, Eq. (60) is not quite correct because in a gravitational field the speed of propagation is not exactly c. Also, the Euclidean length ρ_2 does not take account of any bending of the signal's path in the field. However, this gravitational effect is negligible with respect to the *Mariner II* data, and for the present the only relativistic terms in the Doppler equation result from the transformation to proper time by Eq. (57) and (58).

Now from Eq. (60), the derivative dt_p/dt_2 is

$$\frac{dt_p}{dt_2} = 1 - \frac{1}{c}\frac{d\rho_2}{dt_2} \tag{62}$$

and from the definition $\rho_2^2 = \boldsymbol{\rho}_2 \cdot \boldsymbol{\rho}_2$, the derivative of ρ_2 with respect to t_2 is

$$\rho_2 \frac{d\rho_2}{dt_2} = \boldsymbol{\rho}_2 \cdot \frac{d\boldsymbol{\rho}_2}{dt_2} \tag{63}$$

where from Eq. (61),

$$\frac{d\boldsymbol{\rho}_2}{dt_2} = \frac{d\mathbf{r}_p}{dt_p}\frac{dt_p}{dt_2} - \frac{d\mathbf{r}_2}{dt_2} \tag{64}$$

or

$$\frac{d\boldsymbol{\rho}_2}{dt_2} = \dot{\mathbf{r}}_p(t_p)\frac{dt_p}{dt_2} - \dot{\mathbf{r}}_2(t_2) \tag{65}$$

Combine Eq. (62), (63) and (65) and solve for dt_p/dt_2.

$$\frac{dt_p}{dt_2} = \frac{1 + \dfrac{\boldsymbol{\rho}_2}{\rho_2} \cdot \dfrac{\dot{\mathbf{r}}_2(t_2)}{c}}{1 + \dfrac{\boldsymbol{\rho}_2}{\rho_2} \cdot \dfrac{\dot{\mathbf{r}}(t_p)}{c}} \tag{66}$$

The vector $\dot{\mathbf{r}}_2(t_2)$ and $\dot{\mathbf{r}}_p(t_p)$ represent the inertial velocities of the station and spacecraft respectively at the times indicated by their arguments.

To obtain the ratio of received to spacecraft transmitted frequencies all that is required is to substitute dt_p/dt_2 from Eq. (66) into Eq. (59). Thus

$$\frac{\nu_2(\tau_2)}{\nu_p(\tau_p)} = 1 + \frac{1}{2c^2}(\dot{s}_2^2 - \dot{s}_p^2) - \frac{1}{c^2}(\Phi_2 - \Phi_p) \times \frac{1 + \dfrac{\dot{s}_2}{c}\cos\theta_2}{1 + \dfrac{\dot{s}_p}{c}\cos\theta_p} \tag{67}$$

Where the angles θ_2 and θ_p are defined by the scaler products in Eq. (66) and $\dot{s}_2$ and $\dot{s}_p$ are the magnitudes of the velocity vectors $\dot{\mathbf{r}}_2(t_2)$ and $\dot{\mathbf{r}}_p(t_p)$, respectively.

The other leg of the radar transmission from the transmitting station S, to the spacecraft P can be obtained immediately from Eq. (67) by replacing the subscript p by 1 for the transmitted frequency ν, and the subscript 2 by p for the received frequency ν_p' at the spacecraft. Also to keep the sense of the range vector $\boldsymbol{\rho}$ always directed toward the spacecraft, the sign in the $\boldsymbol{\rho}$ expressions is changed. Then the ratio $\nu_p'(\tau_p)/\nu_1(t_1)$ is given by

$$\frac{\nu_p'(\tau_p)}{\nu_1(\tau_1)} = \left[1 + \frac{1}{2c^2}(\dot{s}_p^2 - \dot{s}_1^2) - \frac{1}{c^2}(\Phi_p - \Phi_1)\right] \times \frac{1 - \dfrac{\dot{s}_p}{c}\cos\phi_p}{1 - \dfrac{\dot{s}_1}{c}\cos\phi_1} \tag{68}$$

where the angles Φ_p and Φ_1 are defined by

$$\rho_1 \dot{s}_p \cos\phi_p = \boldsymbol{\rho}_1 \cdot \dot{\mathbf{r}}_p(t_p) \tag{69}$$

$$\rho_1 \dot{s}_1 \cos\phi_1 = \boldsymbol{\rho}_1 \cdot \dot{\mathbf{r}}_p(t_1) \tag{70}$$

and ρ_1 is the magnitude of the light-time corrected range vector $\boldsymbol{\rho}_1$.

$$\boldsymbol{\rho}_1 = \mathbf{r}_p(t_p) - \mathbf{r}_1(t_1) \tag{71}$$

All that remains in the specification of the Doppler shift is to identify the frequencies in the theoretical expressions with the actual frequencies in the Doppler system. This is accomplished as follows:

$\nu_1(\tau_1) \equiv \nu_{tr}$ — the transmitter frequency S_1

$\nu_p(\tau_p) \equiv k'\nu'_p(\tau_p)$ — the frequency transmitted by the spacecraft as a multiple of the received frequency ν'_p (assume instantaneous event)

$\nu_2(\tau_2) \equiv \nu_{ob}$ — the received frequency at S_2

The required frequency ratio for the Doppler formula (Eq. 52) is $\nu_{ob}/k'\nu_{tr}$ which can be written in terms of the theoretical frequencies as

$$\frac{\nu_{ob}}{k'\nu_{tr}} = \frac{\nu_2(\tau_2)}{\nu_p(\tau_p)} \frac{\nu'_p(\tau_p)}{\nu_1(\tau_1)} \tag{72}$$

which is just the product of Eq. (67) and (68). Again carry the relativity terms to order $1/c^2$ and the resulting expression for the function $F(t)$ is

$$F(t_2) = \nu_0 + K'k'\nu_{tr}\left[1 - \frac{\nu_{ob}}{k'\nu_{tr}}\right] \tag{73}$$

where

$$\frac{\nu_{ob}}{k'\nu_{tr}} = \left[1 + \frac{1}{2c^2}(\dot{s}_2^2 - \dot{s}_1^2) - \frac{1}{c^2}(\Phi_2 - \Phi_1)\right] \times \frac{\left(1 + \frac{\dot{s}_2}{c}\cos\theta_2\right)\left(1 - \frac{\dot{s}_p}{c}\cos\phi_p\right)}{\left(1 + \frac{\dot{s}_p}{c}\cos\theta_p\right)\left(1 - \frac{\dot{s}_1}{c}\cos\phi_1\right)} \tag{74}$$

The numerical values of the constants ν_0 and $K'k'$ are for *Mariner II.*

$$\nu_0 = 10^5 \text{ Hz}$$

$$K'k' = 32.359550561$$

Note that independent values of K' and k' are not required in the mathematical representation of the Doppler data.

The transmitted frequency ν_{tr} is given numerically by

$$\nu_{tr} = 29.66 \times 10^6 + \Delta\nu \; Hz$$

where $\Delta\nu$ may differ for various batches of data, but always lies within the interval 8100 to 9200 Hz. Thus, in addition to the three numbers mentioned previously as characterizing a single observation, it is also necessary to specify the value of $\Delta\nu$ associated with each observation.

The resulting formula for the Doppler shift (Eq. 74) is easier to understand if the terms of order $1/c$ are cleared from the denominator. This can be accomplished by multiplying numerator and denominator by the factor

$$\left(1 - \frac{\dot{s}_p}{c}\cos\theta_p\right)\left(1 + \frac{\dot{s}_1}{c}\cos\phi_1\right)$$

to obtain

$$\frac{\nu_{ob}}{k'\nu_{tr}} = 1 + \frac{1}{2c^2}(\dot{s}_2^2 - \dot{s}_1^2) - \frac{1}{c^2}(\Phi_2 - \Phi_1) \times \frac{\left(1 - \frac{\dot{\rho}_2}{c} - \frac{\dot{s}_2\dot{s}_p}{c^2}\cos\theta_2\cos\theta_p\right)\left(1 - \frac{\dot{\rho}_1}{c} - \frac{\dot{s}_1\dot{s}_p}{c^2}\cos\phi_1\cos\phi_p\right)}{\left(1 - \frac{\dot{s}_p^2}{c^2}\cos^2\theta_p\right)\left(1 - \frac{\dot{s}_1^2}{c^2}\cos^2\phi_1\right)} \tag{75}$$

The range rates $\dot{\rho}_1$ for the transmitter and $\dot{\rho}_2$ for the receiver are defined by

$$\rho_1\dot{\rho}_1 = \boldsymbol{\rho}_1 \cdot \dot{\boldsymbol{\rho}}_1 \tag{76}$$

and

$$\rho_2\dot{\rho}_2 = \boldsymbol{\rho}_2 \cdot \dot{\boldsymbol{\rho}}_2 \tag{77}$$

where the range vectors are defined in Eq. (61) and (71) and the range rate vectors are simply their time derivatives.

$$\dot{\rho}_1 = \dot{\mathbf{r}}_p(t_p) - \dot{\mathbf{r}}_1(t_1) \tag{78}$$

$$\dot{\rho}_2 = \dot{\mathbf{r}}_p(t_p) - \dot{\mathbf{r}}_2(t_2) \tag{79}$$

Of course ρ_1 and ρ_2 are the magnitudes of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, respectively.

Now from Eq. (75) it is immediately apparent that to terms in $1/c$ the Doppler frequency ratio is given by

$$\frac{\nu_{ob}}{k'\nu_{tr}} = 1 - \frac{\dot{\rho}_1 + \dot{\rho}_2}{c} + O(1/c^2) \tag{80}$$

which agrees with the expected result written down at the beginning of this development. (Cf. Eq. 61) However, the additional terms in Eq. (75) are important, and they cause considerable complication in computing accurate Doppler data. In fact it is worthwhile to expand Eq. (75) in powers of $1/c$ and to drop terms which are negligible with respect to the accuracy of the *Mariner II* data. At the same time a transformation from inertial to geocentric coordinates can be made in order to cast the Doppler representation in a more convenient coordinate system. The expansion of Eq. (75) and the associated transformation of coordinates is a straightforward matter although the derivation is somewhat tedius. Therefore, only the result is given here and the algebra and arithmetic required to justify the computational formula is given in Appendix A. The result is

$$\frac{\nu_{ob}}{k'\nu_{tr}} = 1 - \frac{\dot{\rho}'_1 + \dot{\rho}'_2}{c} + \frac{1}{c^2}(\dot{\rho}_1\dot{\rho}_2 + \dot{\rho}_2^2 + H) \tag{81}$$

where

$$H = \mathbf{r}'_p(t_p) \cdot \left[\frac{\dot{\rho}_2}{\rho_2}\dot{\mathbf{R}}_2(t_2) - \frac{\dot{\rho}_1}{\rho_1}\dot{\mathbf{R}}_1(t_1)\right] + \frac{1}{2}\omega_\oplus^2(R_2^2\cos^2\phi'_2 - R_1^2\cos^2\phi'_1) \tag{82}$$

and $\omega_\oplus$ is the angular rotation rate of the earth, $\mathbf{R}$ is the geocentric position vector of a radar station, R is its magnitude, $\dot{\mathbf{R}}$ is the geocentric station velocity, ϕ' is the geocentric station latitude, and $\mathbf{r}'_p$ is the geocentric position of the *Mariner II* spacecraft. The range-rates $\dot{\rho}'_1$ and $\dot{\rho}'_2$ are evaluated with geocentric coordinates.

2. ***Representation of cycle count.*** Now that the Doppler shift is available to sufficient accuracy, it is possible to return to the frequency function $F(t)$ defined by Eq. (52) and express it in terms of the topocentric and geocentric quantities of the previous section. Then, the cycle count data represented by f can be evaluated for particular estimates of the *Mariner II* orbit and Goldstone station locations. The next step in the process is to apply corrections and then form *O-C* residuals. From Eq. (52) and (81), the function $F(t)$ is

$$F(t) = \nu_0 + K'k'\frac{\nu_{tr}}{c}\left[\dot{\rho}'_1 + \dot{\rho}'_2 - \frac{1}{c}(\dot{\rho}_1\dot{\rho}_2 + \dot{\rho}_2^2 + H)\right] \tag{83}$$

For the evaluation of the actual normalized cycle count f, the integration of the function $F(t)$ is accomplished by Eq. (55) where $F(t)$ is evaluated at the midpoint t_m of the count interval ($t_{ob} - \tau \leqq t \leqq t_{ob}$). The second and fourth derivatives of $F(t)$ can be obtained quite accurately by neglecting the $1/c^2$ terms in $F(t)$. Then

$$\ddot{F}(t) = K'k'\frac{\nu_{tr}}{c}(\dddot{\rho}_1 + \dddot{\rho}_2) \tag{84}$$

$$F^{(IV)}(t) = K'k'\frac{\nu_{tr}}{c}(\rho_1^{(IV)} + \rho_2^{(IV)}) \tag{85}$$

For *Mariner II*, the $F^{(IV)}(t)$ term is negligible and the final formula for representing Doppler cycle count is

$$f(t_{ob}) = \nu_0 + K'k'\frac{\nu_{tr}}{c}\left[\dot{\rho}'_1 + \dot{\rho}'_2 - \frac{1}{c}(\dot{\rho}_1\dot{\rho}_2 + \dot{\rho}_2^2 + H) + \frac{\tau^2}{24}(\dddot{\rho}_1 + \dddot{\rho}_2)\right] \tag{86}$$

It is understood that all terms in Eq. (86) are evaluated at $t_m = t_{ob} - 1/2\ \tau$. In addition it is still necessary to solve the light time problem because the known time t_m in Eq. (86) is actually the time t_2 in the notation of the previous section. Therefore, the times t_1 and t_p are unknown until the light time associated with the particular value t_m of t_2 is determined. This correction is developed in Section IV-C-1.

Finally the term $\dddot{\rho}$ is obtained from the orbit and station locations through the topocentric range $\boldsymbol{\rho}$, range rate $\dot{\boldsymbol{\rho}}$, acceleration $\ddot{\boldsymbol{\rho}}$ and jerk $\dddot{\boldsymbol{\rho}}$ vectors. The expressions can be developed by successive differentiation as follows

$$\rho_i^2 = \boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i \tag{87}$$

$$\rho_i \dot{\rho}_i = \boldsymbol{\rho}_i \cdot \dot{\boldsymbol{\rho}}_i \tag{88}$$

$$\rho_i \ddot{\rho}_i + \dot{\rho}_i^2 = \boldsymbol{\rho}_i \cdot \ddot{\boldsymbol{\rho}}_i + \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i \tag{89}$$

$$\rho_i \dddot{\rho}_i + 3\dot{\rho}_i \ddot{\rho}_i = \boldsymbol{\rho}_i \cdot \dddot{\boldsymbol{\rho}}_i + 3\dot{\boldsymbol{\rho}}_i \cdot \ddot{\boldsymbol{\rho}}_i \tag{90}$$

C. Corrections to the Doppler Data

The Doppler cycle count recorded at the station S_2 could, in a sense analogous to angular observations, properly be called apparent cycle count. However, there is a distinction in that the doppler data is simply a number, independent of coordinate systems, and it makes no difference whether the computations of the previous section are performed with respect to the true equator and equinox of date, or with respect to any other equator and equinox. Perhaps the easiest conceptual approach to an understanding of the role of the coordinate system is to assume that all vectors required to specify the data are referred to the mean equator and equinox of 1950.0. Actually, in practice this is useful also, because the numerical integration of the equations of motion for the probe are best performed in the 1950.0 system (Cf. Section IV-D). Thus with the coordinates of the probe given in 1950.0 mean coordinates, it is only natural to express the other vectors, namely the station location vectors, in this system also. At this point the dependence of the Doppler data on precession and nutation becomes obvious.

Because the Earth is not a perfect sphere homogeneous in layers, other bodies in the solar system produce torques on it with the result that a coordinate system fixed in the earth undergoes precession and nutation. The station coordinates represented by the position vector $\mathbf{R}$ must be transformed to the mean 1950.0 coordinates to yield the required vector $\mathbf{R}_{1950}$. The earth-fixed coordinates or equivalently the components of $\mathbf{R}$ are given by

$$X = R \cos \phi' \cos \theta \tag{91}$$

$$Y = R \cos \phi' \sin \theta \tag{92}$$

$$Z = R \sin \phi' \tag{93}$$

where R and ϕ' are the geocentric radius and latitude of the station respectively and θ is the local sidereal time. The interpretation and computation of θ are included in Section IV-C-3.

The conversion from mean-coordinates of 1950.0 to mean coordinates $\mathbf{R}_{\mathrm{mean}}$ of date is accomplished by a matrix rotation.

$$\mathbf{R}_{\mathrm{mean}} = A\mathbf{R}_{1950} \tag{94}$$

where the elements of A can be deduced from the expressions given in Ref. 21, pp. 30–34.

$$a_{11} = 1 - 0.000{,}296{,}97T^2 - 0.000{,}000{,}13T^3$$

$$a_{12} = -a_{21} = -0.022{,}349{,}88T - 0.000{,}006{,}76T^2 + 0.000{,}002{,}21T^3$$

$$a_{13} = -a_{31} = -0.009{,}717{,}11T + 0.000{,}002{,}07T^2 + 0.000{,}000{,}96T^3$$

$$a_{22} = 1 - 0.000{,}249{,}76T^2 - 0.000{,}000{,}15T^3$$

$$a_{23} = a_{32} = -0.000{,}108{,}59T^2 - 0.000{,}000{,}03T^3$$

$$a_{33} = 1 - 0.000{,}047{,}21T^2 + 0.000{,}000{,}02T^3$$

The time interval T is the number of Julian centuries of 36,525 days past the epoch 1950.0.

The conversion from $\mathbf{R}_{\mathrm{mean}}$ to true coordinates $\mathbf{R}$ involves the nutation and is again expressed as a matrix rotation

$$\mathbf{R} = N\,\mathbf{R}_{\mathrm{mean}} \tag{95}$$

The elements of the matrix N are computed from tables on pp. 44 and 45 of Ref. 21 which yield corrections to longitude ($\delta\Psi$) and obliquity ($\delta\epsilon$). Then the matrix N is given by

$$N = \begin{bmatrix} 1 & -\delta\Psi \cos\bar{\epsilon} & -\delta\Psi \sin\bar{\epsilon} \\ \delta\Psi \cos\bar{\epsilon} & 1 & -\delta\epsilon \\ \delta\Psi \sin\bar{\epsilon} & \delta\epsilon & 1 \end{bmatrix} \tag{96}$$

where ϵ is the mean obliquity of date. The form of the matrix N in Eq. (96) is an approximation to the rotation from mean to true coordinates and is given on page 43 of Ref. 21.

The complete transformation from mean coordinates of 1950.0 to true coordinates of date is obtained by combining Eq. (94) and (95).

$$\mathbf{R} = N\,A\,\mathbf{R}_{1950} \tag{97}$$

or the inverse is

$$\mathbf{R}_{1950} = A^{-1}\,N^{-1}\,\mathbf{R} \tag{98}$$

Thus, the station location can be referred to the inertial 1950.0 coordinate system in which the orbit is computed.

Other significant corrections which must be included in the Doppler representation are light-time, atmospheric refraction and the conversion of the station time to ephemeris time ET and universal time UT. The problem of a correction caused by the aberration of light was dealt with in the derivation of the Doppler frequency shift where a distinction was made between an inertial time interval and a proper time interval at the station.

1. Light-time correction. In the computation of the cycle count data, three times are important (Cf. Section IV-B-1). The first, from a ray tracing viewpoint, is the time t_1 when the transmitting station S_1 sends a signal to the spacecraft, the second is the time t_p at which the probe receives the signal and the third is t_2 when the signal is received at the second station S_2. Now the only time that is known is the time of reception t_2, and the times t_p and t_1 must be obtained by means of a light time correction of the usual form.

$$t_p = t_2 - \frac{\rho_2}{c} \tag{99}$$

$$t_1 = t_p - \frac{\rho_1}{c} \tag{100}$$

where c is used to convert from the units of the computer program to units of light seconds. The ranges ρ_1 and ρ_2 are themselves functions of t_1 and t_p as well as t_2, and so some iterative procedure is used to find the unknown times. The usual procedure is to apply the method of successive substitutions with the iteration formula left in the form of Eq. (99) and (100). At each iteration the ranges are computed as the magnitudes of $\boldsymbol{\rho}_2 = \mathbf{r}_p(t_p) - \mathbf{r}_2(t_2)$ and $\boldsymbol{\rho}_1 = \mathbf{r}_p(t_p) - \mathbf{r}_1(t_1)$. Since the time t_2 is known, it is unnecessary to recompute $\mathbf{r}_2(t_2)$ after the first evaluation of $\boldsymbol{\rho}_2$ and the most reasonable method of finding t_1 and t_p is to completely solve Eq. (99) for t_p by iterating with successive values of $\mathbf{r}_p(t_p)$. Then with t_p known, the spacecraft position does not require modification in the successive computation of ρ_1 for Eq. (100). Only the location $\mathbf{r}_1(t_1)$ of the transmitter changes from iteration to iteration.

The actual mechanization of the iterative procedure to take advantage of the logical design of the computer program used in this work, can be found in Ref. 4, pp. 12 and 18.

2. Atmospheric refraction. As is the case with most high resolution astronomical data, the fact that the electromagnetic signal must pass through the Earth's atmosphere before it is detected introduces a degradation in the information content of the signal. The *Mariner II* Doppler cycle count data are no exception, although the effect is limited to atmospheric refraction and the associated shift in the Doppler frequency. The procedure followed here is to remove a large portion of the atmospheric effect by computing the frequency shift for a standard atmosphere. The remaining error caused by departures of the actual atmosphere from the standard one at the time of observation is a limiting factor in the inherent accuracy of the *Mariner II* data. However, the results of this section indicate that the atmospheric limit to accuracy is below that of the measurement resolution for *Mariner II*.

Before the correction itself is developed, it will be shown that the refraction effect on counted doppler data can be expressed as the difference in the effect on range ρ at the beginning and end of the count interval. The range is defined as the propagation time multiplied by the speed of propagation c.

If the $1/c^2$ terms in the function $F(t)$ are neglected (Cf. Eq. 83) then the integral for the cycle count data can be evaluated analytically.

$$f(t_{ob}) = \frac{1}{\tau}\int_{t_{ob}-\tau}^{t_{ob}} F(t)dt$$

$$= \nu_o + \frac{K'k'\nu_{tr}}{c\tau}\Big[\rho_1(t_{ob}) + \rho_2(t_{ob}) - \rho_1(t_{ob}-\tau) - \rho_2(t_{ob}-\tau)\Big] \tag{101}$$

Now the refraction correction $\Delta_r f(t_{ob})$ to the cycle count data is written in terms of the correction $\Delta_r \rho$ to range at the beginning and end of the count interval. No distinction is made between the individual corrections in ρ_1 and ρ_2 because the transmitter and receiver at the Goldstone site are only 10.7 km apart. Therefore, the correction formula is

$$\Delta_r f(t_{ob}) = \frac{2K'k'\nu_{tr}}{c\tau}\left[\Delta_r \rho(t_{ob}) - \Delta_r \rho(t_{ob} - \tau)\right] \quad (102)$$

To obtain the correction $\Delta_r \rho$, the path of a ray is numerically traced through a standard atmosphere such that Fermat's principle is satisfied. Then of all possible paths the ray follows the one which makes the time of transmission a minimum. As a model for the atmospheric refraction we choose an index of refraction n which is a function of only the altitude H above the surface of the Earth and which obeys an exponential model as follows:

$$n = 1 + (n_o - 1)e^{-H/H_o} \quad (103)$$

where H_o is the scale height. The constants chosen for the *Mariner II* radar propagation are

$$n_o - 1 = 3.40 \times 10^{-4}$$

$$H_o = 7.315 \text{ km}$$

The details of the ray tracing are carried out in Appendix B and the result for the range correction is plotted as a function of observed elevation angle in Fig. 7. As expected the correction increases rapidly for low elevation angles and amounts to 26 meters at 5 deg above the horizon. At the zenith, where the classical correction to elevation angle is zero, the range correction amounts to about 2.4 meters.

In the actual computation of $\Delta_r \rho$, an interpolation formula is used as an approximation to the numerically determined curve of Fig. 7. Its accuracy can be evaluated by comparing the few points plotted in Fig. 7 against the curve from the ray tracing. The interpolation formula is

$$\Delta_r \rho = 1.8958\,(\sin\gamma + 0.06483)^{-1.4} \text{ meters} \quad (104)$$

where γ is the elevation angle.

Finally, a combination of Eq. (102) and (104) yields the refraction correction to the cycle count data.

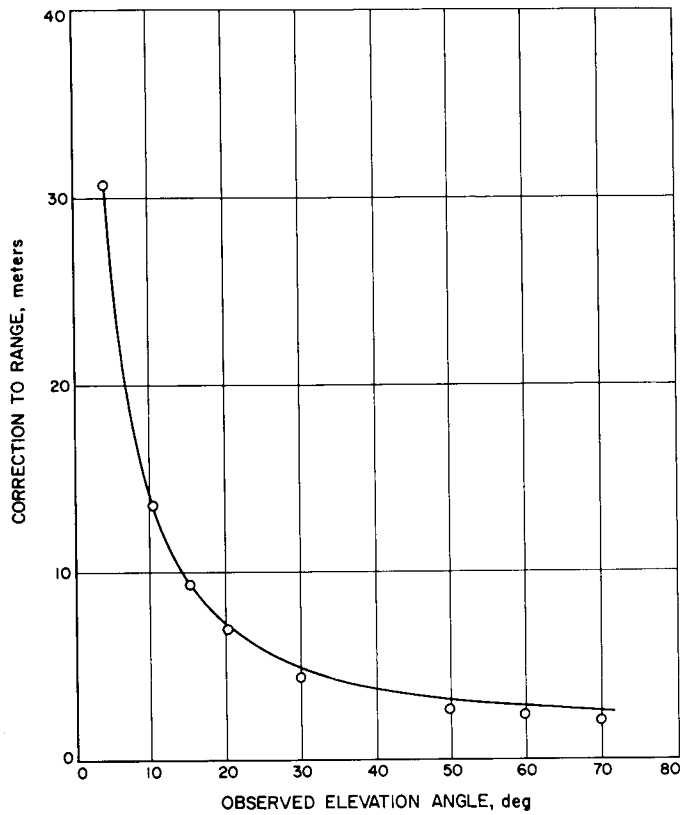

Fig. 7. Refraction correction to range for exponential atmosphere

$$\Delta_r f = \frac{A}{\tau}\left\{\frac{1}{[\sin\gamma\,(t_{ob}) + B]^{1.4}} - \frac{1}{[\sin\gamma\,(t_{ob} - \tau) + B]^{1.4}}\right\} \quad (105)$$

where

$$A = (0.0018958)\,\frac{2(K'k')\nu_{tr}}{c} \quad (106)$$

or

$$A = (0.40926 \times 10^{-6})\nu_{tr} \quad (107)$$

and

$$B = 0.06483 \quad (108)$$

The constants $K'k'$ and ν_{tr} are given in Section IV-B-1.

As in the computation of f itself, all quantities are evaluated at the midpoint t_m of the count interval, and the elevation angles at each end are approximated by

$$\gamma(t_{ob}) = \gamma(t_m) + \frac{1}{2}\tau\dot{\gamma}(t_m) \tag{109}$$

$$\gamma(t_{ob} - \tau) = \gamma(t_m) - \frac{1}{2}\tau\dot{\gamma}(t_m) \tag{110}$$

It is instructive to expand Eq. (105) to the first order in $\tau\ \gamma(t_m)$ by means of the definitions in Eq. (109) and (110). The result is

$$\Delta_r f = -\frac{1.4A\dot{\gamma}(t_m)\cos\gamma(t_m)}{[\sin\gamma(t_m) + B]^{2.4}} \tag{111}$$

Thus, as the spacecraft rises above the station's horizon the Doppler data is shifted to lower frequencies; but later, around the time of meridian crossing, the correction goes through zero, and as the spacecraft sets the data are shifted to higher frequencies. An upper bound to the frequency shift is given by

$$|\Delta_r f| < \frac{1.4A\dot{\gamma}}{B^{2.4}} \tag{112}$$

and with γ set equal to the rotation rate of the Earth $\omega_\oplus$ as an upper bound, and $\nu_{tr} = 29.67 \times 10^6$ Hz, the corresponding upper bound on $\Delta_r f$ is

$$|\Delta_r f| < 0.9 \text{ Hz}$$

The longest count interval used in the *Mariner II* data is 10 min and the accuracy of f is then about 0.005 Hz. Thus in the worst case, the refraction correction is about 180 times the measurement error, and for most of the data an accuracy of two significant figures in the correction should be sufficient to represent the data. Because of the rapid increase in the correction with lower elevation angles, the weights assigned to individual measurements will be a function of elevation angle in the actual data reduction (Cf. Section V), and data at very low elevation angles will essentially be eliminated from the solution.

3. ***Interpretation of station time.*** The meaning of the time label associated with a cycle count measurement has already been introduced in Section IV-B; but the precise relationship of that time to the systems of time used in modern astronomical work has, up to this point, been ignored. A clear understanding of the station time is required to accurately represent the position of the *Mariner II* spacecraft and the station, both being vital to the computation of the Doppler data; and because the station location is fixed with respect to the rotating Earth, while the spacecraft position is given as a function of the ephemerides of other bodies in the solar system, the time argument in the formulas of Section IV-B is not only different for the two cases but in neither is it the station time itself.

The first assumption concerning the interpretation of time is that the station clock is synchronized with WWV transmissions and a pseudo but uniform universal time (UT). The station procedure makes this a fair assumption, because the rate of the WWV transmission is made available in bulletins published by the United States Naval Ovservatory (USNO). Then the station sets the rate of its own clock to the WWV rate which, during the *Mariner II* span of data, defined one second of WWV time as $(9{,}192{,}631{,}770)(1 + 1.3 \times 10^{-8})$ cycles of the cesium resonance (see Ref. 30). Note that the determination of the ephemeris second is 9,192,631,770 $\pm$20 cycles of this resonance (Ref. 17), and thus the WWV rate is slower than the rate of ephemeris time (ET). In order to assure that the station time did not drift during the *Mariner II* period from the WWV time because of errors in setting the station frequency, periodic calibration checks were made at the station by comparing the station time with the WWV transmissions.

Therefore, there is considerable justification for assuming that the time associated with the Doppler data is WWV time or what is referred to as UT2C in the USNO bulletins. However, before this time can be related to the times used in the actual computation of Doppler data, it is necessary to consider the definitions of the various systems of time.

The universal times UT0, UT1 and UT2 are described in Ref. 21. UT0 is the time determined from the meridian crossing of stars of known right ascension where the standard coordinates of the observatory are used in the reduction. UT1 is the universal time corrected for polar-wandering and is thus based on coordinates referred to the true pole at the time of observation. UT2 is a smoothed UT1 where periodic seasonal variations in the rotation rate of the Earth are removed. The WWV time (UT2C) is maintained at about UT2 by periodically incrementing UT2C and by changing the WWV rate at the beginning of the year, if necessary. From January 1, 1961 to June 30, 1965 WWV never differed by more than $0\overset{s}{.}15$ from UT2, and in any case the difference was published in the USNO bulletins covering those years. During the time of the *Mariner II* data, UT2C was not incremented.

Atomic time A.1 is described in Ref. 17. It is a uniform time with the second defined as the best determination of the ephemeris second in terms of the cesium resonance (9,192,631,770 $\pm$20 cps). Thus within the accuracy of this determination, the rate of A.1 is the same as the rate of ET. The calibration of A.1 is accomplished such that at $0^h0^m0^s$ UT2 on January 1, 1958, the value of A.1 is also $0^h0^m0^s$.

As already pointed out in Section III, the ephemeris second is defined by international agreement in terms of the number of ephemeris seconds in 1 tropical year (1900). An equivalent statement is that the rate of the mean Sun (1900) defines the ephemeris second and it makes no difference whether we adopt the length of the tropical year or the rate of the mean Sun as the definition of the rate of ET. In order to establish a zero point for ET, a value for the mean longitude of the Sun (1900) is also adopted. Thus on January 1, 1900 at 12^h ET the mean longitude L of the Sun is taken to be $279°41'48\overset{''}{.}04$ (Ref. 20). At a later time T from this epoch, where T is measured in Julian centuries of 36525 ephemeris days, the mean longitude is

$$L = 279°41'48\overset{''}{.}04 + 129602768\overset{''}{.}13T + 1\overset{''}{.}089T^2 \tag{113}$$

The quadratic term in the expression comes from theory and is the only coefficient subject to change if the theory of the Earth's motion is improved.

If the longitude of the mean sun could be observed, then the ephemeris time T could be determined by inverting the above formula. In practice observations are made of the more rapidly moving Moon and the ephemeris time is determined at the instant of observation from the observed position of the Moon. The assumption here is that the motions of the Sun and Moon as determined from theory are gravitationally consistent. Thus, either body can be used to determine T and the Moon is selected to gain more resolution in the final result for ET.

Now it is possible to decide what times are required for the computation of the Doppler data. Since the time argument in all the ephemerides is ET, it is clear that the *Mariner II* orbit will be given as a function of ET also. Therefore when the probe coordinates are obtained at time t_p, the rigorous interpretation of this is that the coordinates are for a time t_p ET determined from a time t_{ob} ET through the light time solution of Section IV-C-1. Because t_{ob} is given in UT2C time, the correction ET-UT2C is required to correct t_{ob} UT2C to t_{ob} ET. The USNO bulletins give A.1 $-$ UT2C which except for the zero point can be equated with ET $-$ UT2C. However, by the definition of the calibration of A.1 (1958) we can obtain ET $-$ A.1 as the difference in ET $-$ UT2 at the beginning of 1958. It is assumed that the difference ET $-$ UT2 at this time is $32\overset{s}{.}25$, and therefore that ET $=$ A.1 $+$ $32\overset{s}{.}25$ in the *Mariner II* reduction.

When the station coordinates are computed, the ephemeris time is no longer relevant because we are interested in the station location in the coordinate system used to locate the spacecraft. Therefore the non-uniform rotation of the Earth is important. Equations (91), (92) and (93) give the formulae for computing the station location in terms of radius R, geocentric latitude ϕ' and the local sidereal time θ. The latitude and longitude λ of the station are assumed given with respect to the true pole of data. The local sidereal time is then given by

$$\theta = \theta_o + \lambda \tag{114}$$

where θ_o is the Greenwich sidereal time expressed as a function of UT. In particular, at least for *Mariner II*, the universal time of interest is UT1 because this is what locates the true Greenwich meridian of date, and hence the station, in inertial space. The procedure for locating the station in 1950.0 coordinates, for example, is the following:

1. Given the observation time t_{ob} UT2C, convert to t_{ob} UT1. For the transmitting station the light time correction must also be applied to yield t_1. The USNO bulletins give UT2 $-$ UT2C and UT2 $-$ UT1 from which UT1 $-$ UT2C can be obtained.

2. Compute the Greenwich local sidereal time θ_o(UT1) by formulas given in Ref. 21.

3. Apply Eq. (91), (92), (93) and (114) to obtain the position of the station in true coordinates of date. Use values of ϕ' and λ referred to the true pole of date.

4. Compute the mean coordinates (1950) of the station by Eq. (98).

To implement the corrections ET − UT2C and UT1 − UT2C as taken from tables in the USNO bulletins (Ref. 30), polynomial interpolation formulas are used which fit the USNO tables. For the *Mariner II* period, they are given by the following:

$$\dot{\text{E}}\text{T} - \text{UT2C} = 29.221675 + (0.12967819 \times 10^{-7})t \tag{115}$$

$$\text{UT1} - \text{UT2C} = -39.821720 + (0.20287233 \times 10^{-6})t - (0.25818216 \times 10^{-15})t^2 \tag{116}$$

where t is measured in seconds of time from $0^h0^m0^s$ on January 1, 1950. Of course, these polynomials cannot be used outside of the range of time encompassed by the *Mariner II* data. The following table gives values of the time corrections as computed from the polynomials.

Year	Date (0^hUT)	UT1 − UT2C	ET − UT2C
1962	Sept. 2	$0\overset{s}{.}016$	$34\overset{s}{.}406$
↓	Sept. 12	0.014	34.417
	Sept. 22	0.011	34.428
	Oct. 2	0.009	34.439
	Oct. 12	0.006	34.451
	Oct. 22	0.000	34.462
	Nov. 1	−0.006	34.473
	Nov. 11	−0.012	34.484
	Nov. 21	−0.020	34.496
	Dec. 1	−0.026	34.507
	Dec. 11	−0.033	34.518
	Dec. 21	−0.039	34.529

Note that a failure to distinguish between UT1 and UT2C in computing station locations, would result in a drift of about $0\overset{s}{.}055$ in the longitude of the station over the time interval of the *Mariner II* data. This is equivalent to about 21 meters in the Goldstone station location and is significant with respect to the accuracy of the *Mariner II* determination of the longitude.

D. Equations of Motion

The equations of motion for the *Mariner II* spacecraft are expressed in mean equatorial coordinates of 1950.0. They represent a sixth-order system of differential equations where only the coordinates of the spacecraft are obtained by numerical integration. Coordinates of other bodies in the solar system are stored on magnetic tape and are provided by JPL (Ref. 23). The equations of motion are expressed in the relative motion form (Ref. 31, p. 161) which in vector notation are given by

$$\frac{d^2\mathbf{r}_{12}}{dt^2} = -k^2(m_1 + m_2)\frac{\mathbf{r}_{12}}{r^3_{12}} + k^2 \sum_{j=3}^{n} m_j \left(\frac{\mathbf{r}_{2j}}{r^3_{2j}} - \frac{\mathbf{r}_{1j}}{\mathbf{r}^3_{1j}}\right) + \mathbf{P}_2 \tag{117}$$

By convention a position vector $\mathbf{r}_{ij}$ represents the coordinates of the j^{th} body of mass m_j with respect to the i^{th} body of mass m_i. Thus, the first term in Eq. (117) represents the two body acceleration of the *Mariner II* spacecraft with respect to the primary body of mass m_1. The mass of the probe is 198.22 kgm and is negligible with respect to the primary mass m_1. Therefore, m_2 can be set equal to zero in Eq. (117). The second term in the equations represents the contribution to the relative acceleration from other bodies in the solar system. For the *Mariner II* orbit the primary body is either the Sun, the Earth or Venus and the other bodies in the n-body system are the remaining planets and the Moon. The third term $\mathbf{P}_2$ represents perturbative accelerations on the spacecraft which arise from forces aside from the gravitational attraction of the Sun, Moon and planets. In particular $\mathbf{P}_2$ includes effects from solar radiation pressure on the spacecraft and low thrust forces from the spacecraft's attitude control system which operates by releasing cold nitrogen gas through a number of jets. Because neither of these non-gravitational forces has a significant effect on the primary body, the form of $\mathbf{P}_2$ can be equated to the inertial acceleration from solar pressure and low-thrust forces.

If k^2 in Eq. (117) is set equal to the Gaussian gravitational constant, the units are astronomical units, solar masses and ephemeris days in the equations. However the units used in the integration of Eq. (117) are kilometers and ephemeris seconds and $k^2 m_i$ is combined into a single factor GM_i (km³/sec²). The formula for $k^2_{gs} = GS$ in the case of the Sun is given by Eq. (19) and the value of $k^2_{gi} = GM_i$ for any planet whose mass M^s_i is given in solar mass units is

$$k^2_{gi} = (GS)\frac{M_i}{S} = (86400)^{-2}\, k^2 A^3 M^s_i \tag{118}$$

It is understood that the value A of the a.u. in km is based on the adopted value of c because, as discussed in Section III-A, the standard meter is of no consequence in the mathematical representation of the tracking data. As an example of the equations expressed in km and sec., consider the geocentric form and let the geocentric position of the probe be given by $\mathbf{r}$. Then

$$\frac{d^2\mathbf{r}}{dt^2} = -k_{ge}^2 \frac{\mathbf{r}}{r^3} + k_{gm}^2\left(\frac{\mathbf{r}_{p☾}}{r_{p☾}^3} - \frac{\mathbf{r}_☾}{r_☾^3}\right) + k''^2 A^3 \left(\frac{\mathbf{r}_{p\odot}}{r_{p\odot}^3} - \frac{\mathbf{r}_\odot}{r_\odot^3}\right)$$

$$+ k''^2 A^3 \sum_{j=1}^{8} M_j^s \left(\frac{\mathbf{r}_{pj}}{r_{pj}^3} - \frac{\mathbf{r}_j}{r_j^3}\right) + \mathbf{P} \qquad (119)$$

where k'' is the Gaussian constant in unit of seconds, $k'' = (86400)^{-1} k$. All singly subscripted vectors are geocentric and the doubly subscripted vectors can be formed according to the convention that

$$\mathbf{r}_{pj} = \mathbf{r}_j - \mathbf{r} \qquad (120)$$

Note that the lunar and solar terms have been separated from the summation so that the indices $j = 1, 2, \cdots, 8$ occur for all the planets exclusive of the Earth.

The position vectors in Eq. (119) and (120) are still not in a form consistent with the ephemerides of the Sun, Moon, and Planets. Only the lunar ephemeris is given in geocentric coordinates and it is still necessary to scale it by the factor R_{em} as discussed in Sections III and III-A in order to obtain $\mathbf{r}_☾$ in km. Thus the ephemeris values of $\mathbf{r}_☾$, call them $\mathbf{r}_☾$ (ephem), are converted to km by

$$\mathbf{r}_☾ = R_{em}\, \mathbf{r}_☾ \text{ (ephem)} \qquad (121)$$

where R_{em} is given by Eq. (17) and (18). For the lunar ephemeris provided by JPL, the value of R_{em} for the IAU list of constants was given in Section III-A as (Cf. Eq. 18)

$$R_{em} = 6378.327 \text{ km}$$

Then with the notation of Section III-A,

$$R_{em} = R_{em}\,(1 + \hat{R}_{em}) \qquad (122)$$

where

$$\hat{R}_{em} = \hat{a}_☾ \qquad (123)$$

and $\hat{a}_☾$ is given in terms of $\hat{k}_{ge}$ and $\hat{k}_{gm}$ by Eq. (29) and (30). If $\hat{n}_☾^*$ is neglected because of its relatively great accuracy, then $\hat{a}_☾$ is given by

$$\hat{a}_☾ = \frac{2}{3}\frac{1}{1+\mu}\hat{k}_{ge} + \frac{2}{3}\frac{\mu}{1+\mu}\hat{k}_{gm} - 0.31 \times 10^{-6} \qquad (124)$$

Again since we are basing A on the adopted value of c, it is not necessary to consider a correction c to c in any of the equations. Now Eq. (124) gives, with $2k_{ge} = (-5.3 \pm 2.0) \times 10^{-6}$ and $2k_{gm} = (0.0 \pm 120) \times 10^{-6}$,

$$\hat{a}_☾ = (-1.8 \pm 0.8) \times 10^{-6} \qquad (125)$$

This is the value adopted in Section III-A. Thus our best value of R_{em} is not $\widetilde{R}_{em}$ but instead 6378.315 km. The latter number essentially reflects the deviation of the value of k_{ge}^2 determined by the *Ranger* series from that adopted by the IAU.

The basic ephemerides of the Earth-Moon barycenter and the planets referred to the Sun must also be converted to geocentric coordinates in units of kms. Not only is it necessary to multiply all ephemeris positions by A, but also the geocentric location of the Earth-Moon barycenter is required in the conversion. In terms of the lunar position $\mathbf{r}_☾$ given by Eq. (121), the barycenter is located by the formula

$$\mathbf{r}_B = \frac{\mu}{1+\mu}\,\mathbf{r}_☾ \qquad (126)$$

Then with the heliocentric ephemeris of the Earth-Moon barycenter given by $\mathbf{r}_{\odot B}$ (ephem) in a.u.'s, the geocentric coordinates of the sun in kms are

$$\mathbf{r}_\odot = \frac{\mu}{1+\mu} R_{em}\, \mathbf{r}_☾ \text{ (ephem)} - A\mathbf{r}_{\odot B} \text{ (ephem)} \qquad (127)$$

A planet with heliocentric coordinates $\mathbf{r}_{\odot i}$ (ephem) in a.u.'s has geocentric coordinates in km given by $\mathbf{r}_i$, where

$$\mathbf{r}_i = \mathbf{r}_\odot + A\mathbf{r}_{\odot i} \text{ (ephem)} \qquad (128)$$

Now all quantities in the geocentric equations of motion (Eq. 119) have been specified with the exception of the non-gravitational accelerations $\mathbf{P}$. These are treated separately in Section IV-D-1 and IV-D-2.

*1. **Solar radiation pressure.*** The *Mariner II* spacecraft was equipped with two fairly large solar panels to provide power for the instruments on board, and because the spacecraft was attitude controlled to keep these panels directed at the Sun, a component of force arising from solar radiation pressure occurred in a direction radially outward from the Sun. To understand the physical mechanism behind this force let $\mathcal{F}$ be the flux of radiation from the Sun at the distance of the spacecraft, so that the rate that momentum is transferred to the spacecraft with an effective area A_{eff} is $\mathcal{F}A_{eff}/c$ where c is the velocity of light. Also a fraction of the radiation is reflected from the surface according to some reflection law and the resulting momentum added by the reflection can be accounted for by augmenting the radiation flux by a fraction γ so that the total radial force acting on the probe is

$$f = \frac{\mathcal{F}A_{eff}}{c}(1+\gamma) \quad (0 \leqq \gamma \leqq 1) \tag{129}$$

Of course the flux of radiation obeys an inverse square law with respect to the distance $r_{\odot p}$ from the Sun and it is convenient to express the flux as $\mathcal{F} = \mathcal{F}_o/r^2_{\odot p}$ (a.u.) so that for $r_{\odot p}$ (a.u.) equal to unity the flux is simply equal to the solar constant $\mathcal{F}_o$. According to Abbot (Ref. 32) the value of $\mathcal{F}_o$ is 1.374×10^6 erg/cm^2/sec. Now by dividing the force f by the mass m of the probe the perturbative acceleration $\ddot{\mathbf{r}}_{rad}$ which results from the solar pressure is given by

$$\ddot{\mathbf{r}}_{rad} = \frac{\mathcal{F}_o A_{eff}}{cm}(1+\gamma)\frac{\mathbf{r}^3_{\odot p}\,(\text{a.u.})}{\mathbf{r}_{\odot p}\,(\text{a.u.})} \tag{130}$$

For *Mariner II* the values of A_{eff} and m are known to the same order of accuracy as the solar constant $\mathcal{F}_o$ but γ can lie anywhere in the interval from zero to one. We adopt values of $A_{eff} = 3.83 \times 10^4$ cm^2 and $m = 1.9822 \times 10^5$ gm and leave γ as a free parameter to be determined from the least squares solution of Section V. Then the numerical expression of Eq. (130) is

$$\ddot{\mathbf{r}}_{rad} = (0.8856 \times 10^{-10})(1+\gamma)\frac{\mathbf{r}_{\odot p}\,(\text{a.u.})}{\mathbf{r}^3_{\odot p}\,(\text{a.u.})}\ \text{km/sec}^2 \tag{131}$$

The magnitude of the acceleration from solar radiation pressure is significant. For the approximately 100 days of the *Mariner II* data, an assumed propagation of the effect of the acceleration on position according to the square of the time results in a 3500 km effect on the trajectory.

2. ***Low-thrust attitude-control forces.*** The attitude of the *Mariner II* spacecraft was controlled so that the solar panels were always facing the Sun and the high-gain antenna could be directed at the Earth. This control was produced by the release of cold nitrogen gas from a number of jets. In standard operation, the jets were supposed sufficiently coupled so that there could be no significant perturbation of the trajectory of the spacecraft. However, existing least-squares fits to the *Mariner II* Doppler data (Ref. 1, p. 141; Ref. 2, Section 3) indicate that low-thrust forces were present because of a nonstandard operation of the attitude control system. These forces are assumed unknown here and a set of attitude-control parameters is introduced for estimation. An iterative minimization of the sum of squares of the residuals requires that the attitude-control model be incorporated through **P** of Eq. (118) into the equations of motion for the spacecraft.

The basic assumption for constructing the low-thrust part of **P** is that the forces are represented by two physical processes. The first is a slow gas leak from some unknown point in the attitude-control system. The second is a failure of the gas jets to act in couples. With a model of this sort the number of attitude-control parameters can be kept to a minimum. Also, an estimation of physically meaningful parameters can aid in the postflight engineering evaluation of the attitude-control system, although such an evaluation is not a part of this dissertation.

For an ideal slow leak, the perturbative acceleration in the equations of motion acts in an unknown direction with a magnitude proportional to the pressure in the gas reservoir. If the leak is slow enough, the pressure can be considered a constant. Because the spacecraft remains attitude-stabilized over the entire *Mariner II* mission, the leak must not be so fast as to deplete the supply of gas. Rather than assuming a constant pressure, and consequently a constant perturbative acceleration, the leak is represented by a quadratic function in the time from the epoch. The direction of the thrust is assumed fixed with respect to the principal axes of the spacecraft. The reference plane for these axes is the plane containing the Sun, the spacecraft, and the Earth. One of the axes in the reference plane, the roll axis, lies along the Sun-spacecraft line.

Let $\mathbf{U}_{\odot p}$ be the unit vector in the direction of the heliocentric position of the spacecraft. Also let $\mathbf{U}_{\oplus p}$ be the unit vector in the direction of the geocentric position of the spacecraft. Both unit vectors are simply computed as a function of time from geocentric ephemerides of the Sun and spacecraft. The unit vector **N**, normal to the reference plane, is given by a vector product of $\mathbf{U}_{\odot p}$ and $\mathbf{U}_{\oplus p}$

$$\mathbf{N} = \frac{\mathbf{U}_{\odot p} \times \mathbf{U}_{\oplus p}}{|\mathbf{U}_{\odot p} \times \mathbf{U}_{\oplus p}|} \tag{132}$$

where $| \mathbf{U}_{\odot p} \times \mathbf{U}_{\oplus p} |$ is the magnitude of the vector product. The third axis **T**, is computed by the following formula:

$$\mathbf{T} = \mathbf{N} \times \mathbf{U}_{\odot p} \tag{133}$$

Thus ($\mathbf{U}_{\odot p}$, **T**, **N**) defines an orthonormal right-handed coordinate system which is fixed with respect to a completely attitude-controlled spacecraft. The perturbative acceleration for the leak is therefore given by $(1 - \alpha_1 \times \tau - \alpha_2 \tau^2)(f_1 \mathbf{U}_{\odot p} + f_2 \mathbf{T} + f_3 \mathbf{N})$, where τ is the time from the epoch t_o. The parameters in the leak model are (α_1, α_2), the coefficients in the assumed quadratic decrease in the thrust, and (f_1, f_2, f_3), the magnitude of the thrust at the epoch multiplied by the respective direction cosines of the thrust vector in the spacecraft-fixed system of coordinates.

The failure of the gas jets to act in couples introduces a perturbative acceleration which depends on the degree of unbalance between opposing jets, the limit-cycle characteristics of the attitude-control system and the disturbing torques acting on the spacecraft. In the normal limit-cycle operation the net average thrust over the duration of the mission is practically zero. However, if there is a significant unbalance in the individual thrust levels of the jets, then, on the average, a constant low thrust is imparted to the spacecraft. This effect can be absorbed in the model already introduced for the slow-leak thrust. Again the direction of the net thrust is fixed with respect to the principal axes of the spacecraft.

For the situation where the attitude-control system senses the effects of disturbing torques and subsequently opposes them, an unbalance in the jets imparts a thrust which on the average is proportional to the disturbing torque. If this torque is constant then the average thrust produced by the unbalanced jets in opposing the torque is also constant. Therefore, the previous model will suffice. Notice that torques produced by the slow gas leaks can also be represented by the previous model and thus the thrust imparted by the unbalanced jets is absorbed in the coefficients of the thrust from the leak itself.

The only other significant time-varying torque arises from the solar-radiation pressure acting on a center of pressure not coincident with the center of gravity. Then the torque is proportional to the inverse square of the distance $r_{\odot p}$ of the spacecraft from the Sun. The perturbative acceleration which results from the solar-radiation pressure itself was treated in Section IV-D-1 as a radial perturbation along $\mathbf{U}_{\odot p}$ with a magnitude proportional to $r_{\odot p}^{-2}$.

It is impossible to separate this radial acceleration from that produced by the uncoupled jets which react to radiation-pressure torques. However, it is necessary to add a tangential and normal component to the equations of motion in order to represent the general reaction to these torques. Thus the additional term in the acceleration to account for the unbalanced jets is simply $(K/r_{\odot p}^2)(G_T \mathbf{T} + G_N \mathbf{N})$. The constant K is introduced arbitrarily to make G_T and G_N dimensionless parameters. It is assigned a value equal to the constant of proportionality for the acceleration which results from the incident radiation on the *Mariner II* spacecraft and is the numerical coefficient in Eq. (131).

The total non-gravitational perturbative acceleration **P** is now specified by combining the solar radiation perturbation (Eq. 131) with the attitude control model described in this section. The result is

$$\begin{aligned} \mathbf{P} = & \left[f_1 \alpha(\tau) + \frac{K(1+\gamma)}{r_{\odot p}^2 \,(\text{a.u.})} \right] U_{\odot p} \\ & + \left[f_2 \alpha(\tau) + \frac{K G_T}{r_{\odot p}^2 \,(\text{a.u.})} \right] \mathbf{T} \\ & + \left[f_3 \alpha(\tau) + \frac{K G_N}{r_{\odot p}^2 \,(\text{a.u.})} \right] \mathbf{N} \end{aligned} \tag{134}$$

where $\alpha(\tau)$ is the quadratic function

$$\alpha(\tau) = 1 - \alpha_1 \tau - \alpha_2 \tau^2 \tag{135}$$

and K is the numerical coefficient of Eq. (131).

$$K = 0.8856 \times 10^{-10} \,(\text{km})(\text{a.u.})^2(\text{sec})^{-2}$$

E. Differential Coefficients

In order to apply the differential correction formula of the least-squares procedure, in particular Eq. (49) of Section IV-A, the differential coefficients relating variations in the parameters x of the problem to the data z are required. In the terminology of least squares, the linearized observation equations, $dz = A\,dx$ (Cf. Eq. 46), are the equations of condition and the elements of the matrix A are the differential coefficients which are derived in this section. Again, as in Section IV-A, we rely on matrix notation when discussing the general theoretical aspects of the computation of the coefficients in the matrix A, but the detailed formulation of the differential expressions carried out to obtain computational formulas is accomplished without the complete generality of the matrix notation.

The general situation is that the representation of the Doppler data occurs in two stages. In the first stage the Doppler observables are represented as a function $z = z(q, s, t)$, where q represents the six components of the spacecraft's position and velocity, s is the set of constants needed explicitly to represent the data, in particular the station coordinates, and t is the set of observation times indicating when values of the Doppler observable are required. To make the function $z(q, s, t)$ less mysterious it should be compared with Eq. (86) which gives the actual functional form of the Doppler computation. The second stage of the representation involves the expression of the position and velocity q of the spacecraft as a function $q = q(q_o, p, t)$ of the initial position and velocity q_o of the spacecraft at the epoch, the astronomical constants p required to compute the trajectory and again the observation times t. Thus, the total variation of z with respect to the parameter set $x = (q_o \vdots p \vdots s)$ can be obtained by differentiating the two functions of z and q and combining the result. If the differential coefficients are collected in matrices, the resulting expressions can be written in a compact matrix form useful for theoretical considerations. The matrices containing the various differential coefficients are defined through their location in the following general relations.

$$dz = Gdq + Hds \tag{136}$$

$$dq = Udq_o + Vdp \tag{137}$$

A combination of Eq. (136) and (137) yields a partitioned form of the matrix A introduced in Eq. (46).

$$dz = Adx = GUdq_o + GVdp + Hds \tag{138}$$

Thus, A can be expressed by

$$A = (GU \vdots GV \vdots H) \tag{139}$$

Now the matrices G and H can be obtained in closed form by simply differentiating the Doppler formula (Eq. 46) with respect to the position and velocity q of the spacecraft and the station locations s. The resulting differential coefficients which make up G and H are derived in Appendix C. When one considers the evaluation of the other two matrices in Eq. (139), i.e., U and V, it is not so easy to obtain closed form expressions for the differential coefficients because the equations of motion themselves (Eq. 119) cannot be integrated in closed form. Therefore, it is expected that an accurate calculation of U and V must involve numerical integration techniques, just as does the numerical evaluation of the orbit in the form $q = q(q_o, p, t)$. Of course extreme precision in the coefficients of U and V are not required because they are only used to obtain the least squares solution to the parameters q_o and p and the calculation of residuals, which must be precise, is always performed by an accurate numerical solution to the equations of motion, not by a linear correction to nominal or preliminary residuals. Herrick (Ref. 33) has described four methods for obtaining a numerical evaluation of the matrix U and has derived in addition (Ref. 34) two-body expressions for the differential coefficients in U in terms of his "universal variables." However, analytic expressions of this sort are applicable to situations of near two-body motion, and because planetary probes of the *Mariner* type are dominated by three bodies, first the Earth, then the Sun and finally the target planet, their orbits are difficult to approximate by two-body motion; it is necessary to transfer the origin of coordinates from one body to another depending on which is exerting the greatest influence on the spacecraft. Therefore, to avoid the problems associated with this transformation of coordinates and to obtain accurate coefficients even in a region where two bodies exert an equal influence on the probe, both matrices U and V are evaluated by a numerical integration of expressions associated with what Herrick calls the "linearized Encke" method (Ref. 33, pp. 15–18). It involves the integrating of a set of variational equations. This choice is somewhat arbitrary and has been made on the basis of the method being perhaps the most straightforward and the easiest to implement in a computer program. It may not be the most efficient method but a comparison of the various alternatives for the computation of U and V is outside the scope of this work which is oriented strictly toward a meaningful determination of the constants. To this end we will show in Section V-B that the "linearized Encke" method is sufficiently accurate for the reduction of the *Mariner II* data.

A brief explanation of the method is in order, particularly since the procedure for evaluating the matrix V is not immediately obvious. This explanation is based on arguments of variational calculus as applied to matrices. The interested reader who wants to investigate the relationship of the method to a linearization of Encke's method as used with variant calculations or who wants to explore the alternative methods of computation is referred to Ref. 33 and 34.

It is convenient, at least formally, to express the equations of motion for the probe as six first-order differential equations $\dot{q} = \dot{q}\ (q,\ p,\ t)$ in the position and velocity components q and the constants p. Clearly, analytical formulas exist for these equations and if we partition q into the three position components q_{pos} and three velocity components q_{vel} the explicit form is $\dot{q}_{pos} = q_{vel}$ and $\dot{q}_{vel} = \dot{q}_{vel}\ (q_{pos},\ p,\ t)$ where the function $\dot{q}_{vel}\ (q_{pos},\ p,\ t)$ is given by the accelerations in Eq. (119). Now, the equations of motion can be differentiated with respect to q and p to obtain a set of differential coefficients. Thus, formulas can be obtained for the elements of differential coefficient matrices defined by $d\dot{q} = \Phi\ dq + \Theta\ dp$. On the other hand the solution to the equations of motion is of the form $q = q\ (q_o,\ p,\ t)$. It is not important that the function $q\ (q_o,\ p,\ t)$ cannot be obtained in closed form, the only consideration here is that the function exists and that it is determined by the initial conditions q_o, the constants p, and the time t. Then the variation in q which arises from differentiating the solution can be written $dq = Udq_o + Vdp$ as in Eq. (137). Because of the existence of the equations of motion the time derivative of dq exists and a second expression for $d\dot{q}$ is $d\dot{q} = \dot{U}dq_o + \dot{V}dp$ which must be equal to the first expression for $d\dot{q}$. Therefore

$$\dot{U}dq_o + \dot{V}dp = \Phi\ dq + \Theta\ dp \tag{140}$$

and by substituting $dq = Udq_o + Vdp$ on the right-hand side of Eq. (140) and equating coefficients the following differential equations are obtained

$$\dot{U} = \Phi\ U \tag{141}$$

$$\dot{V} = \Phi\ V + \Theta \tag{142}$$

A numerical integration of Eq. (141) and (142) will yield the required matrices U and V subject to the initial conditions $U\ (t_o) = 1$ and $V\ (t_o) = 0$ However we can obtain an integral of the system of equations for U and V. From Eq. (140) the matrix Φ is equal to $\dot{U}\ U^{-1}$ and if this is substituted in Eq. (142) there results

$$\dot{V} = \dot{U}\ U^{-1}\ V + \Theta \tag{143}$$

or by premultiplying by U^{-1}, a new expression is obtained.

$$U^{-1}\ \dot{V} - U^{-1}\ \dot{U}\ U^{-1}\ V = U^{-1}\ \Theta \tag{144}$$

The expression $U^{-1}\ \dot{U}\ U^{-1}$ is simply the negative of the time derivative of U^{-1}. Therefore

$$U^{-1}\ \dot{V} + \left(\frac{d}{dt}\ U^{-1}\right) V = U^{-1}\ \Theta \tag{145}$$

and

$$\frac{d}{dt}\,(U^{-1}\ V) = U^{-1}\ \Theta \tag{146}$$

With the previously stated initial conditions the matrix V is given by an integral of the form

$$V(t) = U\ (t) \int_{t_0}^{t} U^{-1}\ \Theta\ dt \tag{147}$$

The actual procedure used to compute U and V is to integrate Eq. (141) along with the equations of motion by a step-by-step numerical integration procedure and then to evaluate V by numerical quadrature according to Eq. (147). All that is required besides the numerical integration procedure is formulas for the elements of the matrices Φ and Θ which are derived in Appendix D.

V. Numerical Results

In this section, numerical results are presented which lead to a selection of values for the constants determined from the *Mariner II* data. Only the data listed in Appendix E are used, but several solutions for the constants are given in order to explore the effect on the solutions of various assumptions about the nature of the *Mariner II* orbit. Unfortunately, there are more unknowns in the problem than the orbital parameters and the three constants of interest, and it is necessary to consider in addition such matters as the precision of the numerical methods, the behavior of the low-thrust forces, and also uncertainties in the solar radiation force, the Venus ephemeris, and the station locations.

First of all, the numerical accuracy of the numerical integration of the equations of motion is investigated in Section V-A, and then in Section V-B the accuracy of the differential coefficients, as computed by the method described in Section IV-E, is evaluated by comparing the numerical results of that method with those obtained by performing variant calculations with the Cowell form of the equations of motion. The variant calculations also permit the evaluation of the numerical accuracy of the calculation of Doppler residuals. It is concluded, as a result of all this, that the *Mariner II* orbit can be computed to better than seven significant figures, and probably closer to eight, and that the numerical error in the orbit is caused by a rounding of numbers. There is no noticeable error growth with time over the duration of the *Mariner II* orbit. With respect to the differential coefficients, the conclusion is that the method of Section IV-E yields a satisfactory approximation to the coefficients as computed by variant calculations. Further, the accuracy for constants which affect the planetary encounter portion of the *Mariner* orbit, the mass and position of Venus, is maintained even when the epoch for the numerical integration is taken at the beginning of the cruise data, over three months before encounter. However, the agreement for the mass of Venus is not as satisfactory as for the position coordinates. An independence of epoch is important in the solutions for the constants because both epochs of Sept. 5 and Dec. 8, 1962 are used to define *Mariner II* orbital elements in the form of cartesian coordinates. The various solutions, which lead to the values of the constants summarized in Section V-E, are given in Section V-D.

A. Accuracy of Orbit Computation

In Section IV-D the equations of motion for the spacecraft were given and it was pointed out that their solution could be obtained by numerical integration. The computer program used for this integration is that developed by JPL for their trajectory calculations (Ref. 24). It uses an Adams-Moulton method which is applied directly to the equations in the Cowell form. Both predictor and corrector formulae are used with truncation occurring at the 6th difference. The predictor and corrector formulae are given, respectively, as Eq. (20.3) and Eq. (20.5) in Ref. 35.

In order to investigate the accuracy of the numerical integration, the two-body equations of motion are substituted for those of Section IV-D, where the two bodies are the Sun and spacecraft. The numerical integration should, for this situation, produce heliocentric cartesian coordinates which agree with those predicted by the literal solution to the two-body problem. To test whether this is so, the cartesian coordinates from the numerical integration, given to eight significant figures, are converted to the classical Keplerian orbital elements by means of the two-body formulae so that osculating orbital elements are produced as a function of the time from the initial epoch. If the computed orbit is actually a two-body one, then the osculating elements will be constant over the interval of integration; and, more important, any instability in the numerical integration procedure will be evident in a greater and greater deviation of the elements from constancy as the integration progresses step-by-step.

The epoch for the integration is selected as 1962, Sept. 7, $00^h24^m07^s\!.000$ ET and all computations are carried out in mean coordinates of 1950.0. The initial heliocentric equatorial coordinates are

$$x = 1.4299640 \times 10^8 \text{ km}$$
$$y = -0.4064246 \times 10^8 \text{ km}$$
$$z = 0.16906182 \times 10^8 \text{ km}$$
$$\dot{x} = 6.0853449 \text{ km/sec}$$
$$\dot{y} = 23.748898 \text{ km/sec}$$
$$\dot{z} = 11.234259 \text{ km/sec}$$

Using a gravitational constant of $0.13271411 \times 10^{12}$ in the conversion to orbital elements, we find the following set of elements at the epoch:

$$a = 1.2694774 \times 10^8 \text{ km}$$
$$e = 0.19334007$$
$$M_o = 156^\circ\!.58351$$
$$i = 1^\circ\!.8772439$$
$$\Omega = 333^\circ\!.03331$$
$$\omega = 171^\circ\!.71162$$

The orientation elements (i, Ω, ω) are referred to the 1950.0 mean ecliptic and equinox and the epoch for the mean anomaly M_o is 1962, Sept. 5, $00^h24^m7^s\!.000$ ET The orbit corresponding to the elements is a close approximation to the actual *Mariner II* trajectory during the time interval from 1962, Sept. 5 to early December when the Sun dominates the motion of the spacecraft.

The computed elements are given at five-day intervals in Table 2. The time of tabulation is $00^h24^m07^s\!.000$ ET Only the last three digits of each element are tabulated because the first five digits are always the same and are equal to those in the initial values of the elements.

The important observation with respect to the values in the table is that there is no noticeable instability in the integration procedure, and it appears that errors in the *Mariner II* trajectory are predominantly caused by rounding. The cartesian coordinates are accurate to better then seven significant figures.

Table 2. Last three digits in two-body orbital elements as computed by numerical integration

Date (1962)	a	e	M_0	i	Ω	ω
Sept. 7	774	007	351	439	331	162
Sept. 12	774	007	351	439	331	163
Sept. 17	774	013	351	437	331	162
Sept. 22	774	009	351	440	331	163
Sept. 27	774	009	351	434	330	163
Oct. 2	774	005	351	440	331	162
Oct. 7	775	013	350	437	330	163
Oct. 12	774	009	351	434	330	163
Oct. 17	774	007	351	440	330	163
Oct. 22	775	015	351	432	331	163
Oct. 27	774	007	351	438	331	162
Nov. 1	775	011	351	436	331	163
Nov. 6	775	013	350	437	331	163
Nov. 11	774	007	352	437	331	163
Nov. 16	774	001	352	439	331	163
Nov. 21	774	005	353	439	331	163
Nov. 26	775	009	351	435	331	162
Dec. 1	774	013	352	437	331	163
Dec. 6	774	007	352	434	330	163

B. Accuracy of Differential Coefficients

Because the calculation of differential coefficients, which relate variations in the constants and orbital elements to variations in the data, is not particularly straightforward (Cf. Section IV-E), it is advisable to investigate the accuracy of the calculation by comparing the results of the linearized Encke method used in the solutions for the constants, with that of the more straightforward variant calculations. We will perform this comparison for the constants only, but it is clear from Eq. (147) that the method will not produce a favorable comparison if the so-called state transition matrix U for the cartesian coordinates is not reasonably accurate. Thus, a verification of the accuracy of the matrix V, as computed by numerical quadrature according to Eq. (147), will also verify the accuracy of the matrix U, which is obtained by an application of the Adams-Moulton numerical integration procedure to the differential equations in Eq. (141).

First of all, consider the accuracy of the coefficients for the three constants of particular interest, the masses of Venus and the Moon and the astronomical unit. The variations used for the variant calculations method are

$$\Delta A = +\,1000\ \text{km}\ (0.0007\%)$$

$$\Delta k^2_{gm} = +\,1.0\ \text{km}^3/\text{sec}^2\ (0.02\%)$$

$$\Delta M^*_v = +\,1.0 \times 10^{-9}\ (0.04\%)$$

The resulting variations in the Doppler data are reflected in columns (2), (4) and (6) of Table 3. The first column gives the time at which the variation is evaluated. The epoch for the calculations is 1962, Sept. 5, $00^h24^m07^s.000$ ET. A non-entry for the coefficients obtained from variant calculations indicates that the variation in the Doppler observation at that time is not numerically significant. Columns (3), (5) and (7) of Table 3 give the partial derivatives of the Doppler data with respect to the three constants by the linearized Encke method (Eq. 147).

The relatively poor agreement for the mass of Venus suggests that perhaps the coefficients would be more accurate if computed with an epoch near planetary encounter. That this is so is demonstrated in Table 4 where the epoch for the comparison of the methods is 1962 Dec. 7, $00^h.0$ E.T. The interval right around the time of closest approach to Venus, about $20^h.0$ on Dec. 14, is tabulated separately at the bottom of the table. The agreement between the two columns is excellent.

Table 5 gives the variant calculation coefficients in columns (2), (4) and (6) and the linearized Encke coefficients in columns (3), (5) and (7) for the position of Venus on 1962 Dec. 14, $20^h.0$ E.T. The variation in each position component is + 1000 km for the variant calculations method. The epoch for the calculations is Sept. 5. The same calculations with an epoch on Dec. 7 are shown in Table 6. Notice that the two tables are quite similar in the size and comparison of the coefficients; and, hence, any difference in the two methods of computing the coefficients, with regard to non-linearities, seem independent of whether the epoch is Sept. 5 or Dec. 7. This was not the case for the mass M^*_v, but the agreement for the earlier epoch is still acceptable.

C. *A-priori* Information

The least-squares estimation formula (Eq. 49) used in obtaining the constants from the *Mariner II* data has a provision for adding *a-priori* information on the parameters x in the form of values $\tilde{x}$ and an associated covariance matrix $\tilde{\Gamma}_x$. By *a-priori* information is meant assumed knowledge, completely independent of the *Mariner II* data, which bears on the parameters necessary to represent the Doppler observations. Indeed, even if we used

Table 3. Accuracy of differential coefficients in A, k^2_{gm} and M^s_v (Sept. 5 epoch)

Date (21^h UT)	$\frac{\Delta CC3}{\Delta A} \times 10^{-4}$ ($sec^{-1} km^{-1}$)	$\frac{\partial CC3}{\partial A} \times 10^{-4}$ ($sec^{-1} km^{-1}$)	$\frac{\Delta CC3}{\Delta k^2_{gm}}$ (sec km^{-3})	$\frac{\partial CC3}{\partial k^2_{gm}}$ (sec km^{-3})	$\frac{\Delta CC3}{\Delta M^s_v} \times 10^7$ (sec^{-1})	$\frac{\partial CC3}{\partial M^s_v} \times 10^7$ (sec^{-1})
9/10		0.000	−0.020	−0.018		0.00
9/15		0.002	−0.011	−0.011		0.00
9/20		0.004	0.010	0.011		0.00
9/25		0.007	0.020	0.020		0.00
9/30		0.010	0.009	0.010		0.00
10/5		0.011	−0.007	−0.005		0.01
10/10		0.006	−0.006	−0.007		0.01
10/15		−0.007	0.017	0.016		0.02
10/20		−0.034	0.032	0.032		0.02
10/25		−0.079	0.029	0.027		0.03
10/30	−0.156	−0.147	0.012	0.011	0.20	0.05
11/04	−0.234	−0.240	0.002	0.001	0.59	0.08
11/08	−0.371	−0.333	0.010	0.009	0.39	0.11
11/13	−0.508	−0.471	0.029	0.028	0.39	0.18
11/18	−0.664	−0.630	0.027	0.030	0.39	0.29
11/23	−0.859	−0.804	0.014	0.014	0.78	0.50
11/28	−1.016	−0.982	−0.008	−0.006	1.17	0.88
12/3	−1.191	−1.150	−0.014	−0.014	2.15	1.67
12/8	−1.367	−1.315	−0.002	−0.003	4.30	3.79
12/13	−4.746	−4.675	0.000	0.003	28.71	28.03
12/18	−135.0	−132.8	1.379	1.433	147.3	140.7
12/23	−13.09	−11.52	1.137	1.178	59.18	58.71
12/28	142.5	142.6	0.813	0.837	−49.41	−41.30

Table 4. Accuracy of differential coefficients in M^s_v (Dec. 7 epoch)

Date (21^h UT)	$\frac{\Delta CC3}{\Delta M^s_v} \times 10^7$ (sec^{-1})	$\frac{\partial CC3}{\partial M^s_v} \times 10^7$ (sec^{-1})	Date (21^h UT)	$\frac{\Delta CC3}{\Delta M^s_v} \times 10^7$ (sec^{-1})	$\frac{\partial CC3}{\partial M^s_v} \times 10^7$ (sec^{-1})
12/9	1.8	1.8	12/27	−15.2	−15.0
12/11	5.3	5.4	12/29	−57.8	−57.7
12/13	24.0	24.0	12/14		
12/14	545	545	$16^h.0$	185.9	185.8
12/15	196.1	196.1	$17^h.0$	259.4	259.2
12/17	156.6	156.5	$18^h.0$	393.8	393.1
12/19	127.7	127.3	$19^h.0$	620.3	620.3
12/21	96.5	96.4	$20^h.0$	753.1	751.2
12/23	61.9	62.3	$21^h.0$	545.3	545.2
12/25	25.2	25.1	$22^h.0$	393.0	392.7

the classical least-squares procedure, rather than that indicated by Eq. (54), the initial values of the constants and conclusions about their accuracy before and after the introduction of the *Mariner II* data would depend on such *a-priori* information. In fact the additional term in Eq. (49) is used simply as a device to reduce to a systematic procedure the consideration of such *a-priori* information.

Most of the information is contained in the IAU report on constants (Ref. 8) which gives limits on the values of the constants as of 1964. More recent work on the

Table 5. Accuracy of differential coefficients in the position of Venus (Sept. 5 epoch)

Date (21^h UT)	$\frac{\Delta CC3}{\Delta X_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial X_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\Delta CC3}{\Delta Y_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial Y_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\Delta CC3}{\Delta Z_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial Z_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)
12/9	0.029	−0.004	0.057	0.027	0.065	0.053
12/11	−0.010	−0.008	0.065	0.067	0.147	0.150
12/13	0.043	0.043	0.611	0.614	1.38	1.38
12/14	262	261	13.3	19.3	−116	−116
12/15	98.8	98.4	−27.4	−24.5	−19.1	−18.1
12/17	92.5	92.1	−35.9	−33.1	−9.50	−8.43
12/19	85.8	85.2	−47.0	−44.3	0.629	1.73
12/21	77.8	77.3	−59.8	−57.3	12.2	13.3
12/23	68.5	68.2	−74.0	−71.7	25.3	26.3
12/25	58.1	58.1	−89.5	−87.4	39.4	40.4
12/27	46.8	47.1	−106	−104	54.6	55.4
12/29	34.4	35.3	−123	−122	70.8	71.1
12/14						
$16^h.0$	17.0	16.9	22.3	22.3	32.6	32.5
$17^h.0$	34.5	34.2	37.3	37.3	48.1	48.1
$18^h.0$	79.0	78.1	67.8	67.9	71.0	71.4
$19^h.0$	193	188	120	121	79.8	81.8
$20^h.0$	332	326	118	123	−17.5	−13.7
$21^h.0$	262	261	13.3	19.3	−116	−116
$22^h.0$	168	169	−32.5	−28.3	−103	−103

Table 6. Accuracy of differential coefficients in the position of Venus (Dec. 7 epoch)

Date (21^h UT)	$\frac{\Delta CC3}{\Delta X_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial X_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\Delta CC3}{\Delta Y_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial Y_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\Delta CC3}{\Delta Z_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)	$\frac{\partial CC3}{\partial Z_v} \times 10^{-3}$ ($sec^{-1}km^{-1}$)
12/9	−0.006	−0.003	0.010	0.014	0.029	0.032
12/11	−0.010	−0.008	0.055	0.054	0.127	0.129
12/13	0.039	0.040	0.596	0.597	1.36	1.36
12/14	262	262	13.2	19.2	−117	−116
12/15	99.0	98.8	−27.5	−24.7	−19.2	−18.1
12/17	92.8	92.4	−36.0	−33.4	−9.53	−8.48
12/19	85.8	85.6	−47.1	−44.6	0.602	1.72
12/21	77.8	77.5	−59.9	−57.6	12.2	13.4
12/23	68.5	68.3	−74.3	−72.0	25.3	26.5
12/25	58.3	57.9	−89.8	−87.7	39.5	40.8
12/27	46.8	46.5	−106	−104	54.8	56.2
12/29	34.4	34.1	−123	−122	70.8	72.4
12/14						
$16^h.0$	17.0	16.9	22.1	22.1	32.5	32.4
$17^h.0$	34.4	34.2	37.0	37.1	47.9	47.9
$18^h.0$	79.0	78.0	67.3	67.5	70.8	70.9
$19^h.0$	193	189	119	120	79.0	81.0
$20^h.0$	333	326	117	122	−18.1	−14.4
$21^h.0$	262	262	13.2	19.2	−117	−116
$22^h.0$	168	169	−32.5	−28.3	−103	−103

determination of A from planetary radar bounce data* would indicate a value of 149597890 km as more appropriate than the value of 149600000 km in the IAU report. There seems to be good justification for assigning A an *a-priori* standard deviation of ± 100 km. In the solutions for the constants, we sometimes use this value when we wish to introduce the planetary bounce data into the *Mariner II* results, but solutions are also given that assume a much larger *a-priori* uncertainty in order to obtain an independent determination of A from *Mariner II.* For the new value of A, it is necessary to compute a new correction $\hat{A}$ by Eq. (20), which is

$$\hat{A} = (-14.10 \pm 0.67) \times 10^{-6}$$

A-priori numerical values and uncertainties for other constants are given in Section III-E. Therefore, the remainder of this section is devoted to exploring the *a-priori* information about constants peculiar to the *Mariner II* data, in particular constants for the low-thrust forces, solar radiation pressure, and the station locations.

The low-thrust and solar radiation forces are given by Eq. (134) and the constants which are treated as unknowns are $f_1, f_2, f_3, \alpha_1, \alpha_2, \gamma, G_T$, and G_N. The uncertainty in the solar radiation proportionality constant K is absorbed in γ, G_T, and G_N. The most obvious information about this set of constants is that by definition $0 \leqq \gamma \leqq 1$ and so we set the *a-priori* value of γ equal to $\tilde{\gamma} = 0.0 \pm 1.0$.

With respect to the low-thrust constants we will set them all equal to zero initially and assign uncertainties on f_1, f_2, f_3, G_T and G_N equal to 100% of the solar radiation force. A force of this magnitude would be unreasonably large for the *Mariner II* spacecraft. A 100% force yields uncertainties of ± 1.0 for G_T and G_N. The maximum acceleration imparted to the spacecraft from solar radiation pressure is 1.66×10^{-10} km/sec^2 (Cf. Eq. 131) and thus this same value is used for the *a-priori* uncertainties on f_1, f_2, and f_3.

A piece of information about the magnitude of α_1 and α_2 in Eq. (135) is that the spacecraft remained attitude stabilized until at least Dec. 30 and thus did not deplete its supply of cold nitrogen gas. If we assume that a leak occurred through a permanent hole in the system and was present throughout the whole duration of accurate tracking data, then α_1, and α_2 must be bounded so that $\alpha(\tau)$ does not go to zero in the 117-day interval. This is assured if α_1 is less than $1/\tau$ and α_2 less than $1/\tau^2$ where $\tau = 117$ days. Thus α_1 and α_2 are initially set equal to zero with uncertainties of $\pm 10^{-7}$ sec^{-1} and $\pm 10^{-14}$ sec^{-2}, respectively.

The situation with respect to the station locations of the transmitting and receiving antennas is that their relative positions can be determined quite accurately since they are only a few kilometers apart. However, we want to assume a large uncertainty in their absolute locations because the *Mariner II* data can determine the absolute positions better than the *a-priori* survey data. Therefore, we will assume a ± 1 km absolute error in station location and ± 10 m relative error. To define the relative positions of the stations, define two parameters ΔR and $\Delta\lambda$ by

$$\Delta R = R_{11} - R_{12} \qquad (148)$$

$$\Delta\lambda = \lambda_{11} - \lambda_{12} \qquad (149)$$

The latitude is not considered because only two components of the station location can be determined with the *Mariner II* data and the least-squares solutions correct only radius R and longitude λ, not latitude ϕ. Now in the solutions the station location parameters for both stations are R_{11}, λ_{11}, R_{12} and λ_{12} and *a-priori* values for these coordinates are available from survey data. However, because of the assumed high accuracy of the differences ΔR and $\Delta\lambda$, when an *a-priori* covariance matrix is used in the solutions, it must reflect this high accuracy through correlations between the four station coordinates which are obtained by the least-squares solution. A more direct approach would be to solve for R_{11}, λ_{11}, say, and the differences ΔR and $\Delta\lambda$. Then the correlations between the four parameters could be assumed zero and the covariance matrix would have a simple diagonal form. However, the computer program is not set up this way, so the simple diagonal matrix on the set $(R_{11}, \lambda_{11}, \Delta R, \Delta\lambda)$ must be transformed to the covariance matrix on $(R_{11}, \lambda_{11}, R_{12}, \lambda_{12})$. To do this, write

$$R_{12} = R_{11} - \Delta R \qquad (150)$$

$$\lambda_{12} = \lambda_{11} - \Delta\lambda \qquad (151)$$

Then the first-order variations in R_{12} and λ_{12} are

$$\delta R_{12} = \delta R_{11} - \delta \Delta R \qquad (152)$$

$$\delta \lambda_{12} = \delta \lambda_{11} - \Delta\lambda \qquad (153)$$

*Private communication with experimenters at JPL (Melbourne, Muhleman, Holdridge) and Lincoln Laboratories (Ash, Shapiro).

Remembering that the covariances between R_{11}, λ_{11}, ΔR and $\Delta\lambda$ are all zero, the covariances between R_{11}, λ_{11}, R_{12} and λ_{12} can be computed by forming the expected value of various products of δR_{11}, $\delta\lambda_{11}$ and the δR_{12} and $\delta\lambda_{12}$ defined by Eq. (152) and (153). For example, the covariance between R_{11} and R_{12} is

$$\mathcal{E}\{\delta R_{11}\,\delta R_{12}\} = \mathcal{E}\{(\delta R_{11})^2\} - \mathcal{E}\{(\delta \Delta R)\,(\delta R_{11})\} \tag{154}$$

But the second expectation is zero, and Eq. (154) yields the result that

$$\rho_{R_{11},\,R_{12}}\,\sigma_{R_{11}}\,\sigma_{R_{12}} = \sigma^2_{R_{11}} \tag{155}$$

where $\rho_{R_{11},\,R_{12}}$ is the correlation coefficient which can be written

$$\rho_{R_{11},\,R_{12}} = \frac{\sigma_{R_{11}}}{\sigma_{R_{12}}} \tag{156}$$

The variance on R_{12} is obtained by squaring Eq. (152) and again taking the expected value

$$\sigma^2_{R_{12}} = \sigma^2_{R_{11}} + \sigma^2_{\Delta R} \tag{157}$$

The other variances and covariances can be evaluated in a similar fashion to obtain the complete 4×4 covariance matrix on station coordinates

$$\tilde{\Gamma} = \begin{pmatrix} \sigma^2_{R_{11}} & 0 & \sigma^2_{R_{11}} & 0 \\ 0 & \sigma^2_{\lambda_{11}} & 0 & \sigma^2_{\lambda_{11}} \\ \sigma^2_{R_{11}} & 0 & \sigma^2_{R_{11}} + \sigma^2_{\Delta R} & 0 \\ 0 & \sigma^2_{\lambda_{11}} & 0 & \sigma^2_{\lambda_{11}} + \sigma^2_{\Delta\lambda} \end{pmatrix} \tag{158}$$

The uncertainties of 10 m and 1 km in $(\Delta R, \Delta\lambda)$ and (R_{11}, λ_{11}), respectively, yield

$$\sigma_{R_{11}} = R\cos\phi\,\sigma_{\lambda_{11}} = 1.0\ \text{km} \tag{159}$$

$$\sigma_{\Delta R} = R\cos\phi\,\sigma_{\Delta\lambda} = 0.01\ \text{km} \tag{160}$$

and with $R\cos\phi = 5212$ km, the *a-priori* covariance matrix, or actually the more useful inverse covariance matrix, which can be made a part of the whole inverse matrix $\tilde{\Gamma}x^{-1}$ for Eq. (49), is

$$\tilde{\Gamma}^{-1} = \begin{pmatrix} R_{11} & \lambda_{11} & R_{12} & \lambda_{12} \\ 10{,}001 & 0 & -10{,}000 & 0 \\ 0 & 82{,}808{,}280 & 0 & -82{,}800{,}000 \\ -10{,}000 & 0 & 10{,}000 & 0 \\ 0 & -82{,}800{,}000 & 0 & 82{,}800{,}000 \end{pmatrix}$$

D. Numerical Solutions

The methods of Section IV are applied in this section to obtain values for the three constants (μ, M_v^*, A) as well as the coordinates of the transmitting and receiving stations and the position of Venus during the encounter period. Previous solutions (Ref. 1 and 2) have also obtained values for the constants but both suffer from serious defects. The first solution of Ref. 1 does not include the effects of the low-thrust forces or ephemeris errors. The second solution of Ref. 2 includes these effects in the differential correction but does not apply the corrected parameters for a recomputation of residuals. Also, the corrections to the position of Venus are erroneous in Ref. 2 because of an error in the computer program used to compute the differential coefficients. Until the corrections to the Venus position could be applied to the ephemeris for a recomputation of the *Mariner II* orbit, it was not possible to check the computation of the coefficients as done in Section V-B.

The limitations of the previous solutions do not allow the determination of definitive values for the constants, but the solutions are useful in establishing that the *Mariner II* determinations are not sensitive to reasonable errors in the Earth's ephemeris or in the velocity coordinates of Venus. The demonstration of this fact through solutions for the Earth's elements and velocity coordinates of Venus is not duplicated here, but can be found in Ref. 2. Instead, the emphasis of this section is on the selection of a proper model for the spacecraft's low-thrust forces and on displaying a number of solutions for the constants under varying assumptions. As a result, the values of the constants given in Section V-E can be accepted with a fairly high degree of confidence as reflected in the assigned standard deviations for the determined parameters.

For comparison purposes, the value of μ^{-1} obtained from the previous solutions of Ref. 1 and 2 is 81.3012 ±0.0034. The mass ratio $(M_v^*)^{-1}$ from Ref. 1 is 408526 ±30 and from Ref. 2 it is 408587 ±25 where in both cases the values are adjusted for an assumed astronomical unit of 149597900 km. A definitive value of A was not claimed in either of the previous solutions. Again, it is emphasized that these values are subject to systematic errors which are now known and which have been practically eliminated in the current solutions. This improved determination of the constants gives

$$\mu^{-1} = 81.3001 \pm 0.0013$$

$$(M_{r}^{s})^{-1} = 408504.8 \pm 5.5$$

$$A = (149597546 \pm 500)\text{km}$$

1. Determination of low-thrust model. In Section IV-D-2 a model for low-thrust forces is described based on physical processes occurring in the attitude-control system. Consequently besides the solar radiation constant γ, the situation of significant forces because of an unbalance in the firing of attitude control jets is handled by the parameters G_T and G_N. On the other hand, a slow leak from a small hole in the system is handled by the parameters f_1, f_2, f_3, α_1 and α_2 which also take care of the case of unbalanced jets when the dominant torque on the spacecraft is not solar radiation pressure. Thus the parameters G_T and G_N are intended for a very special situation where the jets are unbalanced and the dominant torque is proportional to the inverse square of the spacecraft's distance from the Sun, as for radiation pressure torques.

It is not possible to estimate all of these parameters simultaneously because of the relatively short time interval over which data are available. Therefore the approach of this section is to display two least-squares solutions to the cruise data, defined by data in the time interval from Sept. 5 to Dec. 7, 1962 before Venus dominates the spacecraft's motion, such that the first solution (Solution I) assumes that the forces obey the model described by the parameter set (f_1, f_2, f_3, α_1, α_2) and the second (Solution II) assumes a model described by the set (γ, G_T, G_N). Of course, even in the first solution it is necessary to include the parameter γ because, whether or not there are low-thrust forces present, there is always a direct force component from the solar radiation. In the second solution, the parameter γ is varied to allow both for the direct radiation perturbation on the trajectory and for an indirect perturbation from uncoupled jets which fire because of the solar radiation torque on the spacecraft.

The results of the two solutions are given in Table 7. Parameters in the solutions are given in the first column, the second column gives the assumed initial values of the parameters with their *a-priori* errors as discussed in Section V-C, and the third and fourth columns give, respectively for Solutions I and II, the corrections to the initial values that best fit the data in the least-squares sense. The position and velocity coordinates of the spacecraft represent as initial conditions, the orbital elements referred to the true equator and equinox at the initial epoch (1962, Sept. 5, $00^h24^m07^s.000$ E.T.). If a parameter is not included in the solution, then its *a-priori* error is zero by definition. Thus, for example, k_{ge} is assumed perfectly known in both solutions, while G_T and G_N are known perfectly in Solution I but known *a-priori* to ± 1.0 in Solution II.

The most obvious method of discovering which parameter set best represents the actual low-thrust forces on the spacecraft is to compare the two solutions with respect to their ability to fit the data. One measure of the degree of approximation to the real data is the weighted sum of squares of the residuals, the function $S(x)$ in Eq. (43). Another measure is the function $Q(x)$ defined by Eq. (44) which is minimized by the indicated corrections in the solutions. The residuals for the two solutions yield the following values for the functions.

	Solution I	Solution II
$S(x)$	595.70	515.64
$Q(x)$	596.18	524.76

It is highly doubtful that the difference in $S(x)$ and $Q(x)$ between the two solutions can be considered significant. Note, because of the weighting of the data in the two functions, that they are dimensionless, and their magnitudes depend on the values of the weights. For all solutions in this section the nominal weight assigned to the Doppler data is $3500(Hz^{-2})$, but we will return to a more detailed discussion of weighting later. For the moment the chief concern is the selection of the proper low-thrust model for use in subsequent solutions.

Because $S(x)$ and $Q(x)$ reveal very little about the comparative aspects of the two fits to the data, the residuals themselves are next investigated. However, rather than list the complete set of 1006 residuals, the residuals for each horizon-to-horizon pass of data are compressed into two numbers. The first of these is the mean residual which is obtained by adding together all the residuals of a particular pass and then by dividing by the number of residuals in that pass. The difficulty with this sort of number is that large systematic effects in a single pass of residuals can go undetected if they average to zero over the pass. To avoid missing such systematic effects, the RMS residual is also computed by summing the squares of all the residuals and again dividing by the number in the pass. Admittedly, these two

Table 7. Comparison of low-thrust models

Parameter	*A-priori* value	Solution I correction	Solution II correction
x(km)	−1424206.8 ± 10^6	22.6 ± 49.6	−120.8 ± 42.1
y(km)	−1939477.0 ± 10^6	34.1 ± 67.7	−142.4 ± 58.3
z(km)	−100648.79 ± 10^6	10.6 ± 77.9	99.9 ± 71.8
x(km/sec)	−1.7444904 ± 1.0	(−0.28 ± 1.57) × 10^{-5}	(−1.33 ± 1.26) × 10^{-5}
y(km/sec)	−2.4234005 ± 1.0	(0.31 ± 1.30) × 10^{-5}	(0.22 ± 1.09) × 10^{-5}
z(km/sec)	−0.11009572 ± 1.0	(−1.90 ± 6.15) × 10^{-5}	(18.66 ± 5.36) × 10^{-5}
k^2_{gm}(km^3/sec^2)	4902.8365 ± 10.0	−0.052 ± 0.069	0.114 ± 0.068
k^2_{ge}(km^3/sec^2)	398601.27 ± 0.0	—	—
A(km)	149597890*	−202 ± 970	−6670 ± 1250
f_1(km/sec^2)	0.0 ± 10^{-10}	(−0.03 ± 0.35) × 10^{-10}	—
f_2(km/sec^2)	0.0 ± 10^{-10}	(−0.36 ± 0.02) × 10−10	—
f_3(km/sec^2)	0.0 ± 10^{-10}	(−0.15 ± 0.15) × 10^{-10}	—
α_1(sec^{-1})	0.0 ± 10^{-7}	(0.07 ± 0.29) × 10^{-7}	—
α_2(sec^{-2})	0.0 ± 10^{-14}	(0.81 ± 0.73) × 10^{-14}	—
G_T	0.0 ± 1.0	—	−0.381 ± 0.019
G_N	0.0 ± 1.0	—	0.109 ± 0.027
γ	0.0 ± 1.0	0.00 ± 0.34	0.047 ± 0.057
R_{11}(km)	6372.0149 ± 1.0	−0.003 ± 0.008	−0.014 ± 0.007
λ_{11}	243.°15067 ± 0.°01	(−0.06 ± 0.15) × 10^{-3}	(−0.28 ± 0.13) × 10^{-3}
R_{12}(km)	6371.8805 ± 1.0	0.002 ± 0.007	−0.008 ± 0.007
λ_{12}	243.°19454 ± 0.°01	(−0.05 ± 0.15) × 10^{-3}	(−0.24 ± 0.13) × 10^{-3}

**A-priori* uncertainty in A is ±1000 km for Solution I, ±2000 km for Solution II.

derived or compressed residuals are less informative than the original set of residuals, but the listing of residuals for all the solutions is not reasonable, although all residuals for Solutions III and VII, which represent the best fits to the data, are given in Appendix E along with the data themselves.

The compressed residuals for Solutions I and II are given in Table 8. Note that the first entry for pass 9/5–9/6 is actually a combination of two passes of data over the last 4½ hr as the spacecraft was setting at the station.

It appears as though Solution II provides a slightly better fit to the data than Solution I whose residuals near the end of the cruise data seem biased toward positive values. The negative bias of the residuals in Solution II is less pronounced and the RMS residuals are generally smaller in this period. However, before G_T and G_N are accepted as the parameters to represent the low-thrust forces, it is advisable to look at the corrections to the initial values of the parameters. This shows that there is nothing which conflicts with the *a-priori* uncertainties on the parameters with the exception of the astronomical unit (a.u.) A. A correction of −6670 km would result in a value of 149591220 km for A which seems highly unlikely in view of the results obtained from the bounce experiments.

The conclusion from the comparison of the two low-thrust models is that either will produce a satisfactory fit to the data, but that in order for the parameters G_T and G_N to do so, the value of A must be given an unreasonably low value, a constraint not imposed by the parameters of Solution I. Therefore, the model of Solution I is chosen to represent the *Mariner II* low-thrust forces. At this point it appears that it might be impossible to determine a value of A from the *Mariner II* data without engaging in circular reasoning; for, after all, does not

Table 8. Comparison of compressed residuals for solutions I and II

Pass	Number of obser-vations	Solution I		Solution II	
		Mean residual	Rms residual	Mean residual	Rms residual
Receiver (11)		(Hz)	(Hz)	(Hz)	(Hz)
9/5-9/6	90	0.0006	0.0114	0.0045	0.0124
9/6-9/7	61	−0.0005	0.0146	0.0003	0.0146
9/7-9/8	64	0.0001	0.0142	−0.0009	0.0144
9/8-9/9	65	−0.0013	0.0141	−0.0016	0.0142
9/14-9/15	64	−0.0000	0.0149	−0.0004	0.0151
9/22-9/23	27	0.0041	0.0072	0.0020	0.0064
9/23-9/24	63	−0.0001	0.0148	−0.0025	0.0151
9/29-9/30	43	−0.0025	0.0048	−0.0028	0.0049
10/6-10/7	31	0.0019	0.0066	0.0005	0.0066
10/14-10/15	60	0.0031	0.0169	−0.0019	0.0167
10/24-10/25	44	0.0062	0.0105	0.0018	0.0088
10/27-10/28	45	0.0024	0.0056	−0.0016	0.0051
11/5-11/6	52	0.0032	0.0050	−0.0046	0.0061
11/10-11/11	46	0.0100	0.0112	−0.0059	0.0079
11/17	54	0.0087	0.0103	−0.0081	0.0098
11/26	50	0.0140	0.0148	−0.0003	0.0058
12/1	48	0.0123	0.0190	−0.0013	0.0144
12/7	38	0.0126	0.0140	−0.0111	0.0128
Receiver (12)					
10/14-10/15	53	0.0026	0.0113	−0.0030	0.0112
10/24	8	0.0092	0.0110	0.0049	0.0075

the selection of a low-thrust model based on the radar bounce value of A assure that any subsequent solution will simply result in the same radar bounce value? Fortunately, it does not, but only because of the generality of the model characterized by the f and α parameter set (f_1, f_2, f_3, α_1, α_2). In particular, Eq. (134) and (135) show that the f and α set can produce a perturbative acceleration that either increases or decreases as a function of time, while G_T and G_N can produce only an increasing acceleration for *Mariner II* because the heliocentric distance is always decreasing during the duration of the data. Therefore, the parameters G_T and G_N would only be selected if it were clear that the low-thrust forces obey that sort of model. Then the use of only two parameters to describe the forces would be far superior to using the five parameters of the f and α set to approximate the curve generated by G_T and G_N. However, since there is no clear choice between the two parameter sets, it is reasonable to choose the more general f and α set which can also represent the G_T and G_N curve if required to do so.

The two solutions of this section indicate that if one insists on an increasing model for the level of the low-thrust force, then it is necessary to choose a very low value of A in order to fit the data. However, if the choice of whether the curve should increase or decrease is left free, then a solution for the constants such as Solution I tends toward a decreasing force with a small correction to A. Again, if later solutions with both the cruise and encounter data, which are required for the actual determination of A, indicate that a low value of A and an increasing force provides a better fit to all the data, we have not excluded this possibility by selecting the f and α set or parameters.

In order to see that Solution I really indicates a decreasing force, the magnitude of the low-thrust force as computed by the values of f and α parameters from Solution I is plotted in Fig. 1 along with its one-sigma upper and lower error bounds. The magnitude of the force is defined by $(f_1^2 + f_2^2 + f_3^2)^{1/2}\,\alpha(\tau)$. Notice that even the upper error bound decreases, and hence it is very unlikely that the actual force could not be decreasing also unless the *a-priori* value of A is in serious error.

Another interesting result of Solutions I and II is that in both cases the primary contribution of the low-thrust force is along the negative tangential T axis of the spacecraft (Cf. Section IV-D-2) and is of about the same magnitude in both. The tangential force is plotted for both solutions in Fig. 8 with their respective one-sigma error bounds as derived from the uncertainties given in the solutions. Note that the two curves are significantly different after about 50 days from the epoch. Before this time, the errors on the two curves make them almost indistinguishable.

The *a-posteriori* uncertainties associated with the solutions of this section and those to follow are computed by simply taking the square roots of the diagonal elements in the matrix $(A^T W A + \widetilde{\Gamma}_x^{-1})^{-1}$ as described in Section IV-A. Of course the matrix Γ_x is made up of the *a-priori* uncertainties as discussed in Section V-C and listed, for example, in column 2 of Table 7. The most important factor in the size of the *a-posteriori* uncertainties is the size of the weights in the matrix W which is constrained as a diagonal matrix by the computer program. The weight for an individual doppler observation is taken as the inverse of its variance or the inverse square of its standard deviation. From Table 8 this mean error would seem to lie in the region of 0.015 Hz which is compatible with the expected performance of the Goldstone station during the *Mariner II* period.

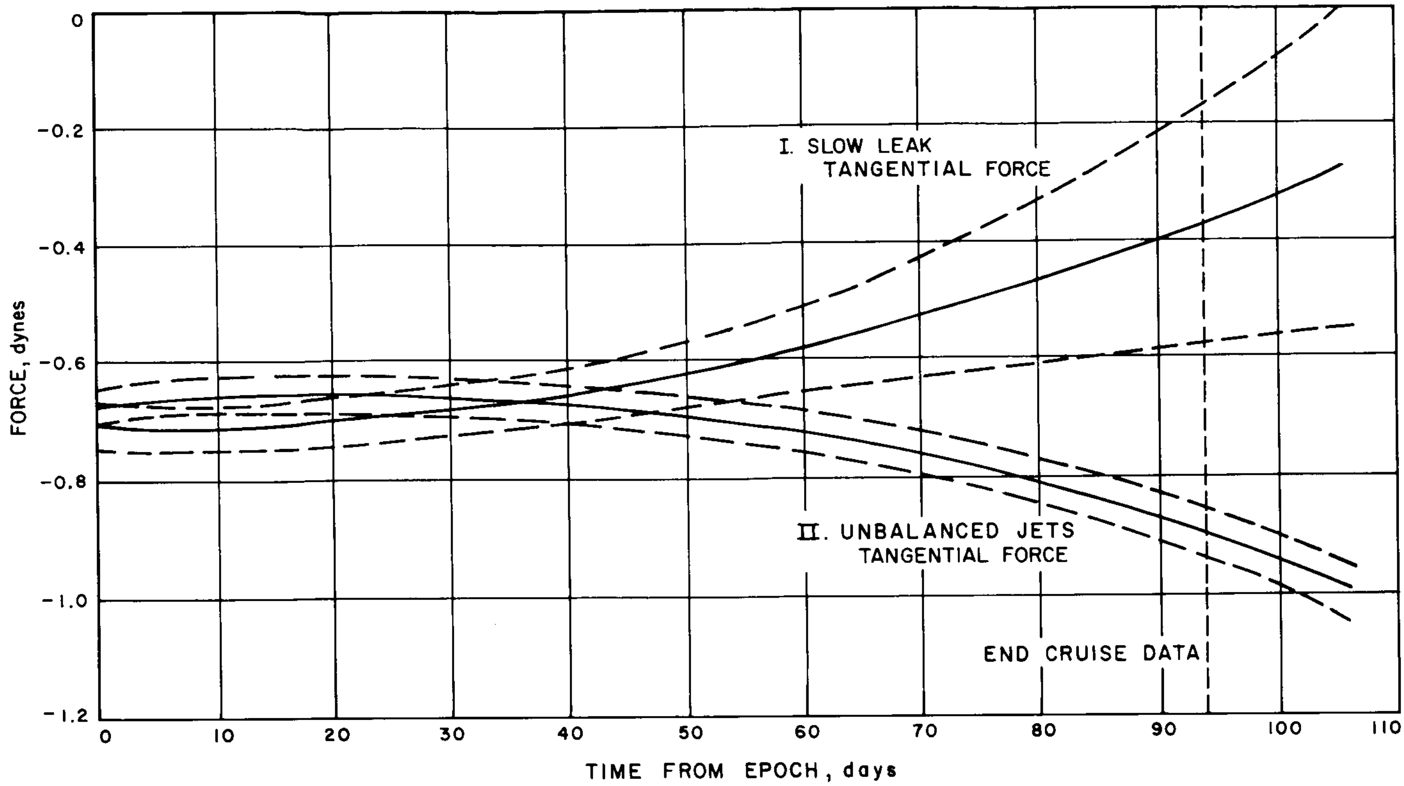

Fig. 8. Comparison of tangential low-thrust force for solutions I and II

However, in all the solutions of this and the following sections, the basic weight for the data is based on a mean error of 0.05 Hz, slightly larger than a three-sigma value, and in addition the data at lower elevation angles are assigned a lower weight according to a formula of D. L. Cain (JPL):

$$\sigma(\gamma) = \left[1 + \frac{18}{(1+\gamma)^2}\right]\sigma \tag{161}$$

where σ is the basic standard deviation and $\sigma(\gamma)$ is the standard deviation as a function of the elevation angle γ in degrees of arc. Thus, for $\gamma = 0$ deg, $\sigma(0) = 19\sigma$ while for $\gamma = 90$ deg, $\sigma(90) = 1.002\sigma$.

The basic standard deviation of 0.05 Hz is for data sampled every minute. Now a look at Appendix E indicates that the data used in these solutions have varying sample and count times, although the cruise data is fairly consistently sampled at 10-min intervals. In fact, the only data sampled at 1-min intervals are those of encounter when the count time is 50 sec. Effectively, there are 10 times as many observations of 50-sec count in the encounter passes as in the cruise passes, and if the data error is random, then a pass of this kind of cruise data should be weighted 1/10 as heavily as a similar pass of encounter data. However, we do not have enough confidence in the randomness of data sampled every one minute to allow this $1/\sqrt{N}$ effect to operate in determining the solutions and their *a-posteriori* uncertainties. Not only systematic errors in the data themselves, but also systematic errors from the numerical representation of the data make independence of the data and the corresponding validity of the $1/\sqrt{N}$ effect a poor assumption. Thus, a pass of data with a 50-sec count sampled every 10 min is weighted exactly the same as a pass of 50-sec data sampled every 1 min. To assure that this is so, the standard deviation used to weight the data is constructed as a function of the sample interval Δt according to the formula

$$\sigma(\Delta t) = \sqrt{\frac{60}{\Delta t}}\,\sigma \tag{162}$$

In practice Eq. (161) and (162) are combining into a single formula which is used for all weighting in these sections

$$\sigma(\Delta t, \gamma) = \left[1 + \frac{18}{(1+\gamma)^2}\right]\sqrt{\frac{60}{\Delta t}}\,\sigma \tag{163}$$

where

$$\sigma = 0.05\,\text{Hz}$$

Thus, practically all the cruise data for moderate values of γ are assigned standard deviations of 0.016 Hz. An alternate way of viewing the weighting of the data is to state that all data are nominally assigned a standard deviation of 0.016, unless the sample interval is less than 10 min, in which case the standard deviation is increased to defeat the $1/\sqrt{N}$ effect. Perhaps this is the simplest statement of the implications of using Eq. (163) to weight the data.

2. *Error formulae.* Following Herrick (Ref. 7), a correction to a reference value of a constant is split into the correction itself, plus the statistical uncertainty. For example, the correction A is written

$$\hat{A} = \frac{\delta A}{A} + \frac{\epsilon A}{A} \tag{164}$$

where $\delta A/A$ is the correction to $\tilde{A}$ and ϵA is taken here as the mean square error. From the *Mariner II* data, the determinations of $\hat{k}_{gm}$, $\hat{A}$ and $\hat{M}^s_v$ yield in addition the associated mean square errors $2\ \epsilon k_{gm}/k_{gm}$, $\epsilon A/A$ and $\epsilon M^s_v/M^s_v$. It is important also to include correlations given by the covariance matrix associated with the determination (see Section IV-A) whenever the results are transformed to other constants such as τ_A, μ, L, a, etc. Designate the three correlations from the *Mariner II* solution by the following:

$\rho_{GM,A}$ — correlation between k^2_{gm} and A

$\rho_{GM,V}$ — correlation between k^2_{gm} and M^s_v

$\rho_{A,V}$ — correlation between A and M^s_v

All other correlations in constants not determined directly by *Mariner II* are assumed zero. Eq. (27) to (33) yield the necessary error expressions. Note that only $\rho_{GM,A}$ appears in them and that the statistics of the constants are quite simple for the *Mariner II* solution. Of course, it is important to realize that constants left out of the least-squares solution (e.g., J_2, k^2_{ge}, p, ϵ, N) are assumed known far better *a-priori* than if they were included, and in fact it is also assumed that their *a-priori* uncertainty has no effect on the statistics associated with the three determinable constants. Thus, the direct contribution of the uncertainty in k^2_{ge} to that in μ is included in this analysis, but its indirect contribution through the *Mariner II* statistics is assumed negligible.

$$\left(\frac{\epsilon\tau_A}{\tau_A}\right)^2 = \left(\frac{\epsilon_A}{A}\right)^2 \tag{165}$$

$$\left(\frac{2\epsilon k_{gs}}{k_{gs}}\right)^2 = 9\left(\frac{\epsilon_A}{A}\right)^2 \tag{166}$$

$$\left(\frac{\epsilon\mu}{\mu}\right)^2 = \left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)^2 + \left(\frac{2\epsilon k_{ge}}{k_{ge}}\right)^2 \tag{167}$$

$$9\left(\frac{\epsilon a_{☾}}{a_{☾}}\right)^2 = \left(\frac{1}{1+\mu}\right)^2\left(\frac{2\epsilon k_{ge}}{k_{ge}}\right)^2 + \left(\frac{\mu}{1+\mu}\right)^2\left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)^2 + 4\left(\frac{\epsilon n^*_{☽}}{n^*_{☾}}\right)^2 \tag{168}$$

$$\left(\frac{\epsilon L}{L}\right)^2 = \frac{1}{9}\left(\frac{3+\mu}{1+\mu}\right)^2\left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)^2 + \frac{4}{9}\left(\frac{1}{1+\mu}\right)^2\left(\frac{2\epsilon k_{ge}}{k_{ge}}\right)^2 + \frac{4}{9}\left(\frac{\epsilon n^*_{☾}}{n^*_{☾}}\right)^2 + \left(\frac{\epsilon_A}{A}\right)^2 - \frac{2}{3}\left(\frac{3+\mu}{1+\mu}\right)\rho_{GM,A}\left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)\left(\frac{\epsilon_A}{A}\right) \tag{169}$$

$$\left(\frac{\epsilon P_{☾}}{P_{☾}}\right)^2 = \frac{1}{9}\mu^2\left(\frac{5+\mu}{1-\mu^2}\right)^2\left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)^2 + \frac{1}{9}\left(\frac{1+5\mu}{1-\mu^2}\right)^2\left(\frac{2\epsilon k_{ge}}{k_{ge}}\right)^2 + \frac{4}{9}\left(\frac{\epsilon n^*_{☾}}{n^*_{☾}}\right)^2 + \left(\frac{\epsilon_A}{A}\right)^2 + \frac{2}{3}\mu\left(\frac{5+\mu}{1-\mu^2}\right)\rho_{GM,A}\left(\frac{2\epsilon k_{gm}}{k_{gm}}\right)\left(\frac{\epsilon_A}{A}\right) \tag{170}$$

$$\left(\frac{\epsilon S}{S}\right)^2 = 9\left(\frac{\epsilon_A}{A}\right)^2 + \left(\frac{\epsilon G}{G}\right)^2 \tag{171}$$

3. *Determination of Earth-Moon mass ratio.* In Section III-F-1 it was pointed out that the mass of the Moon, or actually the Earth-Moon mass ratio μ^{-1}, could be determined by the periodic component in the doppler data resulting from the motion of the Earth about the Earth-Moon barycenter. In this section the least-squares procedure is applied to the cruise data with the low-thrust forces represented by the f and α parameters as indicated in the preceding section. Since the primary goal of these solutions is the determination of μ, the encounter data are excluded from them because uncertainties in the mass and position of Venus and the astronomical unit introduce systematic effects of their own which can be ignored when only the cruise data are considered. Thus, the purpose here is to isolate the effects of μ by choosing data taken before Dec. 8, 1962. The introduction of the encounter data is expected to result in only a slight improvement in μ over the solutions with cruise data only, an expectation verified in the next section, because the neglect of data from Dec. 8 to Dec. 20 represents less than ½ cycle in the periodic component which determines μ. However, the cruise data includes about 3⅓ cycles in the barycentric motion.

Actually, Solution I of the last section represents a determination of μ under the assumption that the geocentric gravitational constant is perfectly known and that the astronomical unit A is known *a-priori* to ±1000 km. The formula for μ^{-1} is

$$\mu^{-1} = \frac{k_{ge}^2}{k_{gm}^2} = \frac{GE}{GM} \quad (172)$$

and the variance on μ^{-1} is given in general by

$$\left(\frac{\sigma_{\mu^{-1}}}{\mu^{-1}}\right)^2 = \left(\frac{\sigma_{GE}}{GE}\right)^2 - 2\rho_{GE,GM}\frac{\sigma_{GE}}{GE}\frac{\sigma_{GM}}{GM} + \left(\frac{\sigma_{GM}}{GM}\right)^2 \quad (173)$$

where $\rho_{GE,GM}$ is the *a-posteriori* correlation between k_{ge}^2 and k_{gm}^2. Of course for Solution I, both σ_{GE} and $\rho_{GE,GM}$ are assumed zero, and the percentage error in μ^{-1} is simply equal to the percentage error in k_{gm}^2.

A second solution (Solution III) for μ is constructed by using a more realistic value of the *a-priori* uncertainty on A, specifically ±100 km, to see if constraining the variation in A has any effect on μ and, more generally, on the whole cruise solution. The results are given in the second column of Table 9 and the mean and RMS residuals are given in Table 10. There are no significant differences between Solutions I and III, either in the values of the parameters or in their *a-posteriori* uncertainties. For example, k_{gm}^2 differs by about 0.060 km³/sec², but the uncertainty based on the previously mentioned weighting of data is 0.069 km³/sec². Therefore the values from the two solutions differ by less than one standard deviation and the value of k_{gm}^2 is probably somewhere in the region defined by the two solutions. However, before deciding on a value for k_{gm}^2 it is well to remember that the quantity actually determined from the *Mariner* data is the mass ratio μ as shown in Section III-F-1, and hence a value for k_{gm}^2 as determined from Solutions I and III depends on the assumed value of the geocentric constant k_{ge}^2. In order to assure that no bias is present in μ because of an erroneous value of k_{ge}^2, two more solutions

Table 9. Comparison of solutions for the Earth-Moon mass ratio

Parameter	Solution III correction	Solution IV correction	Solution V correction
σ_A(km)	100	100	1000
σ_{GE}(km³/sec²)	0.0	10.0	10.0
x(km)	−30.2 ± 49.6	1.7 ± 51.8	5.5 ± 51.8
y(km)	−36.3 ± 67.5	5.4 ± 70.7	10.6 ± 70.7
z(km)	41.8 ± 78.0	−38.3 ± 78.2	15.5 ± 78.5
$\dot{x}$(km/sec)	$(-0.37 \pm 1.58) \times 10^{-5}$	$(-0.33 \pm 1.58) \times 10^{-5}$	$(0.00 \pm 1.58) \times 10^{-5}$
$\dot{y}$(km/sec)	$(0.24 \pm 1.30) \times 10^{-5}$	$(0.35 \pm 1.31) \times 10^{-5}$	$(0.11 \pm 1.31) \times 10^{-5}$
$\dot{z}$(km/sec)	$(2.04 \pm 5.88) \times 10^{-5}$	$(-0.98 \pm 6.12) \times 10^{-5}$	$(-1.07 \pm 6.43) \times 10^{-5}$
k_{gm}^2(km³/sec²)	0.0077 ± 0.067	−0.0541 ± 0.106	−0.0269 ± 0.106
k_{ge}^2(km³/sec²)	—	−2.76 ± 9.52	−3.04 ± 9.52
A(km)	0.0 ± 100	−42 ± 100	−208 ± 969
f_1(km/sec²)	$(0.02 \pm 0.38) \times 10^{-10}$	$(0.03 \pm 0.32) \times 10^{-10}$	$(-0.05 \pm 0.32) \times 10^{-10}$
f_2(km/sec²)	$(-0.34 \pm 0.02) \times 10^{-10}$	$(-0.35 \pm 0.02) \times 10^{-10}$	$(-0.36 \pm 0.02) \times 10^{-10}$
f_3(km/sec²)	$(-0.12 \pm 0.14) \times 10^{-10}$	$(-0.19 \pm 0.17) \times 10^{-10}$	$(-0.16 \pm 0.17) \times 10^{-10}$
α_1(sec⁻¹)	$(0.03 \pm 0.29) \times 10^{-7}$	$(-0.01 \pm 0.28) \times 10^{-7}$	$(-0.06 \pm 0.30) \times 10^{-7}$
α_2(sec⁻²)	$(0.51 \pm 0.73) \times 10^{-14}$	$(0.92 \pm 0.72) \times 10^{-14}$	$(1.04 \pm 0.72) \times 10^{-14}$
γ	−0.03 ± 0.36	−0.05 ± 0.30	0.02 ± 0.30
R_{11}(km)	−0.007 ± 0.008	−0.009 ± 0.008	−0.007 ± 0.008
λ_{11}	$(-0°01 \pm 0°15) \times 10^{-3}$	$(-0°06 \pm 0°15) \times 10^{-3}$	$(-0.03 \pm 0.15) \times 10^{-3}$
R_{12}(km)	−0.001 ± 0.007	0.003 ± 0.007	0.004 ± 0.007
λ_{12}	$(-0°03 \pm 0°15) \times 10^{-3}$	$(-0.04 \pm 0°15) \times 10^{-3}$	$(-0.01 \pm 0.15) \times 10^{-3}$

Table 10. Comparison of compressed residuals for Solutions III, IV and V

Pass	Solution III		Solution IV		Solution V	
	Mean residual (Hz)	RMS residual (Hz)	Mean residual (Hz)	RMS residual (Hz)	Mean residual (Hz)	RMS residual (Hz)
Receiver (11)						
9/5-9/6	0.0008	0.0115	0.0007	0.0115	−0.0001	0.0114
9/6-9/7	−0.0012	0.0145	−0.0023	0.0147	−0.0023	0.0146
9/7-9/8	−0.0014	0.0145	−0.0010	0.0145	−0.0015	0.0145
9/8-9/9	−0.0015	0.0142	−0.0005	0.0140	−0.0005	0.0141
9/14-9/15	0.0003	0.0150	0.0015	0.0151	0.0011	0.0149
9/22-9/23	0.0042	0.0073	0.0036	0.0069	0.0048	0.0077
9/23-9/24	−0.0009	0.0149	−0.0025	0.0149	−0.0012	0.0148
9/29-9/30	−0.0033	0.0052	−0.0040	0.0056	−0.0023	0.0046
10/6-10/7	0.00041	0.0062	0.0015	0.0063	0.0032	.0.0070
10/14-10/15	−0.0010	0.0166	−0.0009	0.0166	0.0002	0.0166
10/24-10/25	0.0018	0.0086	−0.0002	0.0084	0.0001	0.0084
10/27-10/28	−0.0033	0.0060	−0.0027	0.0057	−0.0028	0.0057
11/5-11/6	−0.0015	0.0043	−0.0033	0.0052	−0.0034	0.0055
11/10-11/11	0.0008	0.0056	0.0008	0.0053	0.0001	0.0053
11/17	−0.0018	0.0057	−0.0021	0.0057	−0.0021	0.0057
11/26	−0.0012	0.0050	−0.0009	0.0047	−0.0031	0.0058
12/1	0.0011	0.0142	0.0020	0.0147	−0.0013	0.0147
12/7	0.0008	0.0059	0.0004	0.0063	−0.0008	0.0066
Receiver (12)						
10/14-10/15	−0.0016	0.0109	−C.0018	0.0110	−0.0006	0.0109
10/24	0.0044	0.0076	0.0032	0.0068	0.0033	0.0069

(Solutions IV and V) are displayed in Table 9 where k^2_{ge} is also included in the solution. It is given an *a-priori* uncertainty of 10 km^3/sec^2 which is more than 10 times larger than the expected uncertainty from the *Ranger* determination (Cf. Eq. 26). Thus, the least-squares procedure is free to apply fairly large corrections to both k^2_{gm} and k^2_{ge} in order to fit the cruise data. Certainly, an independent determination of the mass of the Earth plus Moon is not possible because the spacecraft is too far from the Earth at the beginning of the data. Therefore, the *a-posteriori* uncertainties in k^2_{gm} and k^2_{ge} should be quite large for Solutions IV and V, but a computation of the uncertainty on μ or μ^{-1} from these large uncertainties and the associated correlation $\rho_{GE,GM}$ (Eq. 173) should be comparable to that from Soutions I and III.

The two Solutions with k^2_{ge} included as a free parameter differ in that one, Solution IV, assumes a realistic *a-priori* uncertainty on A of ± 100 km and the other, Solution V, assumes an error of ± 1000 km as in Solution I. The latter solution represents a determination of μ with the least *a-priori* information on other constants that significantly affect the representation of data. In this sense it is the most independent determination of μ from the *Mariner II* data.

To summarize the results of the four solutions for μ, the quantities necessary to compute the mass ratio and its uncertainty by Eq. (172) and (173) are given in Table 11. The correlation coefficient is taken from the covariance matrix $(A^T W A + \widetilde{\Gamma}_x^{-1})^{-1}$ associated with each solution. Two significant facts are apparent from the results in Table 11. The first is that the correlation between k^2_{ge} and k^2_{gm} in Solutions IV and V is of the right sign and magnitude to partially cancel the relatively large errors in the two gravitational constants and to produce an error in μ which is almost equal to that from Solutions I and III. This agreement in the four solutions is further evidence that the *Mariner* data provide an independent measurement of the mass ratio μ. The second fact of importance is that the value obtained for μ is quite stable for fairly large variations in k^2_{ge} and A. Therefore, the value of μ^{-1}

Table 11. Computation of mass ratio μ

Parameters	Solution I	Solution III	Solution IV	Solution V
k_{ge}^2(km^3/sec^2)	398601.27	398601.27	398598.51	398598.23
k_{gm}^2(km^3/sec^2)	4902.7840	4902.8442	4902.7824	4902.8096
σ_{GE}/GE	0	0	23.893×10^{-6}	23.882×10^{-6}
σ_{GM}/GM	13.994×10^{-6}	13.756×10^{-6}	21.551×10^{-6}	21.634×10^{-6}
$\rho_{GE,\ GM}$	0	0	0.76480	0.76235
σ_μ/μ	13.994×10^{-6}	13.756×10^{-6}	15.739×10^{-6}	15.831×10^{-6}
μ^{-1}	81.3010 ± 0.0011	81.3000 ± 0.0011	81.3005 ± 0.0013	81.3000 ± 0.0013

from the *Mariner* data lies somewhere in the region 81.300 to 81.301. In selecting a value in this region there is no overwhelming reason to pick one of the values of μ over the others. All fits to the data are quite good as can be seen from the mean and RMS residuals given in Table 10. Also, the functions $S(X)$ and $Q(X)$ offer little help in the selection of a value, except that Solutions II, III and IV are all about equal and seem to fit the data slightly better than Solution I. The values of the least-squares functions are

Solution	$S(X)$	$Q(X)$
III	464.53	464.97
IV	468.18	468.48
V	470.12	470.51

which can be compared with $S(X) = 595.70$ and $Q(X) = 596.18$ for Solution I.

The selection of a value for μ, then, is made on the basis of Solutions III, IV and V by taking a simple average of the three values and assigning the largest of the standard deviations, that from Solutions IV and V, as the *a-posteriori* uncertainty on the ratio. The result is

$$\mu^{-1} = 81.3001 \pm 0.0013$$

4. Determination of astronomical unit and the mass of Venus. In Section III-F-2, the nature of the determination of the mass of Venus M_v^* and the astronomical unit A is discussed in terms of the shape of the Doppler curve during planetary encounter. The mass M_v^* is determined by combining this curve with the data before and after encounter and unless the cruise and encounter data are both used, the mass is dependent on the value chosen for A. With the combination of the cruise and encounter data, the two constants become independent.

As a first solution for the mass, only the encounter data are used and the *a-priori* value of A is included in the determination with its uncertainty of ± 100 km. The result is shown in Table 12 as Solution VI. The *a-priori* values of the parameters are from Solution III with the cartesian coordinates or orbital elements now referred to an epoch of Dec. 8, 12^h0 E.T. instead of the Sept. 5 epoch of the cruise solutions. The epoch is moved closer to the data in order to achieve greater numerical stability in the encounter solutions. Increased stability results because the corrections in the position and velocity at the Dec. 8 epoch must be larger than at the Sept. 5 epoch to produce the same effect on the Doppler residuals. Conversely, numerical rounding errors in the computation of corrections to the coordinates have less effect on the residuals for the later epoch than for the earlier one, and, as a result, the entire least-squares process is less sensitive to numerical errors in the coordinates.

The numerical values of the constants in the second column of Table 12 are slightly different than in Solution III of Table 9 because the values were taken from different iterations in the numerical iterative procedure that minimizes the function $Q(x)$ as described in Section IV-A-1. However the differences are not significant. The *a-priori* uncertainties for the parameters in the solution for M_v^* are all large to allow as much freedom as possible in the determination of a value for M_v^* from the encounter data with an adopted value of A. The low-thrust and solar radiation parameters must be included in the encounter trajectory computation since the cruise solutions have shown they are significant. Their values as determined by Solution III are included so as not to bias the solution for M_v^* with erroneous low-thrust forces. The *a-priori* uncertainties on f_1, f_2, f_3 and γ are about three times the *a-posteriori* uncertainties of Solution III and

Table 12. Solution for the mass of Venus with the encounter data alone

Parameter	*A-priori* value	Solution VI correction
x (km)	-37434344 ± 0.0	—
y (km)	-31350426 ± 0.0	—
z (km)	-10175342 ± 0.0	—
$\dot{x}$ (km/sec)	-7.1521619 ± 1.0	$(-0.59 \pm 0.57) \times 10^{-4}$
$\dot{y}$ (km/sec)	-11.402508 ± 1.0	$(1.09 \pm 0.78) \times 10^{-4}$
$\dot{z}$ (km/sec)	-5.7124045 ± 1.0	$(-1.28 \pm 0.38) \times 10^{-4}$
k^2_{gm} (km^3/sec^2)	4902.8534 ± 0.0	—
M^s_v	$(0.24471118 \pm 1.0) \times 10^{-5}$	$(8.29 \pm 0.11 \times 10^{-10}$
A (km)	149597890 ± 100	-14 ± 100
x_v (km)	0.0 ± 10^8	-517 ± 41
y_v (km)	0.0 ± 10^8	102 ± 48
z_v (km)	0.0 ± 10^8	41 ± 19
f_1 (km/sec^2)	$(-0.013 \pm 0.868) \times 10^{-10}$	$(-0.335 \pm 0.868) \times 10^{-10}$
f_2 (km/sec^2)	$(-0.352 \pm 0.058) \times 10^{-10}$	$(0.002 \pm 0.058) \times 10^{-10}$
f_3 (km/sec^2)	$(-0.154 \pm 0.458) \times 10^{-10}$	$(0.027 \pm 0.458) \times 10^{-10}$
α_1 (sec^{-1})	$(-0.004 \pm 0.0) \times 10^{-7}$	—
α_2 (sec^{-2})	$(0.776 \pm 0.0) \times 10^{-14}$	—
γ	-0.0156 ± 0.549	-0.007 ± 0.549
R_{11} (km)	6372.0104 ± 0.0	—
λ_{11}	$243^\circ.15064 \pm 0.0$	—
R_{12} (km)	6371.8819 ± 0.0	—
λ_{12}	$243^\circ.19452 \pm 0.0$	—

thus are quite conservative. Even so, the encounter data alone cannot improve these constants, but their inclusion in the solution assures a more realistic computation of the covariance matrix for the parameters. The *a-posteriori* uncertainty on M^s_v, therefore, contains a contribution from uncertainties in the low-thrust and solar radiation forces.

The inclusion of the position coordinates of the planet in the solution requires some explanation. Clearly, the encounter data alone are not sufficient to determine the position of both Venus and the spacecraft with respect to the Sun or Earth. Therefore, it makes no difference whether the planet is held fixed and the position of the spacecraft at epoch is varied to fit the data, or whether the geocentric spacecraft position is held fixed and the planet is moved to achieve the same fit to the Doppler curve. The latter alternative is chosen here because the spacecraft position has been determined from the cruise data and it is of interest to see how much the planet must be varied from its nominal position as given by the ephemeris compiled by JPL (Ref. 23) in order to properly represent the *Mariner II* motion. The heliocentric position of Venus in true equatorial coordinates at 1962, Dec. 14, $20^h.0$ E.T. is

$$X_v = -0.14378455 \text{ (a.u.)}$$

$$Y_v = 0.63894646 \text{ (a.u.)}$$

$$Z_v = 0.29684580 \text{ (a.u.)}$$

and the corresponding geocentric coordinates referred to the true equator and equinox of data are

$$\alpha = 14^h 51^m 38^s.382$$

$$\delta = -13^\circ 39' 28''.83$$

$$r = 0.38640258 \text{ (a.u.)}$$

The epoch (Dec. 14, $20^h.0$ E.T.) for the corrections to the position is within a few minutes of the time of closest approach of the spacecraft to the planet. In Appendix D, the method is described which propagates corrections at this periapsis epoch into the ephemeris at other times during the time of the *Mariner II* data. The corrections themselves can be interpreted in terms of corrections to the geocentric true coordinates at the periapsis epoch.

For example, the corrections of Solution VI yield the following:

$$\cos\delta\Delta\alpha = -1''52 \pm 0''18$$

$$\Delta\delta = 0''40 \pm 0''07$$

$$\Delta r = (290 \pm 39) \text{ km}$$

In addition the Sun-Venus mass ratio from the solution is

$$(M_v^s)^{-1} = 408506.68 \pm 1.8$$

It was pointed out in Section III-F-2 that the gravitational constant GM_v (km^3/sec^2) is more directly determined from the *Mariner* data than the ratio M_v^s. Therefore, the result for M_v^s and A can be combined into a determination of $GM_v = k_{gv}^2$

$$k_{gv}^2 = \frac{M_v^s}{k''^2 A^3} \tag{174}$$

and with the correlation coefficient $\rho_{A,V}$ between A and M_v^s given by the covariance matrix $(A^T W A + \widetilde{\Gamma}_x^{-1})^{-1}$, the standard deviation σ_{gv} on k_{gv}^2 can be computed according to

$$\left(\frac{\sigma_{gv}}{k_{gv}^2}\right)^2 = \left(\frac{\sigma_v}{M_v^s}\right)^2 + 6\rho_{A,V}\left(\frac{\sigma_v}{M_v^s}\right)\left(\frac{\sigma_A}{A}\right) + 9\left(\frac{\sigma_A}{A}\right)^2 \tag{175}$$

where σ_v is the standard deviation for M_v^s. Solution VI yields a correlation coefficient $\rho_{A,V}$ of -0.4013, and thus from Eq. (174) and (175) we have

$$k_{gv}^2 = (324872.17 \pm 1.30) \text{ km}^3/\text{sec}^2$$

Two more encounter solutions are included in this section in order to attempt a more definitive determination of M_v^s and an independent determination of A. The procedure used is to combine the encounter solution with those already obtained for the cruise data. Thus the covariance matrices from Solutions I and III are used as *a-priori* matrices $\widetilde{\Gamma}_x$ in the two new encounter solutions, Solutions VII and VIII, where the epoch for the spacecraft coordinates is the Dec. 8 one of Solution VI. Because the covariance matrices for Solutions I and III are referred to the Sept. 5 epoch, it is necessary to map the Sept. 5 covariance matrix to the Dec. 8 epoch by means of the matrix U discussed earlier (Section IV-E). The transformation for the 6×6 coordinate portion Γ_x of the covariance matrix is

$$\widetilde{\Gamma}_x(t) = U\widetilde{\Gamma}_x(t_o)\,U^T \tag{176}$$

where t and t_o represent the Dec. 8 and Sept. 5 epochs, respectively. The effect of using $\widetilde{\Gamma}_x$ from the cruise solutions in these new encounter solutions is to statistically combine the estimates of the parameters as determined from each batch of data. The results should be the same as determining the constants from all the data with the advantage that the solutions are more numerically stable.

Solution VII is given in Table 13 and represents a determination of M_v^s from all the *Mariner II* data with an adopted value of A, in particular the radar bounce value. The first column lists the parameters, the second gives the *a-priori* values and uncertainties from the results of Solution III, and the corrections and *a-posteriori* uncertainties are given in the third column. The encounter parameters (X_v, Y_v, Z_v, M_v^s) that were not included in Solution III are given very large *a-priori* errors. Also, because of limitations imposed by the computer program on the size and selection of parameters, the low thrust parameters, α_1 and α_2, are not included in these encounter solutions, although their values from the cruise solutions are used in the orbit computation. The results of Solution VII can be summarized, as were those of Solution VI, by the following list of correction and values.

Solution VII

$$\cos\delta\Delta\alpha = -1''31 \pm 0''05$$

$$\Delta\delta = 0''86 \pm 0''21$$

$$\Delta r = (409 \pm 39) \text{ km}$$

$$(M_v^s)^{-1} = 408509.95 \pm 1.8$$

$$\rho_{A,V} = -0.4569$$

$$k_{gv}^2 = (324869.41 \pm 1.26) \text{ km}^3/\text{sec}^2$$

The last encounter solution, Solution VIII, uses the covariance matrix from Solution I as *a-priori* information and thus assumes an *a-priori* uncertainty in A of ± 1000 km. The correction from this solution represents

Table 13. Solution for the mass of Venus with *a-priori* information from cruise data

Parameter	*A-priori* value	Solution VII correction
x (km)	-37434344 ± 106	-97 ± 18
y (km)	-31350426 ± 81	-76 ± 22
z (km)	-10175342 ± 221	59 ± 65
$\dot{x}$ (km/sec)	$-7.1521619 \pm 2.85 \times 10^{-5}$	$(1.13 \pm 0.69) \times 10^{-5}$
$\dot{y}$ (km/sec)	$-11.402508 \pm 1.23 \times 10^{-5}$	$(-0.83 \pm 0.57) \times 10^{-5}$
$\dot{z}$ (km/sec)	$-5.7124045 \pm 6.08 \times 10^{-5}$	$(-5.33 \pm 0.98) \times 10^{-5}$
k^2_{gm} (km^3/sec^2)	4902.8534 ± 0.0675	0.0473 ± 0.058
M^s_v	$(0.24471118 \pm 0.24) \times 10^{-5}$	$(8.09 \pm 0.11) \times 10^{-10}$
A (km)	149597890 ± 100	-38 ± 98
x_v (km)	0.0 ± 10^8	-581 ± 19
y_v (km)	0.0 ± 10^8	-42 ± 28
z_v (km)	0.0 ± 10^8	137 ± 64
f_1 (km/sec^2)	$(-0.013 \pm 0.35) \times 10^{-10}$	$(0.035 \pm 0.052) \times 10^{-10}$
f_2 (km/sec^2)	$(-0.35 \pm 0.02) \times 10^{-10}$	$(0.016 \pm 0.014) \times 10^{-10}$
f_3 (km/sec^2)	$(-0.15 \pm 0.15) \times 10^{-10}$	$(0.051 \pm 0.037) \times 10^{-10}$
α_1 (sec^{-1})	$(-0.004 \pm 0.0) \times 10^{-7}$	—
α_2 (sec^{-2})	$(0.776 \pm 0.0) \times 10^{-14}$	—
γ	-0.0156 ± 0.333	0.003 ± 0.061
R_{11} (km)	6372.0103 ± 0.0076	-0.0059 ± 0.0051
λ_{11}	$243.15064 \pm 0.15 \times 10^{-3}$	$(0.07 \pm 0.09) \times 10^{-3}$
R_{12} (km)	6371.8819 ± 0.0069	0.0049 ± 0.0047
λ_{12}	$243.19452 \pm 0.15 \times 10^{-3}$	$(0.08 \pm 0.09) \times 10^{-3}$

a determination of A from the *Mariner* data. Table 14 gives the results for Solution VIII in exactly the same form as Table 13. A tabulation of results for this solution also includes a value for A.

Solution VIII

$$\cos \delta\Delta\alpha = -1''.16 \pm 0''.08$$

$$\Delta\delta = 0''.20 \pm 0''.39$$

$$\Delta r = (659 \pm 143) \text{ km}$$

$$A = (149597032 \pm 485) \text{ km}$$

$$(M'_v)^{-1} = 408505.93 \pm 4.26$$

$$\rho_{A,V} = -0.87770413$$

$$k^2_{gv} = 324867.27 \pm 1.63 \text{ km}^3/\text{sec}^2$$

Note that with an increase in the uncertainty in A the correlation coefficient $\rho_{A,V}$ also increases, which tends to keep the uncertainty in k^2_{gv} nearly equal to that of the preceding two solutions. This fact supports the contention that the *Mariner* data determine k^2_{gv} directly rather than M'_v. In this last solution, where A is relatively uncertain, the percentage error in M'_v is 10.4×10^{-6} as compared to 5.03×10^{-6} for k^2_{gv}. With an uncertainty in A of ± 100 as in Solutions VI and VII, the percentage uncertainty is about 4×10^{-6} in both M'_v and k^2_{gv}.

Again, as in the case of the Earth-Moon mass ratio, it is not easy to select values for the constants from one of the solutions. They all fit the data fairly well as can be seen from the table of mean and RMS residuals (Table 15). The functions $S(x)$ and $Q(x)$ are also about the same.

Solution	$S(x)$	$Q(x)$
VI	243.10	243.23
VII	157.84	161.03
VIII	155.53	157.52

Rather than choose values for M'_v and A from the results of these encounter solutions alone, a determination of the constants from both the cruise and encounter data in one solution is carried out in the next section and values are selected there.

Table 14. Solution for the mass of Venus and the astronomical unit with *a-priori* information from cruise data

Parameter	*A-priori* value	Solution VIII correction
x (km)	-37437345 ± 103	24.0 ± 57
y (km)	-31350423 ± 87	-28.3 ± 46
z (km)	-10175351 ± 220	-169 ± 114
x (km/sec)	$-7.1521562 \pm 3.04 \times 10^{-5}$	$(3.17 \pm 1.06) \times 10^{-5}$
y (km/sec)	$-11.402507 \pm 2.66 \times 10^{-5}$	$(2.68 \pm 0.85) \times 10^{-5}$
z (km/sec)	$5.7124031 \pm 6.80 \times 10^{-5}$	$(-0.75 \pm 2.47) \times 10^{-5}$
k^2_{gm} (km^3/sec^2)	4902.8279 ± 0.0686	0.0065 ± 0.0619
M^s_v	$(0.24471118 \pm 0.24) \times 10^{-5}$	$(8.33 \pm 0.25) \times 10^{-10}$
A (km)	149597740 ± 970	-708 ± 485
x_v (km)	0.0 ± 10^8	-672 ± 74
y_v (km)	0.0 ± 10^8	-212 ± 93
z_v (km)	0.0 ± 10^8	-145 ± 136
f_1 (km/sec^2)	$(-0.019 \pm 0.366) \times 10^{-10}$	$(-0.151 \pm 0.124) \times 10^{-10}$
f_2 (km/sec^2)	$(-0.347 \pm 0.020) \times 10-^{10}$	$(0.001 \times 0.015) \times 10^{-10}$
f_3 (km/sec^2)	$(-0.147 \pm 0.151) \times 10^{-10}$	$(0.114 \pm 0.052) \times 10^{-10}$
α_1 (sec^{-1})	$(-0.0157 \pm 0.0) \times 10^{-7}$	—
α_2 (sec^{-2})	$(0.7601 \pm 0.0) \times 10^{-14}$	—
γ	0.0067 ± 0.351	0.180 ± 0.123
R_{11} (km)	6372.0103 ± 0.0077	0.0009 ± 0.0058
λ_{11}	$243^\circ 15061 \pm 0^\circ 15 \times 10^{-3}$	$(0.^\circ 076 \pm 0.088) \times 10^{-3}$
R_{12} (km)	6371.8823 ± 0.0070	0.0009 ± 0.0053
λ_{12}	$243^\circ 19450 \pm 0.015 \times 10^{-3}$	$(-0.072 \pm 0.090) \times 10^{-3}$

Table 15. Comparison of compressed residuals for Solutions VI, VII and VIII

Pass	No. Data	Solution VI		Solution VII		Solution VIII	
		Mean residual (Hz)	RMS residual (Hz)	Mean residual (Hz)	RMS residual (Hz)	Mean residual (Hz)	RMS residual (Hz)
12/8	50	−0.0003	0.0086	−0.0012	0.0081	−0.0030	0.0082
12/11	57	0.0049	0.0103	0.0045	0.0104	0.0035	0.0105
12/12	184	0.0023	0.0130	−0.0012	0.0134	−0.0021	0.0134
12/13	19	0.0047	0.0126	−0.0033	0.0118	−0.0043	0.0123
12/14	42	−0.0045	0.0231	−0.0063	0.0102	−0.0057	0.0098
12/15	35	−0.0090	0.0169	−0.0018	0.0137	−0.0016	0.0137
12/16	48	−0.0141	0.0207	−0.0089	0.0173	−0.0089	0.0172
12/17	49	−0.0064	0.0184	−0.0020	0.0170	−0.0034	0.0173
12/19	27	−0.0051	0.0078	−0.0033	0.0066	−0.0026	0.0064
12/20	43	−0.0067	0.0103	−0.0070	0.0103	−0.0055	0.0091

5. *Simultaneous solution for all constants.* Solutions VII and VIII of the last section are representative of a least-squares fit to all the data. However, it is necessary to rely on a statistical combination of estimates in order to obtain a solution. If the corrections are so large, as they are in fact, that a linear correction to the *a-priori* values is not possible, then the corrected parameters must be used to recompute residuals and the least-squares procedure applied again iteratively until the function $Q(x)$ is at a minimum value. Unfortunately, the numerical accuracy of the computer program does not permit an accurate computation of residuals during and after the

encounter of the spacecraft with Venus and the iterative process fails. The differential coefficients of Table 5 can be used to demonstrate the sensitivity of the computation of Doppler residuals to numerical errors in the position of Venus.

Suppose, optimistically, that the heliocentric position coordinates of Venus are numerically accurate to eight significant figures. Then, the numerical error in the position is about 0.5 km and since some of the differential coefficients in Table 5 are on the order of 0.2 to 0.3 Hz/km, the numerical error in the residuals can become as large as 0.15 Hz which is about an order of magnitude larger than the expected size of the residuals from the least squares determinations. The effect of all this is that if a solution is attempted with all the *Mariner* data, a set of parameters can be found that actually are very close to those that minimize the function $Q(x)$; but the relatively large and erroneous residuals around planetary encounter and beyond indicate further small corrections to the parameters which, when used to compute new residuals, simply produce another set of erroneous values and the process never converges. In all this, the residuals before encounter behave as well as those of the cruise solutions of Section V-D-1 and V-D-3 and once the parameters are in the region of the least-squares solution they remain small.

In effect, then, a solution with all the data is impossible within the framework established for the solutions of the previous sections. Of course the encounter solutions are affected by this same sort of numerical error in the residuals, but because of the relatively short span of data and the favorable location of the epoch for the spacecraft coordinates it is possible to find values for the parameters that smooth out some of the numerical errors in the residuals. That the parameters of the last section are to some extent based on a fitting to numerical errors is not particularly alarming, because the numerical errors are in all cases two, three or more significant figures beyond the *a-posteriori* uncertainties assigned to the parameters. However, some sort of simultaneous solution for all the data does seem indicated if complete confidence that the parameters are really those which minimize $Q(x)$ is to be obtained.

Fortunately, such a solution is possible by relying on the classical differential correction process for the solution which can be justified since the solutions of the previous sections provide parameters close enough to the simultaneous solution that a linear correction is valid. Also, by fitting all of the data by the iterative procedure, even though convergence cannot be achieved, it is possible to obtain parameters such that their deviations from the least square solution are in the linear region. The procedure for obtaining the solution of this section, therefore, is as follows.

First, the best possible determination of the parameters is accomplished by the iterative method and residuals are computed. Next, a linear correction is computed by Eq. (49) but it is not applied to the parameters for a new computation of residuals. Instead the new residuals are computed by the linear formula

$$\Delta z\,(\text{new}) = \Delta z\,(\text{old}) - A\Delta x \qquad (175)$$

where Δx are the corrections to the parameters and A is the matrix of differential coefficients. The linearized residuals $\Delta z\,(\text{new})$ represent the residuals obtained from the least squares values of the parameters in the absence of numerical errors.

The actual linear solution (Solution IX) for the *Mariner* data is shown in Table 16. The *a-priori* values of the parameters in column (2) represent an iterative solution using all the data and the *a-priori* uncertainties are set quite large to allow an independent solution for the constants and position of Venus. The corrections and *a-posteriori* uncertainties are given in column (3). Note that the corrections are small with respect to the uncertainties. The linearized residuals are listed along with the data in Appendix E. Finally, the correlation matrix associated with this solution is given in Table 17 so that the complete *a-posteriori* covariance matrix is available for future reference.

The results of Solution IX are tabulated here for comparison with previous solutions.

Solution IX

$$k^2_{gm} = (4902.540 \pm 0.060)\ \text{km}^3/\text{sec}^2$$

$$\mu^{-1} = 81.2998 \pm 0.0010$$

$$\cos\delta\Delta\alpha = 1''\!.45 \pm 0.11$$

$$\Delta\delta = 1''\!.11 \pm 0''\!.28$$

$$\Delta r = (456 \pm 95)\ \text{km}$$

$$A = (149597546 \pm 373)\ \text{km}$$

$$(M^{s}_{v})^{-1} = 408503.49 \pm 5.2$$

$$\rho_{A,V} = -0.92174$$

$$k^2_{gv} = (324872.56 \pm 2.14)\ \text{km}^3/\text{sec}^2$$

Table 16. Solution for the constants from cruise and encounter data

Parameters	A-priori value	Solution IX correction
x (km)	-1424212.8 ± 10^6	-13.8 ± 34.2
y (km)	-1939480.1 ± 10^6	-17.0 ± 44.9
z (km)	-100617.21 ± 10^6	27.5 ± 40.6
$\dot{x}$ (km/sec)	-1.7444942 ± 1.0	$(0.54 \pm 0.80) \times 10^{-5}$
$\dot{y}$ (km/sec)	-2.4233973 ± 1.0	$(0.32 \pm 0.60) \times 10^{-5}$
$\dot{z}$ (km/sec)	-0.11009455 ± 1.0	$(1.79 \pm 1.83) \times 10^{-5}$
k^2_{gm} (km³/sec²)	4902.9007 ± 100	-0.0467 ± 0.060
M^s_v	$(0.24479208 \pm 1.0) \times 10^{-5}$	$(0.387 \pm 0.314) \times 10^{-10}$
A (km)	149597850 ± 5000	-304 ± 373
x_v (km)	-581.64 ± 1000	-73.68 ± 77
y_v (km)	-42.05 ± 1000	-12.68 ± 58
z_v (km)	136.88 ± 1000	56.56 ± 82
f_1 (km/sec²)	$(0.022 \pm 1.0) \times 10^{-10}$	$(0.261 \pm 0.149) \times 10^{-10}$
f_2 (km/sec²)	$(-0.336 \pm 1.0) \times 10^{-10}$	$(-0.007 \pm 0.015) \times 10^{-10}$
f_3 (km/sec²)	$(-0.103 \pm 1.0) \times 10^{-10}$	$(-0.111 \pm 0.066) \times 10^{-10}$
α_1 (sec⁻¹)	$(-0.004 \pm 0.0) \times 10^{-7}$	—
α_2 (sec⁻²)	$(0.818 \pm 1.0) \times 10^{-14}$	$(-0.262 \pm 0.173) \times 10^{-14}$
γ	-0.0128 ± 1.0	-0.271 ± 0.149
R_{11} (km)	6372.0044 ± 0.0	—
λ_{11}	243°15057 ± 0°0	—
R_{12} (km)	6371°8770 ± 0°0	—
λ_{12}	243°19444 ± 0°0	—

The results of Solution VI, VII, VIII and IX define the values of A, k^2_{gv}, and the ephemeris corrections that are associated with the *Mariner II* data. Solutions VI, VII and IX are quite consistent in their results and only Solution VIII seems to give significantly different results for the constants, although the disagreement is not much more than a *one-sigma deviation* from the values indicated by the other three solutions. The slight disagreement of Solution VIII is probably caused by the fact that the least-squares fit of Solution I, which is used as *a-priori* information in Solution VIII, is not as good as that of Solution III, which is used as *a-priori* information in Solution VII. Thus, just as Solution I was not considered in arriving at a final value for μ in Section V-D-3, it is also neglected here by ignoring the results of Solution VIII and by averaging the other three solutions to obtain the following values for the constants. The uncertainties are again, as in Section V-D-3, taken as the largest of the individual *a-posteriori* uncertainties from each solution.

$$A = (149597546 \pm 500)\ \text{km}$$

$$k^2_{gv} = (324871.5 \pm 2.5)\ \text{km}^3/\text{sec}^2$$

$$\cos\delta\Delta\alpha = -1''45 \pm 0''2$$

$$\Delta\delta = 0''80 \pm 0''4$$

$$\Delta r = (2.56 \pm 0.97) \times 10^{-6}\ \text{a.u.}$$

The mass ratio $(M^s_v)^{-1}$ from the immediately preceding results is

$$(M^s_v)^{-1} = 408504.8 \pm 5.5$$

and the true geocentric equatorial coordinates of Venus for 1962, Dec. 14, 20ʰ0 ET are, with $\Delta\alpha = -0^s100 \pm 0^s015$:

$$\alpha = 14^h51^m58^s282 \pm 0^s015$$

$$\delta = -13°39'28''03 \pm 0''4$$

$$r = 0.038640514 \pm 0.97 \times 10^{-6}\ \text{a. u.}$$

E. Summary of Results

The values of the constants μ, A and M^s_v as determined from the *Mariner II* data are given in Section V-D-3 for μ

Table 17. Correlation matrix, reduced to its lower half, for Solution IX (all data)

Parameters	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$	GM	M_v^s	A
x	1.00								
y	0.98	1.00							
z	−0.27	−0.25	1.00						
$\dot{x}$	−0.06	0.06	0.16	1.00					
$\dot{y}$	0.09	−0.01	−0.07	−0.99	1.00				
$\dot{z}$	0.83	−0.82	0.00	0.37	0.46	1.00			
GM	−0.33	−0.35	0.04	0.07	0.04	0.22	1.00		
M_v^s	0.93	0.91	−0.23	0.22	0.26	0.84	−0.27	1.00	
A	−0.93	−0.91	0.33	0.20	−0.23	0.86	0.19	−0.92	1.00
x_v	0.84	−0.80	0.32	0.38	−0.40	0.80	0.12	−0.95	0.92
y_v	−0.72	−0.70	0.52	0.17	−0.16	0.63	0.05	−0.72	0.91
z_v	−0.22	−0.21	−0.13	0.05	−0.11	0.42	0.03	−0.06	0.36
f_1	0.40	0.35	−0.17	−0.63	0.64	−0.47	−0.20	0.57	−0.31
f_2	−0.69	−0.75	0.00	−0.62	0.58	0.40	0.22	−0.54	0.61
f_3	−0.68	−0.67	0.08	0.25	−0.29	0.62	0.23	−0.82	0.57
α_2	0.17	0.16	−0.02	0.00	−0.02	−0.07	0.23	0.37	−0.27
γ	−0.57	−0.54	0.18	0.43	−0.46	0.53	0.28	−0.70	0.43
Parameters	x_v	y_v	z_v	f_1	f_2	f_3	α_2	γ	
x_v	1.00								
y_v	0.80	1.00							
z_v	0.05	0.44	1.00						
f_1	−0.57	−0.05	0.47	1.00					
f_2	0.39	0.52	0.30	0.26	1.00				
f_3	0.76	0.26	−0.43	−0.84	0.22	1.00			
α_2	−0.51	−0.30	0.46	0.18	−0.18	−0.46	1.00		
γ	0.64	0.13	−0.46	−0.96	−0.01	0.93	−0.21	1.00	

and in Section V-D-5 for A and M_v^s. Those values are

$$\mu^{-1} = 81.3001 \pm 0.0013$$

$$A = (149597546 \pm 500) \text{ km}$$

$$(M_v^s)^{-1} = 408504.8 \pm 5.5$$

The primary purpose of this section is to investigate what effect these values of the constants have on the system of astronomical constants. In Section III-E and V-D-2 the basis for such an investigation is given and the only additional information, besides the preceding values, needed from the *Mariner* determination is the correlation coefficient $\rho_{gm,A}$ between k_{gm}^2 and A. Then, the formulas of Section III-E and V-D-2 can be used to compute corrections and uncertainties in all constants affected by μ, A and M_v^s. The correlation $\rho_{gm,A}$ is taken from Solution IX as given in Table 17.

$$\rho_{gm,A} = 0.1963$$

The other important correlation $\rho_{A,V}$ between A and M_v^s has already been used to compute the uncertainty in k_{gv}^2 which is actually a more fundamental constant than M_v^s for the *Mariner II* data. The correction and uncertainty in M_v^s, or k_{gv}^2, does not affect any of the derived constants in the IAU list. Therefore the analysis of the mass of Venus is complete, and the value of k_{gv}^2 from the *Mariner* data is as given in Section V-D-5

$$k_{gv}^2 = (324871.5 \pm 2.5) \text{ km}^3/\text{sec}^2$$

The value for the mass, M_v^s can be expressed in the notation of Section III-E along with A and μ according to Eq. (20), (21), and (22).

$$\hat{A} = (-16.40 \pm 3.3) \times 10^{-6}$$

$$\hat{M}_v^s = (83.8 \pm 14) \times 10^{-6}$$

$$\hat{\mu} = (-1.2 \pm 16) \times 10^{-6}$$

where the values $\tilde{A}$, $\tilde{M}_{\text{r}}^{s}$ and $\tilde{\mu}$ which are corrected by $\hat{A}$, $\hat{M}_{\text{r}}^{s}$ and $\hat{\mu}$, respectively, are given in Section III-E.

Now Eq. (25) through (33) are used to compute other corrections as defined in Section III-E, and the corresponding uncertainties are obtained from Eq. (165) through (171). Again, it is important to remember that all corrections and uncertainties are based on an assumed value for the velocity of light c, that was adopted by the IAU, and thus are actually only valid for units of light-seconds. If one is interested in expressing the results in terms of the standard meter, then the uncertainty in c enters into all constants that have a length dimension. This is not done here, however, and the corrections from the *Mariner II* data are the following:

$$2\hat{k}_{ge} = (-5.3 \pm 2.0) \times 10^{-6}$$

$$2\hat{k}_{gm} = (-6.5 \pm 16) \times 10^{-6}$$

$$\hat{\tau}_{A} = (-16.44 \pm 3.3) \times 10^{-6}$$

$$2\hat{k}_{gs} = (50.01 \pm 9.9) \times 10^{-6}$$

$$\hat{a}_{\text{☾}} = (-1.77 \pm 0.67) \times 10^{-6}$$

$$\hat{L} = (13.44 \pm 15.6) \times 10^{-6}$$

$$\hat{P}_{\text{☾}} = (14.60 \pm 3.4) \times 10^{-6}$$

$$\hat{S} = (40 \pm 400) \times 10^{-6}$$

In addition the station coordinates for the transmitter (Station 12) and the receiver (Station 11) as determined by the *Mariner II* data are given as follows:

$$R_{11} = (6372004.4 \pm 5.1)\ \text{m}$$

$$\lambda_{11} = 243°09'02''.05 \pm 0''.32$$

$$R_{12} = (6371877.0 \pm 4.7)\ \text{m}$$

$$\lambda_{12} = 243°11'39''.98 \pm 0''.32$$

Because UT1 is used to compute the local sidereal time (Cf. Section IV-C-3) the longitude is referred to the instantaneous pole during the period of *Mariner II.*

Actually, the radius is not determined directly with the Doppler data, but instead the distance $R \cos \phi'$ of the station from the Earth's spin axis is measured by the diurnal component in the data. The geocentric latitudes used in all the solutions are

$$\phi'_{11} = 35°.208070$$

$$\phi'_{12} = 35°.117382$$

and, as a consequence, the distance from the spin axis for the stations during the period of the *Mariner II* data are

$$R_{11} \cos \phi'_{11} = (5206333.6 \pm 4.2)\ \text{m}$$

$$R_{12} \cos \phi'_{12} = (5212037.6 \pm 3.8)\ \text{m}$$

Finally, the true geocentric equatorial coordinates of Venus for 1962, Dec. 14, 20^{h}.0 ET from Section V-D-5 are repeated here for completeness:

$$\alpha = 14^{h}51^{m}58^{s}.282 \pm 0^{s}.015$$

$$\delta = -13°39'28''.03 \pm 0''.4$$

$$r = 0.38640514 \pm 0.97 \times 10^{-6}\ \text{a. u.}$$

Appendix A

Expansion of Doppler Formula

The rigorous Doppler formula (Eq. 75) derived in Section IV-B-1 is expanded here to yield Eq. (81), the Doppler formula used for the actual numerical representation of the observations. First of all, Eq. (75) is expanded to terms in $1/c^2$ by a straightforward application of the binomial series in the following form.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad (x^2 < 1) \tag{A-1}$$

The series for $\nu_{ob}/k\nu_{tr}$ is accordingly given by

$$\frac{\nu_{ob}}{k\nu_{tr}} = 1 - \frac{\dot{\rho}_1 + \dot{\rho}_2}{c} + \frac{1}{c^2}\Big[\dot{\rho}_1\dot{\rho}_2 - \dot{s}_2\dot{s}_p \cos\theta_2 \cos\theta_p - \dot{s}_1\dot{s}_p \cos\phi_1 \cos\phi_p + \dot{s}_p^2 \cos^2\theta_p + \dot{s}_1^2 \cos^2\phi_1 + \frac{1}{2}(\dot{s}_2^2 - \dot{s}_1^2) - (\Phi_2 - \Phi_1)\Big] \tag{A-2}$$

The relativistic terms in Eq. (A-2) involve the relative motion and potential of the station at the time of transmission and reception (t_1 and t_2) but not the velocity or potential of the spacecraft at t_p. Therefore, their contribution is clearly of higher order than $1/c^2$ and can be neglected. Then, Eq. (A-2) takes on the following form:

$$\frac{\nu_{ob}}{k\nu_{tr}} = 1 - \frac{\dot{\rho}_1 + \dot{\rho}_2}{c} + \frac{1}{c^2}\left[\dot{\rho}_1\dot{\rho}_2 + \dot{s}_p \cos\theta_p (\dot{s}_p \cos\theta_p - \dot{s}_2 \cos\theta_2) - \dot{s}_1 \cos\phi_1 (\dot{s}_p \cos\phi_p - \dot{s}_1 \cos\phi_1)\right] \tag{A-3}$$

We now convert from heliocentric to geocentric coordinates. Designate the heliocentric coordinates of the Earth by $\mathbf{r}_E$ and all geocentric coordinates by primed letters. Then the heliocentric positions of the station at t_1 and t_2 and the spacecraft at tp are given by

$$\mathbf{r}_1(t_1) = \mathbf{r}'_1(t_1) + \mathbf{r}_E(t_1) \tag{A-4}$$

$$\mathbf{r}_2(t_2) = \mathbf{r}'_2(t_2) + \mathbf{r}_E(t_2) \tag{A-5}$$

$$\mathbf{r}_p(t_p) = \mathbf{r}'_p(t_p) + \mathbf{r}_E(t_p) \tag{A-6}$$

The two heliocentric range vectors ρ_1 and ρ_2, which are defined by Eq. (61) and (71), are given in terms of the geocentric range vectors $\boldsymbol{\rho}'_1 = \mathbf{r}'_p(t_p) - \mathbf{r}'_1(t_1)$ and $\boldsymbol{\rho}'_2 = \mathbf{r}'_p(t_p) - \mathbf{r}'_2(t_2)$ by

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho}'_1 + \mathbf{r}_E(t_p) - \mathbf{r}_E(t_1) \tag{A-7}$$

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho}'_2 + \mathbf{r}_E(t_p) - \mathbf{r}_E(t_2) \tag{A-8}$$

or to the first order in $t_p - t$, and $t_p - t_2$,

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho}'_1 + \dot{\mathbf{r}}_E(t_p - t_1) \tag{A-9}$$

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho}'_2 + \dot{\mathbf{r}}_E(t_p - t_2) \tag{A-10}$$

but

$$t_p - t_1 = \frac{\rho_1}{c} \tag{A-11}$$

$$t_p - t_2 = -\frac{\rho_2}{c} \tag{A-12}$$

so that

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho}'_1 + \frac{\rho_1}{c}\dot{\mathbf{r}}_E \tag{A-13}$$

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho}'_2 - \frac{\rho_2}{c}\dot{\mathbf{r}}_E \tag{A-14}$$

Eq. (A-13) and (A-14) represent the transformation between the geocentric and heliocentric range vectors to the order $1/c$. Of course in the $1/c^2$ term of Eq. (A-3), it is not necessary to make the distinction between $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ because the difference is of the order $1/c^3$. However, the $1/c$ terms must be carried in transforming $(\dot{\rho}_1 + \dot{\rho}_2)/c$ to geocentric coordinates.

Consider the terms inside the brackets of Eq. (A-3) first. From the definitions of Section IV-B-1 for $\dot{s}_p \cos\phi_p$, $\dot{s}_1 \cos\phi_1$, $\dot{s}_p \cos\theta_p$, and $\dot{s}_2 \cos\theta_2$, we have

$$\dot{s}_p \cos\theta_p - \dot{s}_2 \cos\theta_2 = \dot{\rho}_2 \tag{A-15}$$

$$\dot{s}_p \cos \phi_p - \dot{s}_1 \cos \phi_1 = \dot{\rho}_1 \qquad \text{(A-16)}$$

also

$$\dot{s}_p \cos \theta_p = \frac{\boldsymbol{\rho}_2}{2} \cdot \left[\dot{\mathbf{r}}'_p (t_p) + \dot{\mathbf{r}}_E(t_p)\right] \qquad \text{(A-17)}$$

$$s_1 \cos \phi_1 = \frac{\boldsymbol{\rho}_1}{\rho_1} \cdot \left[\mathbf{r}'_1 (t_1) + \mathbf{r}_E(t_1)\right] \qquad \text{(A-18)}$$

Substitute Eq. (A-15) for (A-18) in Eq. (A-3) to obtain

$$\frac{\nu_{ob}}{k\nu_{tr}} = 1 - \frac{\dot{\rho}_1 + \dot{\rho}_2}{c} + \frac{1}{c^2}\left\{ \dot{\rho}_1 \dot{\rho}_2 + \frac{\dot{\rho}_2}{\rho_2} \boldsymbol{\rho}_2 \cdot \left[\dot{\mathbf{r}}'_p (t_p) + \dot{\mathbf{r}}_E(t_p)\right] - \frac{\dot{\rho}_1}{\rho_1} \boldsymbol{\rho}_1 \cdot \left[\dot{\mathbf{r}}'_1 (t_1) + \dot{\mathbf{r}}_E(t_1)\right]\right\} \qquad \text{(A-19)}$$

The geocentric expression for $(\dot{\rho}_1 + \dot{\rho}_2)/c$ to terms in $1/c^2$ can be obtained from Eq. (A-13) and (A-14) and their derivatives. First, the magnitudes of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are required, or more appropriately, we want $1/\rho_1$ and $1/\rho_2$ for terms in $1/c$.

$$\frac{1}{\rho_1} = \frac{1}{\rho'_1}\left(1 - \frac{1}{c} \frac{\boldsymbol{\rho}_1}{\rho_1} \cdot \dot{\mathbf{r}}_E\right) \qquad \text{(A-20)}$$

$$\frac{1}{\rho_2} = \frac{1}{\rho'_2}\left(1 + \frac{1}{c} \frac{\boldsymbol{\rho}_2}{\rho_2} \cdot \dot{\mathbf{r}}_E\right) \qquad \text{(A-21)}$$

Then

$$\dot{\rho}_1 = \frac{1}{\rho_1} (\boldsymbol{\rho}_1 \cdot \dot{\boldsymbol{\rho}}_1) = \dot{\rho}'_1 + \frac{1}{c} \frac{d}{dt} (\boldsymbol{\rho}_1 \cdot \dot{\mathbf{r}}_E) - \frac{1}{c} \frac{\dot{\rho}_1}{\rho_1} \boldsymbol{\rho}_1 \cdot \dot{\mathbf{r}}_E \qquad \text{(A-22)}$$

$$\dot{\rho}_2 = \frac{1}{\rho_2} (\boldsymbol{\rho}_2 \cdot \dot{\boldsymbol{\rho}}_2) = \dot{\rho}'_2 - \frac{1}{c} \frac{d}{dt} (\boldsymbol{\rho}_2 \cdot \dot{\mathbf{r}}_E) + \frac{1}{c} \frac{\dot{\rho}_2}{\rho_2} \boldsymbol{\rho}_2 \cdot \dot{\mathbf{r}}_E \qquad \text{(A-23)}$$

Substitute Eq. (A-22) and (A-23) into Eq. (A-19).

$$\frac{\nu_{ob}}{k\nu_{tr}} = 1 - \frac{\dot{\rho}'_1 + \dot{\rho}'_2}{c} - \frac{1}{c^2} \frac{d}{dt} (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) \cdot \dot{\mathbf{r}}_E + \frac{1}{c^2}\left\{\dot{\rho}_1\dot{\rho}_2 + \frac{\dot{\rho}_2}{\rho_2} \boldsymbol{\rho}_2 \cdot \dot{\mathbf{r}}'_p - \frac{\dot{\rho}_1}{\rho_1} \boldsymbol{\rho}_1 \cdot \dot{\mathbf{r}}'_1\right\} \qquad \text{(A-24)}$$

As in the case of the relativity terms the term in Eq. (A-24) involving the small difference vector $\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ is a $1/c^3$ term and is neglected. Therefore, the form of the geocentric calculation of the Doppler shift is exactly like the heliocentric form, at least to terms in $1/c^2$.

In order to bring Eq. (A-24) into agreement with the formula used for computing the Doppler data and residuals, call the geocentric station vectors $\mathbf{R}_1(t_1)$ and $\mathbf{R}_2(t_2)$ instead of $\mathbf{r}'_1(t_1)$ and $\mathbf{r}'_2(t_2)$. Also substitute $\dot{\boldsymbol{\rho}}'_2 + \dot{\mathbf{R}}_2$ for $\dot{\mathbf{r}}'_p$ and set the scalar products $\mathbf{R}_1 \cdot \dot{\mathbf{R}}_1$ and $\mathbf{R}_2 \cdot \dot{\mathbf{R}}_2$ equal to zero. These last two products are zero because the radial rates $\dot{R}_1$ and $\dot{R}_2$ of the stations are zero. Then

$$\frac{\nu_{ob}}{k\nu_{tr}} = 1 - \frac{\dot{\rho}'_1 + \dot{\rho}'_2}{c} + \frac{1}{c^2} (\dot{\rho}_1\dot{\rho}_2 + \dot{\rho}_2^2 + H) \qquad \text{(A-25)}$$

where

$$H = \mathbf{r}'_p \cdot \left(\frac{\dot{\rho}_2}{\rho_2} \dot{\mathbf{R}}_2 - \frac{\dot{\rho}_1}{\rho_1} \dot{\mathbf{R}}_1\right) \qquad \text{(A-26)}$$

The expression for H given in Eq. (82) contains the relativistic term $|\dot{\mathbf{R}}_2|^2 - |\dot{\mathbf{R}}_1|^2$ which was dropped early in the derivation of this appendix. It is included in the actual computational formula, however, but for *Mariner II* the contribution is completely negligible. It is sensible only if the transmitter and receiver are separated by several thousand kilometers.

Appendix B

Path of an Electromagnetic Signal Through the Troposphere

The refraction correction of Section IV-C-2 requires the evaluation of the effect of the atmosphere on the range, or more precisely the time of transmission, between the spacecraft and the radar station. In particular we are seeking a correction $\Delta_r\rho$ to range for the evaluation of the cycle count correction given by Eq. (98).

The first assumption in deriving the refraction correction is that the wave is confined to a plane containing the observer, the spacecraft P, and the center of the Earth C. In other words a signal is sent from P and arrives at S (Fig. B-1) or alternatively is sent from S and arrives at P.

The time required for the signal to travel between these two points is designated by Δt, and if the velocity of propagation is given by c, then clearly in the absence of an atmosphere

$$\Delta t = \frac{\rho}{c} \qquad \text{(B-1)}$$

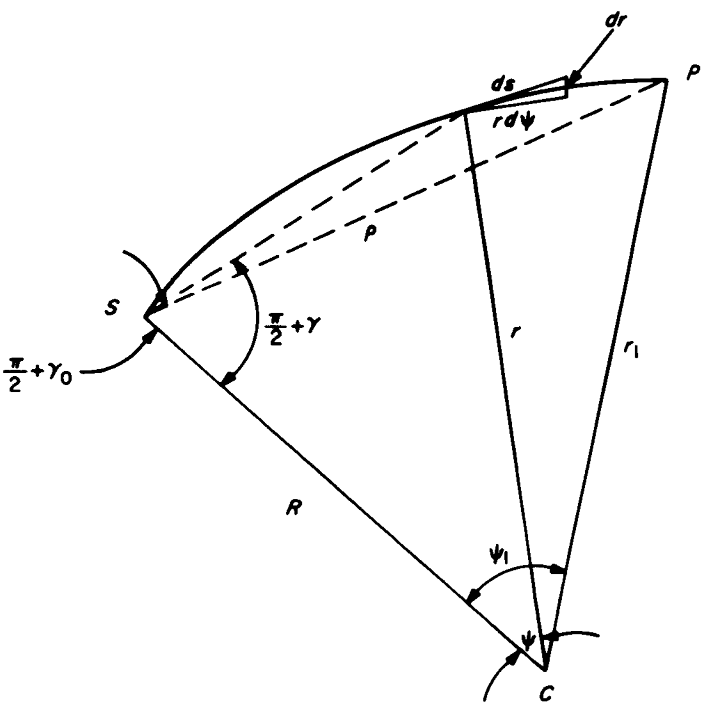

Fig. B-1. Radio propagation geometry

where ρ is the distance between S and P. However, if an atmosphere is introduced, the velocity of propagation will no longer be the constant c but will instead be a variable v. The ratio of c to v is called the index of refraction n, which for empty space is identically equal to unity.

$$n = \frac{c}{v} \qquad \text{(B-2)}$$

For the case where n is a variable the time of transmission Δt is given by

$$\Delta t = \int_S^P \frac{ds}{v} \qquad \text{(B-3)}$$

The element of arc length ds is expressed in terms of the polar coordinates r and ψ of Fig. B-1 by

$$ds^2 = dr^2 + r^2 d\psi^2 \qquad \text{(B-4)}$$

or with

$$\psi_r = \frac{d\psi}{dr} \qquad \text{(B-5)}$$

$$\frac{ds}{dr} = \sqrt{1 + r^2\psi_r^2} \qquad \text{(B-6)}$$

Therefore, the time of transmission can be written as the integral

$$\Delta t = \frac{1}{c}\int_R^{r_1} n\sqrt{1 + r^2\psi_r^2}\; dr \qquad \text{(B-7)}$$

As a matter of interpretation the coordinate r_1 of Fig. B-1 is the geocentric distance to the spacecraft at P. Thus, the altitude H of P above a sphere passing through the station S is

$$H = r_1 - R \qquad \text{(B-8)}$$

The index of refraction n in Eq. (B-7) is simply a function of the physics of the atmosphere and must be chosen once and for all from a consideration of atmospheric measurements. On the other hand, the function ψ_r is arbitrary and for each function selected, a different value of the time of transmission Δt can result. Therefore, in order to specify ψ_r, a physical law is introduced known as Fermat's principle, which states that of all possible paths, a wave will follow the particular path that makes the time of transmission a minimum. From the calculus of variations, the integral (Eq. B-7) is a minimum if the Euler-Lagrange equations are satisfied (Cf. Ref. 36, pp. 326–329).

$$\frac{\partial f}{\partial \psi} - \frac{d}{dr}\frac{\partial f}{\partial \psi_r} = 0 \qquad \text{(B-9)}$$

where the function f is the integrand of Eq. (B-7).

$$f = n\sqrt{1 + r^2\psi_r^2} \qquad \text{(B-10)}$$

The restriction that the index of refraction n is independent of the angle so that f is also independent of ψ is now applied, and the quantity $\partial f/\partial \psi_r$ by Eq. (B-9) is a constant k.

$$\frac{\partial f}{\partial \psi_r} = \frac{Mr^2\psi_r}{\sqrt{1 + r^2\psi_r^2}} = k \qquad \text{(B-11)}$$

An evaluation of this constant at $r = R$ yields

$$k = \frac{n_0 R^2 \psi_{r0}}{\sqrt{1 + R^2\psi_{r0}^2}} \qquad \text{(B-12)}$$

where n_0 is the index of refraction at the station location. To obtain ψ_{r0}, consider Fig. B-1. In terms of the elevation angle γ the law of sines gives the relation

$$R \cos\gamma = r \cos(\gamma + \psi) \qquad \text{(B-13)}$$

and differentiating with respect to the radius r yields

$$r\psi_r \sin(\gamma + \psi) = [R \sin\gamma - r \sin(\gamma + \psi)]\frac{d\gamma}{dr} + \cos(\gamma + \psi) \qquad \text{(B-14)}$$

When $r = R$, Eq. (B-14) reduced to

$$R\psi_{r0} = \text{ctn}\,\gamma_{ob} \qquad \text{(B-15)}$$

where γ_{ob} is the value of γ when the wave reaches S, or in other words, γ_{ob} is the observed elevation angle. The constant k from Eq. (B-12) and (B-15) is

$$k = n_0 R \cos\gamma_{ob} \qquad \text{(B-16)}$$

Substitute this value of k into Eq. (B-11) and solve for ψ_r to obtain the function $d\psi/dr$

$$\frac{d\psi}{dr} = \frac{n_0 R \cos\gamma_{ob}}{r\sqrt{n^2r^2 - n_0^2 R^2 \cos^2\gamma_{ob}}} \qquad \text{(B-17)}$$

The observed range ρ_{ob} is defined by

$$\rho_{ob} = c\Delta t \qquad \text{(B-18)}$$

and using the value of Δt from Eq. (B-7), there results

$$\rho_{ob} = \int_R^{r_1} n\sqrt{1 + r^2\psi_r^2}\,dr \qquad \text{(B-19)}$$

with ψ_r given by Eq. (B-17). An attempt to evaluate the integral of Eq. (B-19) and to form the difference of ρ_{ob} and the computed range ρ leads to numerical difficulties in the subtraction of the two large quantities. However, it is a simple matter to derive the variation

$$\frac{d\Delta_r\rho}{dr} = \frac{d\rho_{ob}}{dr} - \frac{d\rho}{dr} \qquad \text{(B-20)}$$

Using Eq. (B-17) and (B-19) the variation $d\rho_{ob}/dr$ can be written

$$\frac{d\rho_{ob}}{dr} = \frac{n^2r^2\psi_r}{n_0 R \cos\gamma_{ob}} \qquad \text{(B-21)}$$

The variation $d\rho/dr$ is derived from the law of cosines applied to Fig. B-1.

$$\rho^2 = r^2 + R^2 - 2rR\cos\psi \qquad \text{(B-22)}$$

so that

$$\rho\frac{d\rho}{dr} = (r - R\cos\psi) + rR\sin\psi\frac{d\psi}{dr} \qquad \text{(B-23)}$$

To avoid the degeneration of Eq. (B-23) at the point $r = R$ and $\psi = 0$, the elevation angle is used instead of the angle ψ in Eq. (B-23). From Fig. B-1

$$r - R\cos\psi = \frac{\rho}{r}\sqrt{r^2 - R^2\cos^2\gamma} \qquad \text{(B-24)}$$

$$r\sin\psi = \rho\cos\gamma \qquad \text{(B-25)}$$

Thus, the range ρ cancels throughout Eq. (B-23) and

$$\frac{d\rho}{dr} = \psi_r R \cos\gamma + \sqrt{1 - \frac{R^2 \cos^2\gamma}{r^2}} \qquad \text{(B-26)}$$

The variation of Eq. (13-20) follows immediately by subtracting Eq. (B-26) from (B-21):

$$\frac{d\Delta r\rho}{dr} = \psi_r \left(\frac{r^2 n^2}{R\, n_0 \cos\gamma_{ob}} - R\cos\gamma \right) - \sqrt{1 - \frac{R^2\cos^2\gamma}{r^2}} \qquad \text{(B-27)}$$

The presence of both the observed elevation angle γ_{ob} and its computed counterpart in Eq. (B-27) complicates the calculation of $\Delta_r\rho$. However, the difference in the two angles is less than 0°2 even at an elevation angle of 5 deg, and the error committed by failing to differentiate between γ and γ_{ob} in Eq. (B-27) is less than one meter for all elevations above 5 deg. Therefore, Eq. (B-27) can be numerically integrated for various values of γ under the assumption that $\gamma = \gamma_{ob}$. The function ψ_r is given by Eq. (B-17) and an exponential model is chosen for the index of refraction n.

$$n = 1 + (n_0 - 1)\, e^{-H/H_0} \qquad \text{(B-28)}$$

The altitude H is equal to $r - R$ and the selected numerical values for n_0 and H_0 are

$$n_0 - 1 = 3.40 \times 10^{-4}$$

$$H_0 = 7.315 \text{ km}$$

The results of numerically integrating Eq. (B-27) from $H = 0$ to $H = \infty$ are shown in Fig. 7.

Appendix C

Formulae for Matrices G and H

In this appendix, the differential coefficients which comprise the matrices G and H are derived according to the defining relation (Eq. 136) for the two matrices. The differential of the Doppler data is required as a linear combination of differentials in the position and velocity of the probe and in the station coordinates. As a starting point for the development of the coefficients, the formula (Eq. 86) used to compute the Doppler data is differentiated with a neglect of the second order $1/c^2$ and τ^2 terms.

$$df(t_{ob}) = K'k' \frac{\nu_{tr}}{c} (d\dot{\rho}_1 + d\dot{\rho}_2) \qquad \text{(C-1)}$$

From Eq. (87) and (88) the differentials in range rate can be obtained in terms of differentials in the range and range-rate vectors.

A differentiation of Eq. (87) yields

$$\rho_i \, d\rho_i = \boldsymbol{\rho}_i \cdot d\boldsymbol{\rho}_i \qquad \text{(C-2)}$$

and from Eq. (88)

$$\rho_i \, d\dot{\rho}_i + \dot{\rho}_i \, d\rho_i = \boldsymbol{\rho}_i \cdot d\dot{\boldsymbol{\rho}}_i + \dot{\boldsymbol{\rho}}_i \cdot d\boldsymbol{\rho}_i \qquad \text{(C-3)}$$

Designate the unit vector $\boldsymbol{\rho}_i/\rho_i$ by $\mathbf{L}_i$ and combine Eq. (C-2) and (C-3)

$$d\dot{\rho}_i = \frac{1}{\rho_i}(\dot{\boldsymbol{\rho}}_i - \dot{\rho}_i \mathbf{L}_i) \cdot d\boldsymbol{\rho}_i + \mathbf{L} \cdot d\dot{\boldsymbol{\rho}}_i \qquad \text{(C-4)}$$

The differentials in the range and range-rate vectors are now expressed directly in terms of differentials in the geocentric probe coordinates $(\mathbf{r}'_p, \dot{\mathbf{r}}'_p)$ and the station coordinates $(\mathbf{R}_i, \dot{\mathbf{R}}_i)$ by means of the definitions $\boldsymbol{\rho}_i = \mathbf{r}'_p - \mathbf{R}_i$ and $\dot{\boldsymbol{\rho}}_i = \dot{\mathbf{r}}'_p - \dot{\mathbf{R}}_i$

$$d\,\boldsymbol{\rho}_i = d\mathbf{r}'_p - d\mathbf{R}_i \qquad \text{(C-5)}$$

$$d\,\dot{\boldsymbol{\rho}}_i = d\dot{\mathbf{r}}'_p - d\dot{\mathbf{R}}_i \qquad \text{(C-6)}$$

No more is needed to define the differential coefficients with respect to the probe coordinates $\mathbf{r}'_p$ and $\dot{\mathbf{r}}'_p$, but the parameters in the station location vectors $\mathbf{R}_i$ and $\dot{\mathbf{R}}_i$ are not the Cartesian coordinates themselves; instead the geocentric radius R_i and latitude ϕ'_i, and the longitude λ_i are selected for correction by the least squares process. From the transformation equations (Eq. 91, 92, and 93) between R, ϕ' and λ and $\mathbf{R} = (X, Y, Z)$ we obtain differential transformations which are written here without the subscript i:

$$dX = \frac{X}{R} dR - Z \cos\theta d\phi - Y d\lambda \qquad \text{(C-7)}$$

$$dY = \frac{Y}{R} dR - Z \sin\theta d\phi' + X d\lambda \qquad \text{(C-8)}$$

$$dZ = \frac{Z}{R} dR + R \cos\phi' d\phi' \qquad \text{(C-9)}$$

$$d\dot{X} = -\omega dY \qquad \text{(C-10)}$$

$$d\dot{Y} = \omega dX \qquad \text{(C-11)}$$

$$d\dot{Z} = 0 \qquad \text{(C-12)}$$

where the differential in the local sidereal time is assumed equal to the differential in longitude ($d\theta = d\lambda$) which implies that the Greenwich hour angle is known exactly. The angular rate ω is equal to $\dot{\theta}$ and is adopted as the mean sidereal rate of the Earth's rotation.

Now the various differential expressions can be collected together and interpreted as elements of the matrices G and H, but first it is convenient to define a vector $\boldsymbol{\tau}_i$ as the coefficients of $d\boldsymbol{\rho}_i$ in Eq. (C-4)

$$\boldsymbol{\tau}_i = \frac{1}{\rho_i}(\dot{\boldsymbol{\rho}}_i - \dot{\rho}_i \mathbf{L}_i) \qquad \text{(C-13)}$$

Then the required expression for $d\dot{\rho}_i$ becomes

$$\begin{aligned} d\dot{\rho}_i = {} & \tau_{xi}\, dx'_p + \tau_{yi}\, dy'_p + \tau_{zi}\, dz'_p \\ & + L_{xi}\, d\dot{x}'_p + L_{yi}\, d\dot{y}'_p + L_{zi}\, d\dot{z}'_p \qquad \text{(C-14)} \\ & + \dot{\rho}_{Ri}\, dR_i + \dot{\rho}_{\phi i}\, d\phi'_i + \dot{\rho}_{\lambda i}\, d\lambda_i \end{aligned}$$

where

$$\dot{\rho}_{Ri} = -\tau_{xi}\frac{X_i}{R_i} - \tau_{yi}\frac{Y_i}{R_i} - \tau_{zi}\frac{Z_i}{R_i} + \omega L_{xi}\frac{Y_i}{R_i} - \omega L_{yi}\frac{X_i}{R_i} \tag{C-15}$$

$$\dot{\rho}_{\phi i} = \tau_{xi} Z_i \cos\theta_i + \tau_{yi} Z_i \sin\theta_i - \tau_{zi} R_i \cos\phi'_i - \omega L_{xi} Z_i \sin\theta_i + \omega L_{yi} Z_i \cos\theta_i \tag{C-16}$$

$$\dot{\rho}_{\lambda i} = 2\tau_{xi} Y_i - \tau_{yi} X_i + \omega L_{xi} X_i + \omega L_{yi} Y_i \tag{C-17}$$

Of course, the coefficients are functions of the time and are evaluated at the mid-point of the count interval associated with the particular observation of interest. Thus, if coefficients are computed for a series of observations at times $t_1, t_2, \cdots, t_N$, the matrix G has the following form:

$$G = G_1 + G_2 \tag{C-18}$$

where

$$G_i = K'k'\frac{\nu_{tr}}{c}\begin{pmatrix} \tau_{xi}(t_1) & \tau_{yi}(t_1) & \tau_{zi}(t_1) & L_{xi}(t_1) & L_{yi}(t_1) & L_{zi}(t_1) \\ \tau_{xi}(t_2) & \tau_{yi}(t_2) & \tau_{zi}(t_2) & L_{xi}(t_2) & L_{yi}(t_2) & L_{zi}(t_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \tau_{xi}(t_N) & \tau_{yi}(t_N) & \tau_{zi}(t_N) & L_{xi}(t_N) & L_{yi}(t_N) & L_{zi}(t_N) \end{pmatrix}$$

and the elements of the vector **q** in Eq. (136) are arranged as $\mathbf{q} = (x'_p, y'_p, z'_p, \dot{x}'_p, \dot{y}'_p, \dot{z}'_p)$. In a similar fashion, if the elements of the vector **s** in Eq. (136) are arranged according to $\mathbf{s} = (R_1, \phi'_1, \lambda_1, R_2, \phi'_2, \lambda_2)$ then the matrix H is given by

$$H = K'k'\left(\frac{\nu_{tr}}{c}\right) \times \begin{pmatrix} \dot{\rho}_{R1}(t_1) & \dot{\rho}_{\phi 1}(t_1) & \dot{\rho}_{\lambda 1}(t_1) & \dot{\rho}_{R2}(t_1) & \dot{\rho}_{\phi 2}(t_1) & \dot{\rho}_{\lambda 2}(t_1) \\ \dot{\rho}_{R1}(t_2) & \dot{\rho}_{\phi 1}(t_2) & \dot{\rho}_{\lambda 1}(t_1) & \dot{\rho}_{R2}(t_1) & \dot{\rho}_{\phi 2}(t_1) & \dot{\rho}_{\lambda 2}(t_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dot{\rho}_{R1}(t_N) & \dot{\rho}_{\phi 1}(t_N) & \dot{\rho}_{\lambda 1}(t_N) & \dot{\rho}_{R2}(t_N) & \dot{\rho}_{\phi 2}(t_N) & \dot{\rho}_{\lambda 2}(t_N) \end{pmatrix}$$

Appendix D

Formulae for Matrices Θ and Φ

The matrices Θ and Φ defined by Eq. (140) can be derived by forming the first variation of Eq. (119) with respect to the geocentric coordinates $\mathbf{r}$ of the spacecraft and all constants of importance to the orbit computation; but first of all, it is convenient to define a vector function $\mathbf{h}(\mathbf{x})$ of any three dimensional vector $\mathbf{x}$ such that*

$$\mathbf{h}(\mathbf{x}) = -\frac{\mathbf{x}}{x^3}$$

where x is the magnitude of the vector $\mathbf{x}$. Then, a variation in the function $\mathbf{h}$ can be written in terms of a variation in $\mathbf{x}$ as follows:

$$d\mathbf{h}(\mathbf{x}) = J(\mathbf{x})\, d\mathbf{x}$$

where the 3×3 matrix J is given by

$$J(\mathbf{x}) = \frac{3\mathbf{x}\mathbf{x}^T}{x^5} - \frac{1}{x^3} I_3$$

The superscript T indicates the transpose of the column vector $\mathbf{x}$ so that $\mathbf{x}\mathbf{x}^T$ is a 3×3 matrix and I_3 is the unit matrix of order three.

Now the equations of motion (Eq. 119) can be written in a compact form.

$$\frac{d^2\mathbf{r}}{dt^2} = k_{ge}^2\mathbf{h}(\mathbf{r}) + k_{gm}^2 [\mathbf{h}(\mathbf{r}_{☾}) - \mathbf{h}(\mathbf{r}_{p☾})]$$
$$+ k'^2A^3 [\mathbf{h}(\mathbf{r}_{\odot}) - \mathbf{h}(\mathbf{r}_{p\odot})]$$
$$+ k'^2A^3 \sum_{j=1}^{8} M_j^s [\mathbf{h}(\mathbf{r}_j) - \mathbf{h}(\mathbf{r}_{pj})] + \mathbf{P} \qquad \text{(D-1)}$$

where k' is the Gaussian constant in units of a.u. and sec. Consider first the variation of the acceleration of the spacecraft with respect to the position vector $\mathbf{r}$, and obtain the non-trivial part Φ_p of the matrix Φ

$$\partial\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \Phi_p \partial\mathbf{r} \qquad \text{(D-2)}$$

*This definition was suggested by P. R. Peabody.

Neglecting the variation in $\mathbf{P}$, which is zero for no non-gravitational forces and is negligible for small forces, the explicit derivative of Eq. (D-1) for terms containing $\mathbf{r}$ is

$$\partial\left(\frac{d^2\mathbf{r}}{dt^2}\right) = k_{ge}^2 J(\mathbf{r})\partial\mathbf{r} - k_{gm}^2 J(\mathbf{r}_{p☾})\partial\mathbf{r}_{p☾}$$
$$-k'^2A^3 J(\mathbf{r}_{p\odot})\partial\mathbf{r}_{p\odot}$$
$$-k'^2A^3 \sum_{j=1}^{8} M_j^s J(\mathbf{r}_{pj})\partial\mathbf{r}_{pj} \qquad \text{(D-3)}$$

Because the geocentric coordinates of the Sun, Moon and planets are all independent of the spacecraft coordinates $\mathbf{r}$, the variations $\partial\mathbf{r}_{p☾}$, $\partial\mathbf{r}_{p\odot}$ and $\partial\mathbf{r}_{pj}$ are all equal to the negative of the variation $\partial\mathbf{r}$ by Eq. (120). Thus, the 3 x 3 matrix Φ_p is equal to

$$\Phi_p = k_{ge}^2 J(\mathbf{r}) + k_{gm}^2 J(\mathbf{r}_{p☾}) + k'^2A^3 J(\mathbf{r}_{p\odot}) + k'^2A^3 \sum_{j=1}^{8} M_j^s J(\mathbf{r}_{pj}) \qquad \text{(D-4)}$$

The complete 6×6 matrix Φ can now be constructed from the definition $\partial q = \Phi \partial \dot{q}$ where q is the set of six position and velocity coordinates in the spacecraft. Because the coordinates are independent, the variations of $\dot{\mathbf{r}}$ with respect to $\mathbf{r}$ and $\dot{\mathbf{r}}$ are, respectively, the null N_3 and unit I_3 matrices of order three; and also because the equations of motion are independent of $\dot{\mathbf{r}}$, the variations of the accelerations with respect to $\dot{\mathbf{r}}$ produce the null matrix N_3. Therefore, the complete matrix Φ, given in the partitioned form of four 3×3 matrices, is simply

$$\Phi = \begin{pmatrix} N_3 & I_3 \\ \Phi_p & N_3 \end{pmatrix} \qquad \text{(D-5)}$$

The matrix Θ, defined by the variations $\partial\dot{q} = \Theta\partial p$ of the equations of motion with respect to the set p of constants, can be partitioned into a null matrix N_k of order $3\times k$, where k is the number of constants in p, and the $3\times k$ variational matrix Θ_k defined by $\partial\ddot{\mathbf{r}} = \Theta_k\partial p$

$$\Theta = \begin{pmatrix} N_k \\ \Theta_k \end{pmatrix} \qquad \text{(D-6)}$$

It is best to consider the constants in p one at a time when deriving the elements of Θ_k and to remember that each column of Θ_k represents the variation of $\ddot{\mathbf{r}}$ (Eq. 119) with respect to a particular constant. Again, as in the derivation of Φ, the implicit derivatives that occur because of the presence of coordinates of the Sun and spacecraft in $\mathbf{P}$ are neglected, but the explicit derivatives of $\mathbf{P}$ with respect to the attitude control parameters (f_1, f_2, f_3, α_1, α_2, G_T, G_N) and the solar radiation constant γ are included in Θ_p. Now consider the various constants of interest starting with the a.u. to km conversion factor A.

I. Astronomical Unit (a.u.)

In Eq. (119), or better Eq. (D-1), the explicit derivative of A is obvious, but an implicit derivative also enters through the solar and planetary coordinates by means of Eq. (126) and (127).

$$\partial\left(\frac{d^2\mathbf{r}}{dt^2}\right)\Big/\partial A = 3k''^2 A^2 \left[\mathbf{h}(\mathbf{r}_\odot) - \mathbf{h}(\mathbf{r}_{p\odot})\right] + 3k''^2 A^2 \sum_{j=1}^{8} M_j^s \left[\mathbf{h}(\mathbf{r}_j) - \mathbf{h}(\mathbf{r}_{pj})\right] + k''^2 A^3 \left[J(\mathbf{r}_\odot)\frac{\partial \mathbf{r}_\odot}{\partial A} - J(\mathbf{r}_{p\odot})\frac{\partial \mathbf{r}_{p\odot}}{\partial A} + \sum_{j=1}^{8} M_j^s J(\mathbf{r}_j)\frac{\partial \mathbf{r}_j}{\partial A} - \sum_{j=1}^{8} M_j^s J(\mathbf{r}_{pj})\frac{\partial \mathbf{r}_{pj}}{\partial A}\right] \quad \text{(D-7)}$$

where from Eq. (120) and (126)

$$\frac{\partial \mathbf{r}_{p\odot}}{\partial A} = \frac{\partial \mathbf{r}_\odot}{\partial A} = -\mathbf{r}_{\odot B}\,(\text{ephem}) \quad \text{(D-8)}$$

$$\frac{\partial \mathbf{r}_{pj}}{\partial A} = \frac{\partial \mathbf{r}_j}{\partial A} = \mathbf{r}_{\odot j}\,(\text{ephem}) - \mathbf{r}_{\odot B}\,(\text{ephem}) \quad \text{(D-9)}$$

Substituting Eq. (D-8) and (D-9) into (D-7) to obtain the complete variation with respect to A.

$$\partial\left(\frac{d^2\mathbf{r}}{dt^2}\right)\Big/\partial A = 3k''^2 A^2 \Big\{ \mathbf{h}(\mathbf{r}_\odot) - \mathbf{h}(\mathbf{r}_{p\odot}) + \sum_{j=1}^{8} M_j^s \left[\mathbf{h}(\mathbf{r}_j) - \mathbf{h}(\mathbf{r}_{pj})\right] + \left[J(\mathbf{r}_{p\odot}) - J(\mathbf{r}_\odot)\right] A\mathbf{r}_{\odot B}\,(\text{ephem}) + \sum_{j=1}^{8} M_j^s \left[J(\mathbf{r}_j) - J(\mathbf{r}_{pj})\right] \times \left[A\mathbf{r}_{\odot j}\,(\text{ephem}) - A\mathbf{r}_{\odot B}\,(\text{ephem})\right] \Big\} \quad \text{(D-10)}$$

II. Mass of Venus

The variation of Eq. (D-1) with respect to any planetary mass M_i^s is simply the coefficient of that mass in the formula. In particular, for Venus the variation is

$$\frac{\partial}{\partial M_v^s}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = k''^2 A^3 \left[\mathbf{h}(\mathbf{r}_v) - \mathbf{h}(\mathbf{r}_{pv})\right] \quad \text{(D-11)}$$

III. Mass of the Moon

The constant selected for variation is the selenocentric constant k_{gm}^2 which requires, in addition to the explicit derivative, an implicit variation through the scaling R_{em} defined by Eq. (17) and (18), and which occurs through the lunar coordinates according to Eq. (121). It should be noted that the variation in the ephemeris coordinates $\mathbf{r}_{☾}$ (ephem) because of a variation in k_{gm}^2 is neglected just as the effect of the mass of Venus or the ephemeris values of the planetary coordinates were neglected in Eq. (D-11). However, the effect of small variations in the masses on the lunar and planetary ephemerides, which in turn affect the spacecraft coordinates through the equations of motion, is a higher order effect in comparison to the direct variations considered here. Differences between the masses determined with the *Mariner II* data and those used in the lunar and planetary theories could produce noticeable changes to the ephemerides, especially the lunar ephemeris, but these changes would be so small as to have no significant effect on the solution for the constants obtained in Section V-D.

From Eq. (D-1) the partial derivative of the acceleration with respect to k_{gm}^2 is

$$\frac{\partial}{\partial k_{gm}^2}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{h}(\mathbf{r}_{☾}) - \mathbf{h}(\mathbf{r}_{p☾}) + k_{gm}^2 \left[J(\mathbf{r}_{☾}) - J(\mathbf{r}_{p☾})\right]\frac{\partial \mathbf{r}_{☾}}{\partial k_{gm}^2} + k''^2 A^3 \left[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})\right]\frac{\partial \mathbf{r}_\odot}{\partial k_{gm}^2} + k''^2 A^3 \sum_{j=1}^{8} M_j^s \left[J(\mathbf{r}_j) - J(\mathbf{r}_{pj})\right]\frac{\partial \mathbf{r}_j}{\partial k_{gm}^2} \quad \text{(D-12)}$$

From Eq. (121)

$$\frac{\partial \mathbf{r}_{☾}}{\partial k_{gm}^2} = \mathbf{r}_{☾}\,(\text{ephem})\frac{\partial R_{em}}{\partial k_{gm}^2} \quad \text{(D-13)}$$

and from Eq. (127) and (128)

$$\frac{\partial \mathbf{r}_i}{\partial k_{gm}^2} = \frac{\partial \mathbf{r}_\odot}{\partial k_{gm}^2} = \frac{\mu}{1+\mu} \mathbf{r}_☾ \text{(ephem)} \frac{\partial R_{em}}{\partial k_{gm}^2} + \frac{k_{ge}^2}{(k_{ge}^2 + k_{gm}^2)^2} R_{em} \mathbf{r}_☾ \text{(ephem)} \qquad \text{(D-14)}$$

where from Eq. (17) and (18)

$$\frac{\partial R_{em}}{\partial k_{gm}^2} = \frac{1}{3} \frac{R_{em}}{k_{ge}^2 + k_{gm}^2} \qquad \text{(D-15)}$$

Combine Eq. (D-14) and (D-15) to obtain

$$\frac{\partial \mathbf{r}_i}{\partial k_{gm}^2} = \frac{\partial \mathbf{r}_\odot}{\partial k_{gm}^2} = \frac{1}{3} R_{em} \mathbf{r}_☾ \text{(ephem)} \frac{3k_{ge}^2 + k_{gm}^2}{(k_{ge}^2 + k_{gm}^2)^2} \qquad \text{(D-16)}$$

Also, Eq. (D-13) and (D-15) yield

$$\frac{\partial \mathbf{r}_☾}{\partial k_{gm}^2} = \frac{1}{3} R_{em} \mathbf{r}_☾ \text{(ephem)} \frac{1}{k_{ge}^2 + k_{gm}^2} \qquad \text{(D-17)}$$

Finally, substitute Eq. (D-16) and (D-17) into (D-13) to obtain the required partial derivative with respect to k_{gm}^2.

$$\frac{\partial}{\partial k_{gm}^2}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{h}(\mathbf{r}_☾) - \mathbf{h}(\mathbf{r}_{p☾}) + \frac{1}{3}\frac{\mu}{1+\mu}[J(\mathbf{r}_☾) - J(\mathbf{r}_{p☾})] R_{em} \mathbf{r}_☾ \text{(ephem)} + \frac{1}{3} k''^2 A^3 \frac{3k_{ge}^2 + k_{gm}^2}{(k_{ge}^2 + k_{gm}^2)^2} \left\{[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})] + \sum_{j=1}^{8} M_j^s [J(\mathbf{r}_j) - J(\mathbf{r}_{pj})]\right\} R_{em} \mathbf{r}_☾ \text{(ephem)} \qquad \text{(D-18)}$$

IV. Mass of Earth

The derivation of the partial derivative of the accelerations with respect to k_{ge}^2 is very similar to that for k_{gm}^2.

From Eq. (D-1) we have

$$\frac{\partial}{\partial k_{ge}^2}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{h}(\mathbf{r}) + k_{gm}^2 [J(\mathbf{r}_☾) - J(\mathbf{r}_{p☾})] \frac{\partial \mathbf{r}_☾}{2k_{ge}^2} + k''^2 A^3 \left\{[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})] + \sum_{j=1}^{8} M_j^s [J(\mathbf{r}_j) - J(\mathbf{r}_{pj})]\right\} \frac{\partial \mathbf{r}_\odot}{\partial k_{ge}^2} \qquad \text{(D-19)}$$

with

$$\frac{\partial \mathbf{r}}{\partial k_{ge}^2} = \frac{1}{3} R_{em} \mathbf{r}_☾ \text{(ephem)} \frac{1}{k_{ge}^2 + k_{gm}^2} \qquad \text{(D-20)}$$

and

$$\frac{\partial \mathbf{r}_\odot}{\partial k_{ge}^2} = -\frac{2}{3} \frac{k_{gm}^2}{(k_{ge}^2 + k_{gm}^2)^2} R_{em} \mathbf{r}_☾ \text{(ephem)} \qquad \text{(D-21)}$$

The combination of Eq. (D-19), (D-20) and (D-21) is

$$\frac{\partial}{\partial k_{ge}^2}\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{h}(\mathbf{r}) + \frac{1}{3}\frac{\mu}{1+\mu}[J(\mathbf{r}_☾) - J(\mathbf{r}_☾)] R_{em} \mathbf{r}_☾ \text{(ephem)} - \frac{2}{3} \frac{k_{gm}^2}{(k_{ge}^2 + k_{gm}^2)^2} k''^2 A^3 \left\{[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})] + \sum_{j=1}^{8} M_j^s [J(\mathbf{r}_j) - J(\mathbf{r}_{pj})]\right\} R_{em} \mathbf{r}_☾ \text{(ephem)} \qquad \text{(D-22)}$$

V. Solar Radiation and Attitude-Control Parameters

The parameters which can be estimated are f_1, f_2, f_3, α_1, α_2, γ, G_T and G_N, and the required partial derivatives of the spacecraft acceleration vector with respect to these parameters are easily recognized as differential coefficients in the expression for the differential of $\mathbf{P}$. Thus, all that is needed is the expression for $d\mathbf{P}$ which is obtained immediately from Eq. (134).

$$d\mathbf{P} = \alpha(\tau)(\mathbf{U}_{\odot p}\, df_1 + \mathbf{T}\, df_2 + \mathbf{N}\, df_3) + \frac{K}{r_{\odot p}^2 \text{(a.u.)}} (\mathbf{U}_{\odot p}\, d\gamma + \mathbf{T}\, dG_T + \mathbf{N}\, dG_N) - \tau(f_1 \mathbf{U}_{\odot p} + f_2 \mathbf{T} + f_3 \mathbf{N})(d\alpha_1 + \tau\, d\alpha_2) \qquad \text{(D-23)}$$

VI. Earth and Venus Ephemeris

The variations in the spacecraft acceleration with respect to variations in the heliocentric coordinates of the Earth-Moon barycenter $\mathbf{r}_{\odot B}$ (ephem) and Venus $\mathbf{r}_{\odot v}$ (ephem) as they occur in the ephemerides can be obtained immediately from Eq. (D-1) with the use of Eq. (122) and (123). The differential of the acceleration with respect to differentials in $\mathbf{r}_{\odot B}$ (ephem) and $\mathbf{r}_{\odot v}$ (ephem) is

$$d\left(\frac{d^2r}{dt^2}\right) = -k''^2 A^3 \left\{[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})] + \sum_{j=1}^{8} M_j^s [J(\mathbf{r}) - J(\mathbf{r}_{pj})]\right\} A\, d\,\mathbf{r}_{\odot B}\,(\text{ephem}) + k''^2 A^3 M_v^s [J(\mathbf{r}_v) - J(\mathbf{r}_{pv})]\, A\, d\,\mathbf{r}_{\odot v}\,(\text{ephem}) \tag{D-24}$$

Actually, because the spacecraft approaches only Venus, all terms of the summation in Eq. (D-24) except Venus are dropped from the evaluation of the coefficient of $A\, d\,\mathbf{r}_{\odot B}$ (ephem) and Eq. (D-24) is approximated by

$$d\left(\frac{d^2r}{dt^2}\right) = -k''^2 A^3 \left\{[J(\mathbf{r}_\odot) - J(\mathbf{r}_{p\odot})] + M_v^s [J(\mathbf{r}_v) - J(\mathbf{r}_{pv})]\right\} A\, d\,\mathbf{r}_{\odot B}\,(\text{ephem}) + k''^2 A^3 M_v^s [J(\mathbf{r}_v) - J(\mathbf{r}_{pv})]\, A\, d\,\mathbf{r}_{\odot v}\,(\text{ephem}) \tag{D-25}$$

It remains to express the differentials $d\mathbf{r}_{\odot B}$ (ephem) and $d\mathbf{r}_{\odot v}$ (ephem) in terms of a set of orbital elements $\mathbf{E}$ for the Earth and $\mathbf{T}$ for Venus. Four elements of the Earth's orbit are selected for correction according to the ordering as follows:

$\Delta\mathbf{E}_1 = \Delta e'$ (eccentricity correction)

$\Delta\mathbf{E}_2 = \Delta\Psi_1$ (obliquity correction)

$\Delta\mathbf{E}_3 = \Delta 1'_0 + \Delta\Psi'_3$ (mean longitude correction)

$\Delta\mathbf{E}_4 = e'\Delta\Psi'_3$ (longitude of perihelion correction)

The notation on the right-hand side of the definitions for the elements of $\Delta\mathbf{E}$ is from Ref. 28, p. 245. The two elements left out of the total set of elements for the Earth are the mean motion, which is accurately known, and the correction to the equinox, which cannot be determined from radio tracking data without observations against the star background.

The elements of Venus are selected as the cartesian components themselves at some arbitrary epoch because it is the heliocentric position of the planet that must be modified in order to achieve a satisfactory representation of all the data. However, one of the velocity components is eliminated and a constraint is applied to hold the mean distance of the planet in astronomical units a constant. This constaint is derived by linearizing the *vis-viva* integral

$$\dot{\mathbf{r}}_{\odot v} \cdot \dot{\mathbf{r}}_{\odot v} = k''^2(1 + M_v^s)\left(\frac{2}{\mathbf{r}_{\odot v}} - \frac{1}{a_v}\right) \tag{D-26}$$

Hold the mean distance a_v constant in Eq. (D-26) and obtain

$$\dot{\mathbf{r}}_{\odot v} \cdot \Delta\dot{\mathbf{r}}_{\odot v} + k''^2(1 + M_v^s)\frac{\mathbf{r}_{\odot v}}{r_{\odot v}^3} \cdot \Delta\mathbf{r}_{\odot v} = 0 \tag{D-27}$$

In practice, $\dot{x}_{\odot v}$ is eliminated from the elements $\mathbf{T}$ and its correction is computed from the other five elements by rearranging Eq. (D-27) as

$$\dot{x}_{\odot v}\,\Delta\dot{x}_{\odot v} = -k''^2(1 + M_v^s)\frac{1}{r_{\odot v}^3}(x_{\odot v}\,\Delta x_{\odot v} + y_{\odot v}\,\Delta y_{\odot v} + z_{\odot v}\,\Delta z_{\odot v}) - \dot{y}_{\odot v}\,\Delta\dot{y}_{\odot v} - \dot{z}_{\odot v}\,\Delta\dot{z}_{\odot v} \tag{D-28}$$

where the corrections occur at the epoch of osculation for the elements $\mathbf{T}$.

The method for relating corrections $\Delta\mathbf{E}$ and $\Delta\mathbf{T}$ to $\Delta\mathbf{r}_{\odot B}$ and $\Delta\mathbf{r}_{\odot v}$, respectively, is described in detail in Section II-B of Ref. 2 and is not reproduced here. The differential coefficients relating variations in the coordinates to variations in the orbital elements $\mathbf{E}$ and $\mathbf{T}$ are based on two-body formulas which are quite satisfactory for corrections to the planetary ephemeris over short intervals of time.

Appendix E

Doppler Data and Residuals

The Doppler data used to obtain the solutions of Section V-D are listed in Tables E-1 and E-2 along with all other necessary information to completely specify the data. Also the best set of residuals from Solutions III and VII are listed in the fifth column, and in the sixth column the linear residuals from Solution IX are given. The assumed standard deviation on each observation is listed in the fourth column. The inverse square of these numbers represents the weight assigned to each observation in the least-squares solutions.

Table E-1. ***Mariner II*** **Doppler data (Station 12 transmitting, Station 11 receiving)**

Transmitter Frequency = 29.6682 MHz
Pass — Sept. 5, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
01 16 26.0	50		0.0159	−0.0010	−0.0013
01 26 26.0	50	119750.659	0.0159	0.0029	0.0036
01 36 26.0	50	119852.420	0.0159	0.0000	0.0007
01 46 26.0	50	119952.819	0.0159	0.0225	0.0222
01 56 26.0	50	120051.600	0.0159	0.0088	0.0085
02 06 26.0	50	120148.619	0.0159	0.0068	0.0076
02 24 26.0	50	120318.180	0.0159	−0.0049	−0.0051
02 34 26.0	50	120409.239	0.0160	−0.0146	−0.0148
02 44 26.0	50	120497.859	0.0160	0.0010	0.0008
02 54 26.0	50	120583.819	0.0160	−0.0078	−0.0070
03 04 26.0	50	120667.000	0.0160	0.0068	0.0076
03 16 26.0	50	120762.880	0.0161	0.0117	0.0125
03 26 26.0	50	120839.319	0.0161	0.0078	0.0086
03 36 26.0	50	120912.460	0.0160	0.0000	0.0008
03 47 26.0	50	120988.960	0.0161	0.0117	0.0105
03 57 26.0	50	121054.739	0.0162	0.0137	0.0135
04 07 26.0	50	121116.779	0.0162	−0.0137	−0.0139
04 17 26.0	50	121175.039	0.0162	0.0098	0.0095
04 27 26.0	50	121229.319	0.0164	−0.0039	−0.0042
04 37 26.0	50	121279.579	0.0164	0.0078	0.0075
04 48 26.0	50	121330.060	0.0166	0.0039	0.0036
04 59 26.0	50	121375.420	0.0168	0.0088	0.0095
05 09 26.0	50	121412.100	0.0170	−0.0029	−0.0023
05 20 26.0	50	121447.359	0.0175	−0.0234	−0.0238
05 30 26.0	50	121474.760	0.0182	−0.0137	−0.0141
05 40 26.0	50	121497.640	0.0194	−0.0127	−0.0121
05 50 26.0	50	121515.979	0.0218	−0.0010	−0.0014

Transmitter Frequency = 29.6681 MHz
Pass — Sept. 5 and Sept. 6, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 27 26.0	50	116723.979	0.0177	−0.0176	−0.0167
19 37 26.0	50	116754.180	0.0172	0.0186	0.0185
19 47 26.0	50	116788.760	0.0169	0.0059	0.0058
19 57 26.0	50	116827.720	0.0167	0.0107	0.0108
20 07 26.0	50	116870.960	0.0165	0.0068	0.0079
20 17 26.0	50	116918.399	0.0164	−0.0020	−0.0009
20 27 26.0	50	116969.979	0.0163	0.0156	0.0167
20 37 26.0	50	117025.539	0.0162	−0.0020	−0.0008
20 47 26.0	50	117085.020	0.0162	−0.0078	−0.0066
20 57 26.0	50	117148.300	0.0161	−0.0068	−0.0066
21 07 26.0	50	117215.279	0.0161	0.0205	0.0218

Transmitter Frequency = 29.6682 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 20 26.0	50	117307.619	0.0161	−0.0098	−0.0094
21 30 26.0	50	117382.520	0.0161	−0.0049	−0.0045
21 40 26.0	50	117460.640	0.0160	−0.0010	−0.0005
21 50 26.0	50	117541.840	0.0160	0.0137	0.0152
22 02 26.0	50	117643.060	0.0160	−0.0195	−0.0180
22 12 26.0	50	117730.460	0.0160	0.0039	0.0045
22 22 26.0	50	117820.380	0.0159	−0.0029	−0.0033
22 33 26.0	50	117922.060	0.0159	0.0215	0.0212
22 43 26.0	50	118016.720	0.0159	−0.0254	−0.0247
22 53 26.0	50	118113.460	0.0159	0.0137	0.0135
23 03 26.0	50	118211.939	0.0159	−0.0166	−0.0158
23 13 26.0	50	118312.079	0.0159	−0.0039	−0.0030
23 23 26.0	50	118413.619	0.0159	−0.0127	−0.0137
23 33 26.0	50	118516.399	0.0159	−0.0107	−0.0108
23 43 26.0	50	118620.220	0.0159	0.0039	0.0039
23 53 26.0	50	118724.840	0.0159	−0.0078	−0.0078
00 03 26.0	50	118830.119	0.0159	0.0127	0.0118
00 13 26.0	50	118935.800	0.0159	0.0146	0.0148
00 23 26.0	50	119041.680	0.0159	0.0000	−0.0008
00 33 26.0	50	119147.579	0.0159	−0.0059	−0.0066
00 44 26.0	50	119263.840	0.0159	−0.0107	−0.0115
00 54 26.0	50	119369.140	0.0159	0.0254	0.0247
01 04 26.0	50	119473.760	0.0159	0.0049	0.0042
01 14 26.0	50	119577.560	0.0159	−0.0117	−0.0114
01 24 26.0	50	119680.359	0.0159	−0.0010	−0.0026
01 34 26.0	50	119781.920	0.0159	−0.0059	−0.0065
01 44 26.0	50	119882.079	0.0159	0.0078	0.0082
01 54 26.0	50	119980.600	0.0159	0.0000	0.0015
02 04 26.0	50	120077.319	0.0159	−0.0049	−0.0054
02 14 26.0	50	120172.060	0.0159	0.0039	0.0034
02 24 26.0	50	120264.619	0.0159	0.0078	0.0064
02 34 26.0	50	120354.800	0.0160	−0.0137	−0.0151
02 44 26.0	50	120442.500	0.0160	0.0127	0.0123
02 54 26.0	50	120527.479	0.0160	0.0166	0.0162
03 04 26.0	50	120609.579	0.0160	0.0029	0.0025
03 14 26.0	50	120688.680	0.0161	0.0107	0.0103
03 24 26.0	50	120764.579	0.0161	−0.0098	−0.0101
03 34 26.0	50	120837.199	0.0161	0.0107	0.0094
03 44 26.0	50	120906.340	0.0161	0.0098	0.0094
03 54 26.0	50	120971.880	0.0162	0.0010	−0.0004
04 04 26.0	50	121033.720	0.0162	0.0117	0.0104

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 03 26.0	50	117129.359	0.0161	−0.0039	−0.0056
21 13 26.0	50	117199.840	0.0161	0.0078	0.0071
21 23 26.0	50	117273.699	0.0161	−0.0088	−0.0104
21 33 26.0	50	117350.840	0.0161	−0.0107	−0.0123
21 43 26.0	50	117431.100	0.0160	−0.0098	−0.0123
21 53 26.0	50	117514.340	0.0160	0.0078	0.0063
22 03 26.0	50	117600.359	0.0160	0.0049	0.0034
22 15 26.0	50	117707.060	0.0160	0.0127	0.0112
22 25 26.0	50	117798.640	0.0159	−0.0020	−0.0034
22 35 26.0	50	117892.479	0.0159	−0.0098	−0.0112
22 45 26.0	50	117988.399	0.0159	−0.0078	−0.0092
22 55 26.0	50	118086.220	0.0159	0.0088	0.0075
23 05 26.0	50	118185.699	0.0159	−0.0127	−0.0140
23 15 26.0	50	118286.720	0.0159	0.0039	0.0026
23 25 26.0	50	118389.039	0.0159	0.0098	0.0085
23 35 26.0	50	118492.479	0.0159	0.0264	0.0242
23 47 26.0	50	118617.760	0.0159	0.0127	0.0106
23 57 26.0	50	118722.920	0.0159	0.0107	0.0086
00 07 26.0	50	118828.500	0.0159	−0.0381	−0.0401
00 19 26.0	50	118955.619	0.0159	0.0020	0.0009
00 31 26.0	50	119082.720	0.0159	0.0010	0.0009
00 41 26.0	50	119188.380	0.0159	−0.0156	−0.0176
04 14 26.0	50	121091.680	0.0162	−0.0205	−0.0209
04 24 26.0	50	121145.739	0.0164	−0.0029	−0.0043
04 34 26.0	50	121195.739	0.0164	0.0078	0.0065
04 44 26.0	50	121241.579	0.0165	0.0078	0.0075
04 54 26.0	50	121283.180	0.0167	0.0059	0.0055
05 04 26.0	50	121320.460	0.0170	0.0010	−0.0004
05 14 26.0	50	121353.359	0.0174	0.0029	0.0016
05 24 26.0	50	121381.819	0.0180	0.0137	0.0133
05 34 26.0	50	121405.739	0.0190	−0.0117	−0.0131
05 44 26.0	50	121425.159	0.0211	0.0039	0.0035
05 54 26.0	50	121439.960	0.0258	−0.0322	−0.0326

Transmitter Frequency = 29.6682 MHz
Pass — Sept. 6 and Sept. 7, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 31 26.0	50	116662.180	0.0173	−0.0010	−0.0029
19 41 26.0	50	116695.859	0.0170	0.0049	0.0030
19 52 26.0	50	116737.939	0.0167	−0.0107	−0.0116
20 03 26.0	50	116785.239	0.0165	−0.0039	−0.0057
20 13 26.0	50	116832.640	0.0164	−0.0176	−0.0194
20 23 26.0	50	116884.199	0.0163	0.0117	0.0109
20 33 26.0	50	116939.739	0.0162	0.0059	0.0041
20 43 26.0	50	116999.180	0.0162	−0.0098	−0.0105
20 53 26.0	50	117062.439	0.0161	0.0000	0.0017
00 51 26.0	50	119293.640	0.0159	0.0010	−0.0009
01 05 26.0	50	119439.880	0.0159	0.0195	0.0186
01 17 26.0	50	119563.739	0.0159	−0.0371	−0.0380
01 27 26.0	50	119665.760	0.0159	−0.0166	−0.0174
01 37 26.0	50	119766.420	0.0159	0.0020	0.0002
01 47 26.0	50	119865.479	0.0159	−0.0273	−0.0291
01 57 26.0	50	119962.840	0.0159	−0.0117	−0.0125
02 07 26.0	50	120058.260	0.0159	−0.0020	−0.0036
02 17 26.0	50	120151.539	0.0159	−0.0176	−0.0192
02 27 26.0	50	120242.560	0.0160	0.0059	0.0042
02 37 26.0	50	120331.079	0.0160	0.0010	−0.0006
02 47 26.0	50	120416.960	0.0160	0.0020	0.0004
02 57 26.0	50	120500.020	0.0160	−0.0098	−0.0103
03 07 26.0	50	120580.159	0.0161	0.0293	0.0287
03 19 26.0	50	120672.100	0.0161	−0.0117	−0.0132
03 29 26.0	50	120745.140	0.0161	−0.0029	−0.0044
03 39 26.0	50	120814.739	0.0161	0.0078	0.0063
03 49 26.0	50	120880.760	0.0162	0.0156	0.0151
03 59 26.0	50	120943.039	0.0162	−0.0127	−0.0141
04 09 26.0	50	121001.539	0.0162	0.0029	0.0015
04 19 26.0	50	121056.100	0.0163	0.0156	0.0152
04 31 26.0	50	121116.199	0.0164	0.0010	−0.0004
04 41 26.0	50	121161.720	0.0166	−0.0088	−0.0092
04 53 26.0	50	121210.760	0.0168	0.0039	0.0035
05 03 26.0	50	121246.880	0.0171	0.0342	0.0338
05 13 26.0	50	121278.520	0.0175	−0.0088	−0.0101
05 13 26.0	50	121278.520	0.0175	−0.0088	−0.0101
05 23 26.0	50	121305.760	0.0182	0.0137	0.0123
05 33 26.0	50	121328.460	0.0195	0.0107	0.0114
05 43 26.0	50	121346.600	0.0219	0.0010	0.0006
05 53 26.0	50	121360.140	0.0283	−0.0400	−0.0414

Transmitter Frequency = 29.6682 MHz
Pass — Sept. 7 and Sept. 8, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 03 26.0	50	116517.939	0.0193	0.0156	0.0148
19 13 26.0	50	116540.840	0.0182	−0.0029	−0.0038
19 23 26.0	50	116568.279	0.0175	0.0020	0.0011
19 34 26.0	50	116603.600	0.0170	−0.0098	−0.0096
19 44 26.0	50	116640.340	0.0168	−0.0088	−0.0096
19 54 26.0	50	116681.399	0.0166	−0.0176	−0.0184
20 04 26.0	50	116726.739	0.0164	0.0039	0.0041
20 15 26.0	50	116781.399	0.0164	0.0107	0.0109
20 25 26.0	50	116835.340	0.0162	0.0020	0.0021
20 35 26.0	50	116893.260	0.0162	0.0195	0.0197
20 46 26.0	50	116961.380	0.0162	0.0127	0.0129
20 56 26.0	50	117027.220	0.0161	0.0186	0.0198
21 06 26.0	50	117096.619	0.0161	0.0000	0.0003
21 16 26.0	50	117169.500	0.0161	0.0117	0.0120
21 26 26.0	50	117245.659	0.0161	−0.0088	−0.0075
21 36 26.0	50	117325.000	0.0160	−0.0098	−0.0095
21 46 26.0	50	117407.340	0.0160	−0.0225	−0.0212
21 56 26.0	50	117492.579	0.0160	0.0127	0.0140
22 06 26.0	50	117580.460	0.0160	0.0039	0.0043
22 17 26.0	50	117680.039	0.0159	0.0029	0.0033
22 27 26.0	50	117773.000	0.0159	−0.0117	−0.0103
22 37 26.0	50	117868.159	0.0159	0.0293	0.0297
22 47 26.0	50	117965.199	0.0159	−0.0127	−0.0122
22 57 26.0	50	118064.079	0.0159	0.0117	0.0132
23 07 26.0	50	118164.500	0.0159	−0.0059	−0.0044
23 17 26.0	50	118266.340	0.0159	0.0078	0.0084
23 27 26.0	50	118369.359	0.0159	0.0088	0.0094
23 37 26.0	50	118473.359	0.0159	−0.0029	−0.0013
23 47 26.0	50	118578.159	0.0159	−0.0068	−0.0052
23 57 26.0	50	118683.560	0.0159	0.0010	0.0026
00 07 26.0	50	118789.340	0.0159	0.0020	0.0036

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
00 17 26.0	50	118895.300	0.0159	0.0029	0.0046
00 27 26.0	50	119001.239	0.0159	0.0078	0.0086
00 37 26.0	50	119106.939	0.0159	0.0020	0.0027
00 47 26.0	50	119212.220	0.0159	0.0098	0.0106
00 57 26.0	50	119316.819	0.0159	−0.0273	−0.0264
01 07 26.0	50	119420.640	0.0159	−0.0039	−0.0030
01 18 26.0	50	119533.619	0.0159	0.0068	0.0078
01 28 26.0	50	119635.000	0.0159	0.0029	0.0039
01 38 26.0	50	119734.939	0.0159	0.0107	0.0117
01 48 26.0	50	119833.199	0.0159	−0.0137	−0.0127
01 58 26.0	50	119929.659	0.0159	−0.0010	−0.0001
02 08 26.0	50	120024.079	0.0159	−0.0049	−0.0048
02 18 26.0	50	120116.300	0.0159	−0.0039	−0.0038
02 28 26.0	50	120206.140	0.0160	0.0000	0.0011
02 38 26.0	50	120293.420	0.0160	0.0010	0.0022
03 03 26.0	50	120499.340	0.0161	0.0029	0.0022
03 13 26.0	50	120576.357	0.0161	−0.0176	−0.0173
03 23 26.0	50	120650.140	0.0161	−0.0049	−0.0055
03 33 26.0	50	120720.479	0.0161	−0.0205	−0.0201
03 43 26.0	50	120787.300	0 0161	−0.0068	−0.0064
03 53 26.0	50	120850.439	0.0162	0.0029	0.0033
04 03 26.0	50	120909.760	0.0162	−0.0068	−0.0064
04 13 26.0	50	120965.159	0.0163	−0.0244	−0.0249
04 23 26.0	50	121016.619	0.0164	0.0381	0.0376
04 33 26.0	50	121063.880	0.0165	0.0186	0.0191
04 43 26.0	50	121106.920	0.0167	−0.0117	−0.0112
04 53 26.0	50	121145.720	0.0169	0.0098	0.0093
05 03 26.0	50	121180.079	0.0172	−0.0449	−0.0443
05 13 26.0	50	121210.079	0.0178	−0.0273	−0.0267
05 23 26.0	50	121235.619	0.0186	0.0176	0.0182
05 33 26.0	50	121256.539	0.0202	−0.0264	−0.0267
05 43 26.0	50	121272.960	0.0236	−0.0078	−0.0072
05 53 26.0	50	121284.779	0.0336	−0.0234	−0.0228

Transmitter Frequency = 29.6682 MHz
Pass — Sept. 8 and Sept. 9, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 02 26.0	50	116446.720	0.0189	−0.0010	0.0003
19 12 26.0	50	116471.060	0.0179	0.0332	0.0353
19 22 26.0	50	116499.840	0.0173	0.0078	0.0100
19 32 26.0	50	116533.079	0.0170	−0.0068	−0.0047
19 42 26.0	50	116570.720	0.0167	−0.0049	−0.0027
19 52 26.0	50	116612.680	0.0165	0.0039	0.0061
20 02 26.0	50	116658.880	0.0164	0.0186	0.0207
20 12 26.0	50	116709.199	0.0164	0.0098	0.0110
20 22 26.0	50	116763.560	0.0162	−0.0068	−0.0047
20 32 26.0	50	116821.880	0.0162	−0.0059	−0.0037
20 42 26.0	50	116884.000	0.0162	−0.0361	−0.0340
20 52 26.0	50	116949.920	0.0161	0.0215	0.0237
21 02 26.0	50	117019.340	0.0161	−0.0039	−0.0027
21 12 26.0	50	117092.239	0.0161	0.0010	−0.0012
21 22 26.0	50	117176.239	0.0161	−0.0264	−0.0242
21 32 26.0	50	117247.819	0.0160	0.0039	0.0061
21 42 26.0	50	117330.199	0.0160	0.0049	0.0061
21 52 26.0	50	117415.420	0.0160	−0.0059	−0.0036
22 02 26.0	50	117503.319	0.0160	−0.0234	−0.0212
22 12 26.0	50	117593.760	0.0159	−0.0166	−0.0153
22 22 26.0	50	117686.539	0.0159	−0.0146	−0.0124
22 32 26.0	50	117781.479	0.0159	−0.0137	−0.0114
22 42 26.0	50	117878.399	0.0159	−0.0137	−0.0113
22 52 26.0	50	117977.119	0.0159	−0.0068	−0.0055
23 02 26.0	50	118077.420	0.0159	−0.0215	−0.0201
23 12 26.0	50	118179.180	0.0159	0.0146	0.0161
23 22 26.0	50	118282.100	0.0020	−0.0020	0.0005
23 33 26.0	50	118396.479	0.0159	−0.0137	−0.0112
23 43 26.0	50	118501.340	0.0159	0.0156	0.0181
23 53 26.0	50	118606.720	0.0159	−0.0254	−0.0229
00 03 26.0	50	118712.560	0.0159	0.0059	0.0084
00 13 26.0	50	118818.560	0.0159	0.0186	0.0211
00 23 26.0	50	118924.500	0.0159	−0.0049	−0.0033
00 33 26.0	50	119030.239	0.0159	−0.0020	−0.0003
00 43 26.0	50	119135.560	0.0159	0.0146	0.0153
00 53 26.0	50	119240.220	0.0159	0.0059	0.0076
01 03 26.0	50	119344.039	0.0159	−0.0049	−0.0032
01 13 26.0	50	119446.840	0.0159	0.0049	0.0057
01 23 26.0	50	119548.380	0.0159	−0.0088	−0.0070
01 33 26.0	50	119648.520	0.0159	0.0107	0.0126
01 43 26.0	50	119747.000	0.0159	−0.0029	−0.0011
01 53 26.0	50	119843.680	0.0159	0.0010	0.0019
02 03 26.0	50	119938.380	0.0159	0.0283	0.0293
02 13 26.0	50	120030.840	0.0159	0.0020	0.0029
02 23 26.0	50	120120.939	0.0160	−0.0205	−0.0195
02 33 26.0	50	120208.520	0.0160	−0.0234	−0.0214
02 43 26.0	50	120293.420	0.0160	0.0010	0.0030
02 53 26.0	50	120375.439	0.0160	0.0166	0.0187
03 03 26.0	50	120454.399	0.0161	0.0029	0.0041
03 13 26.0	50	120530.199	0.0161	0.0098	0.0119
03 23 26.0	50	120602.659	0.0161	0.0049	0.0061
03 33 26.0	50	120671.659	0.0161	0.0068	0.0091
03 43 26.0	50	120737.060	0.0162	0.0107	0.0120
03 53 26.0	50	120798.720	0.0162	−0.0010	0.0013
04 03 26.0	50	120856.539	0.0162	−0.0078	−0.0055
04 13 26.0	50	120910.399	0.0164	−0.0176	−0.0162
04 23 26.0	50	120960.220	0.0164	−0.0078	−0.0064
04 33 26.0	50	121005.880	0.0166	−0.0029	−0.0015
04 43 26.0	50	121047.300	0.0168	0.0049	0.0064
04 53 26.0	50	121084.399	0.0170	0.0137	0.0152
05 03 26.0	50	121117.100	0.0174	0.0166	0.0182
05 13 26.0	50	121145.319	0.0180	−0.0068	−0.0052
05 23 26.0	50	121169.039	0.0191	−0.0244	−0.0228
05 34 26.0	50	121189.920	0.0215	−0.0010	−0.0003
05 44 26.0	50	121204.060	0.0270	−0.0264	−0.0247

Transmitter Frequency = 29.6681 MHz
Pass — Sept. 14 and Sept. 15, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
18 39 26.0	50	116114.079	0.0186	0.0117	0.0108
18 49 26.0	50	116139.659	0.0177	−0.0098	−0.0097
18 59 26.0	50	116169.779	0.0172	0.0166	0.0157
19 09 26.0	50	116204.300	0.0169	0.0059	0.0059
19 19 26.0	50	116243.199	0.0167	0.0029	0.0029
19 29 26.0	50	116286.399	0.0165	0.0029	0.0029
19 39 26.0	50	116333.800	0.0164	−0.0127	−0.0118

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 49 26.0	50	116385.340	0.0163	−0.0117	−0.0128
19 59 26.0	50	116440.920	0.0162	0.0020	0.0018
20 09 26.0	50	116500.460	0.0162	0.0557	0.0565
20 19 26.0	50	116563.699	0.0161	0.0049	0.0037
20 29 26.0	50	116630.659	0.0161	−0.0107	−0.0110
20 39 26.0	50	116701.180	0.0161	−0.0205	−0.0218
20 49 26.0	50	116775.159	0.0161	0.0078	0.0075
20 59 26.0	50	116852.380	0.0161	0.0010	−0.0003
21 09 26.0	50	116932.760	0.0160	0.0215	0.0212
21 19 26.0	50	117016.060	0.0160	−0.0117	−0.0121
21 29 26.0	50	117102.239	0.0160	0.0195	0.0182
21 39 26.0	50	117191.020	0.0160	0.0010	0.0006
21 50 26.0	50	117291.560	0.0159	0.0068	0.0065
22 00 26.0	50	117385.340	0.0159	−0.0137	−0.0141
22 10 26.0	50	117481.260	0.0159	0.0000	0.0000
22 20 26.0	50	117579.079	0.0159	−0.0059	−0.0062
22 30 26.0	50	117678.640	0.0159	−0.0039	−0.0043
22 41 26.0	50	117789.920	0.0159	−0.0098	−0.0101
22 51 26.0	50	117892.479	0.0159	−0.0195	−0.0189

Transmitter Frequency = 29.6682 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
23 01 26.0	50	117996.239	0.0159	−0.0186	−0.0189
23 11 26.0	50	118100.880	0.0159	−0.0049	−0.0062
23 21 26.0	50	118206.239	0.0159	−0.0020	−0.0023
23 31 26.0	50	118312.140	0.0159	0.0176	0.0173
23 42 26.0	50	118428.979	0.0159	0.0244	0.0241
23 52 26.0	50	118535.260	0.0159	−0.0156	−0.0159
00 02 26.0	50	118641.500	0.0159	0.0117	0.0115
00 12 26.0	50	118747.380	0.0159	−0.0088	−0.0090
00 22 26.0	50	118852.760	0.0159	−0.0117	−0.0109
00 32 26.0	50	118957.420	0.0159	−0.0156	−0.0158
00 42 26.0	50	119061.180	0.0159	0.0020	0.0018
00 52 26.0	50	119163.800	0.0159	−0.0010	−0.0021
01 02 26.0	50	119265.079	0.0159	−0.0264	−0.0264
01 12 26.0	50	119364.899	0.0159	0.0020	0.0029
01 22 26.0	50	119463.000	0.0159	0.0166	0.0166
01 32 26.0	50	119559.180	0.0159	0.0029	0.0030
01 42 26.0	50	119653.319	0.0159	0.0273	0.0274
01 52 26.0	50	119745.159	0.0160	0.0117	0.0119
02 02 26.0	50	119834.579	0.0160	0.0127	0.0129
02 12 26.0	50	119921.359	0.0160	−0.0176	−0.0173
02 22 26.0	50	120005.420	0.0160	0.0078	0.0071
02 32 26.0	50	120086.500	0.0160	−0.0117	−0.0114
02 42 26.0	50	120164.539	0.0161	0.0205	0.0209
02 52 26.0	50	120239.279	0.0161	−0.0049	−0.0045
03 02 26.0	50	120310.680	0.0161	0.0166	0.0171
03 12 26.0	50	120378.539	0.0161	0.0195	0.0201
03 22 26.0	50	120442.739	0.0162	0.0156	0.0162
03 32 26.0	50	120503.140	0.0162	−0.0117	−0.0111
03 42 26.0	50	120559.699	0.0163	0.0117	0.0124
03 52 26.0	50	120612.220	0.0164	−0.0039	−0.0031
04 02 26.0	50	120660.640	0.0165	−0.0195	−0.0177
04 12 26.0	50	120704.899	0.0167	−0.0020	−0.0001
04 22 26.0	50	120744.880	0.0168	0.0156	0.0166
04 32 26.0	50	120780.460	0.0172	−0.0156	−0.0146
04 42 26.0	50	120811.659	0.0176	−0.0049	−0.0038
04 52 26.0	50	120838.359	0.0184	−0.0156	−0.0135
05 02 26.0	50	120860.539	0.0197	−0.0186	−0.0164
05 12 26.0	50	120878.159	0.0225	−0.0186	−0.0183

Transmitter Frequency = 29.6682 MHz
Pass — Sept. 22 and Sept. 23, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
18 13 26.0	50	116082.739	0.0177	0.0078	0.0076
18 26 32.0	180	116123.655	0.0171	−0.0039	−0.0042
18 39 02.0	600	116169.941	0.0167	0.0078	0.0085
18 49 02.0	600	116211.701	0.0165	0.0039	0.0045
19 07 32.0	420	116300.021	0.0164	0.0029	0.0025
19 20 02.0	600	116367.793	0.0162	0.0049	0.0053
20 14 02.0	600	116728.251	0.0161	0.0010	0.0003
21 43 02.0	600	117517.491	0.0159	0.0078	0.0079
21 53 02.0	600	117617.201	0.0159	0.0029	0.0020
22 08 02.0	600	117769.638	0.0159	−0.0029	−0.0028
22 18 02.0	600	117872.901	0.0159	0.0029	0.0030
22 30 32.0	180	118003.428	0.0159	0.0049	0.0050
22 41 02.0	480	118114.125	0.0159	0.0039	0.0040
22 54 02.0	600	118252.069	0.0159	0.0059	0.0069
23 07 02.0	600	118390.585	0.0159	0.0049	0.0060
23 17 02.0	600	118497.251	0.0159	0.0059	0.0050
23 27 02.0	600	118603.805	0.0159	0.0186	0.0177
23 37 02.0	600	118710.003	0.0159	0.0049	0.0050
02 30 32.0	420	120300.466	0.0161	0.0107	0.0116
02 47 02.0	600	120408.359	0.0162	0.0068	0.0068
03 11 02.0	600	120546.873	0.0163	0.0049	0.0049
03 21 02.0	600	120597.784	0.0164	0.0039	0.0040
03 42 02.0	600	120691.086	0.0167	−0.0029	−0.0017
04 00 02.0	240	120756.096	0.0173	0.9186	0.0189
04 12 02.0	600	120791.161	0.9179	−0.0010	−0.0004
04 22 02.0	600	120815.569	0.0190	−0.0010	−0.0005
04 42 02.0	120	120850.883	0.0255	−0.0098	−0.0092

Transmitter Frequency = 29.6681 MHz
Pass — Sept. 23 and Sept. 24, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
18 00 26.0	50	116078.600	0.0183	0.0049	0.0029
18 10 26.0	50	116105.420	0.0176	−0.0371	−0.0382
18 20 26.0	50	116136.819	0.0172	0.0156	0.0145
18 30 26.0	50	116172.560	0.0168	−0.0186	−0.0197
18 40 26.0	50	116212.720	0.0166	0.0068	0.0056
18 50 26.0	50	116257.159	0.0165	0.0273	0.0261
19 01 26.0	50	116310.840	0.0164	−0.0010	−0.0023
19 11 26.0	50	116364.000	0.0163	0.0273	0.0250
19 21 26.0	50	116421.079	0.0162	0.0234	−0.0248
19 31 26.0	50	116482.100	0.0162	−0.0205	−0.0219
19 41 26.0	50	116546.920	0.0161	0.0107	0.0103
19 51 26.0	50	116615.380	0.0161	0.0342	0.0327
20 01 26.0	50	116687.319	0.0161	0.0195	0.0180
20 11 26.0	50	116762.640	0.0161	0.0059	0.0053
20 21 26.0	50	116841.199	0.0160	−0.0039	−0.0045
20 31 26.0	50	116922.840	0.0160	−0.0176	−0.0182
20 41 26.0	50	117007.460	0.0160	0.0195	0.0179
20 51 26.0	50	117094.760	0.0160	−0.0322	−0.0329

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 01 26.0	50	117184.739	0.0159	−0.0029	−0.0036
21 11 26.0	50	117277.119	0.0159	−0.0010	−0.0027
21 21 26.0	50	117371.739	0.0159	−0.0107	−0.0115
21 31 26.0	50	117468.439	0.0159	−0.0088	−0.0095
21 41 26.0	50	117567.020	0.0159	−0.0107	−0.0115
21 51 26.0	50	117667.319	0.0159	0.0107	0.0100
22 01 26.0	50	117769.100	0.0159	0.0098	0.0090
22 12 26.0	50	117882.539	0.0159	−0.0137	−0.0144
22 22 26.0	50	117986.859	0.0159	0.0078	0.0070
22 32 26.0	50	118092.039	0.0159	−0.0029	−0.0037
22 42 26.0	50	118197.920	0.0159	−0.0020	−0.0017
22 52 26.0	50	118304.300	0.0159	0.0137	0.0139
23 02 26.0	50	118410.920	0.0159	−0.0127	−0.0134
23 12 26.0	50	118517.640	0.0159	−0.0176	−0.0173
23 22 26.0	50	118624.260	0.0159	0.0059	0.0051
23 32 26.0	50	118730.520	0.0159	0.0000	−0.0007
23 42 26.0	50	118836.239	0.0159	−0.0107	−0.0104
23 52 26.0	50	118941.239	0.0159	−0.0029	−0.0036
00 03 26.0	50	119055.640	0.0159	−0.0039	−0.0036
00 13 26.0	50	119158.439	0.0159	0.0068	0.0072
00 24 26.0	50	119269.920	0.0159	−0.0068	−0.0074
00 34 26.0	50	119369.659	0.0159	0.0146	0.0141
00 44 26.0	50	119467.619	0.0159	0.0166	0.0171
00 54 26.0	50	119563.619	0.0159	0.0068	0.0073
01 04 26.0	50	119657.500	0.0159	0.0098	0.0103
01 14 26.0	50	119749.060	0.0160	0.0049	0.0055
01 24 26.0	50	119838.140	0.0160	0.0059	0.0065
01 34 26.0	50	119924.539	0.0160	−0.0156	−0.0150
01 44 26.0	50	120008.159	0.0160	0.0068	0.0075
01 54 26.0	50	120088.760	0.0161	−0.0068	−0.0061
02 05 26.0	50	120173.800	0.0161	−0.0137	−0.0129
02 15 26.0	50	120247.680	0.0161	0.0107	0.0116
02 25 26.0	50	120318.100	0.0161	0.0166	0.0175
02 35 26.0	50	120384.920	0.0161	−0.0020	−0.0010
02 45 26.0	50	120448.020	0.0162	−0.0381	−0.0380
02 55 26.0	50	120507.359	0.0162	−0.0107	−0.0097
03 05 26.0	50	120562.760	0.0163	0.0146	0.0158
03 15 26.0	50	120614.060	0.0164	−0.0186	−0.0174
03 25 26.0	50	120661.279	0.0165	0.0088	0.0090
03 35 26.0	50	120704.220	0.0167	−0.0146	−0.0134
03 45 26.0	50	120742.880	0.0169	−0.0059	−0.0055
03 55 26.0	50	120777.159	0.0173	0.0039	0.0053
04 05 26.0	50	120806.979	0.0178	0.0039	0.0054
04 15 26.0	50	120832.279	0.0187	−0.0146	−0.0141
04 25 26.0	50	120853.060	0.0204	−0.0068	−0.0053

Transmitter Frequency = 29.6681 MHz
Pass — Sept. 29 and Sept. 30, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17 23 26.0	50	116351.039	0.0192	0.0029	0.0021
17 40 02.0	600	116393.029	0.0176	−0.0020	−0.0028
17 54 02.0	600	116437.918	0.0170	−0.0049	−0.0068
18 08 02.0	600	116491.471	0.0167	0.0068	0.0068
18 22 02.0	600	116553.448	0.0165	−0.0078	−0.0079
18 38 02.0	480	116634.295	0.0163	−0.0029	−0.0031
18 50 02.0	600	116701.841	0.0162	−0.0098	−0.0100
19 11 02.0	600	116833.409	0.0161	−0.0010	−0.0003
19 21 02.0	600	116901.807	0.0161	−0.0010	−0.0023
19 37 02.0	600	117018.541	0.0161	−0.0068	−0.0072
19 47 02.0	600	117095.852	0.0161	−0.0039	−0.0043
20 07 02.0	600	117259.812	0.0160	−0.0098	−0.0092
20 17 02.0	600	117346.161	0.0160	−0.0059	−0.0053
20 36 02.0	600	117517.489	0.0159	−0.0078	−0.0073
20 53 32.0	300	117682.819	0.0159	0.0059	0.0053
21 06 02.0	600	117804.913	0.0159	0.0059	0.0063
21 16 02.0	600	117904.590	0.0159	−0.0010	−0.0006
21 38 02.0	600	118129.345	0.0159	−0.0029	−0.0025
21 48 02.0	600	118233.546	0.0159	−0.0049	−0.0045
22 07 02.0	600	118434.123	0.0159	0.0000	0.0014
22 19 02.0	600	118562.031	0.0159	−0.0010	0.0004
22 38 02.0	600	118765.505	0.0159	−0.0039	0.0025
22 48 02.0	600	118872.659	0.0159	−0.0059	−0.0045
23 08 02.0	600	119086.107	0.0159	−0.0029	−0.0015
23 18 02.0	600	119191.989	0.0159	0.0000	0.0005
23 42 02.0	600	119442.321	0.0159	−0.0059	−0.0054
23 52 02.0	600	119544.564	0.0159	−0.0020	−0.0005
00 17 02.0	600	119793.161	0.0159	−0.0020	−0.0014
00 40 02.0	600	120011.024	0.0159	−0.0020	−0.0003
00 50 02.0	600	120101.921	0.0160	−0.0059	−0.0052
01 13 02.0	600	120300.977	0.0160	−0.0020	−0.0012
01 23 02.0	600	120382.778	0.0160	0.0000	0.0018
01 38 02.0	600	120499.611	0.0161	−0.0029	−0.0021
01 48 02.0	600	120573.380	0.0161	0.0000	0.0009
02 08 02.0	600	120710.378	0.0162	−0.0078	−0.0058
02 18 02.0	600	120773.356	0.0162	−0.0098	−0.0077
02 28 02.0	600	120832.498	0.0162	−0.0049	−0.0028
02 38 02.0	600	120887.678	0.0164	−0.0059	−0.0037
02 56 02.0	600	120976.704	0.0165	−0.0088	−0.0076
03 14 02.0	600	121052.048	0.0168	−0.0020	−0.0003
03 24 02.0	600	121087.812	0.0172	−0.0029	−0.0006
03 41 32.0	420	121139.790	0.0182	−0.0049	−0.0025
03 53 02.0	600	121166.250	0.0197	−0.0098	−0.0083

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 6 and Oct. 7, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17 36 02.0	600	117553.937	0.0166	0.0127	0.0122
17 46 02.0	600	117600.010	0.0165	0.0146	0.0141
20 28 32.0	300	118852.526	0.0159	0.0020	−0.0009
21 01 02.0	240	119184.782	0.0159	0.0059	0.0059
21 13 02.0	600	119311.107	0.0159	−0.0029	−0.0029
21 23 02.0	600	119417.442	0.0159	−0.0068	−0.0058
21 33 02.0	600	119524.564	0.0159	−0.0059	−0.0049
21 43 02.0	600	119632.270	0.0159	−0.0010	−0.0010
22 04 02.0	600	119859.416	0.0159	−0.0059	−0.0049
22 14 02.0	600	119967.616	0.0159	−0.0068	−0.0058
22 24 02.0	600	120075.562	0.0159	−0.0020	−0.0019
22 34 02.0	600	120183.039	0.0159	−0.0059	−0.0048
22 44 02.0	600	120289.856	0.0159	−0.0039	−0.0029
23 07 32.0	180	120537.160	0.0159	0.0039	0.0059
23 18 32.0	60	120650.517	0.0159	−0.0078	−0.0067
23 36 02.0	360	120746.933	0.0159	0.0010	0.0021

Table E-1 (contd)

Observation time (UT2C) h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
23	52	32.0	60	120915.410	0.0159	−0.0059	−0.0047
00	06	32.0	60	121059.229	0.0160	−0.0010	0.0002
00	23	32.0	180	121188.719	0.0160	−0.0020	−0.0007
00	35	02.0	600	121373.340	0.0160	0.0029	0.0042
00	47	32.0	540	121477.300	0.0160	0.0049	0.0062
01	04	02.0	240	121607.100	0.0161	0.0010	0.0023
01	18	02.0	600	121709.923	0.0161	0.0117	0.0131
01	28	02.0	600	121779.208	0.0161	0.0068	0.0083
01	38	02.0	600	121844.826	0.0162	0.0078	0.0083
02	07	02.0	600	122013.244	0.0164	0.0068	0.0094
02	17	02.0	600	122063.430	0.0165	0.0029	0.0055
02	27	02.0	600	122109.416	0.0167	0.0020	0.0036
02	42	02.0	600	122170.335	0.0170	0.0010	0.0017
03	03	32.0	420	122240.442	0.0180	−0.0059	−0.0050
03	16	32.0	420	122272.676	0.0196	−0.0078	−0.0060

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 14 and Oct. 15, 1962

Observation time (UT2C) h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
16	25	26.0	50	120280.760	0.0174	0.0049	0.0052
16	35	26.0	50	120316.699	0.0170	0.0000	−0.0007
16	45	26.0	50	120357.060	0.0167	0.0010	0.0002
16	55	26.0	50	120401.779	0.0165	0.0156	0.0148
17	05	26.0	50	120450.760	0.0164	0.0234	0.0236
17	16	26.0	50	120509.439	0.0164	0.0117	0.0109
17	26	26.0	50	120567.079	0.0162	0.0127	0.0128
17	36	26.0	50	120628.640	0.0162	−0.0352	−0.0360
17	46	26.0	50	120694.140	0.0162	−0.0029	−0.0038
17	56	26.0	50	120763.359	0.0161	0.0078	0.0079
18	06	26.0	50	120836.159	0.0161	−0.0137	−0.0136
18	18	26.0	50	120928.119	0.0161	−0.0215	−0.0215
18	28	26.0	50	121008.420	0.0161	−0.0156	−0.0147
18	38	26.0	50	121091.899	0.0160	0.0059	0.0058
18	48	26.0	50	121178.359	0.0160	−0.0049	−0.0039
18	58	26.0	50	121267.680	0.0160	−0.0049	−0.0049
19	08	26.0	50	121359.699	0.0160	0.0059	0.0068
19	18	26.0	50	121454.220	0.0159	0.0010	0.0019
19	28	26.0	50	121551.060	0.0159	−0.0244	−0.0254
19	38	26.0	50	121650.100	0.0159	−0.0146	−0.0138
19	48	26.0	50	121751.100	0.0159	−0.0225	−0.0216
19	58	26.0	50	121853.920	0.0159	−0.0010	−0.0001
20	08	26.0	50	121958.319	0.0159	−0.0010	−0.0011
20	18	26.0	50	122064.119	0.0159	−0.0078	−0.0070
20	39	26.0	50	122290.000	0.0159	−0.0215	−0.0216
20	49	26.0	50	122398.920	0.0159	−0.0098	−0.0089
20	59	26.0	50	122508.460	0.0159	0.0352	0.0360
21	09	26.0	50	122618.300	0.0159	−0.0039	−0.0031
21	19	26.0	50	122728.300	0.0159	−0.0615	−0.0597
21	29	26.0	50	122838.399	0.0159	0.0059	0.0057
21	39	26.0	50	122948.180	0.0159	−0.0156	−0.0148
21	49	26.0	50	123057.560	0.0159	−0.0039	−0.0021
21	59	26.0	50	123166.300	0.0159	0.0078	0.0077
22	09	26.0	50	123274.199	0.0159	0.0186	0.0194
22	19	26.0	50	123381.020	0.0159	−0.0098	−0.0089
22	29	26.0	50	123486.640	0.0159	0.0010	0.0019
22	39	26.0	50	123590.819	0.0159	0.0078	0.0068
22	49	26.0	50	123693.359	0.0159	0.0049	0.0058
22	59	26.0	50	123794.100	0.0159	0.0205	0.0224
23	09	26.0	50	123892.800	0.0159	0.0029	0.0049
23	19	26.0	50	123989.319	0.0159	−0.0068	−0.0068
23	29	26.0	50	124083.500	0.0160	0.0127	0.0147
23	39	26.0	50	124175.100	0.0160	−0.0068	−0.0048
23	49	26.0	50	124264.020	0.0160	0.0059	0.0069
23	59	26.0	50	124350.039	0.0160	−0.0078	−0.0058
00	09	26.0	50	124433.060	0.0161	0.0127	0.0148
00	19	26.0	50	124512.880	0.0161	0.0205	0.0226
00	29	26.0	50	124589.380	0.0161	0.0410	0.0422
00	39	26.0	50	124662.359	0.0161	0.0127	0.0139
00	49	26.0	50	124731.739	0.0162	−0.0088	−0.0076
00	59	26.0	50	124797.420	0.0162	0.0039	0.0051
01	31	26.0	50	124981.119	0.0165	−0.0137	−0.0124
01	41	26.0	50	125029.920	0.0166	0.0098	0.0111
01	51	26.0	50	125074.420	0.0168	0.0000	0.0014
02	01	26.0	50	125114.600	0.0171	0.0166	0.0180
02	11	26.0	50	125150.319	0.0175	−0.0088	−0.0074
02	21	26.0	50	125181.579	0.0182	−0.0186	−0.0171
02	31	26.0	50	125208.319	0.0193	−0.0205	−0.0190
02	41	26.0	50	125230.500	0.0217	−0.0156	−0.0141

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 24 and Oct. 25, 1962

Observation time (UT2C) h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
15	25	02.0	120	126608.675	0.0178	0.0322	0.0322
15	42	32.0	420	126674.135	0.0170	0.0156	0.0146
16	01	32.0	420	126760.366	0.0166	0.0186	0.0174
16	19	02.0	600	126853.448	0.0164	0.0010	0.0008
16	29	32.0	60	126915.116	0.0163	−0.0215	−0.0217
16	56	02.0	600	127090.659	0.0161	−0.0049	−0.0061
17	12	32.0	180	127212.905	0.0161	−0.0039	−0.0042
17	23	02.0	600	127295.857	0.0161	−0.0010	−0.0012
17	33	02.0	600	127378.128	0.0161	−0.0039	−0.0042
17	43	02.0	600	127463.569	0.0160	−0.0010	−0.0022
18	05	02.0	600	127661.920	0.0160	0.0049	0.0046
18	25	02.0	600	127853.421	0.0159	0.0000	−0.0003
18	35	02.0	600	127952.716	0.0159	−0.0088	−0.0091
19	01	02.0	600	128220.411	0.0159	−0.0039	−0.0053
19	15	02.0	600	128369.357	0.0159	−0.0088	−0.0092
19	35	32.0	60	128592.133	0.0159	0.0010	0.0016
19	52	02.0	600	128774.526	0.0159	−0.0010	−0.0014
20	13	02.0	600	129008.986	0.0159	−0.0010	−0.0014
20	23	02.0	600	129121.069	0.0159	0.0010	0.0006
20	45	02.0	360	129367.369	0.0159	−0.0029	−0.0014
21	01	02.0	600	129545.236	0.0159	−0.0020	−0.0024
21	11	02.0	600	129655.458	0.0159	−0.0010	−0.0014
21	21	02.0	600	129764.706	0.0159	0.0010	0.0006
21	31	02.0	600	129872.776	0.0159	−0.0010	−0.0004
21	47	02.0	600	130042.760	0.0159	0.0000	0.0006
22	00	02.0	600	130177.739	0.0159	−0.0010	−0.0014
22	10	02.0	600	130279.380	0.0159	0.0010	0.0006
22	20	02.0	600	130378.904	0.0159	−0.0029	−0.0033
22	30	02.0	600	130476.149	0.0160	0.0029	0.0036
22	40	02.0	600	130570.930	0.0160	0.0107	0.0104
22	55	02.0	240	130708.191	0.0160	0.0088	0.0094

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
23 06 02.0	600	130804.581	0.0160	0.0088	0.0085
23 16 02.0	600	130889.098	0.0161	0.0177	0.0104
23 26 02.0	600	130970.399	0.0161	0.0088	0.0085
23 47 02.0	600	131130.057	0.0161	0.0117	0.0115
23 57 02.0	600	131200.514	0.0162	0.0059	0.0056
00 07 02.0	600	130267.215	0.0162	0.0078	0.0076
00 20 02.0	360	131348.207	0.0163	0.0020	0.0017
00 39 02.0	600	131454.039	0.0165	0.0000	−0.0002
00 49 02.0	600	131503.746	0.0167	0.0039	0.0037
01 03 32.0	540	131568.219	0.0170	0.0059	0.0057
01 14 02.0	600	131609.080	0.0175	−0.0020	−0.0021
01 35 02.0	600	131676.072	0.0195	−0.0039	−0.0040
01 45 02.0	600	131700.928	0.0219	−0.0098	−0.0098

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 27 and Oct. 28, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
15 41 02.0	240	129243.291	0.0167	−0.0059	−0.0057
15 52 02.0	600	129299.154	0.0165	−0.0078	−0.0077
16 05 02.0	480	129371.480	0.0164	0.0010	0.0011
16 19 02.0	600	129457.161	0.0162	−0.0088	−0.0087
16 29 02.0	600	129523.048	0.0162	−0.0088	−0.0078
16 49 02.0	120	129666.000	0.0161	0.0107	0.0108
16 59 02.0	600	129743.067	0.0161	−0.0029	−0.0029
17 15 02.0	600	129873.223	0.0161	−0.0049	−0.0049
17 34 02.0	360	130038.264	0.0160	−0.0049	−0.0030
17 56 02.0	600	130242.508	0.0160	−0.0098	−0.0089
18 06 02.0	600	130339.473	0.0159	−0.0088	−0.0079
18 16 02.0	600	130438.795	0.0159	−0.0049	−0.0040
18 32 32.0	60	130607.283	0.0159	0.0078	0.0097
18 46 32.0	60	130754.333	0.0159	0.0010	0.0018
19 01 32.0	180	130915.389	0.0159	0.0059	0.0067
19 20 02.0	600	131118.045	0.0159	−0.0039	−0.0021
19 30 02.0	600	131229.008	0.0159	0.0020	0.0038
19 40 02.0	600	131340.711	0.0159	0.0000	0.0018
19 50 02.0	600	131452.955	0.0159	−0.0078	−0.0060
20 00 02.0	600	131565.555	0.0159	−0.0039	−0.0021
20 10 02.0	600	131678.293	0.0159	0.0000	0.0018
20 27 02.0	600	131869.676	0.0159	−0.0020	−0.0002
20 37 02.0	600	131981.760	0.0159	−0.0059	−0.0041
20 47 02.0	600	132093.229	0.0159	−0.0059	−0.0041
20 57 02.0	240	132203.920	0.0159	−0.0039	−0.0021

Transmitter Frequency = 29.6682 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 18 02.0	600	132432.816	0.0159	0.0039	0.0057
21 28 02.0	600	132539.658	0.0159	−0.0039	−0.0041
21 38 02.0	600	132644.883	0.0159	−0.0059	−0.0041
21 57 02.0	600	132839.658	0.0159	0.0020	0.0038
22 19 02.0	600	133055.322	0.0160	−0.0039	−0.0021
22 29 02.0	600	133149.375	0.0160	−0.0059	−0.0040
22 39 02.0	600	133240.707	0.0160	−0.0039	−0.0040
22 49 02.0	600	133329.150	0.0160	−0.0078	−0.0060
23 14 02.0	600	133536.613	0.0161	−0.0078	−0.0059
23 24 02.0	600	133613.756	0.0161	−0.0039	−0.0020
23 34 02.0	600	133687.352	0.0162	0.0020	0.0038
23 44 02.0	600	133757.268	0.0162	0.0039	0.0058
23 54 02.0	600	133823.375	0.0162	−0.0020	0.0000
00 13 02.0	600	133938.121	0.0164	0.0000	0.0019
00 23 02.0	600	133992.592	0.0165	−0.0039	−0.0020
00 33 02.0	600	134042.867	0.0167	0.0020	0.0039
00 43 02.0	600	134088.840	0.0170	−0.0078	−0.0058
00 53 02.0	600	134130.463	0.0173	−0.0059	−0.0039
01 03 02.0	600	134167.646	0.0178	−0.0117	−0.0117
01 13 02.0	600	134200.350	0.0187	−0.0137	−0.0116

Transmitter Frequency = 29.6682 MHz
Pass — Nov. 5 and Nov. 6, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
14 41 02.0	600	138470.283	0.0171	0.0078	0.0051
14 51 02.0	600	138516.943	0.0168	−0.0078	−0.0086
15 01 02.0	600	138567.926	0.0166	−0.0020	−0.0047
15 21 02.0	480	138682.395	0.0164	−0.0059	−0.0067
15 33 02.0	600	138758.969	0.0162	−0.0078	−0.0086
15 43 02.0	600	138827.037	0.0162	−0.0098	−0.0087
15 53 02.0	600	138898.908	0.0162	0.0000	−0.0009
16 03 02.0	600	138974.437	0.0161	−0.0020	−0.0028
16 13 02.0	600	139053.506	0.0161	−0.0098	−0.0087
16 23 02.0	600	139136.000	0.0161	0.0039	0.0049
16 33 02.0	600	139221.738	0.0161	−0.0020	−0.0029
16 43 02.0	600	139310.598	0.0160	−0.0039	−0.0029
16 53 02.0	600	139402.420	0.0160	0.0000	−0.0010
17 03 02.0	600	139497.037	0.0160	−0.0059	−0.0049
17 13 02.0	600	139594.293	0.0160	−0.0059	−0.0049
17 23 02.0	600	139694.016	0.0160	−0.0039	−0.0030
17 39 02.0	600	139858.256	0.0159	−0.0020	−0.0010
17 51 02.0	600	139984.836	0.0159	0.0000	0.0009
18 06 02.0	240	140146.562	0.0159	−0.0020	−0.0030
18 16 02.0	600	140256.371	0.0159	0.0000	0.0009
18 26 02.0	600	140367.422	0.0159	0.0000	0.0028
18 44 02.0	600	140569.941	0.0159	0.0000	0.0009
18 54 02.0	600	140683.559	0.0159	0.0000	0.0009
19 04 02.0	600	140797.713	0.0159	0.0000	0.0008

Transmitter Frequency = 29.6683 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 31 02.0	600	141107.152	0.0159	0.0020	0.0047
19 41 02.0	600	141221.518	0.0159	0.0039	0.0047
19 51 02.0	600	141335.461	0.0159	0.0020	0.0028
20 01 02.0	600	141448.783	0.0159	0.0039	0.0047
20 11 02.0	600	141561.275	0.0159	0.0020	0.0028
20 21 02.0	600	141672.740	0.0159	0.0000	−0.0012
20 31 02.0	600	141782.992	0.0159	0.0059	0.0067
20 47 02.0	600	141956.348	0.0159	0.0020	0.0027
20 57 02.0	600	142062.500	0.0159	−0.0020	−0.0012
21 07 02.0	600	142166.746	0.0159	0.0020	0.0028
21 17 02.0	600	142268.885	0.0160	−0.0039	−0.0012
21 34 02.0	600	142437.211	0.0160	−0.0020	−0.0012
21 44 02.0	600	142532.807	0.0160	0.0000	0.0028
21 54 02.0	600	142625.658	0.0160	−0.0039	−0.0011
22 05 02.0	600	142724.441	0.0161	0.0000	0.0028

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
22 15 02.0	600	142811.018	0.0161	−0.0020	0.0008
22 25 02.0	600	142894.373	0.0161	0.0020	0.0028
22 35 02.0	600	142974.363	0.0161	0.0059	0.0086
22 45 02.0	600	143050.834	0.0162	0.0000	0.0008
22 55 02.0	600	143123.680	0.0162	0.0020	0.0048
23 12 02.0	600	143238.816	0.0163	−0.0020	−0.0011
23 26 02.0	600	143325.084	0.0165	−0.0039	−0.0030
23 36 02.0	600	143381.807	0.0166	−0.0020	0.0009
23 46 02.0	600	143434.340	0.0168	−0.0020	−0.0010
23 56 02.0	600	143482.600	0.0170	−0.0020	−0.0010
00 06 02.0	600	143526.510	0.0175	−0.0059	−0.0030
00 16 02.0	600	143566.002	0.0181	−0.0078	−0.0049
00 26 02.0	600	143601.021	0.0192	−0.0017	−0.0088

Transmitter Frequency = 29.6683 MHz
Pass — Nov. 10 and Nov. 11, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
13 51 32.0	180	144654.260	0.0190	0.0137	0.0104
14 08 32.0	300	144718.590	0.0176	−0.0039	−0.0072
14 20 02.0	120	144769.340	0.0171	0.0098	0.0084
14 31 02.0	600	144823.445	0.0168	−0.0059	−0.0073
14 47 02.0	600	144911.098	0.0165	0.0000	−0.0034
14 57 02.0	600	144971.268	0.0164	0.0000	0.0005
15 07 02.0	600	145035.459	0.0163	−0.0020	−0.0034
15 17 02.0	600	145103.578	0.0162	−0.0039	−0.0054
15 38 02.0	600	145258.887	0.0161	−0.0039	−0.0055
15 59 02.0	600	145429.854	0.0161	0.0059	0.0043
16 22 02.0	600	145633.521	0.0160	−0.0039	−0.0036
16 39 02.0	600	145794.066	0.0160	0.0000	0.0003
16 58 32.0	600	145987.520	0.0160	0.0039	0.0042
17 10 02.0	480	146105.781	0.0159	0.0059	0.0061
17 29 32.0	300	146312.473	0.0159	0.0020	0.0022
17 49 02.0	600	146525.816	0.0159	0.0020	0.0021
18 10 02.0	600	146761.227	0.0159	0.0000	0.0002
18 20 02.0	600	146874.898	0.0159	0.0000	0.0001
18 30 02.0	600	146989.312	0.0159	0.0020	0.0021
18 45 02.0	600	147161.875	0.0159	0.0020	0.0001
19 04 02.0	600	147381.125	0.0159	0.0000	−0.0019
19 23 02.0	600	147599.828	0.0159	0.0020	0.0001
19 33 02.0	600	147714.246	0.0159	0.0039	0.0040
19 53 32.0	540	147946.270	0.0159	−0.0039	−0.0038
20 07 02.0	600	148096.533	0.0159	0.0000	0.0020
20 17 02.0	600	148206.211	0.0159	0.0020	0.0020
20 27 02.0	600	148314.287	0.0159	0.0020	0.0040
20 37 02.0	600	148420.566	0.0159	0.0020	0.0020
20 47 02.0	600	148524.871	0.0160	0.0059	0.0040
20 57 02.0	600	148627.008	0.0160	0.0020	0.0020
21 07 02.0	600	148726.807	0.0160	0.0000	0.0001
21 17 02.0	600	148824.094	0.0160	0.0000	0.0001
21 27 02.0	600	148918.697	0.0160	−0.0039	−0.0039
21 37 02.0	600	149010.463	0.0161	0.0000	0.0001
21 47 02.0	600	149099.225	0.0161	0.0039	0.0020
21 57 02.0	600	149184.828	0.0161	0.0039	0.0020
22 07 02.0	600	149267.129	0.0161	0.0020	0.0020
22 17 02.0	600	149345.988	0.0162	0.0000	−0.0019
22 32 02.0	480	149457.598	0.0162	0.0020	0.0020
22 45 02.0	600	149547.385	0.0163	0.0000	−0.0018
22 55 02.0	600	149612.020	0.0164	0.0059	0.0040
23 14 02.0	600	149723.678	0.0167	0.0137	0.0119
23 24 02.0	600	149776.389	0.0169	0.0039	0.0021

Transmitter Frequency = 29.6684 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
23 52 02.0	600	149901.174	0.0182	−0.0039	−0.0037
00 13 02.0	600	149971.770	0.0225	−0.0098	−0.0076
00 23 02.0	600	149998.363	0.0298	−0.0215	−0.0212

Transmitter Frequency = 29.6685 MHz
Pass — Nov. 17, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
13 26 02.0	120	154375.750	0.0189	0.0078	0.0081
13 38 02.0	600	154423.225	0.0178	−0.0039	−0.0017
13 48 02.0	600	154467.498	0.0173	−0.0039	−0.0037
21 07 26.0	50	117215.279	0.0161	0.0205	0.0218

Transmitter Frequency = 29.6682 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 20 26.0	50	117307.619	0.0161	−0.0098	−0.0074
21 30 26.0	50	117382.520	0.0161	−0.0049	−0.0045
21 40 26.0	50	117460.640	0.0160	−0.0010	−0.0005
21 50 26.0	50	117541.840	0.0160	0.0137	0.0152
22 02 26.0	50	117643.060	0.0160	−0.0195	−0.0180
22 12 26.0	50	117730.460	0.0160	0.0039	0.0045
22 22 26.0	50	117820.380	0.0159	−0.0029	−0.0033
22 33 26.0	50	117922.060	0.0159	0.0215	0.0212
22 43 26.0	50	118016.720	0.0159	−0.0254	−0.0247
22 53 26.0	50	118113.460	0.0159	0.0137	0.0135
23 03 26.0	50	118211.939	0.0159	−0.0166	−0.0158
23 13 26.0	50	118312.079	0.0159	−0.0039	−0.0030
23 23 26.0	50	118413.619	0.0159	−0.0127	−0.0137
23 33 26.0	50	118516.399	0.0159	−0.0107	−0.0108
23 43 26.0	50	118620.220	0.0159	0.0039	0.0039
23 53 26.0	50	118724.840	0.0159	−0.0078	−0.0078
00 03 26.0	50	118830.119	0.0159	0.0127	0.0118
00 13 26.0	50	118935.800	0.0159	0.0146	0.0148
00 23 26.0	50	119041.680	0.0159	0.0000	−0.0008
13 58 02.0	600	154516.115	0.0170	−0.0039	−0.0057
14 08 02.0	600	154569.014	0.0167	0.0039	0.0060
14 33 02.0	480	154719.305	0.0164	−0.0020	0.0001
14 46 02.0	600	154807.396	0.0163	−0.0078	−0.0077
14 56 02.0	600	154879.521	0.0162	0.0020	0.0040
15 06 02.0	600	154955.318	0.0162	−0.0020	−0.0019
15 16 02.0	600	155034.686	0.0161	0.0020	0.0020
15 31 32.0	540	155164.387	0.0161	−0.0137	−0.0118
15 47 02.0	600	155301.875	0.0161	−0.0039	−0.0020
15 57 02.0	600	155394.381	0.0160	−0.0059	−0.0060
16 12 32.0	540	155543.264	0.0160	−0.0059	−0.0060
16 27 02.0	600	155688.170	0.0160	−0.0039	−0.0021
16 47 02.0	240	155895.812	0.0160	0.0020	0.0017
16 57 02.0	600	156002.748	0.0159	−0.0078	−0.0041
17 07 02.0	600	156111.396	0.0159	−0.0020	−0.0022
17 17 02.0	600	156221.619	0.0159	−0.0020	−0.0003
17 27 02.0	600	156333.227	0.0159	−0.0078	−0.0081

Table E-1 (contd)

Observation time (UT2C) h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17	37	02.0	600	156446.043	0.0159	0.0059	0.0075
17	49	02.0	120	156582.666	0.0159	−0.0039	−0.0043
18	04	02.0	600	156755.027	0.0159	0.0059	0.0074
18	14	02.0	600	156870.521	0.0159	0.0078	0.0094
18	24	02.0	600	156986.266	0.0159	0.0039	0.0054
18	34	32.0	540	157107.848	0.0159	−0.0020	−0.0004
18	49	02.0	600	157275.410	0.0159	−0.0020	−0.0024
18	59	02.0	600	157390.457	0.0159	0.0020	0.0054
19	09	02.0	600	157504.836	0.0159	−0.0020	−0.0024
19	19	02.0	600	157618.369	0.0159	0.0000	0.0015
19	29	02.0	600	157730.852	0.0159	0.0059	0.0054
19	39	02.0	600	157842.074	0.0159	0.0020	0.0015
19	49	02.0	600	157951.861	0.0159	0.0078	0.0073
19	59	02.0	600	158060.004	0.0159	0.0020	0.0034
20	09	02.0	600	158166.326	0.0160	−0.0020	−0.0005
20	19	02.0	600	158270.643	0.0160	−0.0020	−0.0025
20	29	02.0	600	158372.775	0.0160	0.0020	0.0014
20	39	02.0	600	158472.541	0.0160	0.0020	0.0034
20	49	02.0	600	158569.760	0.0160	−0.0078	−0.0083
20	59	02.0	600	158664.287	0.0161	−0.0059	−0.0044
21	09	02.0	600	158755.953	0.0161	−0.0020	−0.0005
21	19	02.0	600	158844.592	0.0161	−0.0020	−0.0025
21	29	02.0	600	158930.059	0.0161	−0.0039	−0.0044
21	39	02.0	600	159012.217	0.0162	0.0039	0.0034
21	59	02.0	600	159166.021	0.0162	−0.0020	−0.0025
22	09	02.0	600	159237.430	0.0163	0.0039	0.0034
22	19	02.0	600	159305.000	0.0164	0.0000	0.0015
22	29	02.0	600	159368.637	0.0165	0.0000	−0.0005
22	53	02.0	600	159504.691	0.0170	−0.0117	−0.0122
23	03	02.0	600	159554.227	0.0173	−0.0078	−0.0082
23	13	02.0	600	159599.434	0.0179	−0.0039	−0.0024
23	26	02.0	600	159651.615	0.0215	−0.0098	−0.0082
23	46	02.0	600	159717.146	0.0269	−0.0176	−0.0179

Transmitter Frequency = 29.6686 MHz
Pass — Nov. 26, 1962

h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
13	32	32.0	60	167779.066	0.0170	−0.0117	−0.0090
13	49	02.0	600	167873.725	0.0166	0.0039	0.0065
13	59	02.0	600	167936.352	0.0165	−0.0039	−0.0013
14	09	02.0	600	168002.920	0.0164	−0.0098	−0.0072
14	19	02.0	600	168073.334	0.0163	−0.0020	0.0006
14	29	02.0	600	168147.465	0.0162	0.0000	0.0025
14	39	02.0	600	168225.182	0.0162	−0.0039	−0.0015
14	49	02.0	600	168306.375	0.0161	−0.0020	0.0005
14	59	02.0	600	168390.904	0.0161	0.0020	0.0043
15	09	02.0	600	168478.619	0.0161	0.0020	0.0043
15	19	02.0	600	168569.373	0.0161	0.0000	0.0004
15	29	02.0	600	168663.021	0.0161	0.0000	0.0003
15	39	02.0	600	168759.396	0.0160	−0.0020	0.0003
15	49	02.0	600	168858.336	0.0160	−0.0059	−0.0036
15	59	02.0	600	168959.680	0.0160	0.0000	0.0002
16	09	02.0	600	169063.248	0.0160	0.0059	0.0080
16	19	02.0	600	169168.846	0.0160	−0.0020	0.0002
16	29	02.0	600	169276.316	0.0160	0.0020	0.0021
16	44	02.0	600	169440.604	0.0159	0.0020	0.0040
16	54	02.0	600	169551.908	0.0159	0.0039	0.0040
17	16	02.0	600	169800.719	0.0159	0.0039	0.0039
17	26	02.0	600	169915.168	0.0159	0.0039	0.0058
17	36	02.0	600	170030.182	0.0159	0.0000	0.0019
17	46	02.0	600	170145.568	0.0159	0.0020	0.0019
17	56	02.0	600	170261.119	0.0159	0.0020	0.0038
18	06	02.0	600	170376.629	0.0159	−0.0039	−0.0021
18	23	02.0	600	170572.363	0.0159	0.0000	0.0018
18	33	02.0	600	170686.779	0.0159	−0.0159	−0.0021
18	43	02.0	600	170800.426	0.0159	0.0000	0.0018
18	53	02.0	600	170913.102	0.0159	0.0000	0.0037
19	03	02.0	600	171024.602	0.0159	0.0000	0.0018
19	13	02.0	600	171134.748	0.0160	0.0039	0.0037
19	23	02.0	600	171243.340	0.0160	0.0039	0.0056
19	38	02.0	600	171402.902	0.0160	0.0039	0.0037
19	48	02.0	600	171506.799	0.0160	0.0039	0.0056
19	58	02.0	600	171608.500	0.0160	−0.0039	0.0056
20	09	02.0	600	171707.820	0.0160	−0.0078	−0.0022
20	18	02.0	60	171809.500	0.0161	−0.0020	0.0017
20	37	02.0	600	171980.941	0.0161	−0.0039	−0.0002
20	47	02.0	600	172069.453	0.0161	−0.0039	−0.0022
20	57	02.0	600	172154.816	0.0162	−0.0039	−0.0022
21	07	02.0	600	172236.889	0.0162	0.0039	0.0056

Transmitter Frequency = 29.6687 MHz

h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21	27	32.0	60	172394.633	0.0163	−0.0059	−0.0041
21	46	02.0	600	172523.205	0.0165	−0.0039	−0.0041
22	09	02.0	600	172664.553	0.0168	−0.0020	0.0018
22	19	02.0	600	172719.264	0.0172	−0.0020	−0.0002
22	29	02.0	600	172769.762	0.0176	−0.0078	−0.0060
22	39	02.0	600	172815.986	0.0184	−0.0039	−0.0021
22	49	02.0	600	172857.857	0.0197	0.0000	0.0019
22	59	02.0	600	172895.301	0.0222	−0.0215	−0.0196

Transmitter Frequency = 29.6687 MHz
Pass — Dec. 1, 1962

h	m	s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12	38	26.0	50	175094.340	0.0209	0.0137	0.0150
12	48	26.0	50	175133.359	0.0190	0.0176	0.0208
12	58	26.0	50	175176.760	0.0180	0.0371	0.0364
13	08	26.0	50	175224.420	0.0175	0.0078	0.0090
13	18	26.0	50	175276.359	0.0170	0.0176	0.0207
13	28	26.0	50	175332.420	0.0168	−0.0078	−0.0067
13	38	26.0	50	175392.578	0.0166	−0.0098	−0.0106
13	48	26.0	50	175456.719	0.0165	−0.0117	−0.0106
13	58	26.0	50	175524.760	0.0164	0.0117	0.0128
14	08	26.0	50	175596.500	0.0163	−0.0332	−0.0322
14	18	26.0	50	175671.959	0.0162	−0.0078	−0.0068
14	28	26.0	50	175750.920	0.0162	−0.0059	−0.0049
14	39	26.0	50	175841.699	0.0161	0.0020	0.0029
14	49	26.0	50	175927.658	0.0161	0.0313	0.0341
14	59	26.0	50	176016.639	0.0161	−0.0176	−0.0149
15	09	26.0	50	176108.639	0.0161	0.0078	0.0106
15	19	26.0	50	176203.398	0.0161	0.0020	0.0027
15	29	26.0	50	176300.799	0.0161	0.0039	0.0046
15	39	26.0	50	176400.658	0.0160	−0.0020	0.0007

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
15 49 26.0	50	176502.818	0.0160	0.0020	0.0026
15 59 26.0	50	176607.100	0.0160	0.0078	0.0084
16 09 26.0	50	176713.299	0.0160	−0.0098	−0.0072
16 19 26.0	50	176821.260	0.0160	−0.0195	−0.0170
16 29 26.0	50	176930.840	0.0160	0.0195	0.0220
16 39 26.0	50	177041.719	0.0159	−0.0215	−0.0190
16 49 26.0	50	177153.859	0.0159	0.0137	0.0141
16 59 26.0	50	177266.939	0.0159	−0.0078	−0.0074
17 09 26.0	50	177380.859	0.0159	0.0137	0.0141
17 19 26.0	50	177495.340	0.0159	−0.0039	−0.0035
17 29 26.0	50	177610.238	0.0159	−0.0020	−0.0016
17 39 26.0	50	177725.340	0.0159	0.0020	0.0043
17 49 26.0	50	177840.439	0.0159	0.0000	0.0023
17 59 26.0	50	177955.340	0.0159	0.0039	0.0062
18 09 26.0	50	178069.840	0.0159	0.0039	0.0061
18 19 26.0	50	178183.738	0.0159	0.0020	0.0042
18 29 26.0	50	178296.859	0.0159	0.0176	0.0178
18 39 26.0	50	178408.939	0.0160	−0.0137	−0.0115
18 49 26.0	50	178519.898	0.0160	0.0176	0.0198
18 59 26.0	50	178629.398	0.0160	−0.0273	−0.0271
19 09 26.0	50	178737.398	0.0160	−0.0039	−0.0037
19 19 26.0	50	178843.639	0.0160	0.0098	0.0139
19 29 26.0	50	178947.920	0.0160	0.0078	0.0100
19 39 26.0	50	179050.059	0.0160	−0.0215	−0.0213
19 49 26.0	50	179149.959	0.0161	0.0020	0.0060
19 59 26.0	50	179247.359	0.0161	−0.0078	−0.0057
20 09 26.0	50	179342.139	0.0161	−0.0078	−0.0076
20 19 26.0	50	179434.139	0.0161	0.0000	0.0021
20 29 26.0	50	179523.199	0.0161	0.0176	0.0217

Transmitter Frequency = 29.6689 MHz
Pass — Dec. 7, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 28 32.0	180	184027.955	0.0209	0.0117	0.0069
12 40 32.0	540	184077.623	0.0188	0.0078	0.0029
13 00 02.0	600	184171.289	0.0174	0.0098	0.0048
13 14 02.0	600	184248.285	0.0169	0.0059	0.0008
13 35 02.0	600	184378.553	0.0165	0.0098	0.0047
13 59 02.0	600	184548.031	0.0163	−0.0020	−0.0052
14 09 02.0	600	184624.795	0.0162	0.0020	−0.0013
14 19 02.0	600	184704.973	0.0162	−0.0059	−0.0072
14 29 02.0	600	184788.463	0.0162	0.0039	0.0006
14 39 02.0	600	184875.092	0.0161	−0.0020	−0.0014
14 49 02.0	600	184964.736	0.0161	0.0020	0.0005
15 02 32.0	60	185090.133	0.0161	−0.0078	−0.0113
15 15 32.0	180	185215.582	0.0161	0.0020	0.0004
15 32 32.0	540	185385.797	0.0160	−0.0059	−0.0094
15 45 02.0	240	185514.832	0.0160	−0.0117	−0.0153
16 02 02.0	600	185695.219	0.0160	0.0117	0.0080
16 24 02.0	600	185935.266	0.0160	0.0000	−0.0037
16 38 02.0	600	186091.141	0.0160	−0.0039	−0.0096
16 53 32.0	60	186265.783	0.0159	0.0117	0.0098
17 12 02.0	600	186476.113	0.0159	0.0000	0.0000
17 22 02.0	600	186590.227	0.0159	0.0000	0.0000
17 40 02.0	600	186795.613	0.0159	−0.0020	−0.0039
17 50 02.0	600	186909.367	0.0159	0.0078	0.0058
18 00 02.0	120	187022.625	0.0160	0.0020	−0.0020
18 19 02.0	600	187235.635	0.0160	−0.0039	−0.0060
18 34 02.0	360	187401.187	0.0160	−0.0039	−0.0040
18 53 32.0	60	187611.715	0.0160	−0.0020	−0.0079
19 19 02.0	600	187876.797	0.0160	0.0020	−0.0002
19 31 02.0	600	187996.844	0.0160	0.0078	0.0077
19 41 02.0	600	188094.254	0.0161	0.0000	−0.0041
19 51 02.0	600	188189.117	0.0161	0.0020	−0.0002
20 07 32.0	420	188339.676	0.0161	−0.0059	−0.0099
20 22 02.0	600	188465.215	0.0162	−0.0059	−0.0099
20 32 02.0	600	188547.967	0.0162	0.0020	−0.0021
20 42 02.0	600	188627.383	0.0163	−0.0039	−0.0099
20 52 02.0	600	188703.352	0.0162	−0.0039	−0.0040
21 07 02.0	600	188810.515	0.0165	0.0039	0.0019
21 17 02.0	600	188877.336	0.0167	−0.0039	−0.0079

Transmitter Frequency = 29.6689 MHz
Pass — Dec. 8, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 29 02.0	240	185494.461	0.0347	−0.0020	−0.0025
12 36 32.0	300	185525.510	0.0292	0.0020	−0.0006
12 43 02.0	120	185554.316	0.0503	0.0000	−0.0026
12 55 32.0	180	185614.855	0.0358	0.0098	0.0091
13 05 02.0	600	185665.352	0.0173	−0.0059	−0.0066
13 11 02.0	120	185698.908	0.0464	−0.0020	−0.0007
13 16 02.0	120	185728.107	0.0461	−0.0078	−0.0086
13 24 02.0	120	185776.916	0.0457	−0.0020	−0.0047
13 33 02.0	480	185834.910	0.0192	−0.0039	−0.0067
13 40 32.0	60	185885.432	0.0520	−0.0039	−0.0028
13 48 02.0	360	185938.207	0.0225	−0.0059	−0.0048
14 45 02.0	120	186402.250	0.0441	0.0254	0.0224
14 51 32.0	180	186461.609	0.0328	0.0098	0.0087
14 59 32.0	60	186536.266	0.0508	0.0000	−0.0011
15 04 32.0	180	186583.811	0.0327	−0.0039	−0.0031
15 12 32.0	180	186661.205	0.0327	0.0039	0.0027
15 19 32.0	60	186730.199	0.0507	0.0059	0.0047
16 10 32.0	180	187263.393	0.0326	−0.0059	−0.0092
16 20 02.0	600	187367.523	0.0161	0.0020	−0.0034
16 26 02.0	120	187433.773	0.0437	−0.0117	−0.0112
17 53 02.0	600	188418.500	0.0161	−0.0059	−0.0114
18 03 02.0	600	188531.180	0.0161	−0.0078	−0.0134
18 13 02.0	600	186643.057	0.0161	−0.0039	−0.0114
18 23 02.0	600	188753.941	0.0161	0.0039	−0.0017
18 28 32.0	60	188814.482	0.0505	0.0059	0.0042
18 37 02.0	600	188907.113	0.0161	0.0000	−0.0056
18 45 02.0	360	188993.424	0.0219	−0.0117	−0.0154
18 58 02.0	600	189131.357	0.0161	0.0020	−0.0056
19 08 02.0	600	189235.377	0.0161	−0.0039	−0.0076
19 14 02.0	120	189296.932	0.0438	0.0156	0.0080
19 17 32.0	180	189332.416	0.0327	0.0059	−0.0017
19 25 02.0	360	189407.535	0.0220	0.0020	−0.0056
19 36 02.0	600	189515.299	0.0162	−0.0078	−0.0115
19 46 02.0	600	189610.703	0.0162	−0.0039	−0.0076
19 56 02.0	600	189703.428	0.0162	−0.0078	−0.0154
20 06 02.0	600	189793.336	0.0162	−0.0039	−0.0037
20 25 32.0	300	189960.033	0.0248	−0.0117	−0.0193
20 37 32.0	540	190056.393	0.0173	−0.0078	−0.0134
21 00 02.0	600	190223.723	0.0166	−0.0020	−0.0095

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
21 10 02.0	600	190292.182	0.0167	0.0098	0.0062
21 20 02.0	600	190356.826	0.0169	0.0156	0.0081
21 30 02.0	600	190417.537	0.0172	0.0000	−0.0055
21 40 02.0	600	190474.246	0.0175	−0.0059	−0.0114
21 50 02.0	600	190526.869	0.0181	−0.0020	−0.0074
22 00 02.0	600	190575.307	0.0189	−0.0039	−0.0132
22 10 02.0	600	190619.498	0.0206	−0.0039	−0.0152
22 27 02.0	240	190684.861	0.0507	−0.0234	−0.0308
Transmitter Frequency = 29.6689 MHz					
Pass — Dec. 11, 1962					
12 28 32.0	60	189813.865	0.0627	−0.0020	−0.0012
12 37 02.0	120	189851.057	0.0510	0.0176	0.0164
12 46 02.0	600	189893.906	0.0180	0.0078	0.0085
12 56 02.0	600	189945.305	0.0175	0.0059	0.0065
13 06 02.0	600	190000.807	0.0172	0.0078	0.0065
13 16 02.0	600	190060.318	0.0169	0.0039	0.0006
13 26 02.0	600	190123.762	0.0167	0.0098	0.0084
13 35 32.0	540	190187.533	0.0176	0.0098	0.0083
13 46 02.0	120	190261.840	0.0448	0.0117	0.0122
13 55 02.0	600	190328.920	0.0164	0.0039	0.0044
14 05 02.0	600	190406.611	0.0164	0.0098	0.0121
14 15 02.0	600	190487.641	0.0163	0.0020	0.0004
14 25 02.0	600	190571.900	0.0163	0.0078	0.0062
14 35 02.0	600	190659.229	0.0162	0.0039	0.0023
14 44 32.0	540	190744.893	0.0172	0.0156	0.0139
14 55 32.0	420	190847.187	0.0201	0.0078	0.0080
15 12 02.0	600	191006.582	0.0162	0.0039	0.0021
15 22 02.0	600	191106.295	0.0161	0.0039	0.0040
15 32 02.0	600	191208.166	0.0161	0.0020	0.0001
15 42 02.0	600	191312.033	0.0161	0.0078	0.0040
15 56 02.0	600	191460.428	0.0161	0.0039	0.0020
16 02 02.0	120	191524.916	0.0438	−0.0117	−0.0137
16 11 02.0	600	191622.783	0.0161	0.0000	−0.0020
16 21 02.0	600	191732.646	0.0161	0.0000	0.0000
16 27 32.0	180	191804.615	0.0326	0.0039	0.0038
Transmitter Frequency = 29.6690 MHz					
17 19 02.0	600	192385.285	0.0161	0.0078	0.0076
17 29 02.0	600	192498.633	0.0161	0.0098	0.0115
17 39 02.0	600	192611.715	0.0161	0.0039	0.0037
17 49 02.0	600	192724.350	0.0161	0.0078	0.0076
17 59 02.0	600	192836.330	0.0161	0.0000	0.0017
18 09 02.0	600	192947.477	0.0161	0.0059	0.0075
18 19 02.0	600	193057.590	0.0161	0.0176	0.0173
18 29 02.0	600	193166.453	0.0161	0.0059	0.0036
18 39 02.0	600	193273.914	0.0161	0.0078	0.0075
18 49 02.0	600	193379.773	0.0161	0.0137	0.0114
18 58 32.0	540	193478.709	0.0171	0.0234	0.0231
19 13 02.0	600	193626.186	0.0162	0.0098	0.0075
19 26 02.0	600	193754.502	0.0162	0.0137	0.0153
19 36 02.0	600	193850.426	0.0162	0.0137	0.0134
19 46 02.0	600	193943.748	0.0162	0.0117	0.0095
19 52 02.0	120	193998.549	0.0442	0.0156	0.0153
20 01 02.0	600	194078.514	0.0163	0.0000	0.0036
20 11 02.0	600	194164.666	0.0163	0.0039	0.0017
20 21 02.0	600	194247.689	0.0164	0.0059	0.0056
20 31 02.0	600	194327.451	0.0164	0.0156	0.0134
20 41 02.0	600	194403.793	0.0165	0.0078	0.0076
20 51 02.0	600	194476.605	0.0166	0.0000	−0.0002
21 05 02.0	600	194572.396	0.0168	−0.0020	−0.0041
21 15 02.0	600	194636.268	0.0170	−0.0078	−0.0061
21 25 02.0	600	194696.240	0.0173	0.0000	−0.0002
21 35 02.0	600	194752.215	0.0177	0.0098	0.0077
21 45 02.0	600	194804.090	0.0184	0.0098	0.0116
21 55 02.0	600	194851.781	0.0195	−0.0098	−0.0079
22 03 02.0	360	194887.008	0.0288	−0.0156	−0.0177
22 10 02.0	120	194915.549	0.0647	−0.0176	−0.0176
22 16 02.0	240	194938.258	0.0471	−0.0234	−0.0254
22 23 32.0	180	194964.510	0.0818	−0.0234	−0.0215
Transmitter Frequency = 29.6690 MHz					
Pass — Dec. 12, 1962					
12 45 32.0	60	191333.482	0.0565	0.0098	0.0113
12 51 02.0	240	191361.670	0.0298	0.0039	0.0054
Transmitter Frequency = 29.6689 MHz					
13 13 32.0	60	191489.398	0.0531	0.0039	0.0073
13 26 02.0	360	191569.252	0.0227	0.0039	0.0053
13 39 02.0	600	191658.623	0.0165	0.0020	0.0013
13 49 02.0	600	191731.508	0.0165	0.0020	0.0033
13 59 02.0	600	191807.924	0.0164	0.0020	0.0032
14 09 02.0	600	191887.738	0.0164	−0.0059	−0.0046
14 19 02.0	600	191970.834	0.0163	−0.0020	−0.0008
14 29 02.0	600	192057.059	0.0162	−0.0039	−0.0027
14 36 32.0	300	192123.605	0.0247	−0.0039	−0.0028
14 47 02.0	600	192219.687	0.0162	−0.0078	−0.0087
14 57 02.0	600	192313.898	0.0162	−0.0039	−0.0048
15 07 02.0	600	192410.658	0.0162	−0.0020	0.0010
15 17 02.0	600	192509.801	0.0161	−0.0020	−0.0048
15 27 02.0	600	192611.158	0.0161	−0.0059	−0.0049
15 37 02.0	600	192714.566	0.0161	0.0020	0.0010
15 47 02.0	600	192819.828	0.0161	−0.0020	−0.0030
15 57 02.0	600	192926.777	0.0161	−0.0020	−0.0010
16 07 02.0	600	193035.225	0.0161	−0.0039	−0.0011
16 13 02.0	120	193100.891	0.0438	0.0039	0.0028
Transmitter Frequency = 29.6690 MHz					
16 31 32.0	60	193306.365	0.0505	0.0156	0.0145
16 41 02.0	480	193412.912	0.0185	0.0039	0.0047
16 52 32.0	300	193542.643	0.0245	−0.0020	0.0008
17 05 02.0	600	193684.311	0.0161	−0.0020	−0.0012
17 15 02.0	600	193797.859	0.0161	−0.0039	−0.0051
17 25 02.0	600	193911.395	0.0161	−0.0078	−0.0051
17 35 02.0	600	194024.723	0.0161	−0.0078	−0.0091
17 41 02.0	120	194092.566	0.0437	0.0020	0.0027
17 52 02.0	600	194216.340	0.0161	−0.0098	−0.0071
18 02 02.0	600	194328.121	0.0161	−0.0059	−0.0071

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
18 12 02.0	600	194438.975	0.0161	−0.0020	−0.0013
18 20 32.0	420	194532.346	0.0200	−0.0020	0.0007
18 30 02.0	600	194635.531	0.0161	−0.0039	−0.0052
18 40 02.0	600	194742.750	0.0161	−0.0059	−0.0071
18 50 02.0	600	194848.316	0.0161	−0.0098	−0.0091
19 00 02.0	600	194952.066	0.0161	0.0039	0.0026
19 10 02.0	600	195053.797	0.0162	0.0078	0.0104
19 18 02.0	360	195133.666	0.0220	−0.0039	−0.0013
19 23 26.0	50	195186.738	0.0508*	−0.0098	−0.0091
19 35 26.0	50	195302.078	0.0509	0.0078	0.0105
19 46 26.0	50	195404.500	0.0510	0.0137	0.0144
20 01 26.0	50	195538.699	0.0511	0.0215	0.0203
20 14 26.0	50	195649.500	0.0513	−0.0254	−0.0227
20 27 26.0	50	195754.979	0.0515	0.0117	0.0125
20 41 26.0	50	195862.139	0.0519	−0.0098	−0.0090
20 53 26.0	50	195948.500	0.0524	0.0039	0.0047
21 07 26.0	50	196042.500	0.0531	−0.0059	−0.0050
21 21 26.0	50	196129.000	0.0542	0.0059	0.0048
21 32 26.0	50	196191.500	0.0557	0.0020	0.0048
21 44 26.0	50	196254.039	0.0583	−0.0176	−0.0166
21 54 26.0	50	196301.619	0.0625	0.0039	0.0049
22 05 26.0	50	196349.020	0.0723	−0.0176	−0.0165
22 15 26.0	50	196387.658	0.0957	0.0039	0.0030

Transmitter Frequency = 29.6690 MHz
Pass — Dec. 13, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 24 02.0	600	192763.854	0.0205	0.0117	0.0138
12 34 02.0	600	192809.088	0.0189	0.0039	0.0060
12 40 02.0	120	192838.115	0.0498	0.0117	0.0158
12 52 02.0	600	192901.174	0.0176	0.0059	0.0080
12 59 02.0	240	192940.465	0.0292	0.0020	0.0041
19 49 02.0	600	197049.248	0.0162	0.0000	−0.0004
20 02 02.0	600	197167.805	0.0163	−0.0098	−0.0101
20 10 32.0	420	197242.682	0.0203	−0.0117	−0.0100
20 23 02.0	600	197348.531	0.0164	−0.0059	−0.0059
20 33 02.0	600	197429.590	0.0165	0.0000	0.0000
20 41 32.0	420	197495.887	0.0206	0.0000	0.0001
20 55 02.0	600	197595.760	0.0167	0.0137	0.0159
21 05 02.0	600	197665.488	0.0169	0.0059	0.0082
21 15 02.0	600	197731.441	0.0172	−0.0039	−0.0034
21 25 02.0	600	197793.537	0.0175	−0.0039	−0.0013
21 35 02.0	600	197851.672	0.0181	−0.0117	−0.0090
21 45 02.0	600	197905.760	0.0190	−0.0234	−0.0205
21 55 02.0	600	197955.746	0.0206	−0.0254	−0.0243
22 04 32.0	540	197999.410	0.0253	−0.0215	−0.0183

Transmitter Frequency = 29.6690 MHz
Pass — Dec. 14, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
14 16 32.0	300	196786.809	0.0248	−0.0098	−0.0101
14 28 02.0	600	196966.967	0.0162	−0.0020	0.0010
14 38 02.0	600	197130.971	0.0162	0.0000	0.0018
14 44 32.0	180	197241.139	0.0328	−0.0059	−0.0064
14 54 02.0	600	197408.449	0.0162	0.0020	0.0015
15 04 02.0	600	197591.656	0.0162	0.0000	0.0013
15 14 02.0	600	197782.719	0.0162	0.0020	0.0045
15 22 02.0	360	197941.207	0.0220	−0.0039	−0.0039
15 31 02.0	120	198126.125	0.0438	−0.0039	−0.0035
15 42 02.0	600	198362.418	0.0161	0.0000	−0.0012
15 52 02.0	600	198586.738	0.0161	0.0000	−0.0025
16 02 02.0	600	198820.998	0.0161	−0.0059	−0.0051
16 12 02.0	600	199065.846	0.0161	−0.0020	−0.0029
16 22 02.0	600	199322.012	0.0161	−0.0020	−0.0032
16 27 32.0	60	199467.516	0.0505	0.0078	0.0076

Transmitter Frequency = 29.6691 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
16 58 02.0	600	200352.441	0.0161	0.0078	0.0075
17 08 02.0	600	200673.355	0.0161	0.0078	0.0104
17 18 02.0	600	201011.875	0.0161	0.0000	−0.0029
17 24 32.0	180	201241.326	0.0326	−0.0039	−0.0020
17 56 32.0	300	202508.271	0.0245	0.0000	0.0014
18 13 02.0	600	203261.273	0.0161	0.0000	0.0045
18 27 02.0	600	203957.930	0.0161	−0.0039	0.0018
18 37 02.0	600	204488.713	0.0161	−0.0156	−0.0094
18 47 02.0	600	205045.697	0.0161	−0.0039	0.0014
18 57 02.0	600	205625.619	0.0161	−0.0059	−0.0012
19 02 32.0	60	205951.883	0.0508	−0.0156	−0.0079
19 13 02.0	600	206586.289	0.0162	−0.0176	−0.0039
19 23 02.0	600	207192.859	0.0162	−0.0137	−0.0035
19 33 02.0	600	207789.252	0.0162	−0.0098	0.0065
19 43 02.0	600	208360.557	0.0162	−0.0098	0.0047
19 53 02.0	600	208891.203	0.0163	−0.0137	0.0014
20 03 02.0	600	209366.773	0.0163	−0.0176	−0.0020
20 08 32.0	60	209603.516	0.0514	−0.0234	−0.0072

Transmitter Frequency = 29.6692 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
20 57 32.0	300	210726.760	0.0256	−0.0137	0.0014
21 08 02.0	600	210773.148	0.0170	−0.0195	−0.0052
21 18 02.0	600	210776.246	0.0173	−0.0176	−0.0079
21 28 02.0	600	210746.910	0.0178	−0.0020	0.0065
21 38 02.0	600	210692.910	0.0186	−0.0039	0.0059
21 48 02.0	600	210620.730	0.0198	−0.0039	0.0040
21 58 02.0	600	210535.521	0.0223	−0.0059	0.0013
22 08 02.0	600	210441.271	0.0284	−0.0137	−0.0062
22 14 32.0	180	210376.793	0.0789	−0.0234	−0.0142

Transmitter Frequency = 29.6691 MHz
Pass — Dec. 15, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 54 32.0	60	203074.365	0.0546	−0.0059	0.0009
15 08 02.0	360	203988.371	0.0220	0.0020	0.0062
15 18 32.0	540	204083.123	0.0171	−0.0039	0.0040
15 31 02.0	120	204198.932	0.0438	−0.0273	−0.0231
15 41 02.0	600	204293.979	0.0161	−0.0039	0.0003
15 51 02.0	600	204390.713	0.0161	−0.0078	−0.0019

*Data during the Dec. 12 pass with a count time of 50 sec. are sampled every minute for the solutions of Section V. The error weight is appropriately larger as a result (Cf. Eq. 163). Not every measurement with a 50-sec. count time is listed for this pass. Instead, every tenth point is given as a representative sample.

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
16 01 02.0	600	204489.043	0.0161	−0.0137	−0.0120
16 11 02.0	600	204588.807	0.0161	−0.0039	0.0015
16 21 02.0	600	204689.777	0.0161	−0.0039	0.0013
16 31 02.0	600	204791.783	0.0161	0.0039	0.0078
16 41 02.0	600	204894.607	0.0161	−0.0020	0.0021
16 51 02.0	600	204998.074	0.0161	0.0020	0.0058
17 01 02.0	600	205101.980	0.0161	0.0039	0.0119
17 11 02.0	600	205206.109	0.0161	−0.0039	−0.0001
17 17 32.0	180	205273.826	0.0326	−0.0137	−0.0096
17 35 02.0	600	205455.807	0.0161	−0.0215	−0.0158
17 41 02.0	120	205517.965	0.0437	−0.0273	−0.0224
17 52 32.0	60	205636.500	0.0505	−0.0137	−0.0108

Transmitter Frequency = 29.6692 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 30 02.0	240	206571.490	0.0273	0.0176	0.0232
19 41 02.0	600	206664.609	0.0162	0.0078	0.0134
19 53 02.0	600	206762.541	0.0163	0.0059	0.0130
20 03 02.0	600	206840.965	0.0164	0.0020	0.0072
20 10 32.0	300	206897.883	0.0249	−0.0098	−0.0027
20 21 32.0	420	206977.982	0.0204	−0.0020	0.0011
20 35 02.0	600	207070.771	0.0165	0.0078	0.0125
20 42 02.0	240	207116.574	0.0281	0.0176	0.0221
20 53 02.0	600	207184.656	0.0168	−0.0059	0.0026
20 58 32.0	60	207217.266	0.0531	0.0020	0.0044
21 09 02.0	600	207275.895	0.0172	0.0000	0.0025
21 19 02.0	600	207327.932	0.0175	−0.0078	−0.0014
21 29 02.0	600	207376.027	0.0180	−0.0020	0.0021
21 39 02.0	600	207420.078	0.0189	−0.0117	−0.0061
21 49 02.0	600	207460.035	0.0205	−0.0059	0.0020
21 59 02.0	600	207495.822	0.0239	0.0117	0.0153
22 04 32.0	60	207513.916	0.0867	0.0508	0.0545

Transmitter Frequency = 29.6691 MHz
Pass — Dec. 16, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 43 02.0	120	203464.207	0.0486	−0.0156	−0.0106
12 52 02.0	600	203508.217	0.0175	−0.0059	−0.0014
13 02 02.0	600	203560.760	0.0171	−0.0020	0.0001
13 12 02.0	600	203617.266	0.0168	0.0000	0.0061
13 22 02.0	600	203677.609	0.0167	−0.0078	−0.0020
13 32 02.0	600	203741.721	0.0166	−0.0039	0.0040
13 42 02.0	600	203809.457	0.0165	−0.0098	−0.0041
13 52 02.0	600	203880.727	0.0164	−0.0020	0.0054
13 57 32.0	60	203921.215	0.0514	−0.0234	−0.0162
14 07 02.0	600	203993.896	0.0164	−0.0449	−0.0376
14 17 02.0	600	204073.406	0.0163	−0.0234	−0.0183
14 27 02.0	600	204155.937	0.0162	−0.0313	−0.0281
14 36 32.0	540	204237.051	0.0172	0.0059	0.0088
14 50 02.0	600	204356.553	0.0162	−0.0098	−0.0034
15 00 02.0	600	204448.057	0.0162	−0.0234	−0.0213
15 10 02.0	600	204541.957	0.0162	−0.0195	−0.0153
15 20 02.0	600	204638.062	0.0161	−0.0156	−0.0097
15 29 32.0	540	204731.217	0.0171	−0.0234	−0.0159
15 44 02.0	600	204876.641	0.0161	−0.0215	−0.0160
15 54 02.0	600	204978.914	0.0161	−0.0195	−0.0159

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
16 04 02.0	600	205082.590	0.0161	−0.0313	−0.0297
16 14 02.0	600	205187.516	0.0161	−0.0195	−0.0147
16 24 02.0	600	205293.479	0.0161	−0.0059	−0.0030
16 34 02.0	600	205400.277	0.0161	0.0020	0.0064
16 41 02.0	240	205475.391	0.0272	−0.0156	−0.0133
16 52 02.0	600	205593.980	0.0161	−0.0195	−0.0172
17 02 02.0	600	205702.104	0.0161	−0.0137	−0.0077
17 12 02.0	600	205810.305	0.0161	−0.0273	−0.0233
17 22 02.0	600	205918.408	0.0161	−0.0391	−0.0353
17 32 02.0	600	206026.252	0.0161	−0.0234	−0.0179
17 42 02.0	600	206133.578	0.0161	−0.0313	−0.0300
17 48 02.0	120	206197.732	0.0438	0.0059	0.0091

Transmitter Frequency = 29.6692 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
19 12 02.0	240	207046.516	0.0273	0.0078	0.0107
19 22 02.0	600	207138.504	0.0162	0.0156	0.0182
19 32 02.0	600	207228.068	0.0162	0.0195	0.0199
19 42 02.0	600	207314.953	0.0162	0.0039	0.0099
19 49 02.0	240	207374.221	0.0275	0.0039	0.0119
20 00 02.0	600	207464.166	0.0164	−0.0039	0.0021
20 13 02.0	600	207565.777	0.0164	0.0059	0.0119
20 22 02.0	480	207632.953	0.0189	0.0020	0.0078
20 35 32.0	540	207728.459	0.0176	0.0059	0.0133
20 46 02.0	240	207798.395	0.0282	0.0078	0.0130
20 58 02.0	600	207873.107	0.0169	0.0039	0.0090
21 08 02.0	600	207931.301	0.0172	0.0078	0.0105
21 18 02.0	600	207985.580	0.0176	−0.0020	0.0028
21 28 02.0	600	208035.877	0.0181	0.0020	0.0065
21 38 02.0	600	208082.098	0.0191	0.0117	0.0179
21 46 02.0	360	208116.172	0.0278	0.0039	0.0019

Transmitter Frequency = 29.6691 MHz
Pass — Dec. 17, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 12 32.0	60	204127.398	0.0710	0.0273	0.0310
12 21 02.0	600	204159.180	0.0201	0.0039	0.0072
12 31 02.0	600	204200.318	0.0187	0.0156	0.0208
12 41 02.0	600	204245.600	0.0179	−0.0039	0.0033
12 51 02.0	600	204295.055	0.0174	0.0488	0.0517
13 01 02.0	600	204348.430	0.0171	−0.0078	−0.0012
13 11 02.0	600	204405.809	0.0168	0.0117	0.0182
13 21 02.0	600	204466.990	0.0167	0.0039	0.0085
13 31 02.0	600	204531.908	0.0166	−0.0020	0.0022
13 37 02.0	120	204572.475	0.0449	0.0117	0.0118
13 48 02.0	600	204650.533	0.0164	0.0234	0.0253
13 58 02.0	600	204724.943	0.0164	0.0273	0.0333
14 08 02.0	600	204802.609	0.0163	0.0156	0.0212
14 18 02.0	600	204883.430	0.0163	0.0176	0.0230
14 28 02.0	600	204967.227	0.0162	0.0020	0.0070
14 38 02.0	600	205053.873	0.0162	−0.0098	−0.0068
14 48 02.0	600	205143.230	0.0162	−0.0039	0.0009
14 58 02.0	600	205235.107	0.0162	−0.0039	0.0005
15 08 02.0	600	205329.334	0.0162	−0.0215	−0.0169
15 18 02.0	600	205425.779	0.0161	−0.0215	−0.0171
15 28 02.0	600	205524.260	0.0161	−0.0059	−0.0017

Table E-1 (contd)

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
15 38 02.0	600	205624.564	0.0161	−0.0137	−0.0100
15 48 02.0	600	205726.543	0.0161	−0.0117	−0.0076
15 56 02.0	360	205809.168	0.0219	−0.0078	−0.0041

Transmitter Frequency = 29.6692 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
16 18 02.0	600	206040.977	0.0161	−0.0059	−0.0012
16 28 02.0	600	206147.768	0.0161	0.0117	0.0207
16 38 02.0	600	206255.260	0.0161	0.0117	0.0142
16 48 02.0	600	206363.264	0.0161	−0.0039	0.0004
16 58 02.0	600	206471.623	0.0161	0.0078	0.0101
17 07 02.0	480	206569.256	0.0185	0.0000	0.0040
17 42 32.0	540	206952.695	0.0170	0.0195	0.0230
18 18 02.0	600	207326.826	0.0161	−0.0039	0.0006
18 28 02.0	600	207429.375	0.0161	0.0000	0.0041
18 38 02.0	600	207530.328	0.0161	0.0020	0.0056
18 48 02.0	600	207629.492	0.0162	−0.0039	0.0000
18 58 02.0	600	207726.701	0.0162	−0.0078	−0.0024
19 08 02.0	600	207821.783	0.0162	−0.0098	−0.0046
19 18 02.0	600	207914.547	0.0162	−0.0195	−0.0146
19 28 02.0	600	208004.861	0.0162	−0.0137	−0.0108
19 38 02.0	600	208092.529	0.0162	−0.0215	−0.0151
19 43 32.0	60	208139.699	0.0511	−0.0137	−0.0094
19 53 02.0	600	208218.777	0.0163	−0.0137	−0.0075
20 03 02.0	600	208299.213	0.0164	0.0020	0.0057
20 13 02.0	600	208376.477	0.0164	−0.0020	−0.0035
20 20 32.0	300	208432.379	0.0250	−0.0039	0.0015
20 27 02.0	360	208479.182	0.0225	−0.0059	−0.0010
21 46 02.0	600	208921.740	0.0210	−0.0391	−0.0381
21 52 02.0	120	208945.258	0.0624	−0.0391	−0.0343
21 59 02.0	120	208970.557	0.0740	−0.0410	−0.0405

Transmitter Frequency = 29.6692 MHz
Pass — Dec. 19, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17 15 02.0	600	208252.398	0.0161	−0.0020	0.0040
17 25 02.0	600	208359.875	0.0161	0.0020	0.0061
17 35 02.0	600	208466.879	0.0161	0.0000	0.0057
17 45 02.0	600	208573.230	0.0161	0.0059	0.0113
17 55 02.0	600	208678.717	0.0161	−0.0039	0.0033
18 05 02.0	600	208783.170	0.0161	0.0000	0.0067
18 15 02.0	600	208886.381	0.0161	−0.0039	0.0008
18 25 02.0	600	208988.174	0.0161	−0.0059	−0.0015
18 35 02.0	600	209088.361	0.0161	−0.0078	−0.0037
18 45 02.0	600	209186.760	0.0162	−0.0098	−0.0060
18 55 02.0	600	209283.191	0.0162	−0.0156	−0.0120
19 05 02.0	600	209377.498	0.0162	−0.0078	−0.0026
19 15 02.0	600	209469.496	0.0162	0.0000	0.0069
19 24 32.0	600	209554.699	0.0510	−0.0020	0.0044
19 35 02.0	600	209645.881	0.0162	0.0000	0.0042
19 45 02.0	600	209729.967	0.0163	0.0000	0.0044
19 55 02.0	600	209811.111	0.0164	0.0039	0.0097
20 05 02.0	600	209889.158	0.0164	−0.0020	0.0053
20 15 02.0	600	209963.988	0.0165	0.0020	0.0073
20 25 02.0	600	210035.457	0.0165	0.0039	0.0090
20 33 02.0	360	210090.221	0.0226	0.0039	0.0087

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
20 46 02.0	600	210174.027	0.0168	−0.0078	−0.0035
20 53 32.0	300	210219.705	0.0259	−0.0137	−0.0078
21 06 02.0	600	210290.744	0.0174	−0.0078	−0.0023
21 16 02.0	600	210343.262	0.0178	0.0020	0.0092
21 30 02.0	600	210409.988	0.0190	−0.0078	−0.0010
21 39 02.0	480	210448.680	0.0236	−0.0137	−0.0095

Transmitter Frequency = 29.6691 MHz
Pass — Dec. 20, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
12 33 32.0	180	206531.889	0.0369	−0.0039	0.0023
12 40 02.0	120	206562.350	0.0483	0.0156	0.0217
12 45 02.0	120	206586.932	0.0477	0.0059	0.0118
12 55 02.0	600	206639.262	0.0172	−0.0020	0.0018
13 02 02.0	240	206677.994	0.0287	−0.0059	0.0020
13 14 02.0	600	206749.137	0.0168	−0.0059	0.0034
13 24 02.0	600	206812.432	0.0166	0.0039	0 0074
13 34 02.0	600	206879.340	0.0165	0.0000	0.0072
13 44 02.0	600	206949.746	0.0164	−0.0059	−0.0008
13 54 02.0	600	207023.539	0.0164	−0.0059	−0.0015
14 04 02.0	600	207100.578	0.0163	−0.0059	−0.0015
14 14 02.0	600	207180.729	0.0163	−0.0059	−0.0037
14 24 02.0	600	207263.848	0.0162	−0.0059	0.0000
14 34 02.0	600	207349.781	0.0162	−0.0059	0.0035
14 44 02.0	600	207438.369	0.0162	−0.0078	−0.0062
14 58 02.0	600	207566.572	0:0162	0.0000	0.0054
15 08 02.0	600	207660.889	0.0162	−0.0059	−0.0023
15 16 02.0	360	207737.807	0.0220	−0.0098	−0.0008

Transmitter Frequency = 29.6692 MHz

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17 52 02.0	600	209372.205	0.0161	−0.0059	−0.0005
18 02 02.0	600	209476.324	0.0161	−0.0117	−0.0048
18 12 02.0	600	209579.238	0.0161	−0.0098	−0.0031
18 22 02.0	600	209680.760	0.0161	−0.0098	−0.0033
18 32 02.0	600	209780.707	0.0161	−0.0039	0.0041
18 40 32.0	420	209864.312	0.0201	−0.0137	−0.0018
18 53 02.0	600	209984.631	0.0162	−0.0098	−0.0041
19 03 02.0	600	210078.543	0.0162	−0.0059	0.0011
19 13 02.0	600	210170.135	0.0162	−0.0137	−0.0086
19 23 02.0	600	210259.266	0.0162	−0.0215	−0.0149
19 33 02.0	600	210345.781	0.0162	−0.0098	−0.0094
19 43 02.0	600	210429.508	0.0163	−0.0098	−0.0036
19 53 02.0	600	210510.295	0.0164	−0.0156	−0.0079
20 03 02.0	600	210588.000	0.0164	−0.0215	−0.0144
20 13 02.0	600	210662.496	0.0165	−0.0137	−0.0108
20 23 02.0	600	210733.635	0.0165	−0.0156	−0.0125
20 33 02.0	600	210801.301	0.0167	−0.0098	−0.0009
20 43 02.0	600	210865.367	0.0168	−0.0020	0.0041
20 53 02.0	600	210925.730	0.0170	0.0137	0.0176
21 03 02.0	600	210982.244	0.0173	−0.0078	0.0036
21 13 02.0	600	211034.861	0.0178	−0.0059	0.0033
21 23 02.0	600	211083.465	0.0186	−0.0117	−0.0032
21 33 02.0	600	211127.986	0.0198	−0.0078	−0.0016
21 43 02.0	600	211168.352	0.0222	−0.0039	0.0042
21 52 02.0	480	211201.119	0.0313	−0.0234	−0.0178

Table E-2. *Mariner II* Doppler data (Station 12 transmitting, Station 12 receiving)

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 14 and Oct. 15, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
17 36 26.0	50	120700.299	0.0162	−0.0039	−0.0038
17 47 26.0	50	120700.399	0.0161	−0.0010	0.0001
17 57 26.0	50	120770.039	0.0161	−0.0039	−0.0039
18 07 26.0	50	120843.279	0.0161	−0.0088	−0.0088
18 18 26.0	50	120927.859	0.0161	0.0010	0.0020
18 28 26.0	50	121008.199	0.0161	−0.0234	−0.0225
18 38 26.0	50	121091.739	0.0160	−0.0127	−0.0127
18 48 26.0	50	121178.300	0.0160	0.0059	−0.0039
18 58 26.0	50	121267.680	0.0160	−0.0059	−0.0059
19 08 26.0	50	121359.760	0.0160	−0.0059	−0.0049
19 18 26.0	50	121454.359	0.0159	−0.0029	−0.0011
19 28 26.0	50	121551.279	0.0159	−0.0195	−0.0196
19 38 26.0	50	121650.399	0.0159	0.0010	0.0019
19 48 26.0	50	121751.460	0.0159	−0.0166	−0.0157
19 58 26.0	50	121854.340	0.0159	−0.0049	−0.0040
20 08 26.0	50	121958.800	0.0159	−0.0137	−0.0128
20 18 26.0	50	122064.680	0.0159	−0.0068	−0.0070
20 28 26.0	50	122171.760	0.0159	−0.0107	−0.0089
20 39 26.0	50	122290.720	0.0159	0.0000	0.0008
20 49 26.0	50	122399.680	0.0159	−0.0117	−0.0109
20 59 26.0	50	122509.239	0.0159	−0.0098	−0.0089
21 09 26.0	50	122619.180	0.0159	−0.0088	−0.0070
21 19 26.0	50	122729.300	0.0159	−0.0059	−0.0040
21 29 26.0	50	122839.399	0.0159	0.0049	0.0057
21 39 26.0	50	122949.260	0.0159	0.0078	0.0087
21 49 26.0	50	123058.680	0.0159	0.0059	0.0077
21 59 26.0	50	123167.460	0.0159	0.0068	0.0077
22 09 26.0	50	123275.399	0.0159	0.0078	0.0097
22 19 26.0	50	123382.279	0.0159	−0.0068	−0.0050
22 29 26.0	50	123487.939	0.0159	0.0000	0.0019
22 39 26.0	50	123592.159	0.0159	0.0059	0.0068
22 49 26.0	50	123694.760	0.0159	0.0234	0.0244
22 59 26.0	50	123795.500	0.0159	0.0029	0.0049
23 09 26.0	50	123894.260	0.0159	0.0117	0.0127
23 19 26.0	50	123990.800	0.0159	−0.0088	−0.0068
23 29 26.0	50	124085.000	0.0160	0.0029	0.0039
23 39 26.0	50	124176.659	0.0160	0.0166	0.0196
23 49 26.0	50	124265.579	0.0160	0.0078	0.0089
23 59 26.0	50	124351.619	0.0160	−0.0039	−0.0028
00 09 26.0	50	124434.659	0.0161	0.0186	0.0148
00 19 26.0	50	124514.479	0.0161	0.0137	0.0158
00 29 26.0	50	124590.979	0.0161	0.0234	0.0256
00 39 26.0	50	124663.979	0.0161	0.0088	0.0109
00 49 26.0	50	124733.380	0.0162	0.0049	0.0061
00 59 26.0	50	124799.039	0.0162	−0.0049	−0.0037
01 31 26.0	50	124982.720	0.0165	−0.0225	−0.0212
01 42 26.0	50	125036.140	0.0166	−0.0117	−0.0094
01 52 26.0	50	125080.199	0.0168	−0.0146	−0.0123
02 02 26.0	50	125119.920	0.0171	0.0020	0.0033
02 12 26.0	50	125155.199	0.0176	0.0020	0.0034
02 22 26.0	50	125186.000	0.0182	0.0088	0.0102
02 32 26.0	50	125212.239	0.0195	−0.0107	−0.0102
02 42 26.0	50	125233.920	0.0222	−0.0186	−0.0170

Transmitter Frequency = 29.6681 MHz
Pass — Oct. 24, 1962

Observation time (UT2C) h m s	Count time (sec)	Doppler data (Hz)	Error weight (Hz)	Residual (Hz)	Linear residual (Hz)
15 42 26.0	50	126672.680	0.0170	0.0010	0.0009
15 52 26.0	50	126716.140	0.0167	0.0088	0.0087
16 02 26.0	50	126763.920	0.0165	0.0098	0.0077
16 12 26.0	50	126815.939	0.0164	0.0000	−0.0011
16 22 26.0	50	126872.119	0.0164	−0.0068	−0.0060
16 32 26.0	50	126932.380	0.0162	0.0068	0.0067
16 42 26.0	50	126996.579	0.0162	0.0020	0.0008
16 52 26.0	50	127064.640	0.0162	0.0137	0.0135

Nomenclature

A as a scalar; number of km in one astronomical unit (a.u.)

A as a matrix; matrix of differential coefficients defined by $dz = Adx$

A_{eff} effective area of spacecraft for solar radiation pressure

$C_{n,m}$ harmonic coefficient of cos $m\lambda$ in expansion of potential function

E mass of Earth (gms)

$\mathbf{E}$ vector of orbital elements for Earth

$F(t)dt$ number of Doppler cycles occurring in interval t to $t + dt$

F_2 factor for modified Kepler's third law in lunar motion

$\mathcal{F}$ flux of radiation from Sun at spacecraft distance

$\mathcal{F}_\odot$ solar constant

G as a scalar; universal gravitational constant (laboratory units)

G as a matrix; matrix of differential coefficients defined by $dz = Gdq + Hds$

(G_T, G_N) parameters for tangential and normal low-thrust forces for unbalanced gas jets

H as a scalar; altitude above surface of Earth

H as a matrix; matrix of differential coefficients defined by $dz = Gdq + Hds$

H_o atmospheric scale height

(I_x, I_y, I_z) principal moments of inertia about body-fixed axes

J_n coefficient of nth order zonal harmonic in expansion of potential function

K orbital parameter defined by $na\,(1-e^2)^{-1/2} \sin i$

K' multiplying factor for received frequency

L lunar inequality

M as a scalar; mass of Moon (gms)

M as a matrix; matrix to rotate from 1950.0 mean equatorial coordinates to mean equatorial coordinates of date

M_o mean anomaly at the epoch

M_v mass of Venus (gms)

M_v^s mass of Venus (solar mass units)

N as a scalar; constant of nutation

N as a matrix; matrix to rotate from mean equatorial coordinates of date to true equatorial coordinates of date

$\mathbf{N}$ unit vector normal to Earth-Sun-spacecraft plane

N_3 null matrix of order three

N_ν number of Doppler cycles accumulated by electronic cycle counting device

$P_\oplus$ heliocentric period of Earth

$P_☾$ constant of parallactic inequality

$\mathbf{P}_2$ non-gravitational perturbative acceleration

$Q(x)$ weighted sum of squares of residuals plus *a-priori* term

$\mathbf{R}$ geocentric position vector of tracking station

R_1 mean radius of Earth at geocentric latitude $\phi = \sin^{-1}\sqrt{1/3}$

R_B mean distance of Earth from Earth-Moon barycenter

R_{em} scaling factor to convert the lunar ephemeris to km

S mass of Sun (gms)

$S(x)$ weighted sum of squares of residuals

$S_{n,m}$ harmonic coefficient of sin $m\lambda$ in expansion of potential function

$\mathbf{T}$ unit tangential vector

T_c interval of time over which integrated cycle count is accumulated

$\mathbf{T}_v$ set of orbital elements for Venus

U as a scalar; potential function

U as a matrix; matrix of differential coefficients in expression $dq = Udq_o + Vdp$

$\mathbf{U}$ unit position vector

V as a scalar; geocentric radial velocity of Venus

V as a matrix; matrix of differential coefficients in expression $dq = Udq_o + Vdp$

V_L linear velocity of Earth about Earth-Moon barycenter

V_∞ hyperbolic velocity at infinite distance

W diagonal weighting matrix for set of data

Nomenclature (contd)

(X, Y, Z) inertial ecliptic coordinates

a semi-major axis

a_e mean equatorial radius of Earth

$a_\oplus$ mean Sun-Earth distance

$a_☾$ mean Earth-Moon distance

b semi-minor axis

c velocity of light

e eccentricity

f Doppler observation

(f_1, f_2, f_3) components of acceleration from low-thrust forces

g_1 mean acceleration of gravity at geocentric latitude $\phi = \sin^{-1}\sqrt{1/3}$

i inclination

k gaussian gravitational constant

k' multiplying factor for spacecraft frequency

k^2_{ge} geocentric gravitational constant

k^2_{gm} selenocentric gravitational constant

k^2_{gs} heliocentric gravitational constant

k^2_{gv} Venus-centered gravitational constant

ℓ generic longitude

m generic mass

n as an orbital element; mean motion

n as a physical constant; index of refraction

p as a scalar; general precessional constant

p as a column matrix; set of astronomical constants

q set of six position and velocity coordinates

$\mathbf{r}$ position vector

$\dot{\mathbf{r}}$ velocity vector

(r, ψ) polar coordinates

s as a scalar; number of ephemeris seconds in one tropical year (1900)

s as a column matrix; set of station coordinates

$\overset{\circ}{s}$ magnitude of velocity vector

t time parameter

t_m time at midpoint of cycle count time interval

t_{ob} observation time

v true anomaly

x set of parameters

$\tilde{x}$ set of *a-priori* parameters

(x, y, z) principal coordinate axes

$\hat{z}$ set of observations

$z(x)$ set of computed observations

$\tilde{\Gamma}_x$ covariance matrix for *a-priori* parameters

$\Delta_r f$ refraction correction to Doppler data

Δt time interval between successive Doppler observations

$\Delta_r \rho$ refraction correction to topocentric range

Θ matrix of differential coefficients in expression $d\dot{q} = \Phi dq + \Theta dp$

Φ as a scalar; gravitational potential energy

Φ as a matrix; matrix of differential coefficients in expression $d\dot{q} = \Phi dq + \Theta dp$

Ω longitude of ascending node

α right ascension

(α_1, α_2) coefficients in quadratic model for low-thrust forces

$\alpha(\tau)$ quadratic function $(1 - \alpha_1\tau - \alpha_2\tau^2)$

β generic latitude

γ unknown parameter in solar radiation pressure model

γ as an angle; elevation angle above horizon

δ declination

ϵ obliquity of ecliptic

θ local sidereal time

θ_0 Greenwich true sidereal time

(θ, ϕ, ψ) Eulerian angles

λ geocentric longitude

μ mass ratio of Moon to Earth (M/E)

ν generic frequency

ν_{ob} received frequency

ν_{tr} transmitted frequency

$\pi_\odot$ solar parallax

Nomenclature (contd)

$\pi_☾$ lunar parallax

ρ topocentric range

$\dot{\rho}$ topocentric range rate

$\boldsymbol{\rho}$ topocentric range vector

$\rho_{x,y}$ correlation coefficient between parameters x and y

σ standard deviation

τ time interval from epoch

τ_A number of light-seconds in one astronomical unit

τ_r light time for distance r

ϕ' geocentric latitude

ψ angle in polar coordinates (r, ψ)

ψ_r the derivative $d\psi/dr$

ω argument of periapsis

$\omega_\oplus$ angular rotation rate of Earth

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