NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1044

Expressions for the Precession Quantities and Their Partial Derivatives

Jay Lieske

Approved by:

Robert J. Jirka, Manager
Scientific Programming Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

June 15, 1967

TECHNICAL REPORT 32-1044

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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Abstract

The purpose of this paper is to investigate the functional dependence of precession quantities such as ζ_0, z, θ upon the fundamental constants. The effect of small changes of the fundamental constants upon the precession quantities is derived, and numerical partial derivatives are given as power series in time from an arbitrary epoch.

Expressions for the Precession Quantities and Their Partial Derivatives

I. Introduction

Comparison of planetary observations with ephemeris positions, such as is necessary for orbit improvement or refinement of astronomical quantities, requires that corrections be applied to either the observed or ephemeris positions in order to refer both sets of coordinates to the same reference system. These corrections include such well-known effects as nutation, aberration, geocentric parallax, and the precession of the equinox.

Although rather lengthy numerical expressions are usually given for the parameters describing these various effects (the precession matrix is a good example), the expressions in fact depend upon a rather limited set of basic parameters, often called the fundamental astronomical constants.

The purpose of this paper is to examine the basic parameters involved in the expressions for the mean obliquity of the ecliptic of date and for the elements of the matrix which is often used to account for the effects of precession.

It will be shown that the mean obliquity of the ecliptic at 1900.0, the speed of the general precession in longitude at 1900.0, and the system of planetary masses constitute the set of basic parameters upon which the lengthy polynomials defining the mean obliquity of date and the elements of the precession matrix depend.

Since the functional relations between the basic parameters and the derived expressions for obliquity and precession are quite complex, numerical expressions are given which show the effects of small changes in the basic parameter set on the derived quantities. Thus, for example, one can determine the effect of a change in the mass of Venus on the value of the mean obliquity of date and on the precession matrix.

If one has the heliocentric rectangular coordinates of a body referred to some fixed equator and equinox (e.g., 1950.0) and wishes to find the coordinates referred to the mean equator and equinox of date, the transformation used is

$$(x, y, z)_{Date} = (x_0, y_0, z_0)_{\substack{Initial \\ Epoch}} R(-\zeta_0)\, Q(\theta)\, R(-z) \tag{1}$$

where

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

and where ζ_0, θ, and z are parameters describing precession. Performing the multiplications in (1) we find

$$(x, y, z) = (x_0, y_0, z_0)\, A \text{ where } A = R(-\zeta_0)\, Q(\theta)\, R(-z)$$

Thus the elements of matrix A are the following:

$$a_{11} = \cos\zeta_0 \cos\theta \cos z - \sin\zeta_0 \sin z$$

$$a_{12} = \cos\zeta_0 \cos\theta \sin z + \sin\zeta_0 \cos z$$

$$a_{13} = \cos\zeta_0 \sin\theta$$

$$a_{21} = -\sin\zeta_0 \cos\theta \cos z - \cos\zeta_0 \sin z$$

$$a_{22} = -\sin\zeta_0 \cos\theta \sin z + \cos\zeta_0 \cos z$$

$$a_{23} = -\sin\zeta_0 \sin\theta$$

$$a_{31} = -\sin\theta \cos z$$

$$a_{32} = -\sin\theta \sin z$$

$$a_{33} = \cos\theta$$

These expressions are given in Ref. 1, p. 31.

Usually the parameters ζ_0, θ, and z are written as polynomials in powers of time from some fundamental epoch. The remainder of this paper is a discussion of ζ_0, θ, and z and related quantities and their dependence upon fundamental constants. Numerical partial derivatives of precession quantities with respect to the fundamental constants are given.

II. Symbols and Nomenclature

The principal symbols used herein are defined as follows (refer to Fig. 1 for the geometry):

P_0 = celestial pole at T_0

C_0 = ecliptic pole at T_0

γ_0 = equinox at T_0

$\gamma_1\gamma_0E_0$ = ecliptic at T_0 (fixed)

γ_0QA_0 = equator at T_0

ε_0 = obliquity of ecliptic of T_0 on equator at $T_0 = \angle E_0\gamma_0A_0$

P = celestial pole at T_1

C = ecliptic pole at T_1

γ = equinox at T_1

γN_1E = ecliptic at T_1

γQA = equator at T_1

$\varepsilon = \angle E\gamma A$ = obliquity of ecliptic of T_1 on equator of T_1

$\varepsilon_1 = \angle E_0\gamma_1A$ = obliquity of equator of T_1 on ecliptic of T_0

$\gamma_0\gamma_1 \equiv \Psi$ = luni-solar precession (including geodesic)

$\gamma_1\gamma \equiv \lambda$ = planetary precession

$\gamma_0N_1 \equiv \Pi_1$ = longitude of ascending node of ecliptic at T_1 on ecliptic at T_0, measured from fixed equinox γ_0 along fixed ecliptic of T_0

$\gamma N_1 \equiv \Lambda$ = longitude of ascending node of ecliptic at T_1 on ecliptic at T_0, measured from mean equinox γ of T_1 along mean ecliptic of T_1

$\pi_1 \equiv \angle E_0N_1E$ = angle between ecliptics of T_0 and T_1

ζ_0 = angle at P_0 of T_0 between the great circles joining P_0 with γ_0 and P_0 with P

$90^\circ - \zeta_0$ = right ascension of node of equator at T_1 on fixed equator at T_0 measured from γ_0 of T_0 along equator of T_0

$90^\circ + z$ = right ascension of node of equator at T_1 on fixed equator at T_0 measured from γ of T_1 along equator of T_1

$\theta = \angle AQA_0$ = angles between equators of T_1 and T_0

III. Geometry of the Problem

If C is the pole of the ecliptic and P the pole of the equator, then the equinox γ is defined by the intersection of the planes of the equator and ecliptic. Since P and C are continuously in motion, the equinox also is in motion. The actual poles are described by the position of a mean pole plus a small oscillation (nutation) of the actual pole about the mean pole.

The precessional motion of the mean equinox is due to the combined motions of the two poles that define it.

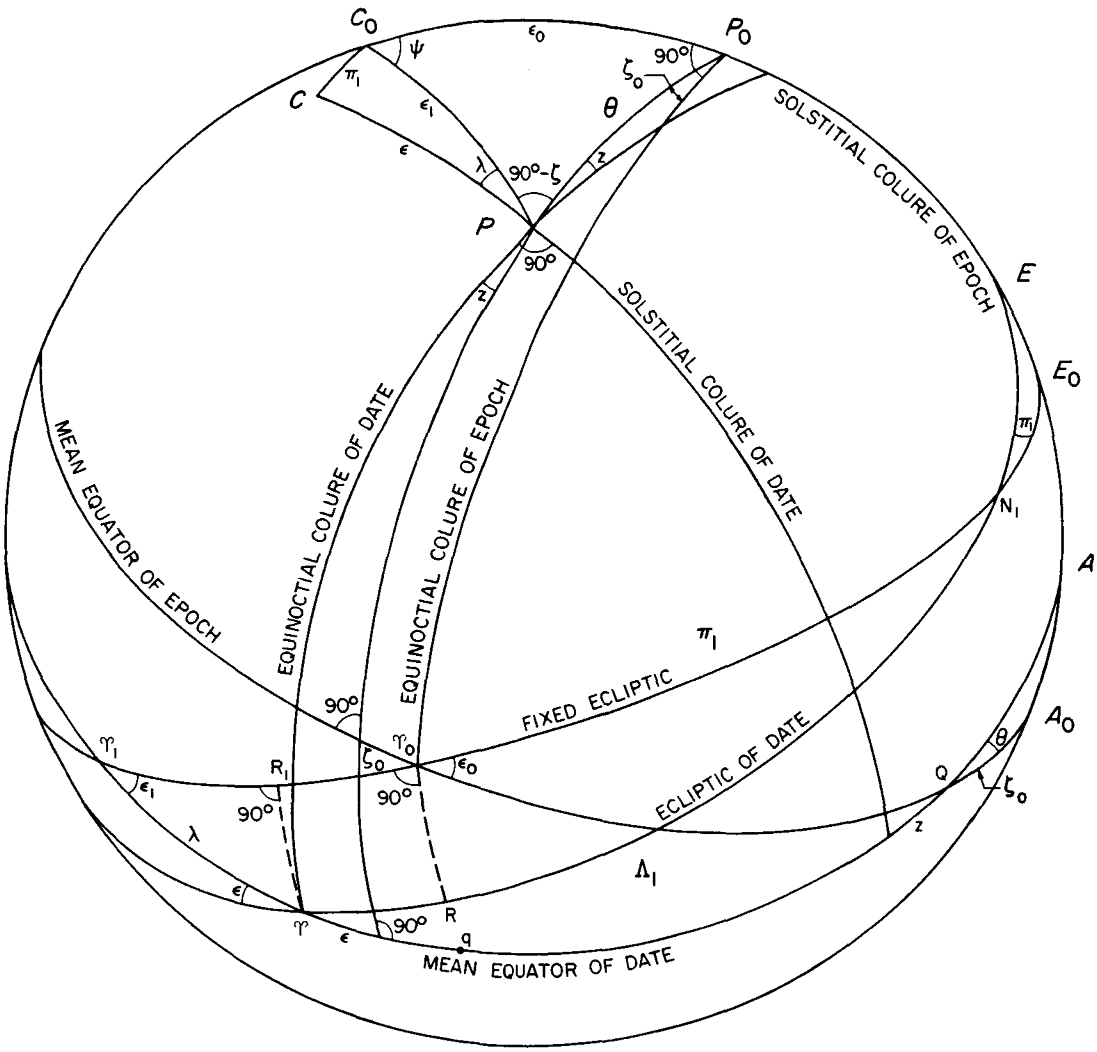


Fig. 1. Geometric system associated with precession (from Ref. 2)

The precessional motion due to the motion of the celestial pole is called luni-solar precession and is caused by the action of the sun and moon on the earth's equatorial bulge. A small relativistic effect called the geodesic precession is also included in the motion of the celestial pole. It is a direct slipping of the mean equinox of date along a fixed ecliptic, at the rate of 1″9 per century. The part caused by motion of the ecliptic pole is called planetary precession. Luni-solar precession slides the equinox along a fixed ecliptic while planetary precession changes the plane of the ecliptic.

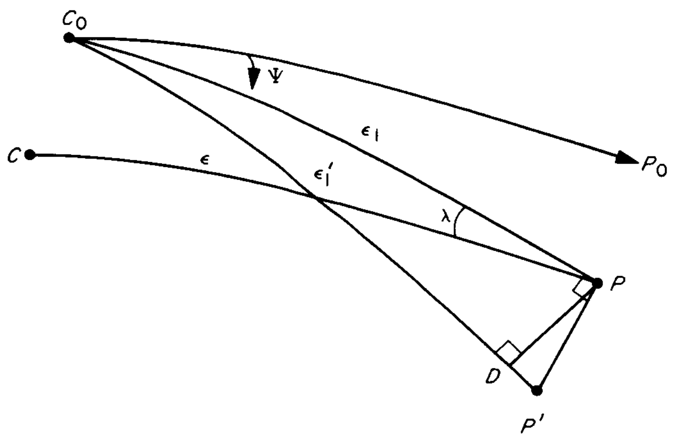
Owing to luni-solar attraction on the earth's equatorial bulge, the mean celestial pole moves continuously toward the mean equinox of the moment with a speed given by $P \sin \varepsilon \cos \varepsilon$, where P is Newcomb's "precession constant." It is a function of the mechanical ellipticity of the earth and the elements of the orbits of the earth and moon. P is not strictly a constant, but has a small secular term of $-0''.0036$ per century due mainly to a secular change in the earth's eccentricity. The speed of geodesic precession in the plane of the fixed ecliptic is $-pg$: hence the speed of the celestial pole toward the mean equinox of date is $-pg \sin \varepsilon$. Thus the speed of the celestial pole toward the mean equinox of date is given by

$$(P \cos \varepsilon - pg) \sin \varepsilon \tag{2}$$

Knowing that the celestial pole P at T_1 moves toward the equinox γ of T_1 with speed $(P \cos \varepsilon - pg) \sin \varepsilon$, we can derive expressions for the rates of change of Ψ and ε_1.

Let

$$P' = \text{celestial pole at } T_1 + \Delta T$$
$$C' = \text{ecliptic pole at } T_1 + \Delta T$$



Then

$$CP = \varepsilon$$

$$C_0P = \varepsilon_1$$

$C_0P' = \varepsilon_1'$ (obliquity of equator of $T_1 + \Delta T$ on ecliptic of T_0)

$$< P_0C_0P = \Psi$$

$$< PC_0P' = d\Psi$$

$$< C_0PC = \lambda$$

$$PP' = (P \cos \varepsilon - pg) \sin \varepsilon \, dt$$

$< CPP' = 90°$ (pole P moves in circle about C)

If one draws a small circle about C_0 through the point P, intersecting the arc C_0P' at D, then

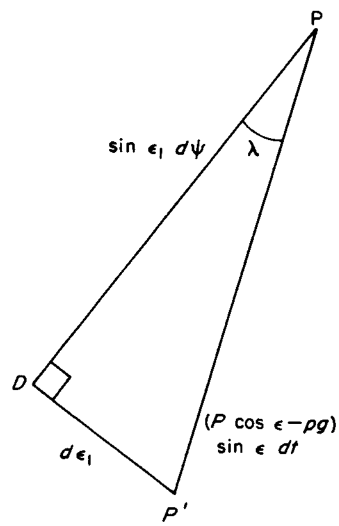
$$C_0PD = 90°$$
$$C_0DP = 90°$$

and arc PD is of length $d\Psi \sin \varepsilon_1$. Thus

$$< DPP' = \lambda$$

$$P'D = d\varepsilon_1$$

$$< PC_0P' = d\Psi$$
$$PD = \sin \varepsilon_1 \, d\Psi$$
$$PP' = (P \cos \varepsilon - pg) \sin \varepsilon \, dt$$



and subsequently

$$\frac{\sin \varepsilon_1 \, d\Psi}{\sin\left(\frac{\pi}{2} - \lambda\right)} = \frac{(P \cos \varepsilon - pg) \sin \varepsilon \, dt}{\sin \frac{\pi}{2}} = \frac{d\varepsilon_1}{\sin \lambda}$$

$$\sin \varepsilon_1 \frac{d\Psi}{dt} = (P \cos \varepsilon - pg) \sin \varepsilon \cos \lambda \tag{3}$$

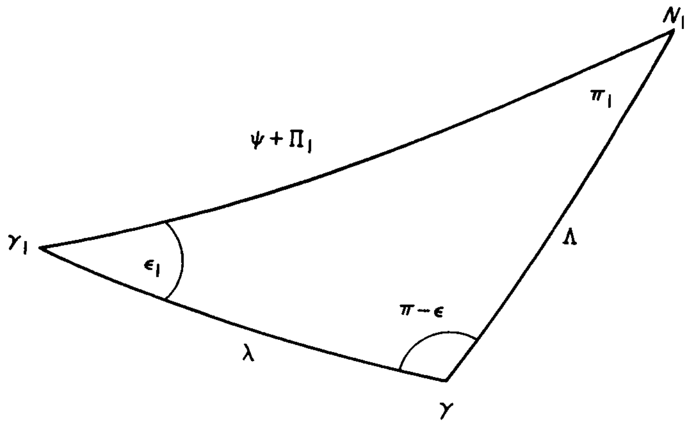
$$\frac{d\varepsilon_1}{dt} = (P \cos \varepsilon - pg) \sin \varepsilon \sin \lambda \tag{4}$$

In the triangle $\gamma\gamma_1N_1$, differential spherical trigonometry yields

$$d\varepsilon = \cos \lambda \, d\varepsilon_1 + \cos \Lambda \, d\pi_1 - \sin \lambda \sin \varepsilon_1 \, d\Psi$$
$$- \sin \Lambda \sin \pi_1 \, d\Pi_1$$

or

$$\frac{d\varepsilon}{dt} = \cos \lambda \frac{d\varepsilon_1}{dt} + \cos \Lambda \frac{d\pi_1}{dt} - \sin \lambda \sin \varepsilon_1 \frac{d\Psi}{dt}$$
$$- \sin \Lambda \sin \pi_1 \frac{d\Pi_1}{dt}$$



Using the expressions (3) and (4) in the above, one obtains

$$\frac{d\varepsilon}{dt} = \cos\Lambda \frac{d\pi_1}{dt} - \sin\Lambda \sin\pi_1 \frac{d\Pi_1}{dt}$$

which may be written as

$$\frac{d\varepsilon}{dt} = \cos(\Lambda - \pi_1)\frac{d}{dt}(\sin\pi_1 \cos\Pi_1) - \sin(\Lambda - \Pi_1) \times \frac{d}{dt}(\sin\pi_1 \sin\Pi_1) + 2\sin^2\frac{\pi_1}{2}\cos\Lambda\frac{d\pi_1}{dt} \qquad (5)$$

We also have the following relations:

$$\frac{\sin\lambda}{\sin\pi_1} = \frac{\sin(\Pi_1 + \Psi)}{\sin\varepsilon} = \frac{\sin\Lambda}{\sin\varepsilon_1}$$

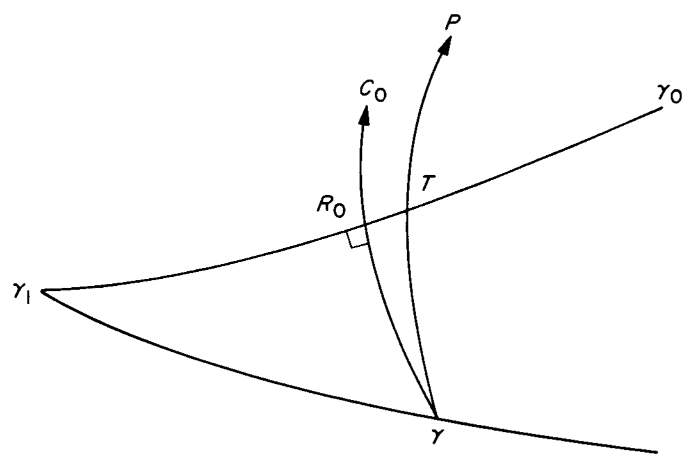
$$\left.\begin{aligned} \cos\varepsilon &= \cos\varepsilon_1 \cos\pi_1 - \sin\varepsilon_1 \sin\pi_1 \cos(\Pi_1 + \Psi) \\ \sin\lambda\cos\varepsilon &= -\sin\Lambda\cos(\Pi_1 + \Psi) + \cos\Lambda\cos\pi_1 \sin(\Pi_1 + \Psi) \\ \tan\frac{\Lambda - \Pi_1 - \Psi}{2} &= -\tan\frac{\lambda}{2}\,\frac{\cos\frac{\varepsilon + \varepsilon_1}{2}}{\cos\frac{\varepsilon - \varepsilon_1}{2}} \end{aligned}\right\} \qquad (6)$$

IV. General Precession in Longitude

The general precession in longitude is a result of the luni-solar precession westward along the fixed ecliptic of T_0 and an eastward motion of the equinox along the moving equator of T_1 which is known as planetary precession. There are several measures of the general precession in longitude. One definition is that the general precession in longitude is the difference between the great circle arcs joining the equinox of T_1 to the node N_1 and the equinox of T_0 to N_1—i.e., $\Lambda - \Pi_1$. This definition is used by Andoyer, Tisserand, and Chauvenet, among others.

Another definition is that of Newcomb. He defines the general precession in longitude as the motion of the mean equinox of T_1 along the moving ecliptic of T_1 and adopts as its measurement the orthogonal projection of this motion onto the fixed ecliptic of T_0 (i.e., intersection of great circle joining γ and C_0 with ecliptic of T_0).

Thus Andoyer's general precession in longitude is $\gamma_0 T$ in the figure below, whereas Newcomb's is $\gamma_0 R_0$.



The numerical difference between the two expressions is about $0''.001\,T^2$.

The Newcomb definition may be then defined as the longitude of the mean equinox of T_1 referred to the fixed mean equinox and ecliptic of T_0 (measured westward).

One usually expresses Newcomb's "precession constant" as

$$P = P_0 + P_1 T_1 \qquad (7)$$

The expression for P_1 is given by deSitter and Brouwer (Ref. 3).

From the theory of secular perturbations in planetary theory one obtains the quantities (see Appendix A):

$$\left.\begin{aligned}\sin\pi_1 \sin \Pi_1 &= sT_1 + s'T_1^2 + s''T_1^3\\ \sin\pi_1 \cos \Pi_1 &= cT_1 + c'T_1^2 + c''T_1^3\end{aligned}\right\} \qquad (8)$$

where T_1 is the time in tropical centuries from some basic epoch.

Using the observationally determined values of the speed of general precession in longitude at T_0, the obliquity ε_0 at T_0, the values P_1 and pg (also given by deSitter and Brouwer), the expressions $\sin\pi_1 {}^{\sin}_{\cos}\,\Pi_1$ and the theoretical relations given by Eqs. (3)–(6), one can determine $\Lambda - \Pi_1$, λ, Ψ, ε, ε_1, etc.

Set

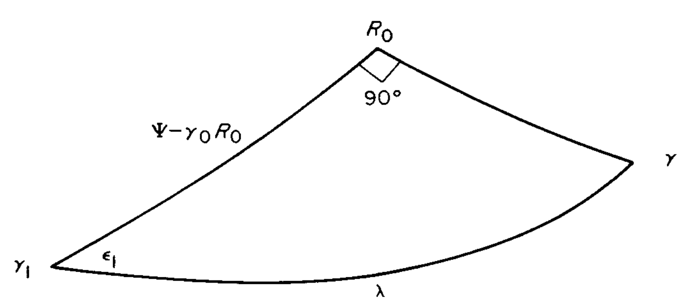
$$\left.\begin{aligned}\varepsilon &= \varepsilon_0 + aT_1 + a'T_1^2 + a''T_1^3\\ \varepsilon_1 &= \varepsilon_0 + b'T_1^2 + b''T_1^3\\ \Psi &= fT_1 + f'T_1^2 + f''T_1^3\\ \lambda &= gT_1 + g'T_1^2 + g''T_1^3\\ \Lambda - \Pi_1 &= hT_1 + h'T_1^2 + h''T_1^3\end{aligned}\right\} \qquad (9)$$

Using Eqs. (3)–(9) we get, similarly to Andoyer (Ref. 4),

$$\left.\begin{aligned}
&a = c, \qquad f = P_0\cos\varepsilon_0 - pg, \qquad g = s\csc\varepsilon_0, \qquad h = f - g\cos\varepsilon_0\\
&a' = c' - \frac{sh}{2}, \qquad b' = \frac{sf}{2}, \qquad f' = \frac{1}{2}P_1\cos\varepsilon_0 + \frac{1}{2}cP_0\cos 2\varepsilon_0\csc\varepsilon_0 - \frac{1}{2}cpg\cot\varepsilon_0\\
&g' = (s' + ch)\csc\varepsilon_0, \qquad h' = f' - g'\cos\varepsilon_0 + \frac{sc}{2}\\
&a'' = c'' - \frac{1}{3}(2s'h + sh') - \frac{c}{6}(h^2 - s^2 - c^2) \qquad b'' = \frac{\sin\varepsilon_0}{3}(2f'g + fg')\\
&f'' = \frac{aP_1}{3}\cos 2\varepsilon_0\csc\varepsilon_0 + \frac{P_0}{3}\left[a'\cos 2\varepsilon_0\csc\varepsilon_0 - b'\cos^2\varepsilon_0\csc\varepsilon_0 - \left(2a^2 + \frac{g^2}{2}\right)\cos\varepsilon_0\right]\\
&\qquad + \frac{pg}{3}\left[(b' - a')\cot\varepsilon_0 + \frac{1}{2}(a^2 + g^2)\right]\\
&g'' = \left(s'' + c'h + ch' - sb'\cot\varepsilon_0 - \frac{sh^2}{2}\right)\csc\varepsilon_0 + \frac{g^3}{6}\\
&h'' = f'' - g''\cos\varepsilon_0 + \frac{\sin\varepsilon_0}{2}[(a' + b')g + ag'] - \frac{s^3}{12}\cot\varepsilon_0
\end{aligned}\right\} \qquad (10)$$

Since h, pg, and ε_0 are given, we can find P_0 from $P_0 = (h + pg + s\cot\varepsilon_0)\cdot(\sec\varepsilon_0)$ and then all the other quantities.

To find the relation between Newcomb's and Andoyer's measures of general precession in longitude consider the triangle $\gamma_1 R_0 \gamma$



As mentioned earlier, Newcomb's general precession in longitude is $\gamma_0 R_0$ and Andoyer's is $\Lambda - \Pi_1$. Since $\gamma_1\gamma_0 = \Psi$, then $\gamma_1 R_0 = \Psi - \gamma_0 R_0$. Then in the preceding figure we have

$$\tan(\Psi - \gamma_0 R_0) = \cos\varepsilon_1 \tan\lambda$$

Using the expressions

$$\varepsilon_1 = \varepsilon_0 + b'T_1^2 + b''T_1^3 \qquad \Psi = fT_1 + f'T_1^2 + g''T_1^3$$

$$\lambda = gT_1 + g'T_1^2 + g''T_1^3 \qquad \Lambda - \Pi_1 = hT_1 + h'T_1^2 + h''T_1^3$$

and supposing that

$$\gamma_0 R_0 = \alpha T_1 + \alpha' T_1^2 + \alpha'' T_1^3$$

we easily find

$$\left.\begin{aligned} \alpha &= h \\ \alpha' &= h' - \frac{sc}{2} \\ \alpha'' &= h'' + \frac{1}{2}h(s^2 - c^2) - \frac{1}{2}(sc' + cs') \end{aligned}\right\} \tag{11}$$

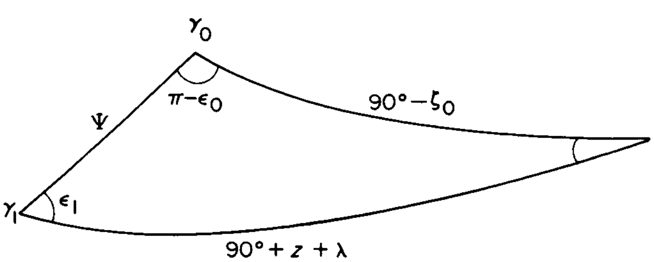
So

$$\gamma_0 R_0 = \Lambda - \Pi_1 - \frac{sc}{2}T_1^2 + \left[\frac{1}{2}h(s^2 - c^2) - \frac{1}{2}(sc' + cs')\right]T_1^3 \tag{12}$$

V. Equatorial Precession Elements

We now shall consider the effects of precession in the equatorial frame and the determination of the quantities ζ_0, z, θ, which are used in (1).

From Fig. 1, the triangle $\Delta\gamma_1\gamma_0 Q$ gives the differential relation



$$\frac{d\theta}{dt} = \sin(90° + z + \lambda)\sin\varepsilon_1 \frac{d\Psi}{dt} - \cos(90° - \zeta_0)\frac{d(-\varepsilon_0)}{dt} - \cos(90° + z + \lambda)\frac{d\varepsilon_1}{dt}$$

and inserting $\sin\varepsilon_1 (d\Psi/dt)$ from (3) and $d\varepsilon_1/dt$ from (4) we find

$$\frac{d\theta}{dt} = (P\cos\varepsilon - pg)\sin\varepsilon\cos z \tag{13}$$

Similarly, from $\sin Adb = \cos c \sin Bda + \sin cdB + \cos A \sin bdC$,

Or, if one writes $\rho = 90° - \zeta_0$, $\mu = 90° + z$ we have

$$\frac{d\theta}{dt} = (P\cos\varepsilon - pg)\sin\varepsilon\sin\mu \tag{15}$$

$$\sin\theta\frac{d\rho}{dt} = (P\cos\varepsilon - pg)\sin\varepsilon\cos\mu \tag{16}$$

$$\sin\theta\frac{d\zeta_0}{dt} = (P\cos\varepsilon - pg)\sin\varepsilon\sin z \tag{14}$$

In $\Delta\gamma_1\gamma_0 Q$ we also have the following relations

$$\left.\begin{aligned} &\frac{\sin\theta}{\sin\Psi} = \frac{\sin\varepsilon_0}{\sin(\mu+\lambda)} = \frac{\sin\varepsilon_1}{\sin\rho} \\ &\cos\theta = \cos\varepsilon_0\cos\varepsilon_1 + \sin\varepsilon_0\sin\varepsilon_1\cos\Psi \\ &\sin\Psi\cos\varepsilon_0 = -\sin\rho\cos(\mu+\lambda) + \cos\rho\cos\theta\sin(\mu+\lambda) \\ &\tan\frac{\mu+\rho+\lambda}{2} = \frac{-\tan\frac{\Psi}{2}\sin\frac{1}{2}(\varepsilon_1+\varepsilon_0)}{\sin\frac{1}{2}(\varepsilon_1-\varepsilon_0)} \\ &\tan\frac{\mu-\rho-\lambda}{2} = \frac{\tan\frac{\Psi}{2}\cos\frac{1}{2}(\varepsilon_1+\varepsilon_0)}{\cos\frac{1}{2}(\varepsilon_1-\varepsilon_0)} \end{aligned}\right\} \tag{17}$$

If one supposes that

$$\mu + \rho = 180^\circ + u'T_1^2 + u''T_1^3 = 180^\circ + z - \zeta_0$$

$$\mu - \rho = rT_1 + r'T_1^2 + r''T_1^3 = z + \zeta_0$$

$$\theta = wT_1 + w'T_1^2 + w''T_1^3$$

from (15)–(17) one finds, similarly to Andoyer,

$$\left.\begin{aligned}
&r = f\cos\varepsilon_0 - g\,, \qquad w = f\sin\varepsilon_0 \\
&u' = \frac{f'g - fg'}{3f}\,, \qquad r' = f'\cos\varepsilon_0 - g'\,, \qquad w' = f'\sin\varepsilon_0 \\
&r'' = f''\cos\varepsilon_0 - g'' + \frac{f^2\sin^2\varepsilon_0}{12}(f\cos\varepsilon_0 - 3g) \\
&w'' = f''\sin\varepsilon_0 + \frac{f\sin\varepsilon_0}{24}(3g^2 + 6fg\cos\varepsilon_0 - f^2\cos^2\varepsilon_0)
\end{aligned}\right\} \tag{18}$$

The quantity u'' will be found later.

Thus, with Eqs. (10) and (11) one can find ε, ε_1, Ψ, λ, $\Lambda - \Pi_1$ and Newcomb's expression for the general precession in longitude for time T_1 from fundamental epoch T_0. And from Eqs. (18) one can find $z - \zeta_0$, $z + \zeta_0$, and θ for epoch T_1 referred to epoch T_0. From (18) we also have

$$\left.\begin{aligned}
&\zeta_0 = \frac{1}{2}rT_1 + \frac{1}{2}(r' - u')T_1^2 + \frac{1}{2}(r'' - u'')T_1^3 \\
&z = \frac{1}{2}rT_1 + \frac{1}{2}(r' + u')T_1^2 + \frac{1}{2}(r'' + u'')T_1^3 \\
&\theta = wT_1 + w'T_1^2 + w''T_1^3
\end{aligned}\right\} \tag{19}$$

for equatorial precession parameters.

VI. Precession from Arbitrary Epoch

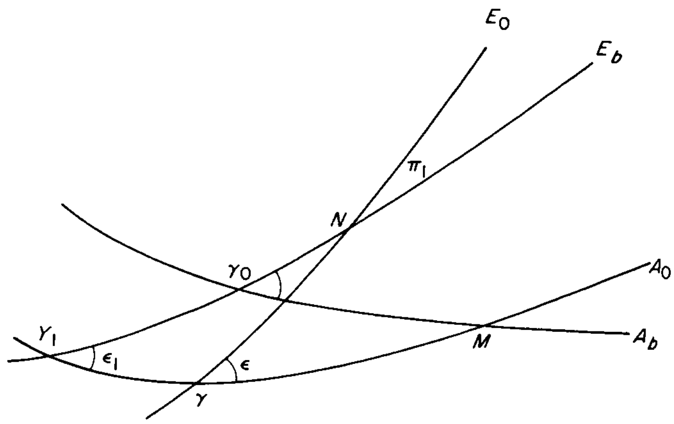
Frequently, one wants the precession quantities referred to some abitrary epoch. In the preceding sections, we expressed the quantities from the epoch which is associated with the values of $\sin\pi_1 {}^{\sin}_{\cos}\Pi_1$, ε_0, etc. We now shall derive the quantities for any arbitrary epoch. Let T_0 refer to the basic epoch to which the quantities $\sin\pi_1 {}^{\sin}_{\cos}\Pi_1$, ε_0 refer (1900, say), T_1 the new fundamental epoch from which we want to express the precessional quantities and T the epoch of date from T_1 in some unit of time (centuries, say). Let T' be the elapsed time from basic epoch to date.

Then $T' = T + T_1$ is the elapsed time in centuries from the original basic epoch to the epoch of date.

What we essentially wish to do is to derive the quantities a, a', etc., for the arbitrary epoch T_1. In other words, we want $\sin\pi_1 {}^{\sin}_{\cos}\Pi_1$ in terms of T from T_1 rather than T' from T_0.

Let a prime denote the value of π_1, Π_1, ε, ε_1 for the epoch of date on the original basic epoch (1900). Let a bar denote the quantities for the epoch of date on the new arbitrary epoch (hereafter called the fundamental epoch), and let no bar or prime refer to the quantities for the fundamental epoch referred to the basic epoch.

From Fig. 1 we have the following equator–ecliptic configurations for the basic epoch and the new fundamental epoch.



where

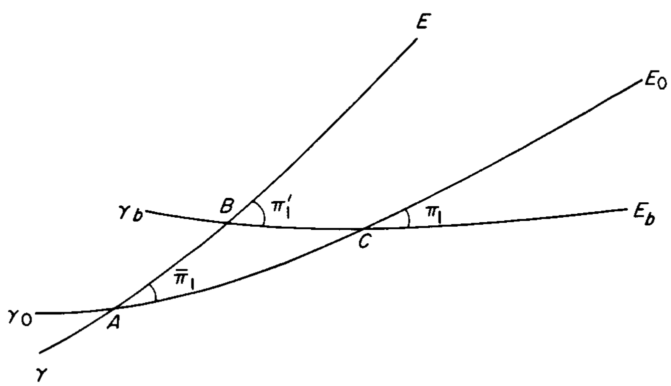
E_0 = ecliptic at fundamental epoch

E_b = ecliptic at basic (1900) epoch

A_0 = equator at fundamental epoch

A_b = equator at zero epoch

Then for three dates (basic, fundamental, date) we have the following ecliptic configuration



where

E_b = ecliptic at basic epoch (1900)

E_0 = ecliptic at fundamental epoch (arbitrary e.g. 1950) (T_1 from T_0)

E = ecliptic of date (T from T_1)

γ_b = equinox at time T_0 (basic)

γ_0 = equinox at time T_1 from T_0 (new fundamental epoch)

γ = equinox of date (T from T_1 or $T' = T + T_1$ from T_0)

π_1 = angle between ecliptics of T_0 and T_1 = $< E_0CE_b$

π_1' = angle between ecliptics of T' and T_0 = $< EBC$

$\bar{\pi}_1$ = angle between ecliptics of T and T_1 = $< EAE_0$

Then

Π_1 = basic equinox to fundamental node on basic ecliptic = $\gamma_b C$

Π_1' = basic equinox to date node on basic ecliptic = $\gamma_b B$

$\bar{\Pi}_1$ = fundamental equinox to date node on fundamental ecliptic = $\gamma_0 A$

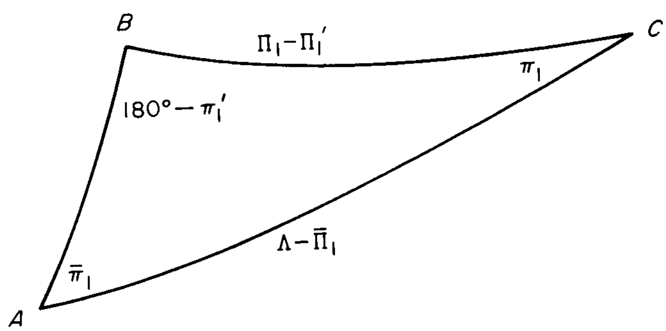
Λ = fundamental equinox to fundamental node on basic ecliptic = $\gamma_0 C$

Then

$$BC = \gamma_b C - \gamma_b B = \Pi_1 - \Pi_1'$$

$$AC = \gamma_0 C - \gamma_0 A = \Lambda - \bar{\Pi}_1'$$

So the triangle formed by the three equinoxes is



Hence

$$\begin{aligned} \sin \bar{\pi}_1 \sin (\bar{\Pi}_1 - \Lambda) &= \sin \pi_1' \sin (\Pi_1' - \Pi_1) \\ \sin \bar{\pi}_1 \cos (\bar{\Pi}_1 - \Lambda) &= \sin \pi_1' \cos \pi_1 \cos (\Pi_1' - \Pi_1) - \cos \pi_1' \sin \pi_1 \end{aligned} \tag{20}$$

which may be written

$$\left.\begin{aligned}\sin\bar{\pi}_1\sin\bar{\Pi}_1 &= \sin\pi_1'\sin(\Pi_1'+\Lambda-\Pi_1)-\sin\pi_1\sin\Lambda\\ &\quad+2\sin\Lambda\sin\pi_1\sin^2\frac{\pi_1'}{2}-2\sin\Lambda\sin\pi_1'\cdot\\ &\quad\sin^2\frac{\pi_1}{2}\cos(\Pi_1'-\Pi_1)\\ \sin\bar{\pi}_1\cos\bar{\Pi}_1 &= \sin\pi_1'\cos(\Pi_1'+\Lambda-\Pi_1)-\sin\pi_1\cos(\Pi_1+\Lambda-\Pi_1)\\ &\quad+2\cos\Lambda\sin\pi_1\sin^2\frac{\pi_1'}{2}-2\cos\Lambda\sin\pi_1'\cdot\\ &\quad\sin^2\frac{\pi_1}{2}\cos(\Pi_1'-\Pi_1)\end{aligned}\right\}\qquad(21)$$

Now

T_1 = time in centuries of new fundamental epoch from basic epoch

T = time in centuries of date epoch from fundamental epoch

T' = time in centuries of date from basic epoch $= T_1 + T$

So

$$T = T' - T_1$$

Now we know that

$$\left.\begin{aligned}\sin\pi_1\sin\Pi_1 &= sT_1 + s'T_1^2 + s''T_1^3\\ \sin\pi_1\cos\Pi_1 &= cT_1 + c'T_1^2 + c''T_1^3\end{aligned}\right.\quad \textit{Fundamental epoch from basic epoch}$$

$$\left.\begin{aligned}\sin\pi_1'\sin\Pi_1' &= s(T_1+T) + s'(T_1+T)^2 + s''(T_1+T)^3\\ \sin\pi_1'\cos\Pi_1' &= c(T_1+T) + c'(T_1+T)^2 + c''(T_1+T)^3\end{aligned}\right.\quad \textit{Date from basic epoch}$$

and we desire to find

$$\left.\begin{aligned}\sin\bar{\pi}_1\sin\bar{\Pi}_1 &= \bar{s}T + \bar{s}'T^2 + \bar{s}''T^3\\ \sin\bar{\pi}_1\cos\bar{\Pi}_1 &= \bar{c}T + \bar{c}'T^2 + \bar{c}''T^3\end{aligned}\right.\quad \textit{Date from fundamental epoch}$$

We also know

$$\Lambda - \Pi_1 = hT_1 + h'T_1^2 + h''T_1^3$$

Inserting the above quantities into (21) we get

$$\left.\begin{aligned}\sin\bar{\pi}_1\sin\bar{\Pi}_1 &= \bar{s}T + \bar{s}'T^2 + \bar{s}''T^3\\ \sin\bar{\pi}_1\cos\bar{\Pi}_1 &= \bar{c}T + \bar{c}'T^2 + \bar{c}''T^3\end{aligned}\right\}\qquad(22)$$

where

$$
\left.\begin{array}{lll}
\bar{s} = s + s_1 T_1 + s_2 T_1^2 & \bar{s}' = s' + s_1' T_1 & \bar{s}'' = s'' \\
\bar{c} = c + c_1 T_1 + c_2 T_1^2 & \bar{c}' = c' + c_1' T_1 & \bar{c}'' = c''
\end{array}\right\}
$$

with

$$
\left.\begin{array}{l}
s_1 = 2s' + ch \\
s_2 = 3s'' + 2c'h + ch' - \dfrac{s}{2}(h^2 - s^2 - c^2) \\
s_1' = 3s'' + c'h + \dfrac{s}{2}(s^2 + c^2) \\
c_1 = 2c' - sh \\
c_2 = 3c'' - 2s'h - sh' - \dfrac{c}{2}(h^2 - s^2 - c^2) = 3a'' \\
c_1' = 3c'' - s'h + \dfrac{c}{2}(s^2 + c^2)
\end{array}\right\} \qquad (23)
$$

Thus if we have $\sin \pi_1 \, {}^{\sin}_{\cos} \Pi_1$ for time T from basic (1900) epoch and if we have h and h' for $\Lambda - \Pi_1$ for T_1 from basic epoch, we can compute $\sin \bar{\pi}_1 \, {}^{\sin}_{\cos} \bar{\Pi}_1$ for time T from arbitrary epoch T_1 via (23).

Now that we have

$$\sin \bar{\pi}_1 \sin \bar{\Pi}_1 = \bar{s}T + \bar{s}'T^2 + \bar{s}''T^3$$

$$\sin \bar{\pi}_1 \cos \bar{\Pi}_1 = \bar{c}T + \bar{c}'T^2 + \bar{c}''T^3$$

and knowing that

$$\bar{P}_0 = P_0 + P_1 T_1$$

$$\bar{P}_1 = P_1$$

$$\bar{\varepsilon}_0 = \varepsilon(T_1) = \varepsilon_0 + aT_1 + a'T_1^2 + a''T_1^3 \qquad (23b)$$

one can compute $\bar{\varepsilon}$, $\bar{\varepsilon}_1$, $\bar{\Psi}$, $\bar{\lambda}$, $\bar{\Lambda} - \bar{\Pi}_1$ by a process similar to that in the first part of the paper.

If

$$\bar{\varepsilon} = \bar{\varepsilon}_0 + \bar{a}T + \bar{a}'T^2 + \bar{a}''T^3$$

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_0 + \bar{b}'T^2 + \bar{b}''T^3$$

$$\bar{\Psi} = \bar{f}T + \bar{f}'T^2 + \bar{f}''T^3$$

$$\bar{\Lambda} - \bar{\Pi}_1 = \bar{h}T + \bar{h}'T^2 + \bar{h}''T^3$$

For Newcomb's general précession in longitude, expressed in the form $\bar{\alpha}T + \bar{\alpha}'T^2 + \bar{\alpha}''T^3$, we find

$$\bar{a} = \bar{c} = c + c_1 T_1 + c_2 T_1^2$$

$$\bar{f} = f + f_1 T_1 + f_2 T_1^2$$

$$\bar{g} = g + g_1 T_1 + g_2 T_1^2$$

$$\bar{h} = h + h_1 T_1 + h_2 T_1^2$$

Also $\bar{\alpha} = \alpha + \alpha_1 T_1 + \alpha_2 T_1^2$

$$
\begin{array}{ll}
\bar{a}' = a' + a_1' T_1 & \bar{a}'' = a'' \\
\bar{b}' = b' + b_1' T_1 & \bar{b}'' = b'' \\
\bar{f}' = f' + f_1' T_1 & \bar{f}'' = f'' \\
\bar{g}' = g' + g_1 T_1 & \bar{g}'' = g'' \\
\bar{h}' = h' + h_1' T_1 & \bar{h}'' = h'' \\
\bar{\alpha}' = \alpha' + \alpha_1' T_1 & \bar{\alpha}'' = \alpha''
\end{array}
$$

where $\bar{\varepsilon}_0$, $\bar{P}_1$, $\bar{P}_0$ are given in (23b); where s_1, s_2, s_1', c_1, c_2, c_1' are given in (23); where a, f, g, h, a', b', etc., are given in (10) and (11), and where

$$
\left.
\begin{aligned}
f_1 &= -aP_0 \sin\varepsilon_0 + P_1 \cos\varepsilon_0 \\
f_2 &= -\frac{1}{2}a^2P_0\cos\varepsilon_0 - a'P_0\sin\varepsilon_0 - aP_1\sin\varepsilon_0 \\
g_1 &= s_1\csc\varepsilon_0 - ag\cot\varepsilon_0 \\
g_2 &= s_2\csc\varepsilon_0 - ag_1\cot\varepsilon_0 + g\left(\frac{1}{2}a^2 - a'\cot\varepsilon_0\right) \\
h_1 &= f_1 - g_1\cos\varepsilon_0 + ag\sin\varepsilon_0 \\
h_2 &= f_2 - g_2\cos\varepsilon_0 + ag_1\sin\varepsilon_0 + a'g\sin\varepsilon_0 + \frac{1}{2}a^2g\cos\varepsilon_0 \\
\alpha_1 &= h_1 \\
\alpha_2 &= h_2 \\
a_1' &= c_1' - \frac{1}{2}(s_1h + sh_1) \\
b_1' &= \frac{1}{2}(sf_1 + s_1f) \\
f_1' &= \frac{1}{2}P_0\csc\varepsilon_0\,[c_1\cos 2\varepsilon_0 - a^2(\cot\varepsilon_0 + \sin 2\varepsilon_0)] \\
&\quad + \frac{1}{2}aP_1\csc\varepsilon_0(\cos 2\varepsilon_0 - \sin^2\varepsilon_0) \\
&\quad + \frac{1}{2}pg\csc\varepsilon_0(a^2\csc\varepsilon_0 - c_1\cos\varepsilon_0) \\
g_1' &= (s_1' + ch_1 + c_1h)\csc\varepsilon_0 - a\cot\varepsilon_0\csc\varepsilon_0(s' + ch) \\
h_1' &= f_1' + ag'\sin\varepsilon_0 - g_1'\cos\varepsilon_0 + \frac{1}{2}s_1c + \frac{1}{2}c_1s \\
\alpha_1' &= h_1' - \frac{1}{2}(s_1c + c_1s)
\end{aligned}
\right\}
\qquad (25)
$$

Thus, the planetary precession from arbitrary epoch T_1 to date T is

$$\bar{\lambda} = (g + g_1T_1 + g_2T_1^2)\,T + (g' + g_1'T_1)\,T^2 + g''T^3$$

and similarly for the luni-solar precession.

For the equatorial precessional elements we have

$$
\begin{aligned}
\bar{r} &= r + r_1T_1 + r_2T_1^2 \\
\bar{w} &= w + w_1T_1 + w_2T_1^2 \\
\bar{r}' &= r' + r_1'T_1 \\
\bar{w}' &= w' + w_1'T_1 \\
\bar{u}' &= u' + u_1'T_1 \\
\bar{r}'' &= r'' \\
\bar{w}'' &= w'' \\
\bar{u}'' &= u''
\end{aligned}
$$

where r, w, r', w', u', r'', w'' are given by (18) and

$$\left.\begin{aligned}
r_1 &= f_1 \cos \varepsilon_0 - af \sin \varepsilon_0 - g_1 \\
r_2 &= f_2 \cos \varepsilon_0 - af_1 \sin \varepsilon_0 - g_2 - \frac{1}{2} a^2 f \cos \varepsilon_0 - a' f \sin \varepsilon_0 \\
w_1 &= f_1 \sin \varepsilon_0 + af \cos \varepsilon_0 \\
w_2 &= f_2 \sin \varepsilon_0 + af_1 \cos \varepsilon_0 + a' f \cos \varepsilon_0 - \frac{1}{2} a^2 f \sin \varepsilon_0 \\
u'_1 &= \frac{1}{3f}\left[g\left(f'_1 - \frac{f_1}{f} f'\right) + f' g_1 - f g'_1 \right] \\
r'_1 &= f' \cos \varepsilon_0 - af' \sin \varepsilon_0 - g'_1 \\
w'_1 &= f'_1 \sin \varepsilon_0 - af' \cos \varepsilon_0 \\
u'' &= \frac{1}{2} u'_1
\end{aligned}\right\} \quad (26)$$

Note that

$$w' = \frac{1}{2} w_1 \qquad r' = \frac{1}{2} r_1$$

$$w_2 = w'_1 \qquad r_2 = r'_1$$

The quantity u'' (which was not computed earlier) may be determined by the fact that since $\mu + \rho = 180° + u'T_1^2 + u''T_1^3$ gives $\mu + \rho$ at T_1 from zero epoch and $\bar{\mu} + \bar{\rho} = 180° + (u' + u'_1 T_1) T^2 + u'' T^3$ gives $\bar{\mu} + \bar{\rho}$ at T from T_1 epoch, then if $T = -T_1$ we have $\mu + \rho$ for zero from T_1 epoch $= \mu + \rho (T_1)$.

Thus we get

$$\mu + \rho (T_1 \text{ from } 0) = 180° + u'T_1^2 + u''T_1^3$$

$$\bar{\mu} + \bar{\rho} (0 \text{ from } T_1) = 180° + u'T_1^2 + (u'_1 - u'') T_1^3$$

or

$$u'' = \frac{1}{2} u'_1$$

Finally, we get

$$\left.\begin{aligned}
\zeta_0 &= \frac{1}{2}(r + r_1 T_1 + r_2 T_1^2) T + \frac{1}{2}[(r' - u') + (r'_1 - u'_1) T_1] T^2 + \frac{1}{2}(r'' - u'') T^3 \\
z &= \frac{1}{2}(r + r_1 T_1 + r_2 T_1^2) T + \frac{1}{2}[(r' + u') + (r'_1 + u'_1) T_1] T^2 + \frac{1}{2}(r'' + u'') T^3 \\
\theta &= (w + w_1 T_1 + w_2 T_1^2) T + (w' + w'_1 T_1) T^2 + w'' T^3
\end{aligned}\right\} \quad (27)$$

where all the quantities are previously given.

We now have explicit expressions for precession quantities and their relationship to the fundamental quantities P_0 or h, ε_0 and the system of planetary masses. Hence, we can compute partial derivatives of these derived quantities with respect to the fundamental constants and thus get an idea of how the derived quantities vary with small changes in the fundamental constants. The partial derivatives are given in Appendix B.

The reader may consult Refs. 5–8 for a more detailed discussion of the problems presented by the necessity of determining the precession constants.

Appendix A

Expressions for sin $\pi_1 {}^{\sin}_{\cos} \Pi_1$

In Ref. 8 (p. 377), Newcomb lists values of a quantity $\kappa {}^{\sin}_{\cos} L$ for the planets Mercury through Neptune at the epochs 1600, 1850, 2100. These expressions are the components from each planet of the time derivative of $\pi_1 {}^{\sin}_{\cos} \Pi_1$ per Julian century. Clemence (Ref. 9) lists $d/dt\,(\pi_1 {}^{\sin}_{\cos} \Pi_1)$ for Pluto at the same epochs.

From the tabular values of $d/dt\,(\pi_1 {}^{\sin}_{\cos} \Pi_1)$ for epochs 1600, 1850, 2100 one can form a second-degree polynomial in time for $d/dt\,(\pi_1 {}^{\sin}_{\cos} \Pi_1)$ and by integration one gets $\pi_1 {}^{\sin}_{\cos} \Pi_1$ for time T centuries from 1850. However, since the perturbations $\kappa {}^{\sin}_{\cos} L$ have been found by multiplying a quantity involving the elements by the mass of the disturbing planet, the quantities $d/dt\,(\pi_1 {}^{\sin}_{\cos} \Pi_1)$ will change whenever the system of planetary masses is changed.

If α_{-1}, α_0, α_1 are the quantities $d/dt\,(\pi_1 \sin \Pi_1)$ or $d/dt\,(\pi_1 \cos \Pi_1)$ for the epochs 1600, 1850, 2100, then the value of $\pi_1 \sin \Pi_1$ or $\pi_1 \cos \Pi_1$ is found from

$$\pi_1 {}^{\sin}_{\cos} \Pi_1 = \alpha_0 T_1 + \frac{1}{10}(\alpha_1 - \alpha_{-1})\, T_1^2 + \frac{2}{75}(\alpha_1 - 2\alpha_0 + \alpha_{-1})\, T_1^3 \tag{28}$$

where the unit of time is the Julian Century. If the expressions are desired for tropical centuries, substitute 0.9999 78641 T_1 for T_1 in (28).

If m_i are the reciprocal masses of the planets which one uses in integrations, then from Newcomb's data and system of masses (Ref. 8, p. 336), one has

For 1600 (α_{-1})

$$\begin{aligned}\frac{d}{dt}\pi_1 {}^{\sin}_{\cos} \Pi_1 = {} & \frac{7{,}500{,}000}{m_{Mercury}}\begin{pmatrix}+0''.247\\-0''.212\end{pmatrix} + \frac{410{,}000}{m_{Venus}}\begin{pmatrix}+6.790\\-28.473\end{pmatrix}\\ & + \frac{3{,}093{,}500}{m_{Mars}}\begin{pmatrix}+0.617\\-0.735\end{pmatrix} + \frac{1047.88}{m_{Jupiter}}\begin{pmatrix}-2.804\\-16.170\end{pmatrix} + \frac{3501.6}{m_{Saturn}}\begin{pmatrix}-0.574\\-1.310\end{pmatrix}\\ & + \frac{22{,}756}{m_{Uranus}}\begin{pmatrix}+0.002\\-0.008\end{pmatrix} + \frac{19{,}540}{m_{Neptune}}\begin{pmatrix}-0.004\\-0.004\end{pmatrix} + \frac{360{,}000}{m_{Pluto}}\begin{pmatrix}-0.0004\\-0.0012\end{pmatrix}\end{aligned} \tag{29}$$

For 1850 (α_0)

$$\begin{aligned}\frac{d}{dt}(\pi_1 {}^{\sin}_{\cos} \pi_1) = {} & \frac{7{,}500{,}000}{Mercury}\begin{pmatrix}+0.251\\-0.210\end{pmatrix} + \frac{410{,}000}{Venus}\begin{pmatrix}+7.412\\-28.332\end{pmatrix}\\ & + \frac{3{,}093{,}500}{Mars}\begin{pmatrix}+0.634\\-0.719\end{pmatrix} + \frac{1047.88}{Jupiter}\begin{pmatrix}-2.511\\-16.047\end{pmatrix} + \frac{3501.6}{Saturn}\begin{pmatrix}-0.542\\-1.318\end{pmatrix}\\ & + \frac{22{,}756}{Uranus}\begin{pmatrix}+0.002\\-0.008\end{pmatrix} + \frac{19{,}540}{Neptune}\begin{pmatrix}-0.004\\-0.004\end{pmatrix} + \frac{360{,}000}{Pluto}\begin{pmatrix}-0.0004\\-0.0012\end{pmatrix}\end{aligned} \tag{30}$$

For 2100 (α_1)

$$\begin{aligned}\frac{d}{dt}(\pi_1 {}^{\sin}_{\cos} \Pi_1) = {} & \frac{7{,}500{,}000}{Mercury}\begin{pmatrix}+0.254\\-0.208\end{pmatrix} + \frac{410{,}000}{Venus}\begin{pmatrix}+8.032\\-28.185\end{pmatrix}\\ & + \frac{3{,}093{,}500}{Mars}\begin{pmatrix}+0.651\\-0.703\end{pmatrix} + \frac{1047.88}{Jupiter}\begin{pmatrix}-2.224\\-15.919\end{pmatrix} + \frac{3501.6}{Saturn}\begin{pmatrix}-0.510\\-1.325\end{pmatrix}\\ & + \frac{22{,}756}{Uranus}\begin{pmatrix}+0.003\\-0.008\end{pmatrix} + \frac{19{,}540}{Neptune}\begin{pmatrix}-0.004\\-0.004\end{pmatrix} + \frac{360{,}000}{Pluto}\begin{pmatrix}-0.0004\\-0.0012\end{pmatrix}\end{aligned} \tag{31}$$

where the unit of time is the Julian Century and the quantities are in seconds of arc.

For an example: JPL, at the time of this writing uses Clemence's masses (except for earth–moon) given in Ref. 9.

Mercury	6,000,000
Venus	408,000
Earth–Moon	329,390
Mars	3,093,500
Jupiter	1047.355
Saturn	3501.6
Uranus	22,869.0
Neptune	19,314
Pluto	360,000

Hence using (29), (30), and (31) or taking the quantities from Clemence (Ref. 9, p. 175)

	1600	1850	2100	(per Julian century)
$\frac{d}{dt}(\pi_1 \sin \Pi_1)$	$+4''.3674$	$+5''.3395$	$+6''.3035$	
$\frac{d}{dt}(\pi_1 \cos \Pi_1)$	-47.1143	-46.8390	-46.5518	

and so

$$\pi_1 \sin \Pi_1 = +5.3394 T_1 + 0''.19361 T_1^2 - 0''.000216 T_1^3$$

$$\pi_1 \cos \Pi_1 = -46.8380 T_1 - 0''.05625 T_1^2 + 0''.000317 T_1^3$$

where T_1 is in centuries (tropical) from 1850.

In the present work, all expansions are made in terms of $\sin \pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1$ rather than $\tan \pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1$ or $\pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1$, which may sometimes be furnished by planetary theory.

Employing the relations

$$\tan x = \sin x + \frac{1}{2}\sin^3 x$$

$$x = \sin x + \frac{1}{6}\sin^3 x$$

it can easily be shown that if the planetary theory furnishes $\tan \pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1$ then one should substitute

$$s'' - \frac{1}{2}(s^2 + c^2)\, s \text{ for } s''$$

and

$$c'' - \frac{1}{2}(s^2 + c^2)\, c \text{ for } c''$$

in the formulae of the preceding sections.

On the other hand, if the planetary theory furnishes $\pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1$, then one should substitute

$$s'' - \frac{1}{6}(s^2 + c^2)\, s \text{ for } s''$$

and

$$c'' - \frac{1}{6}(s^2 + c^2)\, c \text{ for } c''$$

in the preceding formulae. However, since $s^2 + c^2 \sim 5 \times 10^{-8}$ rad, and s and c are less than $50''$, we have $(s^2 + c^2) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} < 3'' \times 10^{-6}$, which may be ignored since the quantities $d/dt\,(\pi_1 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi_1)$ are only given to 4 or 5 figures anyway.

So from Newcomb's values of $\kappa \frac{\sin}{\cos} L$ we get $\sin \pi_1 \frac{\sin}{\cos} \Pi_1$ as time series from 1850 as shown above. However, usually we update the zero epoch from 1850 to 1900. This involves (22) and (23). But in (23) we used

$$\Lambda - \Pi_1 = h_0 T_1 + h' T_1^2 + h'' T_1^3 \qquad T_1 \text{ from } 1850$$

and if we have

$$\frac{d}{dt}(\Lambda - \Pi_1) = h \text{ at } 1900 \qquad \text{Note } h = h_0 + h' + \frac{3}{4} h''$$

as the speed of general precession in longitude, then in formula (23) we set $h' = 0$, $T_1 = 1/2$ and $h =$ general precession in longitude at 1900 to get $\sin \pi_1 \frac{\sin}{\cos} \Pi_1$ for 1900. Or, one could calculate h' at 1900 by (10), estimate $h_{1850} = h_{1900} - 1/2\, h'_{1900}$ and then recalculate h'_{1850}. By using (23) one then gets $\pi_1 \frac{\sin}{\cos} \Pi_1$ for 1900.

Thus, using Clemence's $\pi_1 \frac{\sin}{\cos} \Pi_1$ for 1850, and taking at 1900 $\varepsilon_0 = 23°27'08\overset{''}{.}26$, $h = 5025.64$, we get, from (23):

$$\sin \pi_1 \sin \Pi_1 = 4.9625T + 0.1940T^2 - 0.00022T^3$$

$$\sin \pi_1 \cos \Pi_1 = -46.845T + 0.0544T^2 + 0.0032T^3$$

as $\sin \pi_1 \frac{\sin}{\cos} \Pi_1$ terms of tropical centuries from 1900.

Thus 1900 now becomes our zero epoch with s, s', s'', c, c', c'' given above, and $\varepsilon_0 = 23°27'08\overset{''}{.}26$, $h = 5025\overset{''}{.}64$.

With P_1, pg at 1900, we can compute all the quantities for time T derived in the paper for arbitrary epoch T_1 tropical (or Julian) centuries from 1900. This is given in Appendix B.

Appendix B

Numerical Values of Precession Quantities

Using the JPL masses (Appendix A) and the following observed quantities for 1900 (per tropical century):

$$h = 5025''.64$$

$$P_1 = -0''.0036$$

$$pg = 1''.921$$

and the values of $\pi_1 {}^{\sin}_{\cos} \Pi_1$ for 1850, we get for T_1 tropical centuries after 1900 and T tropical centuries (date) after T_1 the following values:

P_0 @ 1900,	zero epoch 1900
P_1 @ 1900,	new fundamental epoch T_1
pg @ 1900,	date $\begin{cases} T_1 + T \text{ from } 1900 \\ T \text{ from } T_1 \end{cases}$

$$\sin \pi_1 \sin \Pi_1 = (s + s_1 T_1 + s_2 T_1^2)\, T + (s' + s_1' T_1)\, T^2 + s'' T^3$$

$$\sin \pi_1 \cos \Pi_1 = (c + c_1 T_1 + c_2 T_1^2)\, T + (c' + c_1' T_1)\, T^2 + c'' T^3$$

$$\bar{\varepsilon}_0 \,(\text{at } T_1) = \varepsilon_0\,(1900) + aT_1 + a'T_1^2 + a''T_1^3$$

$$\bar{\varepsilon}\,(\text{at } T) = \varepsilon_0\,(1900) + aT_1 + a'T_1^2 + a''T_1^3 + (a + a_1 T_1 + a_2 T_1^2)\,\mathrm{T} + (a' + a_1' T_1)\, T^2 + a'' T^3$$

$$\bar{\varepsilon}_1\,(T \text{ on } T_1) = \bar{\varepsilon}_0 + (b' + b_1' T_1)\, T^2 + b'' T^3 = \varepsilon_0 + aT_1 + a'T_1^2 + a''T_1^3 + (b' + b_1' T_1)\, T^2 + b'' T^3$$

$$\Psi \text{ luni-solar-geodesic} = (f + f_1 T_1 + f_2 T_1^2)\, T + (f' + f_1' T_1)\, T^2 + f'' T^3$$

$$\text{Planetary} = (g + g_1 T_1 + g_2 T_1^2)\, T + (g' + g_1' T_1) T^2 + g'' T^3$$

Andoyer's general precession in longitude $= \Lambda - \Pi_1$

$$= (h + h_1 T_1 + h_2 T_1^2)\, T + (h' + h_1' T_1)\, T^2 + h'' T^3$$

$$\zeta_0 = (x + x_1 T_1 + x_2 T_1^2)\, T + (x' + x_1' T_1)\, T^2 + x'' T^3$$

$$z = (z_0 + z_1 T_1 + z_2 T_1^2)\, T + (z' + z_1' T_1)\, T^2 + z'' T^3$$

$$\theta = (w + w_1 T_1 + w_2 T_1^2)\, T + (w' + w_1' T_1)\, T^2 + w'' T^3$$

It should be noted that de Sitter's (Ref. 3) and Clemence's (Ref. 9) quantities for p_1, λ, and p correspond to the coefficients of T (first power only) in $\bar{\Psi}$, $\bar{\lambda}$, $\bar{\Lambda} - \bar{\Pi}_1$ in our development. Also, several of Clemence's second-order terms in T_1 are in error. (They use T as time since 1900 whereas we call it T_1.)

In the following tables, the first line contains the coefficients of the powers of T and T_1, using the JPL masses and basic constants given above. Subsequent lines give the partial derivatives with respect to general precession in longitude at 1900, obliquity at 1900, and the system of planetary masses. Units for the quantities are seconds of arc, and for the partial derivatives the corrections for Δh, $\Delta\varepsilon_0$ are assumed to be in seconds of arc while those of the masses are pure numbers $\Delta m/m$. The unit of time is the tropical century. T_1 is the time from 1900.0 to the fundamental epoch (e.g., 1950.0), and T is the time from the fundamental epoch to date.

For the partial derivatives we have listed only the terms which affect the quantities involved to 10^{-4} sec of arc. It was assumed that the reasonable sizes of corrections are:

Δh	$1''$
$\Delta\varepsilon_0$	$\frac{1}{2}''$
θ_1 (Mercury)	10^{-2}
θ_2 (Venus)	2×10^{-3}
θ_3 (Mars)	10^{-2}
θ_4 (Jupiter)	5×10^{-4}
θ_5 (Saturn)	10^{-3}
θ_6 (Uranus)	3×10^{-3}
θ_7 (Neptune)	2×10^{-2}
θ_8 (Pluto)	10^{-1}

Table B-1. $x = \sin \pi_1 \sin \Pi_1$

x	(4″.9624	$-$ 0″.7534T_1	+ 0″.00026T_1^2)T	+ (0″.1940	+ 0″.0007T_1)T^2	$-$ 0″.00022T^3
$\partial x/\partial h$ (″)	($-$0.00011	$-$ 0.00023T_1	0T_1^2)T + (	0	0T_1)T^2	0T^3
$\partial x/\partial \varepsilon_0$ (″)	0	0	0	0	0	0
$\partial x/\partial \theta_1$	0″.311	0	0	0	0	0
$\partial x/\partial \theta_2$	7″.23	$-$0″.44	0	0″.12	0	0
$\partial x/\partial \theta_3$	0″.629	$-$0″.011	0	0	0	0
$\partial x/\partial \theta_4$	$-$2″.65	$-$0″.27	0	0	0	0
$\partial x/\partial \theta_5$	$-$0″.55	0	0	0	0	0
$\partial x/\partial \theta_6$	0	0	0	0	0	0
$\partial x/\partial \theta_7$	0	0	0	0	0	0
$\partial x/\partial \theta_8$	0	0	0	0	0	0

Table B-2. $x = \sin \pi_1 \cos \Pi_1$

x	($-$46″.845	$-$ 0″.0122T_1	+ 0.0054T_1^2)T	+ (0.0544	$-$ 0.0038T_1)T^2	+ 0.0032T^3
$\partial x/\partial h$	0	0	0	0	0	0
$\partial x/\partial \varepsilon_0$	0	0	0	0	0	0
$\partial x/\partial \theta_1$	$-$0″.266	0	0	0	0	0
$\partial x/\partial \theta_2$	$-$28.53	$-$0″.12	0	0	0	0
$\partial x/\partial \theta_3$	$-$0″.723	0	0	0	0	0
$\partial x/\partial \theta_4$	16.00	0″.11	0	0	0	0
$\partial x/\partial \theta_5$	$-$1″.31	0	0	0	0	0
$\partial x/\partial \theta_6$	0	0	0	0	0	0
$\partial x/\partial \theta_7$	0	0	0	0	0	0
$\partial x/\partial \theta_8$	0	0	0	0	0	0

Table B-3. $x = \bar{\varepsilon}_0$ (ε_0 = 23° 27′ 08″.26)

x	ε_0	$-$ 46″.845T_1	$-$ 0.0061T_1^2	+ 0″.0018T_1^3
$\partial x/\partial h$	0	0	0	0
$\partial x/\partial \varepsilon_0$	1	0	0	0
$\partial x/\partial \theta_1$	0	$-$0″.266	0	0
$\partial x/\partial \theta_2$	0	$-$28″.53	0	0
$\partial x/\partial \theta_3$	0	$-$0″.723	0	0
$\partial x/\partial \theta_4$	0	$-$16″.00	0	0
$\partial x/\partial \theta_5$	0	$-$1″.31	0	0
$\partial x/\partial \theta_6$	0	0	0	0
$\partial x/\partial \theta_7$	0	0	0	0
$\partial x/\partial \theta_8$	0	0	0	0

Table B-4. $\bar{\varepsilon} = \bar{\varepsilon}_0 + x$ (see Table 3)

x	($-$46″.845	$-$ 0″.0122T_1	+ 0″.0054T_1^2)T	+ ($-$0″.0061	+ 0″.0054T_1)T^2	+ 0″.0018T^3
$\partial x/\partial h$	0	0	0	0	0	0
$\partial x/\partial \varepsilon_0$	0	0	0	0	0	0
$\partial x/\partial \theta_1$	$-$0″.266	0	0	0	0	0
$\partial x/\partial \theta_2$	$-$28″.53	$-$0″.12	0	0	0	0
$\partial x/\partial \theta_3$	$-$0″.723	0	0	0	0	0
$\partial x/\partial \theta_4$	$-$16″.00	$-$0″.11	0	0	0	0
$\partial x/\partial \theta_5$	$-$1″.31	0	0	0	0	0
$\partial x/\partial \theta_6$	0	0	0	0	0	0
$\partial x/\partial \theta_7$	0	0	0	0	0	0
$\partial x/\partial \theta_8$	0	0	0	0	0	0

Table B-5. $\bar{\varepsilon}_1 = \bar{\varepsilon}_0 + x$ **(see Table 3)**

x	(0.0606	− 0.00919T_1)T^2	− 0.00771T^3
$\partial x/\partial h$	0	0	0
$\partial x/\partial \varepsilon_0$	0	0	0
$\partial x/\partial \theta_1$	0	0	0
$\partial x/\partial \theta_2$	0	0	0
$\partial x/\partial \theta_3$	0	0	0
$\partial x/\partial \theta_4$	0	0	0
$\partial x/\partial \theta_5$	0	0	0
$\partial x/\partial \theta_6$	0	0	0
$\partial x/\partial \theta_7$	0	0	0
$\partial x/\partial \theta_8$	0	0	0

Table B-6. $x = \bar{\Psi}$

x	(5037″079	+ 0″4932T_1	− 0″000066T_1^2)T	+ (−1″0719	− 0″00136T_1)T^2	− 0″00141T^3
$\partial x/\partial h$	0.99974	0	0	−0.0002	0	0
$\partial x/\partial \varepsilon_0$	−0.00015	0	0	0	0	0
$\partial x/\partial \theta_1$	0″718	0	0	0	0	0
$\partial x/\partial \theta_2$	1″66	0″30	0	−0″66	0	0
$\partial x/\partial \theta_3$	1″449	0	0	−0″017	0	0
$\partial x/\partial \theta_4$	−6″11	0″17	0	−0″36	0	0
$\partial x/\partial \theta_5$	−1″27	0	0	0	0	0
$\partial x/\partial \theta_6$	0	0	0	0	0	0
$\partial x/\partial \theta_7$	0	0	0	0	0	0
$\partial x/\partial \theta_8$	0	0	0	0	0	0

Table B-7. $x = \bar{\lambda}$

x	(12″469	−1″8866T_1	− 0″00032T_1^2)T	+ (−2″3805	− 0″00159T_1)T^2	− 0″00157T^3
$\partial x/\partial h$	−0.00028	−0.00057	0	−0.00057	0	0
$\partial x/\partial \varepsilon_0$	−0.00014	0	0	0	0	0
$\partial x/\partial \theta_1$	−0″782	−0″011	0	−0″014	0	0
$\partial x/\partial \theta_2$	18″16	−1″10	0	−1″43	0	0
$\partial x/\partial \theta_3$	1″579	0″026	0	−0″036	0	0
$\partial x/\partial \theta_4$	−6″66	−0″69	0	−0″83	0	0
$\partial x/\partial \theta_5$	−1″39	0	0	0	0	0
$\partial x/\partial \theta_6$	0	0	0	0	0	0
$\partial x/\partial \theta_7$	−0″010	0	0	0	0	0
$\partial x/\partial \theta_8$	−0″001	0	0	0	0	0

Table B-8. General precession in longitude

$x_1 = \Lambda - \Pi_1$ **(Andoyer)**

$x_2 = P$ **(Newcomb)**

x_1	(5025″64	+ 2″2228T_1	+ 0″00040T_1^2)T	+ (1″1114	+ 0″00040T_1)T^2	+ 0.00014T^3
x_2	5025.64	2.2228	0.00040	1.1120	0.00032	0.00003
$\partial x/\partial h$	1.000	0.00062	0	0.00031	0	0
$\partial x/\partial \varepsilon_0$	0	0	0	0	0	0
$\partial x/\partial \theta_1$	0	0″013	0	0	0	0
$\partial x/\partial \theta_2$	0	1″31	0	0″66	0	0
$\partial x/\partial \theta_3$	0	0″032	0	0″016	0	0
$\partial x/\partial \theta_4$	0	0″80	0	0″40	0	0

Note: (a) Partials of x_1 and x_2 are identical; (b) Partials not listed are zero.

Table B-9. $x = \zeta_0$

x	$(2304''253 + 1''3972T_1 + 0''000125T_1^2)T + (0''3023 - 0''000211T_1)T^2 + 0''0180T^3$					
$\partial x/\partial h$	0.45872	0.00038	0	0	0	0
$\partial x/\partial \varepsilon_0$	−0.00486	0	0	0	0	0
$\partial x/\partial \theta_1$	−0″062	0	0	0	0	0
$\partial x/\partial \theta_2$	−1″44	0″83	0	0″18	0	0
$\partial x/\partial \theta_3$	−0″125	0″020	0	0	0	0
$\partial x/\partial \theta_4$	0″53	0″50	0	0″11	0	0
$\partial x/\partial \theta_5$	0.11	0	0	0	0	0

Table B-10. $x = z$

x	$(2304''253 + 1''3972T_1 + 0''000125T_1^2)T + (1''0949 + 0''00046T_1)T^2 + 0''0183T^3$					
$\partial x/\partial h$	0.45872	0.00038	0	0.00028	0	0
$\partial x/\partial \varepsilon_0$	−0.00486	0	0	0	0	0
$\partial x/\partial \theta_1$	−0″062	0	0	0	0	0
$\partial x/\partial \theta_2$	−1″44	0″83	0	0″65	0	0
$\partial x/\partial \theta_3$	−0″125	0″020	0	0″016	0	0
$\partial x/\partial \theta_4$	0″53	0″50	0	0″39	0	0
$\partial x/\partial \theta_5$	0″11	0	0	0	0	0

Table B-11. $x = \theta$

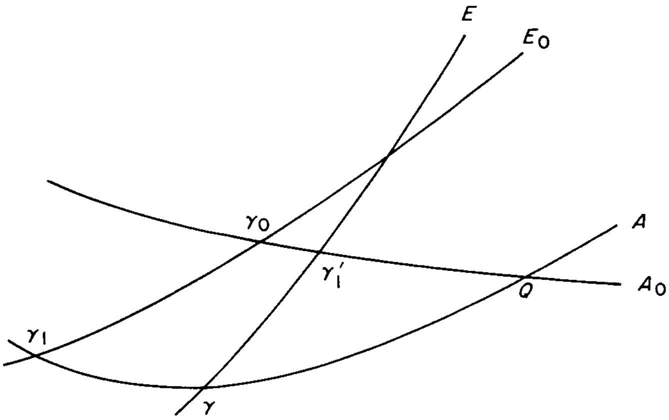
x	$(2004.684 - 0.8532T_1 - 0.000317T_1^2)T + (-0.4266 - 0.00032T_1)T^2 - 0.0418T^3$					
$\partial x/\partial h$	0.39788	−0.00017	0	0	0	0
$\partial x/\partial \varepsilon_0$	0.02234	0	0	0	0	0
$\partial x/\partial \theta_1$	0″286	0	0	0	0	0
$\partial x/\partial \theta_2$	6″63	−0″52	0	0−0″26	0	0
$\partial x/\partial \theta_3$	0″577	−0″013	0	0	0	0
$\partial x/\partial \theta_4$	−2″43	−0″29	0	−0″14	0	0
$\partial x/\partial \theta_5$	−0.51	0	0	0	0	0

From the preceding tables, one can evaluate the quantities for 1950.0 and have the precession quantities expressed in time from 1950. The partial derivatives can also be evaluated at 1950.0 by setting $T_1 = 1/2$. One then has a power series expression in time from 1950.0 for the effect of a change in a fundamental quantity upon the precession numbers used in practice.

Appendix C

Relations Between Forward and Backward Precession Elements

Draw the equator–ecliptic configurations for times T_0 and $T_0 + T$.



If E_0, A_0 are ecliptic and equator for time T_1, and E, A are ecliptic and equator for time $T_1 + T$, then

$$\gamma_0\gamma_1 = \Psi(T_1, T)$$
$$\gamma Q = 90° + z(T_1, T)$$
$$\gamma_0 Q = 90° - \zeta_4(T_1, T)$$

where

$$(T_1, T) = > \text{times } T \text{ from epoch } T_1$$

However, if we consider E to be the fixed ecliptic (Epoch $T_1 + T$), then E_0 is the moving ecliptic at time $-T$ from Epoch $T_1 + T$.

But by definition

$$\gamma Q = 90° - \zeta_0(T_1 + T, -T)$$
$$\gamma_0 Q = 90° + z(T_1 + T, -T)$$

and

$$\angle AQA_0 = \theta(T_1, T) = -\theta(T_1 + T, -T)$$

Hence we have

$$\zeta_0(T_1, T) = -z(T_1 + T, -T)$$
$$z(T_1, T) = -\zeta_0(T_1 + T, -T)$$
$$\theta(T_1, T) = -\theta(T_1 + T, -T)$$

These relations are sometimes useful in reducing the volume of tabular data required for manual data reduction.

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