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AN INVARIANCE PRINCIPLE IN THE THEORY OF STABILITY

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1. Introduction.

The purpose of this paper is to give a unified presentation of Liapunov's theory of stability that includes the classical Liapunov theorems on stability and instability as well as their more recent extensions. The idea being exploited here had its beginnings some time ago. It was, however, the use made of this idea by Yoshizawa in [1] in his study of nonautonomous differential equations and by Hale in [2] in his study of autonomous functional differential equations that caused the author to return to this subject and to adopt the general approach and point of view of this paper. This produces some new results for dynamical systems defined by ordinary differential equations which demonstrate the essential nature of a Liapunov function and which may be useful in applications. Of greater importance, however, is the possibility, as already indicated by Hale's results for functional differential equations,

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that these ideas can be extended to more general classes of dynamical systems. It is hoped, for instance, that it may be possible to do this for some special types of dynamical systems defined by partial differential equations.

In section 2 we present some basic results for ordinary differential equations. Theorem 1 is a fundamental stability theorem for nonautonomous systems and is a modified version of Yoshizawa's Theorem 6 in [1]. A simple example shows that the conclusion of this theorem is the best possible. However, whenever the limit sets of solutions are known to have an invariance property then sharper results can be obtained. This "invariance principle" explains the title of this paper. It had its origin for autonomous and periodic systems in [3] - [5], although we present here improved versions of those results. Miller in [6] has established an invariance property for almost periodic systems and obtains thereby a similar stability theorem for almost periodic systems. Since little attention has been paid to theorems which make possible estimates of regions of attraction (regions of asymptotic stability) for nonautonomous systems results of this type are included. Section 3 is devoted to a brief discussion of some of Hale's recent results [2] for autonomous functional differential equations.

2. Ordinary differential equations.

Consider the system

$$\dot{x} = f(t,x) \qquad (1)$$

where x is an n-vector, f is a continuous function on R^{n+1} to R^n and satisfies any one of the conditions guaranteeing uniqueness of solutions. For each x in R^n we define $|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$; and for E a closed set in R^n we define $d(x,E) = \text{Min}\{|x-y| : y \text{ in } E\}$. Since we do not wish to confine ourselves to bounded solutions, we introduce the point at ∞ and define $d(x,\infty) = |x|^{-1}$. Thus when we write $E^* = E \cup \{\infty\}$, we shall mean $d(x,E^*) = \text{Min}\{d(x,E), d(x,\infty)\}$. If $x(t)$ is a solution of (1), we say that $x(t)$ approaches E as $t \to \infty$ if $d(x(t),E) \to 0$ as $t \to \infty$. If we can find such a set E, we have obtained information about the asymptotic behavior of $x(t)$ as $t \to \infty$. The best that we could hope to do is to find the smallest closed set Ω that $x(t)$ approaches as $t \to \infty$. This set Ω is called the <u>positive limit set</u> of $x(t)$ and the points p in Ω are called the <u>positive limit points</u> of $x(t)$. In exactly the same way one defines $x(t) \to E$ as $t \to -\infty$, negative limit sets, and negative limit points. This is exactly G. D. Birkhoff's concept of limit sets. A point p is a positive limit point of $x(t)$ if and only if there is a sequence of times t_n approaching ∞ as $n \to \infty$ and such that $x(t_n) \to p$ as $n \to \infty$. In the above it may be that the maximal interval of definition of $x(t)$ is $[0,\tau)$. This causes no difficulty since in the results to be presented here we need only with respect to time t replace ∞ by τ. We usually ignore this possibility and speak as though our solutions are defined on $[0,\infty)$ or $(-\infty,\infty)$.

Let $V(t,x)$ be a C^1 function on $[0,\infty) \times R^n$ to R, and let G be any set in R^n. We shall say that V is a Liapunov function on G for equation (1) if $V(t,x) \geqq 0$ and $\dot{V}(t,x) \leqq -W(x) \leqq 0$ for all $t > 0$ and all x in G where W is continuous on R^n to R and

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i .$$

We define ($\overline{G}$ is the closure of G)

$$E = \{x,\ W(x) = 0,\ x \text{ in } \overline{G}\}.$$

The following result is then a modified but closely related version of Yoshizawa's Theorem 6 in [1].

THEOREM 1. If V is a Liapunov function on G for equation (1), then each solution $x(t)$ of (1) that remains in G for all $t > t_0 \geqq 0$ approaches $E^* = E \cup \{\infty\}$ as $t \to \infty$, provided one of the following conditions is satisfied:

(i) For each p in $\overline{G}$ there is a neighborhood N of p such that $|f(t,x)|$ is bounded for all $t > 0$ and all x in N.

(ii) W is C^1 and $\dot{W}$ is bounded from above or below along each solution which remains in G for all $t > t_0 \geqq 0$.

If E is bounded, then each solution of (1) that remains in G for $t > t_o \geqq 0$ either approaches E or ∞ as $t \to \infty$.

Thus this theorem explains precisely the nature of the information given by a Liapunov function. A Liapunov function relative to a set G defines a set E which under the conditions of the theorem contains (locates) all the positive limit sets of solutions which for positive time remain in G. The problem in applying the result is to find "good" Liapunov functions. For instance, the zero function $V = 0$ is a Liapunov function for the whole space R^n and condition (ii) is satisfied but gives no in-information since $E = R^n$. It is trivial but useful for applications to note that if V_1 and V_2 are Liapunov functions on G, then $V = V_1 + V_2$ is also a Liapunov function and $E = E_1 \cap E_2$. If E is smaller than either E_1 or E_2, then V is a "better" Liapunov function than either E_1 or E_2 and is always at least as "good" as either of the two.

Condition (i) of Theorem 1 is essentially the one used by Yoshizawa. We now look at a simple example where condition (ii) is satisfied and condition (i) is not. The example also shows that the conclusion of the theorem is the best possible. Consider $\ddot{x} + p(t)\dot{x} + x = 0$ where $p(t) \geqq \delta > 0$. Define $2V = x^2 + y^2$, where $y = \dot{x}$. Then $V = -p(t)y^2 \leqq - \delta y^2$ and V is a Liapunov function on R^2. Now $W = \delta y^2$ and $\dot{W} = 2\delta y\dot{y} = -2\delta(xy + p(t)y^2) \leqq -2\delta xy$. Since all solutions are evidently bounded for all $t > 0$, condition (ii) is satisfied. Here E is the x-axis $(y = 0)$ and for each solution $x(t)$, $y(t) = \dot{x}(t) \to 0$ as $t \to \infty$. Noting that the equation $\ddot{x} + (2 + e^{t})\dot{x} + x = 0$ has a solution $x(t) = 1 + e^{-t}$, we see that this is the best possible result without further restrictions on p.

In order to use Theorem 1 there must be some means of determining which solutions remain in G. The following corollary, which is an obvious consequence of Theorem 1, gives one way of doing this and also provides for nonautonomous systems a method for estimating regions of attraction.

Corollary 1. Assume that there exist continuous functions $u(x)$ and $v(x)$ on R^n to R such that $u(x) \leqq V(t,x) \leqq v(x)$ for all $t \geqq 0$. Define $Q_\eta^+ = \{x \,;\, u(x) < \eta\}$ and let G^+ be a component of Q_η^+. Let G denote the component of $Q_\eta = \{x \,;\, v(x) < \eta\}$ containing G^+. If V is a Liapunov function on G for (1) and the conditions of Theorem 1 are satisfied, then each solution of (1) starting in G^+ at any time $t_o \geqq 0$ remains in G for all $t > t_o$ and approaches E^* as $t \to \infty$. If G is bounded and $E^o = \overline{E \cap G} \subset G^+$, then E^o is an attractor and G^+ is in its region of attraction.

In general we know that if $x(t)$ is a solution of (1)--in fact, if $x(t)$ is any continuous function on R to R^n--then its positive limit set is closed and connected. If $x(t)$ is bounded, then its positive limit set is compact. There are, however, special classes of differential equations where the limit sets of solutions have an additional invariance property which makes possible a refinement of Theorem 1. The first of these are the autonomous systems

$$\dot{x} = f(x) \tag{3}$$

The limit sets of solutions of (3) are invariant sets. If $x(t)$ is defined on $[0,\infty)$ and if p is a positive limit point of $x(t)$, then the points on the solution through p on its maximal interval of definition are positive limit points of $x(t)$. If $x(t)$ is bounded for $t > 0$, then it is defined on $[0,\infty)$, its positive limit set Ω is compact, nonempty and solutions through points p of Ω are defined on $(-\infty,\infty)$ (i.e., Ω is invariant). If the maximal domain of definition of $x(t)$ for $t > 0$ is finite, then $x(t)$ has no finite positive limit points: that is, if the maximal interval of definition of $x(t)$ for $t > 0$ is $[0,\beta)$, then $x(t) \to \infty$ as $t \to \beta$. As we have said before, we will always speak as though our solutions are defined on $(-\infty,\infty)$ and it should be remembered that finite escape time is always a possibility unless there is, as for example in Corollary 2 below, some condition that rules it out. In Corollary 3 below, the solutions might well go to infinity in finite time.

The invariance property of the limit sets of solutions of autonomous systems (3) now enables us to refine Theorem 1. Let V be a C^1 function on R^n to R. If G is any arbitrary set in R^n, we say that V is a <u>Liapunov function on</u> G for equation (3) if $\dot{V} = (\text{grad } V) \cdot f$ does not change sign on G. Define $E = \{ x \, ; \, \dot{V}(x) = 0 \, , \, x \text{ in } \bar{G} \}$, where $\bar{G}$ is the closure of G. Let M be the largest invariant set in E. M will be a closed set. The fundamental stability theorem for autonomous systems is then the following:

THEOREM 2. If V is a Liapunov function on G for (3), then each solution $x(t)$ of (3) that remains in G for all $t > 0$ $(t < 0)$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$ $(t \to -\infty)$. If M is bounded, then either $x(t) \to M$ or $x(t) \to \infty$ as $t \to \infty$ $(t \to -\infty)$.

This one theorem contains all of the usual Liapunov like theorems on stability and instability of autonomous systems. Here however, there are no conditions of definiteness for V or $\dot{V}$, and it is often possible to obtain stability information about a system with these more general types of Liapunov functions. The first corollary below is a stability result which for applications has been quite useful and the second illustrates how one obtains information on instability. Četaev's instability theorem is similarly an immediate consequence of Theorem 2 (see section 3).

COROLLARY 2. Let G be a component of $Q_\eta = \{ x \, ; \, V(x) < \eta \}$. Assume that G is bounded, $\dot{V} \leq 0$ on G, and $M^o = \overline{M \cap G} \subset G$. Then M^o is an attractor and G is in its region of attraction. If, in addition, V is constant on the boundary of M^o, then

M^o is a stable attractor.

Note that if M^o consists of a single point p, then p is asymptotically stable and G provides an estimate of its region of asymptotic stability.

COROLLARY 3. Assume that relative to (3) that $V\dot{V} > 0$ on G and on the boundary of G that $V = 0$. Then each solution of (3) starting in G approaches ∞ as $t \to \infty$ (or possibly in finite time).

There are also some special classes of nonautonomous systems where the limit sets of solutions have an invariance property. The simplest of these are periodic systems (see [3]).

$$\dot{x} = f(t,x) \quad , \; f(t + T,x) = f(t) \quad \text{for all } t \text{ and } x \,. \qquad (4)$$

Here in order to avoid introducing the concept of a periodic approach of a solution of (4) to a set and the concept of a periodic limit point let us confine ourselves to solutions x(t) of (4) which are bounded for $t > 0$. Let Ω be the positive limit set of such a solution x(t), and let p be a point in Ω. Then there is a solution of (4) starting at p which remains in Ω for all t in $(-\infty,\infty)$; that is, if one starts at p at the proper time the solution remains in Ω for all time. This is the sense now in which Ω is an invariant set. Let V(t,x) be C^1 on $R \times R^n$ and periodic in t of period T. For an arbitrary set G of R^n we say that V is a <u>Liapunov function on</u> G for

for the periodic system (4) if $\dot{V}$ does not change sign for all t and all x in G. Define $E = \{ (t,x);\ \dot{V}(t,x) = 0,\ x \text{ in } \bar{G} \}$ and let M be the union of all solutions $x(t)$ of (4) with the property that $(t,x(t))$ is in E for all t. M could be called "the largest invariant set relative to E". One then obtains the following version of Theorem 2 for periodic systems:

THEOREM 3. If V is a Liapunov function on G for the periodic system (4), then each solution of (4) that is bounded and remains in G for all $t > 0$ $(t < 0)$ approaches M as $t \to \infty$ $(t \to -\infty)$.

In [6] Miller showed that the limit sets of solutions of almost periodic systems have a similar invariance property and from this he obtains a result quite like Theorem 3 for almost periodic systems. This then yields for periodic and almost periodic systems a whole chain of theorems on stability and instability quite similar to that for autonomous systems. For example, one has

COROLLARY 4. Let $Q_\eta^+ = \{ x;\ V(t,x) < \eta, \text{ all } t \text{ in } [0,T] \}$, and let G^+ be a component of Q_η^+. Let G be the component of $Q_\eta = \{ x;\ V(t,x) < \eta \text{ for some } t \text{ in } [0,T] \}$ containing G^+. If G is bounded, $\dot{V} \leq 0$ for all t and all x in G, and if $M^0 = \overline{M \cap G} \subset G^+$, then M^0 is an attractor and G^+ is in its region of attraction. If $V(t,x) = \varphi(t)$ for all t and all x on the boundary of M^0, then M^0 is a stable attractor.

Our last example of an invariance principle for ordinary differential equations is that due to Yoshizawa in [1] for "asymptotically autonomous" systems. It is a consequence of Theorem 1 and results by Markus and Opial (see [1] for references) on the limit sets of such systems. A system of the form

$$\dot{x} = F(x) + g(t,x) + h(t,x) \qquad (5)$$

is said to be <u>asymptotically autonomous</u> if (i) $g(t,x) \to 0$ as $t \to \infty$ uniformly for x in an arbitrary compact set of R^n, (ii) $\int_0^\infty |h(t,\varphi(t))|\, dt < \infty$ for all φ bounded and continuous on $[0,\infty)$ to R^n. The combined results of Markus and Opial then state that the positive limit sets of solutions of (5) are invariant sets of $\dot{x} = F(x)$. Using this, Yoshizawa then improved Theorem 1 for asymptotically autonomous systems.

It turns out to be useful, as we shall illustrate in a moment on the simplest possible example, in studying systems (1) which are not necessarily asymptotically autonomous to state the theorem in the following manner:

THEOREM 4. If, in addition to the conditions of Theorem 1, it is known that a solution $x(t)$ of (1) remains in G for $t > 0$ and is also a solution of an asymptotically autonomous system (5), then $x(t)$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$, where M is the largest invariant set of $\dot{x} = F(x)$ in E.

It can happen that the system (1) is itself asymptotically autonomous in which case the above theorem can be applied. However, as the following example illustrates, the original system may not itself be asymptotically autonomous but it still may be possible to construct for each solution of (1) an asymptotically autonomous system (5) which it also satisfies.

Consider again the example

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - p(t)y \quad , \quad 0 < \delta \leqq p(t) \leqq m \\ & \qquad\qquad\qquad\qquad \text{for all} \quad t > 0 \end{aligned} \tag{6}$$

Now we have the additional assumption that $p(t)$ is bounded from above. Let $(\bar{x}(t), \bar{y}(t))$ be any solution of (6). As was argued previously below Theorem 1, all solutions are bounded and $\bar{y}(t) \to 0$ as $t \to \infty$. Now $(\bar{x}(t), \bar{y}(t))$ satisfies $\dot{x} = y$, $y = -x - p(t)\bar{y}(t)$, and this system is asymptotically autonomous to (*) $\dot{x} = y$, $\dot{y} = -x$. With the same Liapunov function as before, E is the x-axis and the largest invariant set of (*) in E is the origin. Thus for (6) the origin is asymptotically stable in the large.

3. Autonomous functional differential equation.

Difference differential equations of the form

$$\dot{x}(t) = f(t,x(t),x(t-r)) \quad , \quad r > 0 \tag{7}$$

have been studied almost as long as ordinary differential equations and these as well as other types of systems are of the general form

$$\dot{x}(t) = f(t,x_t) \tag{8}$$

where x is in R^n and x_t is the function defined on $[-r,0]$ by $x_t(\tau) = x(t+\tau)$, $-r \leqq \tau \leqq 0$. Thus x_t is the function that describes the past history of the system on the interval $[t-r,t]$ and in order to consider it as an element in the space C of continuous functions all defined on the same interval $[-r,0]$, x_t is taken to be the function whose graph is the translation of the graph of x on the interval $[t-r,t]$ to the interval $[-r,0]$. Since such equations have had a long history it seems surprising that it is only within the last 10 years or so that the geometric theory of ordinary differential equations has been successfully carried over to functional differential equations. Krasovskii [8] has demonstrated the effectiveness of a geometric approach in extending the classical Liapunov theory, including the converse theorems, to functional differential equations. An account of other aspects of their theory which have yielded to this geometric approach can be found in the paper [9] by Hale. What we wish to do here is to present Hale's extension in [2] of the results of Section 2 of this paper to autonomous functional differential equations

$$\dot{x} = f(x_t) . \tag{9}$$

It is this extension that has had so far the greatest success in studying stability properties of the solutions of systems (9), and it is possible that this may lead to a similar theory for special classes of systems defined by partial differential equations.

With $r \geqq 0$ the space C is the space of continuous functions φ on $[-r,0]$ to R^n with $\|\varphi\| = \max\{|\varphi(\tau)|; -r \leqq \tau \leqq 0\}$. Convergence in C is uniform convergence on $[-r,0]$. A function x defined on $[-r,\infty)$ to R^n to said to be a <u>solution</u> of (9) <u>satisfying the initial condition</u> φ at time $t = 0$ if there is an $a > 0$ such that $\dot{x}(t) = f(x_t)$ for all t in $[0,a)$ and $x_0 = \varphi$. Remember $x_0 = \varphi$ means $x(\tau) = \varphi(\tau)$, $-r \leqq \tau \leqq 0$. At $t = 0$, $\dot{x}$ is the right hand derivative. The existence uniqueness theorems are quite similar to those for ordinary differential equations. If f is locally Lipschitzian on C, then for each φ in C there is one and only one solution of (9) and the solution depends continuously on φ. The solution can also be extended in C for $t > 0$ as long as it remains bounded. As in Section 2, we will always speak as though solutions are defined on $[-r,\infty)$. The space C is now the state space of (9) and through each point φ of C there is the motion or flow x_t starting at φ defined by the solution $x(t)$ of (9) satisfying at time $t = 0$ the initial condition φ; x_t, $0 \leqq t < \infty$, is a curve in C which starts at time $t = 0$ at φ. In analogy to Section 2 with C replacing R^n, x_t replacing $x(t)$, and $\|x_t\|$ replacing $|x(t)|$, we define the <u>distance</u> $d(x_t,E)$ of x_t from a closed set E of C to be $d(x_t,E) = \min\{\|x_t-\psi\|; \psi \in E\}$. The positive limit set of x_t is then defined in a manner completely analogous to Section 2. Because there are some important differences we shall be satisfied here with restricting ourselves to motions

x_t bounded for $t > 0$. One of the differences here is that in C closed and bounded sets are not always compact. Another is that although we have uniqueness of solutions in the future two motions starting from different initial conditions can come together in finite time $t_o > 0$; after this they coincide for $t \geqq t_o$. (The motions define semi-groups and not necessarily groups.)

Hale in [2] has, however, shown that the positive limit sets Ω of bounded motions x_t are nonempty, compact, connected, invariant sets in C. Invariance here is in the sense that, if x_t is a motion starting at a point of Ω, then there is an extension onto $(-\infty,-r]$ such that $x(t)$ is a solution of (9) for all t in $(-\infty,\infty)$ and x_t remains in Ω for all t. With this result he is then able to obtain a result which is similar to Corollary 1 of Section 2.

For $\varphi \in C$ let $x_t(\varphi)$ denote the motion defined by (9) starting at φ. For V a continuous function on C to R define $\dot{V}$ and Q_ℓ by

$$\dot{V}(\varphi) = \overline{\lim_{\tau \to 0+}} \frac{1}{\tau} [V(x_\tau(\varphi))-V(\varphi)]. \tag{10}$$

and

$$Q_\ell = \{\varphi \; ; \; V(\varphi) < \ell\} .$$

THEOREM 5. If V is a Liapunov function on G for (9) and x_t is a trajectory of (9) which remains in G and is bounded for $t > 0$, then $x_t \to M$ as $t \to \infty$.

Hale has also given the following more useful version of this result.

COROLLARY 5. Define $Q_\eta = \{\varphi;\ V(\varphi) < \eta\}$ and let G be Q_η or a component of Q_η. Assume that V is a Liapunov function on G for (9) and that either (i) G is bounded or (iii) $|\varphi(0)|$ is bounded for φ in G. Then each trajectory starting in G approaches M as $t \to \infty$.

The following is an extension of Četaev's instability theorem. This is a somewhat simplified version of Hale's Theorem 4 in [2], which should have stated "$V(\varphi) > 0$ on U when $\varphi \neq 0$ and $V(0) = 0$" and at the end "... intersect the boundary of C_γ ...". This is clear from his proof and is necessary since he wanted to generalize the usual statment of Četaev's theorem to include the possibility that the equilibrium point be inside U as well as on its boundary.

COROLLARY 6. Let $p \in C$ be an equilibrium point of (9) contained in the closure of an open set U and let N be a neighborhood of p. Assume that (i) V is a Liapunov function on $G = U \cap N$, (ii) $M \cap G$ is either the empty set or p, (iii) $V(\varphi) < \eta$ on G when $\varphi \neq p$, and (iv) $V(p) = \eta$ and $V(\varphi) = \eta$ on that part of the boundary of G inside N. Then p is unstable. In fact, if N_0 is a bounded neighborhood of p properly contained in N then each trajectory starting at a point of $G_0 = G \cap N_0$ other than p leaves N_0 in finite time.

Proof. By the conditions of the corollary and Theorem 6 each trajectory starting inside G_0 at a point other than p must either leave G_0, approach its boundary or approach p. Conditions (i) and (iv) imply that it cannot reach or approach that part of the boundary of G_0 inside N_0 nor can it approach p as $t \to \infty$. Now (ii) states that there are no points of M on that part of the boundary of N_0 inside G. Hence each such trajectory must leave N_0 in finite time. Since p is either in the interior or on the boundary of G, each neighborhood of p contains such trajectories, and p is therefore unstable.

In [2] it was shown that the equilibrium point $\varphi = 0$ of

$$\dot{x}(t) = ax^3(t) + bx^3(t-r)$$

was unstable if $a > 0$ and $|b| < |a|$. Using the same Liapunov function and Theorem 6 we can show a bit more. With

$$V(\varphi) = -\frac{\varphi^4(0)}{4a} + \frac{1}{2}\int_{-r}^{0} \varphi^6(\theta)d\theta ,$$

$$V(x_t) = -\frac{x^4(t)}{4a} + \frac{1}{2}\int_{t-r}^{t} x^6(\theta)d\theta$$

and

$$\dot{V}(\varphi) = -\frac{1}{2}(\varphi^6(0) + 2\,\frac{b}{a}\,\varphi^3(0)\varphi^3(-r) + \varphi^6(-r))$$

which is nonpositive when $|b| < |a|$ (negative definite with respect to $\varphi(0)$ and $\varphi(-r)$); that is, V is a Liapunov function on C and $E = \{\varphi;\ \varphi(0) = \varphi(-r) = 0\}$. Therefore M is simply the null function $\varphi = 0$. If $a > 0$, the region $G = \{\varphi;\ V(\varphi) < 0\}$

is nonempty, and no trajectory starting in G can have $\varphi = 0$ as a positive limit point nor can it leave G . Hence by Theorem 5 each trajectory starting in G must be unbounded. Since $\varphi = 0$ is a boundary point of G, it is unstable. It is also easily seen [2] that if $a < 0$ and $|b| < |a|$, then $\varphi = 0$ is asymptotically stable in the large.

In [2] Hale has also extended this theory for systems with infinite lag $(r = \infty)$, and in that same paper gives a number of significant examples of the applications of this theory.

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