A STUDY OF DECISION REGION RECEIVERS FOR THE SUNBLAZER SPACE EXPERIMENT

by

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ABSTRACT

For a Rayleigh channel with additive bandlimited noise and time delay, several types of optimum receivers are developed. These include the 3-ary case for noise, and forward and backward Barker codes, and the optimum binary case for signal and noise. Expressions for the error probability in each case are derived, and the results are compared for various values of signal to noise ratio. The theoretical realization for these receivers requires an infinite number of correlators; and since any physical system can have only a finite number, an investigation into the effect of this restriction on receiver performance is made. The concept of receiver "guessing" for small signal to noise ratios is also explored in some detail.

In the optimum binary case the expression for the error probability cannot be evaluated without the aid of a piecewise linear approximation to the decision curves. The effect of varying the number of linear segments in the approximation is examined.

Finally, a simple coding scheme is used to illustrate some of the advantages of block coding as compared to bit by bit signaling.

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Title: Professor of Electrical Engineering.

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CHAPTER 1

INTRODUCTION

1.1 A Problem in Space Communications

The design of an "optimum" receiver for a communications problem is always dependent on the nature of the channel through which the messages must be sent. In most cases a probabilistic description must be chosen since a detailed knowledge of the channel mechanisms are generally unknown. This is particularly true of space communications where the corrupting effects of the channel may include galactic noise, dispersion, and scattering, to name just a few. Little of even a probabilistic nature is known about many of these phenomena, as only a few deep space probes have been launched, and none of these were concerned primarily with the deep space environment. The Center for Space Research at M. I. T. is involved in the construction and launching of a satellite, Sunblazer, whose purpose is to gather information about the space medium itself. This report is directly concerned with the theoretical design and evaluation of the different types of receivers for the Sunblazer Project.

Part of the complexity of this problem is due to the fact that the initial Sunblazer receivers will not have the benefit of any previous experimental data. Thus we are faced with the case of designing a receiver which is fairly versatile in its ability to adapt to unexpected signal conditions.

1.2 Sunblazer Requirements - A More Specific Formulation of the Problem

To be more specific, the Sunblazer satellite will broadcast two signals at different frequencies. By measuring the time delay between these signals, information may be inferred about such phenomena as the electron density of the Suns corona. Furthermore, each time a signal is transmitted it may take on two forms representing either a binary one or zero. The Sunblazer satellite will use "forward" and "backward" Barker codes for these signals. A forward Barker code and its autocorrelation function are shown in Figure 1. (For a backward Barker code simply reverse the n-axis in Figure 1).

In order to approach this problem it is first necessary to develop a general receiver "philosophy".

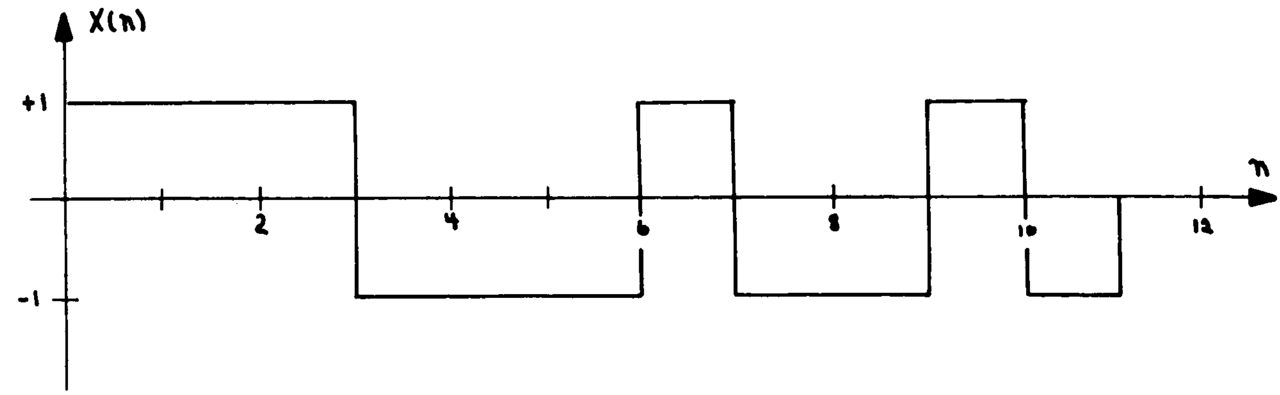
(1) The receiver should decide whether m_1 or m_2 was transmitted and also the signal delay time. This is the maximum amount of information which can be obtained.

(2) If the probability of making a mistake in determining the data for part (1) becomes unacceptable, can we look at only the signal delay (i.e., throw away the telemetry information) and thus reduce our probability of error? Here we are asking whether we can reduce the probability of error by not requiring as much data.

(3) Finally, if the probability of determining any data at all from the signal gets very small, can we at least make a decision as to whether or not signal is received? i.e., for extremely weak signal cases possibly the best that can be determined is whether any signal energy is present in the noise.

The problem is now to design receivers illustrating the characteristics of each of the above three cases, and then to compare their performances.

a) The Eleven-bit Barker Code



b) The Corresponding Autocorrelation Function

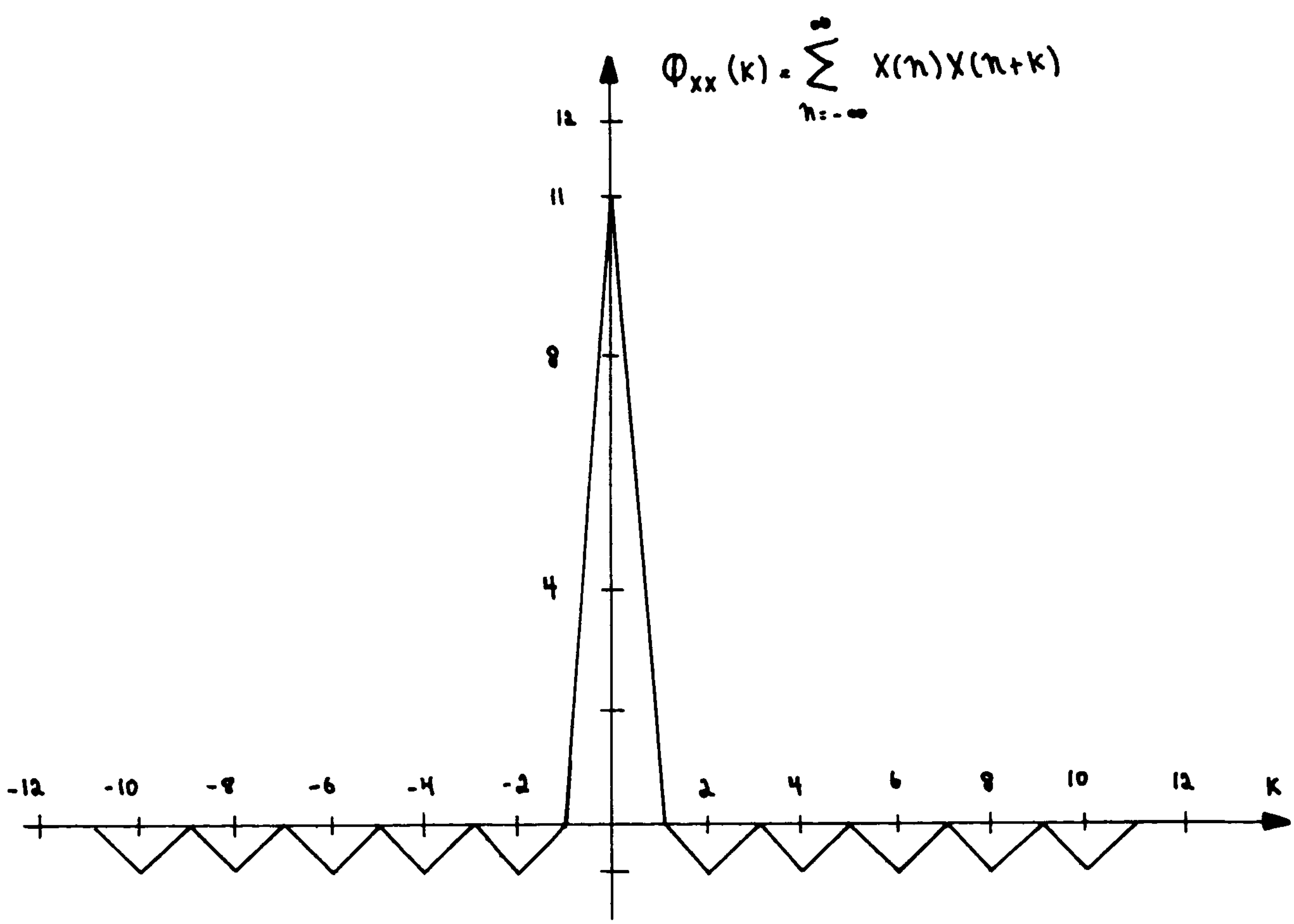


FIGURE 1

1.3 The Estimation Problem

The most straightforward method for estimating the arrival time of the signal is to divide the time scale into many little "baskets" and then check to see into which "basket" the received signal falls. With an infinite number of incrementally small baskets we could expect perfect resolution. However, since no real receiver will be able to use more than a finite number of intervals, it is necessary to check what effect this restriction has on receiver performance. It should also be noted that this estimation method is actually a detection problem; since we are really asking, has the receiver for basket number i detected a signal?

In the next chapter we will begin to develop the models necessary to proceed with the analysis.

CHAPTER 2

THE COMMUNICATIONS SYSTEM

Before any further results can be obtained it is first necessary to model the transmitter, the channel, and that part of the receiver which demodulates the incoming signals. Then the remaining portion of the receiver can be designed to give "optimum" performance with respect to some criteria, as mentioned in the previous chapter.

2.1 The Transmitter

We wish to transmit any one of m_i different messages. Each message has a certain probability P_i of being transmitted. These messages are fed into an encoder (see Figure 2) which is essentially a device which makes a one-to-one transformation between the messages and time-limited, lowpass signals[1]. A sample signal and its frequency spectrum are shown in Figure 2A. A double-sideband suppressed carrier (DSB-SC) amplitude modulation scheme is used which amounts to a frequency translation of the lowpass signal (Figure 2B). For ease of analysis it is assumed that

$$\int_0^T s_i^2(t)\, dt = 1$$

and E_{T_i} is the transmitted energy. Thus the signal at the antenna is of the form

$$s^0(t) = \sqrt{2E_{T_i}}\, s_i(t) \cos \omega_0 t$$

[1] Since a strictly bandlimited, time limited signal doesn't exist, a definition for bandwidth that requires that 90% of the signal energy be between ± w is used.

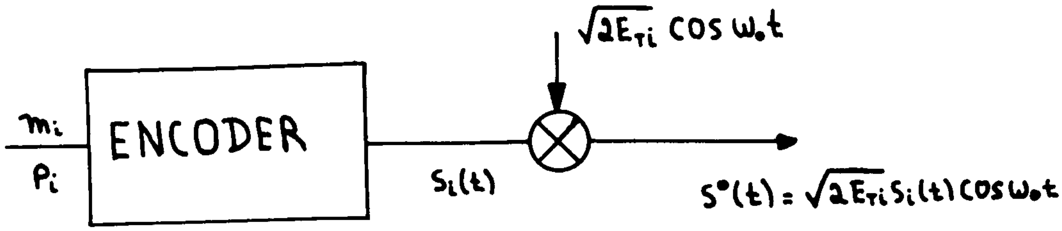


FIGURE 2. Model of the Transmitter

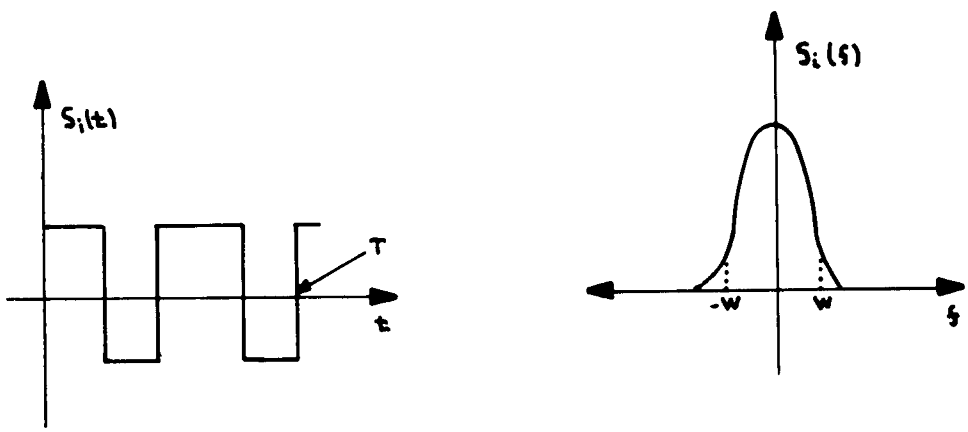


FIGURE 2a. Sample Signal and its Spectrum

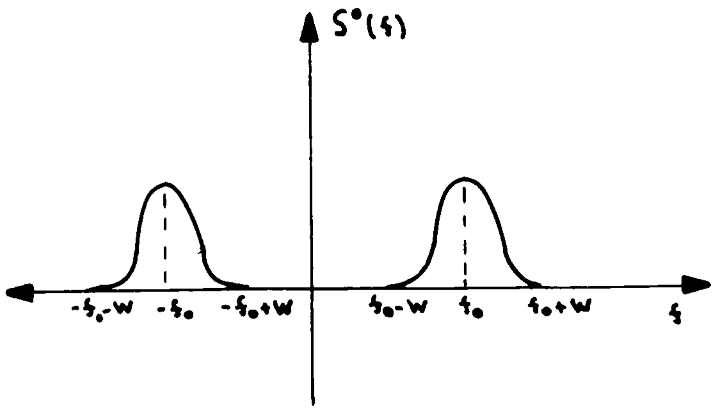


FIGURE 2b. Spectrum of Transmitted Signal

where $\omega_0 = 2\pi f_0$

2.2 The Channel

The channel model used here attenuates the signal by a factor of a, introduces a random phase angle θ into the carrier, and delays the received signal by an amount τ. Both θ and a are random variables which are assumed to be time-invariant over the period of signal transmission[1]. Furthermore, the channel also introduces additive Gaussian noise, n(t). A block diagram is shown in Figure 3.

The obvious question as to what are realistic distributions for a, θ, and n(t) is very hard to answer. Over the frequencies at which Sunblazer is operating (75 mc and 225 mc) the additive noise may be modeled well by band limited white Gaussian noise[2]. See Figure 3A.

However, little seems to be known about the form of either the phase or amplitude distribution. The following assumptions were made;

(1) Random Phase - Slowly varying in time with respect to signal length with a uniform density from 0 to 2π. This assumption is the result of many physical models, and particularly of the scattering and long-path-length models.

(2) Random Amplitude - There is actually very little to go on here. Scattering models lead to Rayleigh distributions, which have allowed closed form expressions for the receiver errors. Results of the assumption of a Rayleigh distribution show that receiver probability of error is approximately directly proportional to 1/(sig to noise ratio).

[1] If this assumption doesn't hold then the whole signaling scheme becomes ineffective as we shall later see.

[2] Gilbert, Some Communication Aspects of Sunblazer, p. 2

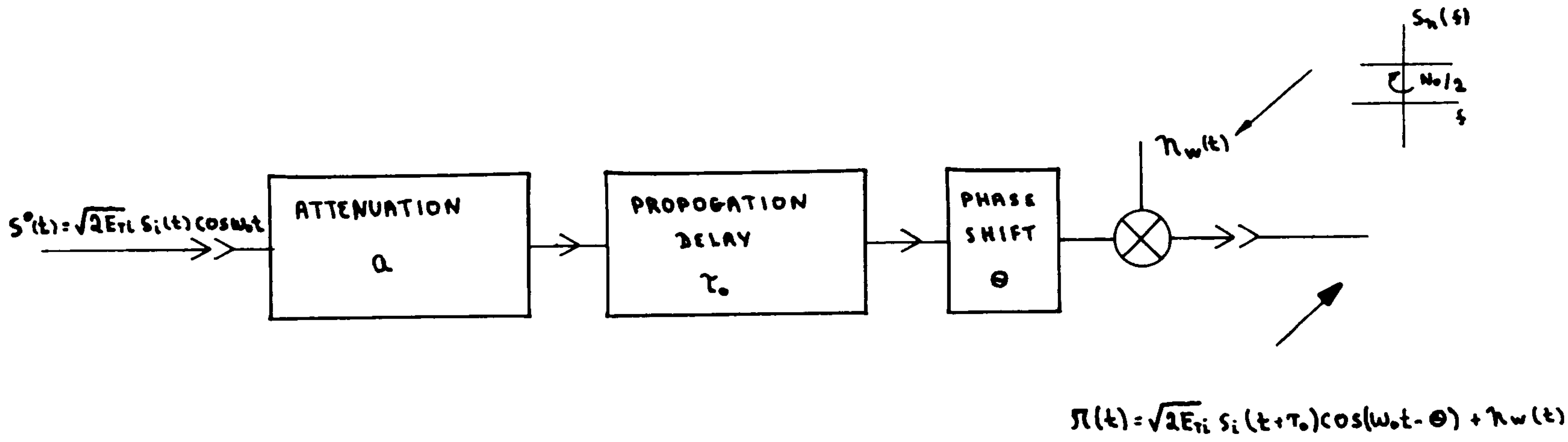


FIGURE 3. CHANNEL MODEL

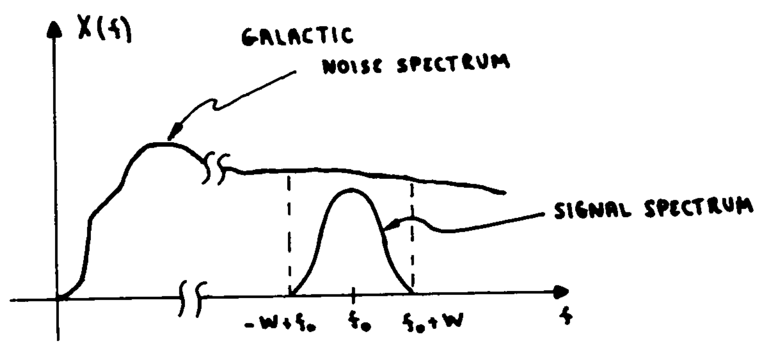


FIGURE 3A. COMPARISON OF GALACTIC NOISE AND SIGNAL SPECTRUMS

If we assume no scattering of the signal, the probability of error decreases exponentially with signal to noise ratio. The actual distribution probably gives results which lie somewhere between these two cases. Also, the results from the Rayleigh case don't differ significantly from the Rayleigh plus specular (direct) component case until the specular component becomes 4 to 5 times larger than the scattered component[1]. Thus for low signal to noise ratios our results should be reasonably valid. In any case, assumption of this distribution allows a comparative study of the different receiver configurations to be made.

Since the time delay τ is also constant for each transmission, and we are estimating its arrival, the detection scheme does not need any probability information concerning it. However, if an average probability of error (with respect to time delay) is desired then certain assumptions about a distribution for τ must be made. But since this is not critical for the detection problem, it can wait until after the first launch when better data will be available.

2.3 The Receiver Front End

Ideally the receiver output should be a lowpass signal of the same form as transmitted. However, the transmitted signal has been corrupted by the channel so this is generally not possible. The receiver input is of the form

[1] Van Trees, Detection and Estimation Theory

$$r'(t) = s'(t) + n_w(t) = \sqrt{2E_{T_i}}\ as_i(t+\tau)\cos(\omega_0 t-\theta) + n_w(t)$$
$$= \left\{\sqrt{2E_{T_i}}\ as_i(t+\tau)\cos\theta\right\}\cos\omega_0 t + \left\{\sqrt{2E_{T_i}}\ as_i(t+\tau)\right.$$
$$\left.\cdot \sin\omega_0 t\right. + n_w(t)$$

From Figure 4 we see that the receiver front end consists first of a bandpass filter whose output is of the form

$$r(t) = s(t) + n_{bp}(t) \tag{2.1}$$

In order to simplify (2.1) we express $n_{bp}(t)$ as a Fourier series.

$$n_{bp}(t) = \sum_{n=1}^{\infty} (x_{cn}\cos n\omega t + x_{sn}\sin n\omega t)$$

where
$$x_{cn} = \frac{2}{T}\int_0^T n_{bp}(t)\cos n\omega t\, dt$$

$$x_{sn} = \frac{2}{T}\int_0^T n_{bp}(t)\sin n\omega t\, dt$$

Now, rewriting $n\omega_0$ as $(n\omega-\omega_0)+\omega_0$ we get

$$n_{bp}(t) = n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)\sqrt{2}\sin\omega_0 t$$

where $n_c(t) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}[x_{cn}\cos(n\omega-\omega_0)t + x_{sn}\sin(n\omega-\omega_0)t]$

and $n_s(t) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}[x_{sn}\cos(n\omega-\omega_0)t - x_{cn}\sin(n\omega-\omega_0)t]$

The only nonvanishing terms in the sums above are the ones for which nf falls between $-w+f_0$ and f_0+w. Thus $n_c(t)$ and $n_s(t)$ are low pass waveforms with a frequency spectrum of width $2w$ centered about zero. Therefore,

$$r(t) = \sqrt{2}\cos\omega_0 t\,[a\sqrt{E_{T_i}}\ si(t+\tau)\cos\theta + n_c(t)]$$
$$+ \sqrt{2}\sin\omega_0 t\,[a\sqrt{E_{T_i}}\ si(t+\tau)\sin\theta + n_s(t)]$$

Nest, using the synchronous demodulation scheme shown in Figure 4, multiplying by $\sin\omega_0 t$ and $\cos\omega_0 t$, and then low pass filtering, we have

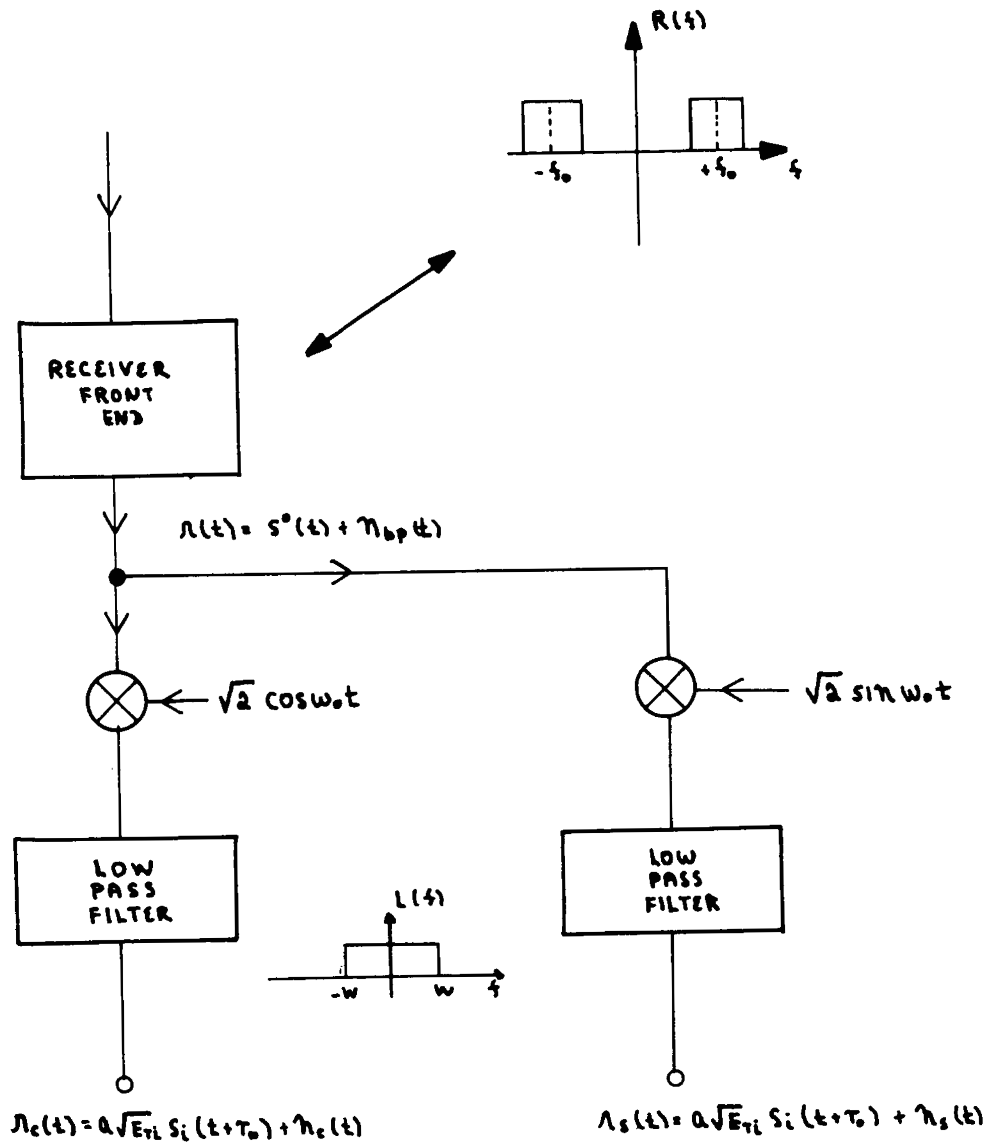


FIGURE 4. THE RECEIVER FRONT END AND DEMODULATOR

for the receiver front end output, two signals

$$r_c(t) = a \sqrt{E_T}\, s_i(t+\tau) \sin\theta + n_s(t)$$

$$r_s(t) = a \sqrt{E_T}\, s_i(t+\tau) \sin\theta + n_s(t)$$

On the basis of $r_c(t)$ and $r_s(t)$ we want to decide which of the m_i was sent. In other words we would like to set the estimated message $\hat{m}$, equal to the transmitted message m_i.

The key to analyzing this problem lies in expressing the signals and noise in terms of a finite dimensional vector space. By using finite dimensional vectors we have to worry only about the statistics for a finite number of components, rather than for an infinite number of times. In the next chapter vector decision rules for the receivers will be developed.

CHAPTER 3

DERIVATION OF THE N-SIGNAL DECISION RULES

In this section general rules are developed for implementing the first two cases of the receiver philosophy. The third case is an extension of case two using a block coding scheme and will be considered in more detail later on in Chapter 7.

3.1 Representation of Signals as Vectors

It can be shown[1] that any time varying set of signals can be represented by vectors in an appropriate signal space. Furthermore, if we have N signals, then an orthogonal basis for this vector space consisting of a maximum of N vectors can be found. Stochastic processes may also be represented as infinite dimension vectors using the Karhunen-Loeve expansion[2]. Thus we would like to represent the received signals as vectors, and on the basis of these vectors make the "best" decision as to which signal was actually sent. Of course, the "best" choice is a function of the assumptions used to generate the decision making rule. An example of one class of decision criteria, the minimum probability of error case, follows.

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[1] Wozancraft & Jacobs, Principles of Communications Engineering, pp. 266-273.

[2] Davenport & Root, Random Signals and Noise, P. 96. Although the noise process requires an infinite dimensional vector for complete characterization, we are generally faced with having to use only the finite number of components lying along the N signal vectors. This means an easier description for the noise than as a time-varying signal.

3.2 The N-Signal Rule

Assume that the cost of saying message i was sent when message j was actually sent is c_{ij}. Next we wish to design a receiver which will minimize the total risk, $\mathcal{R}$, defined as

$$\mathcal{R} = \sum_{i=1}^{N} \sum_{j=1}^{N} \text{Prob [j sent] } c_{ij} \text{ Prob [say i/j sent]}$$

Using a decision space model[1] (see Figure 5), we can represent P [say i/j sent] as

$$\int_{z_i} P_{\frac{\bar{r}}{m}} \left(\frac{\bar{R}}{m_j}\right) d\bar{R}$$

where $\bar{r}$ is the received vector. Calling Prob [j sent] P_j, we have for the risk:

$$\mathcal{R} = \sum_{i=1}^{N} \sum_{j=1}^{N} P_j \, c_{ij} \int_{z_i} P_{\frac{\bar{r}}{m}} \left(\frac{\bar{R}}{m_j}\right) d\bar{R}$$

Now lets choose the cost equal to one if we make a mistake (i.e., say i, when j was sent) and zero if we don't make a mistake (say i, when i was sent).

Let $c_{ij} = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$

Then

$$\mathcal{R} = \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} P_j \int_{z_i} P_{\frac{\bar{r}}{m}} \left(\frac{\bar{R}}{m_j}\right) d\bar{R}$$

But this may be recognized as Prob [making an error] = P[E]

$$P[E] = \sum_{i=1}^{N} \int_{z_i} \sum_{\substack{j=1 \\ j \neq i}} P_j \, P_{\frac{\bar{r}}{m}} \left(\frac{\bar{R}}{m_j}\right) d\bar{R}$$

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[1] Van Trees, p 12.

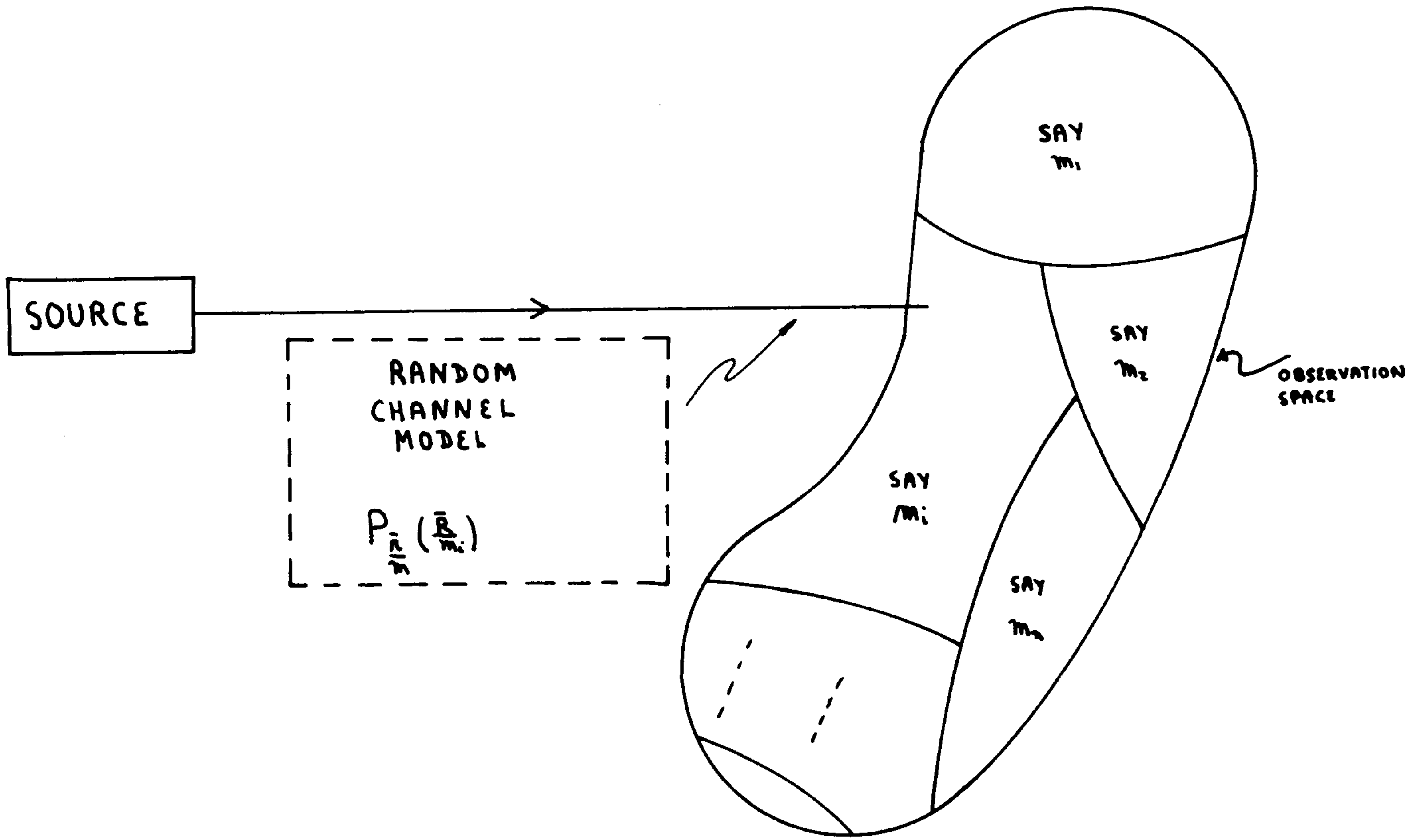

FIGURE 5. DECISION SPACE FOR RANDOM CHANNEL

Let $I_i(\bar{R}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} P_j \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_j}\right)$ and $P[E] = \sum_{i=1}^{N} \int_{z_i} I_i(\bar{R}) \, d\bar{R}$

To minimize $P[E]$ we should assign each point in z to the z_i which minimizes expression (3.1) for $P[E]$. Thus we have the decision rule:

For any $\bar{R}$

if $I_1(\bar{R}) < I_2(\bar{R})$ and $I_3(\bar{R})$ and . . . $I_N(\bar{R})$, Choose m_1

$I_2(\bar{R}) < I_1(\bar{R})$ and $I_3(\bar{R})$ and . . . $I_N(\bar{R})$, Choose m_2

$\vdots$

$I_N(\bar{R}) < I_1(\bar{R})$ and $I_2(\bar{R})$ and . . . $I_{N-1}(\bar{R})$, Choose m_n

Hence, the decision rule is to choose m_i as the message transmitted if

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} P_j \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_j}\right) < \sum_{\substack{j=1 \\ j \neq k}}^{N} P_j \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_j}\right) \quad \text{for all } k \neq i$$

Expanding and canceling out common factor we have

$$P_k \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_k}\right) < P_i \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_i}\right) \quad \text{for all } k \neq i$$

and dividing by $P_{\bar{r}}(\bar{R})$. The decision rule simplifies to choosing message m_j if and only if

$$\frac{P_k \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_k}\right)}{P_{\bar{r}}(\bar{R})} < \frac{P_i \, P_{\underline{r}/\underline{m}} \left(\frac{\bar{R}}{m_i}\right)}{P_{\bar{r}}(\bar{R})} \quad \text{for all } i \neq 1 \qquad (3.2)$$

Or, to state this more compactly, set $\hat{m}$, the estimated message, equal to m_j if and only if $P_i \, P_{\underline{m}/\bar{r}} \left(\frac{m_i}{\bar{R}}\right)$ is a maximum for $i = j$.

Although this model does lead to the smallest possibility of making an error it is somewhat misleading, since to realize this performance we need to know the a priori probabilities, P_j, and the costs c_{ij} exactly. This restriction, however, can be minimized by varying both P_j and c_{ij} and calculating their effect on $P[E]$. Alternately, there are other test procedures which do not require this information, such as minimax tests or Neyman-Pearson tests[1], but these usually produce larger values for $P[E]$, or maximize other quantities of interest such as probability of detection. However, since $P[E]$ is a good measurement of the performance of a communication system the first approach will be used here[2].

3.3 The Binary Decision Rule for N-signals

Next we would like to specialize the result to the case where no penalty is incurred if we confuse any of m_2 through m_n with any other m_2 through m_n. The only cost occurs if we mistake m_1 for one of the messages m_2 to m_n. This will allow us to investigate case 2 of the receiver philosophy; since by setting m_1 equal to noise then we are asking the question is there signal or noise present.

We now have a binary decision problem. That this will reduce the probability of error can be seen if we observe that additional c_{ij}'s are now set equal to zero, and so even if we fix the $I(\bar{R})$'s, the risk,

$$R = \sum \sum P_j \, c_{ij} \int_{z_i} I_j(\bar{R}) \, d\bar{R}$$

---

[1] Hancock & Wentz, Signal Detection Theory, pp. 35, 40, 43.

[2] We shall see later on that by developing the expression for the minimum $P[E]$ case we have practically all the tools needed to evaluate the other cases if we desire.

will decrease. The optimum rule does even better by not only making the additional c_{ij}'s zero, but also by adjusting the $I(\bar{R})$'s to make the integral in each nonzero term a minimum.

$$\text{Let } c_{ij} = 0 \quad i = j \quad \text{or} \quad \left\{\begin{matrix} i > 1 \\ j > 1 \end{matrix} \quad \text{simultaneously}\right\}$$

$$c_{1j} = c_{j1} = 1 \qquad j = 2, 3, \ldots, n$$

therefore

$$P[E] = \sum_{j=1}^{N} P_j \int_{z_1} P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_j}\right) d\bar{R} + \sum_{i=2}^{N} P_1 \int_{z_i} P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_1}\right) d\bar{R}$$

$$= \int_{z_1} \sum_{j=2}^{N} P_j \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_j}\right) d\bar{R} + \int_{\sum_{i=2} z_i} P_1 \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_1}\right) d\bar{R}$$

$$= \int_{z_1} \sum_{j=2}^{N} P_j \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_j}\right) d\bar{R} + \int_{z - z_1} P_1 \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_1}\right) d\bar{R}$$

Then the decision rule to minimize $P[E]$ is to choose;

one of m_2 through m_n sent, if $\sum_{j=2}^{N} P_j \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_j}\right) > P_1 \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_1}\right)$ (3.3)

or m_1, if $\sum_{j=2}^{N} P_j \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_j}\right) < P_1 \, P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_1}\right)$

Both equations (3.2) and (3.3) depend upon P_j which we have, and $P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_i}\right)$ which must be calculated.

In the next section we shall calculate $P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_i}\right)$ for the channel model developed in Chapter 2.

CHAPTER 4.

$P_{\bar{r}/m}(\bar{R}/m_i)$ FOR THE RAYLEIGH CHANNEL

As mentioned earlier the key to analyzing this problem lies in expressing the signals and noise in terms of a finite dimensional vector space. The i^{th} signal may be represented as

$$s_i(t) = \sum_{k=1}^{N} s_{ki}\ \phi_k(t) \quad \text{or} \quad \bar{s}_i = (s_{1i}, \ldots s_{Ni})$$

where

$$s_{ij} = \int_0^T s_i(t)\ \phi_j(t)\ dt \quad \text{and} \quad \int_0^T \phi_k(t)\ \phi_j(t)\ dt = \delta_{kj}$$

We would also like to represent $r_c(t)$ and $r_s(t)$ as vectors so we may apply the results of the previous section. Taking $r_c(t)$ first, we find $\bar{r}_c = (r_{c1}, r_{c2} \ldots, r_{cn})$ by

$$r_{cj} = \int_0^T r_c(t)\ \phi_j(t)\ dt$$

$$= \int_0^T [a\sqrt{E_{T_i}}\ s_i(t+\tau)\ \cos\theta + n_c(t)]\ \phi_j(t)\ dt$$

$$= a\sqrt{E_{T_i}}\ \cos\theta \int_0^T \sum_{k=1}^{N} s_{ik}\ \phi_k(t+\tau)\ \phi_j(t)\ dt + \int_0^T n_c(t)\ \phi_j(t)\ dt$$

$$= a\sqrt{E_{T_i}}\ \cos\theta \sum_{k=1}^{N} s_{ik} \int_0^T \phi_k(t+\tau)\ \phi_j(t)\ dt + \int_0^T n_c(t)\ \phi_j(t)\ dt$$

$$= a\sqrt{E_{T_i}}\ \cos\theta \sum_{k=1}^{N} s_{ik}\ R_{kj}(\tau) + n_{cj}$$

where $R_{kj}(\tau) = \int_0^T \phi_k(t+\tau)\ \phi_j(t)$ and $n_{cj} = \int_0^T n_c(t)\ \phi_j(t)\ dt$

In a similar manner we may find the components of r_{sj} of the vector $\bar{r}_s = (r_{s1}, r_{s2} \ldots r_{sn})$

$$r_{sj} = a\sqrt{E_{Ti}} \sin\theta \sum_{k=1}^{N} s_{ik} R_{kj}(\tau) + n_{sj}$$

Expressing these equations in terms of matrix notation we have, for the vectors, the following

$$\bar{r}_c = a\sqrt{E_{Ti}} \cos\theta \, \bar{s}_i \bar{R}(\tau) + \bar{n}_c$$

$$\bar{r}_s = a\sqrt{E_{Ti}} \sin\theta \, \bar{s}_i \bar{R}(\tau) + \bar{n}_s \qquad (4.1)$$

$$R(\tau) = \begin{bmatrix} R_{11}(\tau) & \ldots & R_{1N}(\tau) \\ \vdots & & \vdots \\ R_{N1}(\tau) & \ldots & R_{NN}(\tau) \end{bmatrix}$$

It is interesting to observe the effect that the channel and demodulator have on the "transmitted vector". For example, the vectors shown in Figure 6A may be mapped into those of 6B. The random phase and amplitude simply cause a scaler multiplication of the received vector while the time delay τ gives rise to a linear transformation. This transformation actually rotates the vector in the signal space. Finally the Gaussian noise effectively adds a random vector to the rotated signal vector as shown in Figure 6B.

One further step remains before we can actually calculate $P_{\frac{\bar{r}}{m}}\left(\frac{R}{m}\right)$ and that is to calculate the statistics of $\bar{n}_c$ and $\bar{n}_s$.

It may be shown[1] that $n_c(t)$ and $n_s(t)$ are Gaussian processes with:

[1] Wozancraft & Jacobs, p. 496.

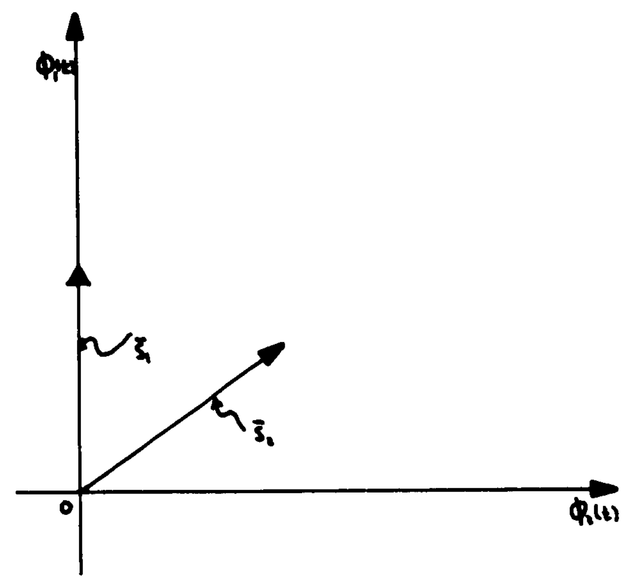

FIGURE 6A. TRANSMITTED SIGNAL VECTOR

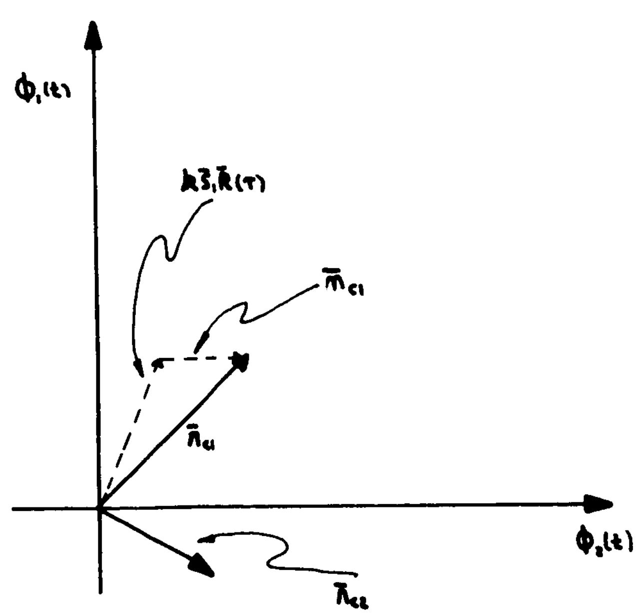

FIGURE 6B. OUTPUT VECTOR AT THE DEMODULATOR

$$E\,[n_c(t)\, n_c(t-\tau)] = R_c(\tau) = \int_{-w}^{w} \frac{N_0}{2} \cos 2\pi f\tau df = \frac{N_0}{2\pi\tau} \sin 2\pi w\tau \qquad (4.2)$$

$$E\,[n_s(t)\, n_s(t-\tau)] = R_s(\tau) = \int_{-w}^{w} \frac{N_0}{2} \cos 2\pi f\tau df = \frac{N_0}{2\pi\tau} \sin 2\pi w\tau$$

$$E\,[n_c(t)\, n_s(t-\tau)] = R_{cs}(\tau) = 0 \quad \text{where} \quad s_{nc}(f) = s_{ns}(f) = \begin{cases} \frac{N_0}{2} & |f| < w \\ 0 & \text{elsewhere} \end{cases}$$

Since $n_s(t)$ and $n_c(t)$ are Gaussian processes then

$$n_{ci} = \int_0^T n_c(t)\, \phi_i(t)\, dt$$

$$n_{si} = \int_0^T n_s(t)\, \phi_i(t)\, dt$$

are Gaussian random variables. In order to find the joint distribution of the n_{ci}'s and n_{si}'s we must determine the following:

$$E[n_{ci}] = \int_0^T E[n_c(t)]\, \phi_i(t)\, dt = 0$$

$$E[n_{si}] = \int_0^T E[n_c(t)]\, \phi_i(t)\, dt = 0 \qquad (4.3)$$

$$E[n_{si}\, n_{cj}] = \int_0^T\!\!\int_0^T E\,[n_c(t)\, n_s(u)]\, \phi_i(t)\, \phi_j(u)\, dt\, dt$$

$$= \int_0^T\!\!\int_0^T R_{cs}(t-u)\, \phi_i(t)\, \phi_j(u)\, dt\, du = 0$$

$$E\,[n_{ci}\, n_{cj}] = \int_0^T\!\!\int_0^T E[n_c(t)\, n_c(u)]\, \phi_i(t)\, \phi_j(u)\, dt\, du$$

$$= \int_0^T\!\!\int_0^T R_c(t-u)\, \phi_i(t)\, \phi_j(u)\, dt\, du$$

Then, using (4.2) we could solve for $E\,[n_{ci}\, n_{cj}]$. However, this is a complicated expression and is dependent on the choice of the $\phi_j(t)$.

Since $n_c(t)$ and $n_s(t)$ have a power density spectrum which is equal to $N_0/2$ for $-w < f < w$, the signal spectrum, we can assume the power spectrum of the noise uniform for all f because of the fact that out-of-band noise doesn't affect the performance of an optimum receiver[1]. This makes $R_c(t-u) = R_s(t-u) = u_0(t-u)\frac{N_0}{2}$ and simplifies (4.3) considerably:

$$E\,[n_{ci}\,n_{cj}] = \int_0^T \int_0^T \frac{N_0}{2}\,u_0(t-u)\;\phi_i(t)\;\phi_j(u)\;dt\;du = \frac{N_0}{2}\,\delta_{ij}$$

Similarly

$$E\,[n_{si}\,n_{sj}] = \int_0^T \int_0^T \frac{N_0}{2}\,u_0(t-u)\;\phi_i(t)\;\phi_j(u)\;dt\;du = \frac{N_0}{2}\,\delta_{ij}$$

Thus n_{si}, n_{ci} $(i = 1,2,\ldots,N)$ are statistically independent random variables and have a joint distribution.

$$P_{n_{ci} \ldots n_{cn},\, n_{si} \ldots n_{sn}}(a_{ci} \ldots a_{cn},\, b_{si} \ldots b_{sn})$$

$$= \frac{1}{(\pi N_0)^{N/2}}\, e^{-\frac{1}{N_0}(a_{ci}^2 + \ldots + a_{cn}^2 + b_{si}^2 + \ldots + b_{sn}^2)}$$

or, in vector notation

$$P_{\bar{n}i,\, \bar{n}s}(\bar{u},\, \bar{v}) = P_{\bar{n}c}(\bar{u})\; P_{\bar{n}s}(\bar{v}) =$$

$$= \frac{1}{(\pi N_0)^{N/2}}\, e^{\left[-\frac{1}{N_0}(|\bar{u}|^2 + |\bar{v}|^2)\right]} \tag{4.4}$$

We are now in a position to calculate $P_{\frac{\bar{r}}{m}}\left(\frac{\bar{R}}{m_i}\right)$ or, for this specific case $P_{\frac{\bar{r}_c,\bar{r}_s}{m}}\left(\frac{\bar{R}_c,\bar{R}_s}{m_i}\right)$

---

[1] Theorem of Irrelevancy. See Wozencraft & Jacobs, p. 220. Sufficient Statistics. See Van Trees, p. 33.

Introducing the following definition to simplify notation

$$P_{\frac{\bar{\alpha}}{\bar{\beta}}}\left(\frac{\bar{a}}{\bar{b}}\right) = \int_{-\infty}^{\infty} P_{\frac{\bar{\alpha}}{\bar{\beta},\bar{\gamma}}}\left(\frac{\bar{a}}{\bar{b},\bar{c}}\right) P_{\bar{\gamma}}(\bar{c})\, d\bar{c} = \overline{P_{\frac{\bar{\alpha}}{\bar{\beta},\bar{\gamma}}}\left(\frac{\bar{a}}{\bar{b},\bar{c}}\right)}^{\gamma}$$

we have:

$$P_{\frac{\bar{r}_c,\bar{r}_s}{m}}\left(\frac{R_c,R_s}{m_i}\right) = \overline{P_{\frac{\bar{r}_c,\bar{r}_s}{m,\theta,a}}\left(\frac{R_c,R_s}{m_i,\theta,A}\right)}^{\theta,a}$$

Now if $\bar{r}_c = \bar{\alpha}$ and $\bar{r}_s = \bar{\beta}$ on a given trial, then

$$\bar{n}_c = \bar{\alpha} - a\sqrt{E_{Ti}}\cos\theta\, \bar{s}_i\, \bar{R}(\tau)$$

$$\bar{n}_s = \bar{\beta} - a\sqrt{E_{Ti}}\sin\theta\, \bar{s}_i\, \bar{R}(\tau)$$

and we may write

$$P_{\frac{\bar{r}_c,\bar{r}_s}{m,\theta,a}}\left(\frac{\bar{\alpha},\bar{\beta}}{m_i,\theta,A}\right) = P_{\frac{\bar{n}_c,\bar{n}_s}{m,\theta,a}}\left(\frac{\bar{\alpha}-A\sqrt{E_{Ti}}\cos\theta\,\bar{s}_i\bar{R}(\tau),\ \bar{\beta}-A\sqrt{E_{Ti}}\sin\theta\,\bar{s}_i\bar{R}(\tau)}{m,\theta,A}\right)$$

$$= P_{\bar{n}_c,\bar{n}_s}\left(\bar{\alpha}-A\sqrt{E_{Ti}}\cos\theta\,\bar{s}_i\bar{R}(\tau),\ \bar{\beta}-A\sqrt{E_{Ti}}\sin\theta\,\bar{s}_i\bar{R}(\tau)\right) \qquad (4.5)$$

where the last line is justified by the assumption that $\bar{n}_c$, $\bar{n}_s$ are statistically independent of m, θ, a. Substituting equation (4.4) into (4.5) we have

$$P_{\frac{\bar{r}_c,\bar{r}_s}{m,\theta,a}}\left(\frac{\bar{\alpha},\bar{\beta}}{m_i,\theta,A}\right) = \frac{1}{(\pi N_0)^{N/2}} \exp -\frac{1}{N_0}\left[\left(\bar{\alpha}-A\sqrt{E_{Ti}}\cos\theta\,\bar{s}_i\bar{R}(\tau)\right)^2 - \left(\bar{\beta}-A\sqrt{E_{Ti}}\sin\theta\,\bar{s}_i\bar{R}(\tau)\right)^2\right]$$

(4.6)

Expanding and factoring out terms which do not depend on i we have, for the right side of equation (4.6):

$$k(\bar{\alpha},\bar{\beta})\,e^{+\frac{2}{N_0}A\sqrt{E_{Ti}}\left[\bar{S}_i\bar{R}(\tau)\bar{\alpha}^T\cos\theta+\bar{S}_i\bar{R}(\tau)\bar{\beta}^T\sin\theta\right]-A^2E_{Ti}\bar{S}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{S}_i/N_0}$$

The probability distributions for a and θ are Rayleigh and uniform respectively (see Figure 7).

$$P_a(A)=\frac{A}{\gamma^2}e^{-\frac{A^2}{2\gamma^2}}\qquad A\geq 0$$

$$P_a(\theta)=\frac{1}{2\pi}\qquad 0\leq\theta<2\pi$$

Letting $k(\bar{\alpha},\bar{\beta})=k$, and averaging with respect to θ and a gives,

$$P_{\frac{\bar{r}_c,\bar{r}_s}{m}}\left(\frac{\bar{\alpha},\bar{\beta}}{m_i}\right)=k\int_0^{2\pi}\int_0^{\infty}\left[e^{+\frac{2}{N_0}A\sqrt{E_{Ti}}\left[\bar{S}_i\bar{R}(\tau)\bar{\alpha}^T\cos\theta+\bar{S}_i\bar{R}(\tau)\bar{\beta}^T\sin\theta\right]}+e^{-\frac{A^2}{N_0}E_{Ti}\bar{S}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{S}_i^T}\right]P_a(A)\,P_\theta(\theta)\,dA\,d\theta$$

$$=\frac{k}{2\pi\gamma^2}\int_0^{2\pi}\int_0^{\infty}A\,e^{+\frac{2}{N_0}A\sqrt{E_{Ti}}\left[\bar{S}_i\bar{R}(\tau)\bar{\alpha}^T\cos\theta+\bar{S}_i\bar{R}(\tau)\bar{\beta}^T\sin\theta\right]-\frac{A^2}{N_0}E_{Ti}\bar{S}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{S}_i^T-\frac{A^2}{2\gamma^2}}\,dA\,d\theta$$

Transforming to cartesian coordinates by the following change of variables,

$$x=A\cos\theta$$

$$y=A\sin\theta\qquad dx\,dy=A\,dA\,d\theta$$

we have;

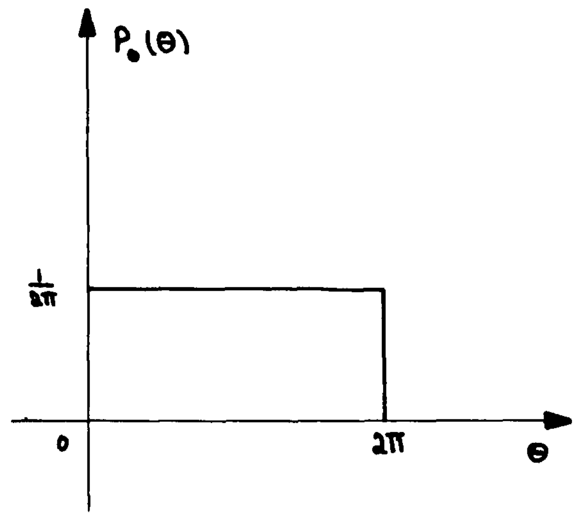

FIGURE 7A. UNIFORM DISTRIBUTION

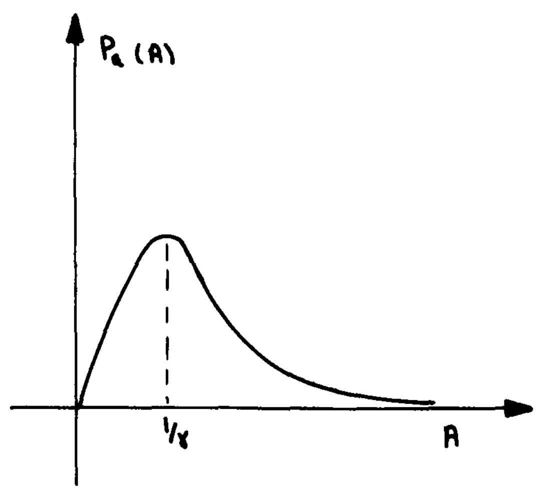

FIGURE 7B. RAYLEIGH DISTRIBUTION

$$= \frac{k}{2\pi\gamma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ell^{+\frac{2}{N_0}\sqrt{E_{Ti}}\left[\bar{s}_i\bar{R}(\tau)\bar{\alpha}^T X + \bar{s}_i\bar{R}(\tau)\bar{\beta}^T Y\right] - (X^2+Y^2)\left(\frac{E_{Ti}\bar{s}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{s}_i^T}{N_0} + \frac{1}{2\gamma^2}\right)} dX\,dY$$

$$= \frac{k}{2\pi\gamma^2}\int_{-\infty}^{\infty} \ell^{-\frac{X^2 E_{Ti}\bar{s}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{s}_i^T}{N_0} + \frac{2\sqrt{E_{Ti}}\left[\bar{s}_i\bar{R}(\tau)\bar{\alpha}^T\right]X}{N_0} - \frac{X^2}{2\gamma^2}} dX$$

$$\cdot \int_{-\infty}^{\infty} \ell^{-\frac{Y^2 E_{Ti}\bar{s}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{s}_i^T}{N_0} + \frac{2\sqrt{E_{Ti}}\left[\bar{s}_i\bar{R}(\tau)\bar{\beta}^T\right]}{N_0} - \frac{Y^2}{2\gamma^2}} dY$$

These integrals are modified forms of the Gaussian distributions and may be evaluated to give

$$P_{\frac{\bar{T}_i,\bar{T}_s}{m}}\left(\frac{\bar{\alpha},\bar{\beta}}{m_i}\right) = \frac{k}{\frac{E_{Ti}}{N_0}2\gamma^2\bar{s}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{s}_i^T + 1}\exp\left\{X_i^2\left[\frac{E_{Ti}}{E_{Ti}\bar{s}_i\bar{R}(\tau)\bar{R}^T(\tau)\bar{s}_i^T N_0 + N_0^2/2\gamma^2}\right]\right\} \qquad (4.7)$$

where $X_i^2 = (\bar{s}_i\bar{R}(\tau)\bar{\alpha}^T) + (\bar{s}_i\bar{R}(\tau)\bar{\beta}^T)$ (4.8)

x_i^2 has its maximum value when τ, the relative delay between the incoming signal and the correlated signal, is zero. This allows us to estimate the arrival time of a signal as the moment when x_i^2 is the largest. At this instant $x_i^2 = (\bar{s}_i \cdot \bar{\alpha})^2 + (\bar{s}_i \cdot \bar{\beta})^2$ and

$$P_{\frac{\bar{r}_c, \bar{r}_s}{m}}\left(\frac{\bar{\alpha}, \bar{\beta}}{m_i}\right) = \frac{k}{\frac{E_{Ti}}{N_0} 2\gamma^2 + 1} \exp\left\{X_i^2 \left(\frac{E_{Ti}}{N_0 E_{Ti} + N_0^2/2\gamma^2}\right)\right\} \qquad (4.9)$$

where $\bar{s}_i \cdot \bar{\alpha}$ is the correlation of s_i and α; $\int_0^T s_i(t)\,\alpha(t)\,dt$ in terms of the time varying signals.

Once we assume a signal has been detected we also wish to determine which of the m_j's it was. By substituting (4.8) into the decision rules of the previous section we see that N-ary decision rule becomes: choose $m = m_j$, iff $P_i A_i\, e^{B_i x_i^2}$ is a max for $i = j$ where

$$A_i = \frac{P_i}{\left(\frac{E_{T_i}}{N_0}\right) 2\gamma^2 + 1}$$

$$B_i = \frac{E_{T_i}}{N_0 E_{T_i} + N_0^2/2\gamma^2}$$

While the modified binary rule becomes;

choose m_2 or m_3 or . . . n_n

$$\text{iff} \sum_{j=2}^{N} P_j\, A_j\, e^{B_j\, x_j^2} > P_1\, A_1\, e^{B_1\, x_1^2}$$

otherwise choose m_1. A receiver which will generate the estimated message is shown in Figure 8. The fact that in practice only a finite number of correlation devices are used, may lead to a case where the maximum value of x_i^2 will not occur when $\tau = 0$. Note that by using multiple correlators the value of τ is now used to indicate the miss time by closest correlator. This is just another way of saying, what

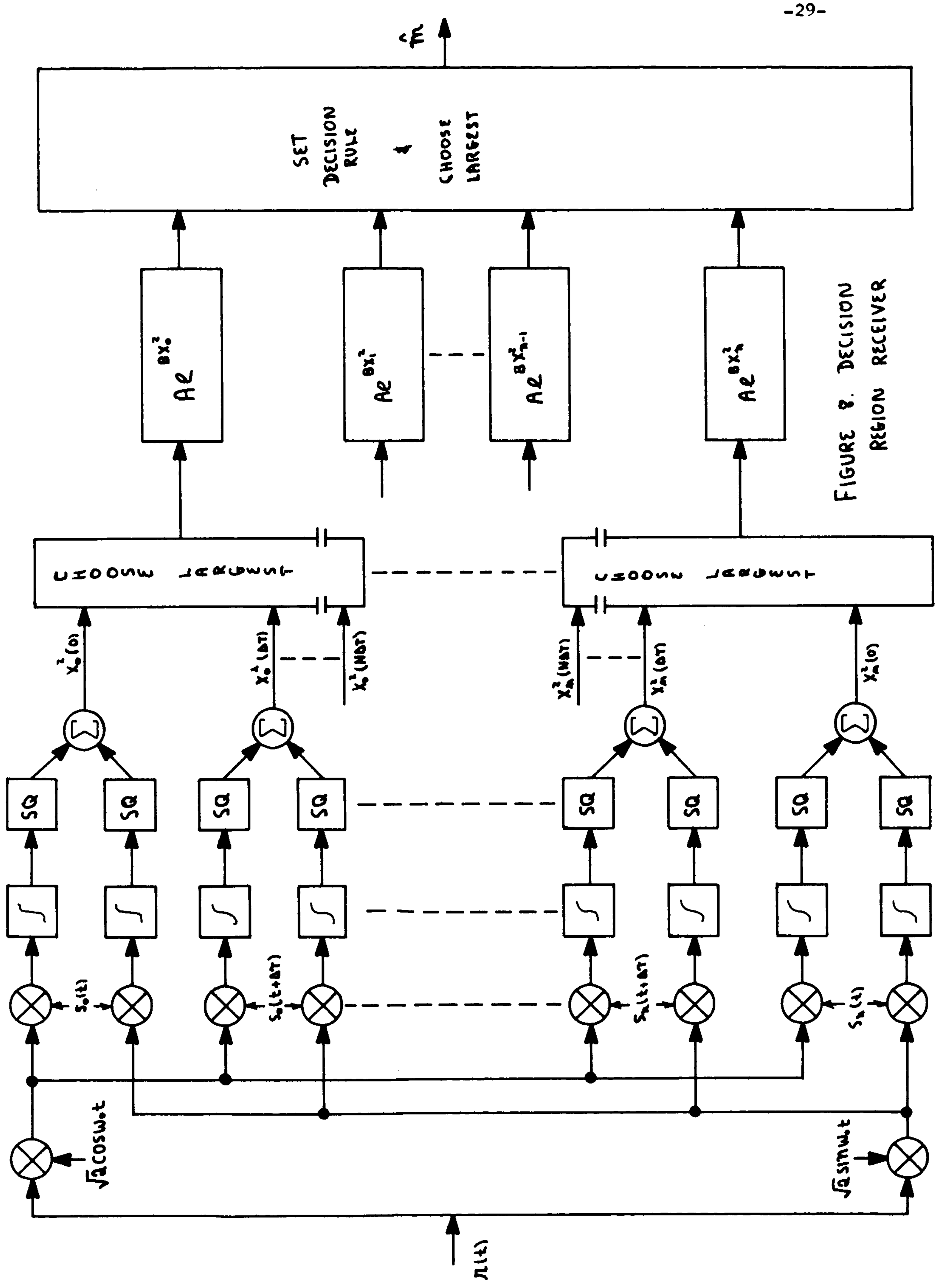

FIGURE 8. DECISION REGION RECEIVER

happens to the probability of error when the receiver misses locking onto the received input by τ seconds. Numerical values of error probability for various τ's are presented in Chapter 7.

CHAPTER 5.

DERIVATION OF THE STATISTICS OF x_i^2

After developing the decision rule for an "optimum" receiver, the question arises as to just how optimum the receiver really is. An answer to this question requires the calculation of P[E] for different values of E_{T_i}, N_0, and τ. The only additional information needed to make these calculations is the statistics for the x_i^2.

We have two cases, the statistics of x_i^2 when message m_i was sent, and the statistics when message m_j $(j \neq i)$ was sent.

Case 1 - given m_i was sent;

$$\bar{\alpha} = a\sqrt{E_{T_i}}\cos\theta\,\bar{s}_i\,\bar{R}(\tau) + \bar{n}_c$$

$$\bar{\beta} = a\sqrt{E_{T_i}}\sin\theta\,\bar{s}_i\,\bar{R}(\tau) + \bar{n}_s$$

$$x_i^2 = (\bar{\alpha}\cdot\bar{s}_i)^2 + (\bar{\beta}\cdot\bar{s}_i)^2$$

$$(\bar{\alpha}\cdot\bar{s}_i) = a\sqrt{E_{T_i}}\cos\theta\,(\bar{s}_i\,\bar{R}(\tau)^T\,\bar{s}_i^T) + \bar{n}_c\,\bar{s}_i^T$$

$$= b_i + \sum_{j=1}^{N} n_{c_j}\,(s_i)_j$$

$$= b_i + \sum_{j=1}^{N} n_{c_j}\,s_{i_j}$$

where n_{c_j} are independent zero mean Gaussian random variables. Since $(\bar{\alpha}\cdot\bar{s}_i)$ is the sum of Gaussian random variables, then it also is a Gaussian random variable. Thus in order to specify $\bar{\alpha}\cdot\bar{s}_i$ completely we need only find its mean variance

$$E\,[\bar{\alpha}\cdot\bar{s}_i] = b_i + \sum_{j=1}^{N} E[n_{c_j}]\,s_{ij} = b_i$$

$$[\bar{\alpha} \cdot \bar{s}_i]^2 = b_i^2 + 2b_i \sum_{j=1}^{N} n_{cj}\, s_{ij} + \sum_{j=1}^{N} \sum_{k=1}^{N} n_{ci}\, n_{ck}\, s_{ij}\, s_{ik}$$

$$E\,[\bar{\alpha} \cdot \bar{s}_i]^2 = b_i^2 + 2b_i \sum_{j=1}^{N} E\,[n_{cj}]\, s_{ij} + \sum_{j=1}^{N} \sum_{k=1}^{N} E\,[n_{ci}\, n_{ck}]\, s_{ij}\, s_{ik}$$

$$= b_i^2 + \sum_{j=1}^{N} \sigma_i^2\, s_{ij}^2 = b_i^2 + \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2$$

Since we have, from equation (4.3A), that $E\,[n_{ci}\, n_{ck}] = \sigma_i^2\, \delta_{ik} = \frac{N_0}{2}\, \delta_{ik}$

Therefore,

$$\sigma^2_{(\bar{\alpha} \cdot \bar{s}_i)} = E\,[\bar{\alpha} \cdot s_i]^2 - [E(\bar{\alpha} \cdot \bar{s}_i]^2 = \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2$$

$$P_{(\bar{\alpha} \cdot \bar{s}_i)}(x) = N\left(b_i, \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2\right)$$

$$b_i = a\sqrt{E_{T_i}}\, \cos\theta\, \bar{s}_i\, \overset{T}{\bar{R}}(\tau)\, \bar{s}_i^T$$

where $N(m, \sigma^2)$ is defined as a Gaussian random variable with mean m and variance σ^2

Similarly:

$$(\bar{\beta} \cdot \bar{s}_i) = a\sqrt{E_{T_i}}\, \sin\theta\, (\bar{s}_i\, \overset{T}{\bar{R}}(\tau)\, \bar{s}_i^T) + \bar{n}_s \cdot \bar{s}_i^T$$

$$P_{(\beta \cdot s_i)}(x) = N\left(c_i, \frac{N_0}{2} \sum_{j=1} s_{ij}^2\right)$$

$$c_i = a\sqrt{E_{T_i}}\, \sin\theta\, \bar{s}_i\, \overset{T}{\bar{R}}(\tau)\, \bar{s}_i^T$$

Hence x_i^2 is the sum of the squares of two normally distributed variables of different means and the same variances. From Papoulis[1], we have that if x is $N(n_1, \sigma^2)$ and y is $N(n_2, \sigma^2)$ then $w = \sqrt{x^2 + y^2}$ is

$$P_w(w) = \frac{w}{\sigma^2}\, e^{-(w^2 + n^2)/2\sigma^2}\, I_0\left(\frac{wn}{\sigma^2}\right) \qquad w > 0$$

---

[1] Papoulis, Probability, etc, p. 195.

where $n = \sqrt{n_1^2 + n_2^2}$ and $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} \, d\theta$

$$= \sum_{m=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2}$$

We want $z = w^2$, therefore

$$P_z(z) = \frac{1}{\left|\frac{dz}{dw}\right|} P_w(\sqrt{z})$$

$$= \frac{1}{2\sigma^2} e^{-(z + n^2)/2} \; I_0 \left(\frac{z^{1/2} n}{\sigma^2}\right) \qquad (5.1)$$

Letting $x = \bar{\alpha} \cdot \bar{s}_i$, $y = \bar{\beta} \cdot \bar{s}_i$, $z = x_i^2$, $n_1 = b_i$, and $n_2 = c_i$, we may use (5.1). Then

$$n = \sqrt{b_i^2 + c_i^2} = \sqrt{a^2 \, E_{T_i} (\bar{s}_i \; \bar{R}^T(\tau) \; \bar{s}_i^T)^2 \; (\cos^2\theta + \sin^2\theta)}$$

$$= a \sqrt{E_{T_i}} \; \bar{s}_i \; \bar{R}(\tau)^T \; \bar{s}_i^T \; , \quad \sigma^2 = \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2$$

So, if m_i was transmitted, the probability distribution for the x_i^2 is;

$$P_{\frac{x_i^2}{m}}\left(\frac{z}{m_i}\right) = \frac{1}{N_0 \sum_{j=1}^{N} s_{ij}^2} \exp\left[-\frac{\left[z + a^2 E_{Ti} (\bar{s}_i \bar{R}^T(\tau) \bar{s}_i^T)^2\right]}{N_0 \sum_{j=1}^{N} s_{ij}^2}\right] I_0 \left[\frac{2 z^{1/2} a \sqrt{E_{Ti}} \, \bar{s}_i \bar{R}^T(\tau) \bar{s}_i^T}{N_0 \sum_{j=1}^{N} s_{ij}^2}\right] \qquad (5.2)$$

$$z > 0$$

<u>Case 2</u> - given m_j was sent;

$$\bar{\alpha} = a \sqrt{E_{T_j}} \cos \theta \; \bar{s}_j \; \bar{R}(\tau) + \bar{n}_c$$

$$\bar{\beta} = a \sqrt{E_{T_j}} \sin \theta \; \bar{s}_j \; R(\tau) + \bar{n}_s$$

Proceeding in the same manner as before we can derive the distributions for $P_{(\bar{\alpha} \cdot \bar{s}_i)}(x)$ and $P_{(\bar{\beta} \cdot \bar{s}_i)}(x)$

$$P_{(\bar{\alpha}\,\cdot\,\bar{s}_i)}(x) = N\left(d_i, \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2\right)$$

$$P_{(\bar{\beta}\,\cdot\,\bar{s}_i)}(x) = N\left(e_i, \frac{N_0}{2} \sum_{j=1}^{N} s_{ij}^2\right)$$

with

$$d_i = a\sqrt{E_{T_j}} \cos\theta \; \bar{s}_i \; \bar{R}(\overset{T}{\tau}) \; \bar{s}_j^T$$

$$e_i = a\sqrt{E_{T_j}} \sin\theta \; \bar{s}_i \; \bar{R}(\overset{T}{\tau}) \; \bar{s}_j^T$$

Thus, when m_j is sent, the probability density function for x_i^2 is

$$P_{x_i^2/m}\left(\frac{z}{m_j}\right) = \frac{1}{N_0 \sum_{j=1}^{N} s_{ij}^2} \exp\left[-\frac{\left[z + a^2 E_{Tj}\left(\bar{s}_i \bar{R}^T(\tau)\bar{s}_j^T\right)^2\right]}{N_0 \sum_{j=1}^{N} s_{ij}^2}\right] I_0\left[\frac{2z^{1/2} a\sqrt{E_{Tj}}\, \bar{s}_i \bar{R}^T(\tau)\bar{s}_j^T}{N_0 \sum_{j=1}^{N} s_{ij}^2}\right] \qquad (5.3)$$

$$z > 0$$

It is easily seen from the derivation of these results that the $P_{x_i^2/m}(z/m_k)$ are independent whenever an orthogonal signal set is chosen. Furthermore, the above equations are valid for unequal signal energy, random carrier phase and amplitude, and arbitrary signal crosscorrelation functions. Thus, in theory anyway, one can set up the decision regions and evaluate the P[E] for any orthogonal N-signal receiver[1]. The next chapter is a specialization of these results to the Sunblazer signal format of two messages and noise.

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[1] Many times the expression for P[E] contains integrals which require a numerical solution.

CHAPTER 6.

SUNBLAZER SIGNAL ANALYSIS

The Sunblazer receiver is faced with making a decision as to whether m_1 (a binary zero), m_2 (a binary one) of m_3 (nothing) was transmitted. Using orthogonal signals we have

m_1	m_2	m_3 (noise case)
$\bar{s}_1 = (1, 0)$	$\bar{s}_2 = (0, 1)$	$\bar{s}_3 = (0, 0)$
$E_{T_1} = E_T$	$E_{T_2} = E_T$	$E_{T_3} = 0$
$P[m_1] = P$	$P[m_2] = P$	$P[m_3] = 1 - 2P$

The signal space is shown in Figure 9. Since, to a good approximation the forward and backward Barker codes are uncorrelated we have for the crosscorrelation matrix, the form

$$\bar{R}(\tau) = \begin{bmatrix} R(\tau) & 0 \\ 0 & R(\tau) \end{bmatrix}$$

Next lets look at the rules for this case.

6.1 3-Ary Signal Rule

From Chapter 4 the decision rule is set $\hat{m} = m_i$ if

$$P_i A_i e^{B_i x_i^2} < P_j A_j e^{B_j x_J^2} \quad \text{for } i \neq j \tag{6.1}$$

$$P_1 A_1 = \frac{kP}{2\frac{E_T}{N_0}\gamma^2 + 1} \qquad B_1 = \frac{E_T}{N_0 E_T + N_0^2/2\gamma^2}$$

$$P_2 A_2 = \frac{kP}{2\frac{E_T}{N_0}\gamma^2 + 1} \qquad B2 = \frac{E_T}{N_0 E_T + N_0^2/2\gamma^2}$$

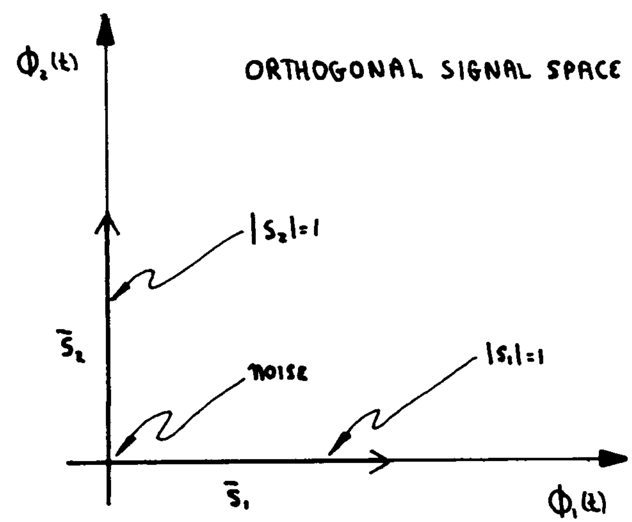


FIGURE 9. SUNBLAZER SIGNAL SET

$$P_3 A_3 = k(1 - 2P) \qquad B_3 = 0$$

Defining $B_1 = B_2 = B$, $A_1 = A_2 = A$, and $Q = {}^{P}A/_{1-P}$ we now can solve equation (6.1) for x_j^2.

$$\ln P_j A_j + B_j x_j^2 > \ln P_i A_i + B_i x_i^2$$

$$x_j^2 > \frac{B_i}{B_j} x_i^2 + \frac{1}{B_j} \ln \frac{P_i A_i}{P_j A_j} \qquad \begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix} \qquad i \neq j$$

These rules can best be summarized by the diagram shown in Figure 10. Note that this diagram applies only to the case for $Q < 1$ where point 1 (see Figure 10) is greater than zero. More will be said about the case where $Q > 1$ later on. In order to evaluate the probability of error we use equations (5.2) and (5.3). The calculation is most conveniently done in three steps by calculating the individual probability of errors for the cases when m_1, m_2 and m_3 are sent.

6.2 Probability of Error Given No Signal Present, $\bar{s}_3 = (0, 0)$ $E_{T_3} = 0$

The receiver makes a mistake in the event that the point (x_1^2, x_2^2) lies outside the shaded area in Figure 11. When no signal is present,

$$\left.\begin{matrix} x_1^2 = n_{c_1}^2 + n_{s_1}^2 \\ x_2^2 = n_{c_2}^2 + n_{s_2}^2 \end{matrix}\right\} \text{note: } x_1^2 \text{ and } x_2^2 \text{ are independent}$$

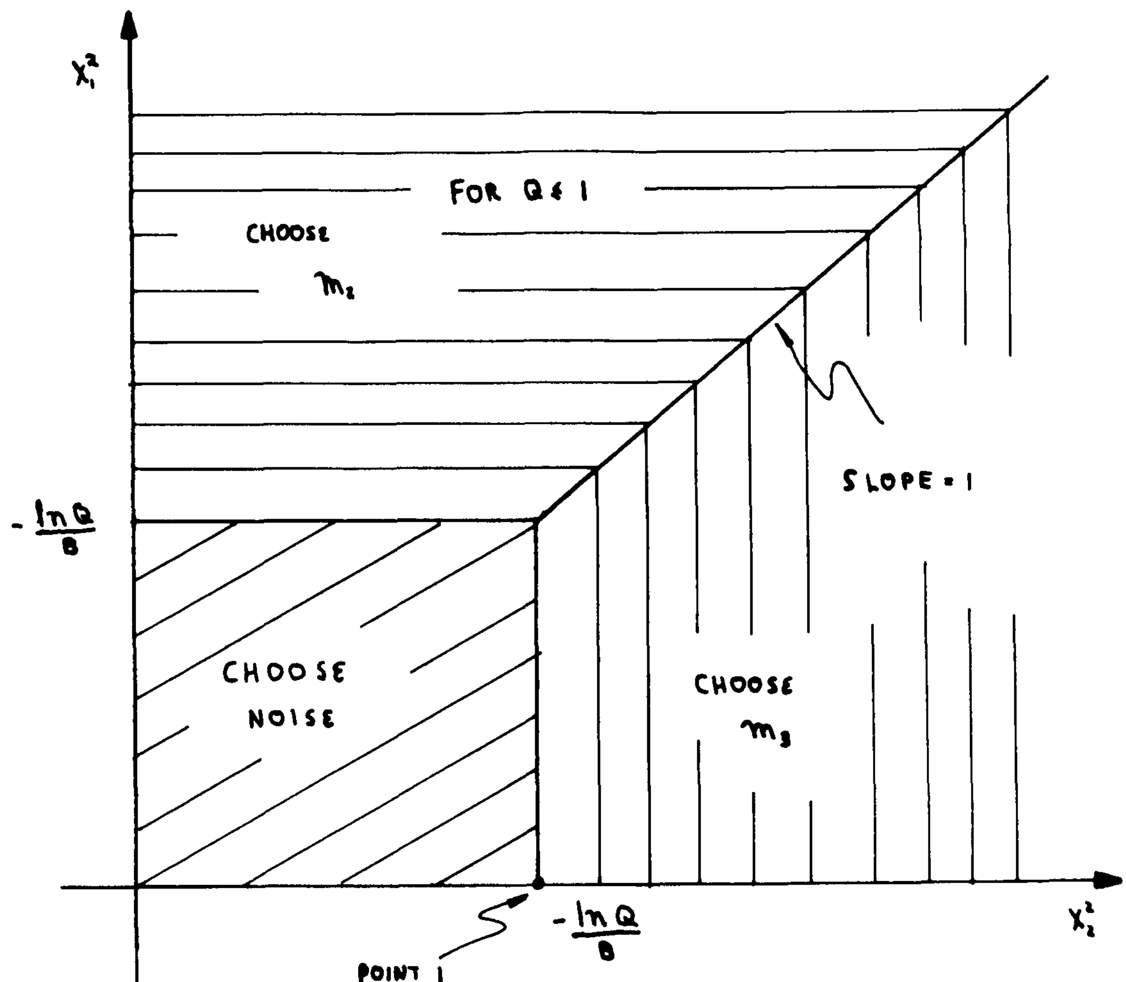


FIGURE 10. DECISION SPACE FOR 3-ARY CASE

and from equation (5.2)

$$P_{x_1^2}(z_1) = \frac{1}{N_0} e^{-z_1/N_0} \quad z_1 > 0$$

$$P_{x_2^2}(z_2) = \frac{1}{N_0} e^{-z_2/N_0} \quad z_2 > 0$$

then

$$P[E/_n] = \text{Prob [Error is made/given noise only]}$$

$$= 1 - \text{Prob [Correct decision/given noise only]}$$

$$= 1 - P\,[c/n]$$

The event $P\,[c/n]$ is shown in Figure 11.

$$P\,[c/n] = \int_0^{-\frac{\ln Q}{B}} \int_0^{-\frac{\ln Q}{B}} P_{x_1}(z_1)\, P_{x_2}(z_2)\, dz_1\, dz_2 \quad Q < 1$$

$$= \left[\int_0^{-\frac{\ln Q}{B}} \frac{1}{N_0} e^{-z/N} dz \right]^2$$

$$= [1 - Q^{1/BN_0}]^2$$

Defining $D = {}^{1/}BN_0$

$$P\,[E/n] = 1 - (1 - Q^D)^2$$

$$= 2Q^D - Q^{2D} \quad \text{for} \quad Q < 1 \qquad (6.2)$$

6.3 Probability of Error Given m_1 was Transmitted, $\bar{s}_1 = (1, 0)$, $E_{T_1} = E_T$

If m_1 was transmitted then we make an error if x_1^2 and x_2^2 lead to a point outside the shaded area in Figure 12. With signal m_1 present then

$$x_1^2 = (a\sqrt{E_T} \cos\theta\, R(\tau) + n_{c_1})^2 + (a\sqrt{E_T} \sin\theta\, R(\tau) + n_{s_1})^2$$

$$x_2^2 = n_{c_2}^2 + n_{s_2}^2$$

Note that x_1^2 and x_2^2 are independent random variables. Substituting into (5.2) for x_1^2 and (5.3) for x_2^2,

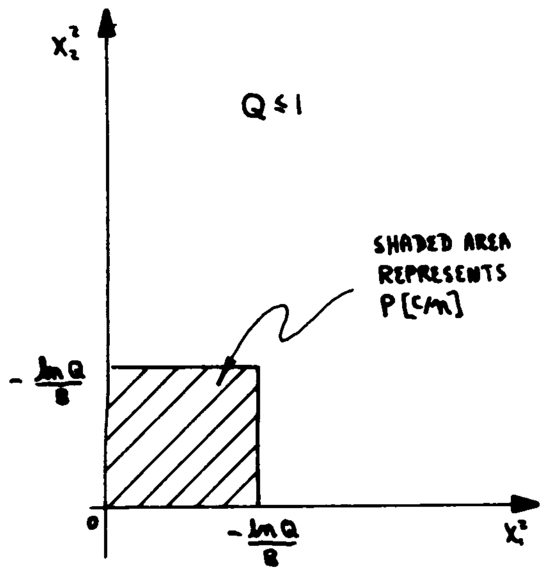


FIGURE 11. DECISION SPACE FOR THE CASE WHEN NOISE IS TRANSMITTED

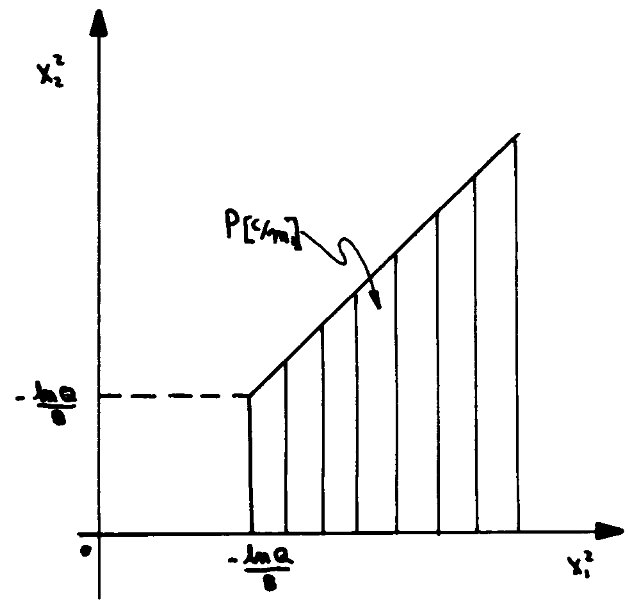


FIGURE 12. DECISION SPACE FOR THE CASE WHEN m_1 IS TRANSMITTED

$$P_{x_1^2}(z_1) = \frac{1}{N_0} \exp\left[-\frac{[z_1 + a^2 E_T R^2(\tau)]}{N_0}\right] I_0 \left[\frac{2z^{1/2}\, a \sqrt{E_T}\, R(\tau)}{N_0}\right] \qquad (6.3)$$

$$P_{x_2^2}(z_2) = \frac{1}{N_0} \exp\left[-\frac{z_2}{N_0}\right] \quad \begin{matrix} z_1 > 0 \\ z_2 > 0 \end{matrix} \qquad (6.4)$$

Then, averaging over the random amplitude (note the phase terms have already cancelled out) gives

$$P[c/m_1] = \overline{\int_{z_1=-\frac{\ln Q}{B}}^{\infty} P_{\chi_1^2}(z_1)\left[\int_{z_2=0}^{z_1} P_{\chi_2^2}(z_2)\,dz_2\right]dz_1}^{\,a}$$

$$= \int_0^{\infty} \frac{A}{2\gamma^2} e^{-A^2/2\gamma^2} \int_{z_1=-\frac{\ln Q}{B}}^{\infty} P_{\chi_1^2}(z_1)\left[\int_{z_2=0}^{z_1} P_{\chi_2^2}(z_2)\,dz_2\right] dz_1\, dA$$

$$= \int_{z_1=-\frac{\ln Q}{B}}^{\infty} \left\{\left[\int_0^{\infty} \frac{A}{\gamma^2} e^{-A^2/2\gamma^2} P_{\chi_1^2}(z_1)\, dA\right] \int_{z_2=0}^{z_1} P_{\chi_2^2}(z_2)\,dz_2\right\} dz_1$$

$$= \int_{z_1=-\frac{\ln Q}{B}}^{\infty} dz_1\, \overline{P_{\chi_1^2}(z_1)}^{\,a} \int_{z_2=0}^{z_1} P_{\chi_2^2}(z_2)\,dz_2$$

(6.5)

where

$$\overline{P_{x_1}^2(z_1)}^{a} \triangleq \int_{a=0}^{\infty} \frac{A}{\gamma^2} e^{-A^2/2\gamma^2} P_{x_1}^2(a)\, dA$$

We would like to evaluate this integral and then substitute it back into (6.5) replacing

$$= \int_{A=0}^{\infty} \frac{A}{N_0\gamma^2} e^{-A^2/2\gamma^2} \left[e^{-z_1/N_0} \cdot e^{-A^2 E_T R^2(\tau)/N_0} \right] I_0 \left[\frac{z^{1/2} A \sqrt{E_T} R(\tau)}{N_0} \right] \tag{6.6}$$

But $I_0(x)$ can be expressed in terms of an infinite series of the form

$$I_0(x) = \sum_{k=0} \frac{(x)^{2k}}{(k!)^2}$$

Therefore, (6.6) becomes

$$\int_{A=0}^{\infty} \frac{A}{N_0\gamma^2} e^{-A^2/2\gamma^2} \left[e^{-z_1/N_0} e^{-A^2 E_T R^2(\tau)/N_0} \right] \sum_{k=0}^{\infty} \frac{(z^{1/2} A \sqrt{E_T} R(\tau)/N_0)^{2k}}{(k!)^2}\, dA$$

$$= \int_{A=0}^{\infty} \frac{e^{-z_1}}{N_0\gamma^2} e^{-A^2 w} \sum_{k=0}^{\infty} \frac{(z^{1/2} A \sqrt{E_T} R(\tau)/N_0)}{(k!)^2}\, dA \tag{6.7}$$

and $w = \dfrac{E_T R^2(\tau)}{N_0} + \dfrac{1}{2\gamma^2}$

Interchanging the integral and summation signs of (6.7)

$$\sum_{k=0}^{\infty} \frac{e^{-z_1/N_0}}{N_0\ \gamma^2} \left(\frac{z^{1/2} \sqrt{E_T}\, R(\tau)}{N_0}\right)^{2k} \cdot \frac{1}{(k!)^2} \int_{A=0}^{\infty} A^{2k+1} e^{-wA^2}\, dA$$

This integral is straightforward and the expression may be evaluated to give

$$= \sum_{k=0}^{\infty} \frac{e^{-z_1}}{N_0 \gamma^2} \left(\frac{z_1 E_T R^2(\tau)}{N_0^2} \right)^k \cdot \frac{1}{(k!)^2} \frac{k!}{2W^{k+1}}$$

$$= \frac{e^{-z_1/N_0}}{2WN_0\gamma^2} \sum_{k=0}^{\infty} \left(\frac{z_1 E_T R^2(\tau)}{WN_0^2} \right)^k \cdot \frac{1}{k!}$$

$$= \frac{e^{-z_1}}{2WN_0\gamma^2} e^{\frac{z_1 E_T R^2(\tau)}{WN_0^2}} = \frac{1}{2WN_0\gamma^2} e^{-z_1/N_0\left(1 - \frac{E_T R^2(\tau)}{WN_0}\right)} \qquad (6.8)$$

Substituting w into (6.8), multiplying out terms and then recombining we have for $\overline{Px_1^2(z)^a}$;

$$\overline{Px_1^2(z_1)^a} = \frac{1}{N_0 c} e^{-\frac{z_1}{N_0 c}}$$

with $c = 2\left(\frac{E_T}{N_0}\right) \gamma^2 R^2(\tau) + 1$

Using this result with equation (6.5) and rearranging gives

$$P[c/m_1] = \int_{z_1 = -\frac{\ln Q}{B}}^{\infty} \int_{z_2=0}^{z_2=z_1} \frac{1}{N_0 c} e^{-\frac{z_1}{N_0 c} - \frac{z_2}{N_0}} dz_2\, dz_1 \qquad Q < 1$$

which may be evaluated, with $D = 1/BN_0$, as

$$P[c/m_1] = Q^{D/c} - \frac{1}{C+1} Q^{D(1+1/c)} \qquad Q < 1$$

or

$$P[E/m_1] = 1 - Q^{D/c} + \frac{Q^{\frac{D(c+1)}{c}}}{C+1} \qquad Q < 1$$

6.4 Probability of Error Given m_2 transmitted $s_2 = (0, 1)$, $E_{T_2} = E_T$

The receiver is incorrect whenever the point (x_1^2, x_2^2) is inside the shaded and dotted areas in Figure 12. When m_2 was transmitted

$$x_1^2 = nc_1^2 + ns_1^2$$

$$x_2^2 = (a \sqrt{E_T} \cos\theta\, R(\tau) + nc_2)^2 + (a \sqrt{E_T} \sin\theta\, R(\tau) + n_{s2})^2$$

and proceeding as in the last section

$$P_{x_1^2}(z_1) = \frac{1}{N_0} \exp\left[- {z_1}/{N_0}\right] \qquad \begin{matrix} z_1 > 0 \\ z_2 > 0 \end{matrix} \tag{6.9}$$

$$P_{x_2^2}(z_2) = \frac{1}{N_0} \exp\left[- \frac{(z_2 + a^2 E_T R^2(\tau))}{N_0}\right] I_0\left[\frac{2 z_2^{1/2} a \sqrt{E_T} R(\tau)}{N_0}\right] \tag{6.10}$$

But it may be observed that this is the same problem we just finished solving in the previous section if x_1^2 and x_2^2 are interchanged. Thus,

$$P[E/m_2] = 1 - Q^{D/c} + \frac{Q}{C+1}^{D(c+1)/c} \qquad Q \leq 1$$

6.5 Total Probability of Error for Case 1 with $Q \leq 1$

The total probability of error is defined as

$$P[E] = \sum_{i=1}^{3} P[m_i]\, P[E/m_i] \tag{6.11}$$

Substituting our previous results into (6.11) we have for the case 1 probability of error the following expression

$$P[E] = (1 - 2P)(2Q^D - Q^{2D}) + 2P \left(1 - Q^{D/c} + \frac{Q}{c+1}^{\frac{D(c+1)}{c}}\right) \qquad Q \leq 1 \tag{6.12}$$

where

$$Q = \frac{P}{\left(2 \frac{E_T}{N_0} \gamma^2 R^2(\tau)\right)(1 - 2P)}$$

$$D = 1 + \frac{1}{2\left(\frac{E_T}{N_0} \gamma^2\right)}$$

$$c = 2\left(\frac{E_T}{N_0} \gamma^2 R^2(\tau)\right) + 1$$

As mentioned previously this result is only valid when $Q \leq 1$. In the next section we look at the case when $Q > 1$.

6.6 P[E] When $Q > 1$ and Summary of Results

When $Q < 1$ point one in Figure 10 moves to the origin (it cannot go negative since $x_1^2 > 0$) and the decision space is shown in Figure 13. Note that the receiver will never say noise when $Q > 1$. This interesting phenomena will be examined in more detail in Chapter 7. Using the same definitions for $P[E/m_i]$ as in the last section we see immediately that

$$P[E/_n] = 1$$

When m_1 is sent, the statistics of x_1^2 and x_2^2 are given by (6.9) and (6.10). An error occurs when we say m_2 rather than m_1, even though m_2 was sent

$$P\,[E/m_1] = \overline{\int_{z_1=0}^{\infty} \int_{z_2=z_1}^{\infty_1} P_{x_1^2}(z_1)\, P_{x_2^2}(z_2)\, dz_2\, dz_1}^{\,a}$$

$$= \int_{z_1=0}^{\infty} \int_{z_2=z_1}^{\infty} \overline{P_{x_1^2}(z_1)}^{\,a}\, P_{x_2^2}(z_2)\, dz_2\, dz_1$$

$$= \int_{z_1=0}^{\infty} \int_{z_2=z_1}^{\infty} \frac{1}{N_0 c}\, e^{-\frac{z_1}{N_0 c}} \cdot \frac{1}{N_0}\, e^{-\frac{z_2}{N_0}}\, dz_2\, dz_1$$

$$= \frac{1}{c+1} \qquad Q > 1$$

By utilizing symmetry it can be shown that

$$P\,[E/m_2] = P\,[E/m_1] = \frac{1}{c+1}$$

Therefore, the probability of error when $Q > 1$ is

$$P\,[E] = \sum_{i=1}^{N} P\,[m_i]\, P\,[E/m_i]$$

$$= 1 - 2P + 2P\left(\frac{1}{c+1}\right) \qquad Q > 1$$

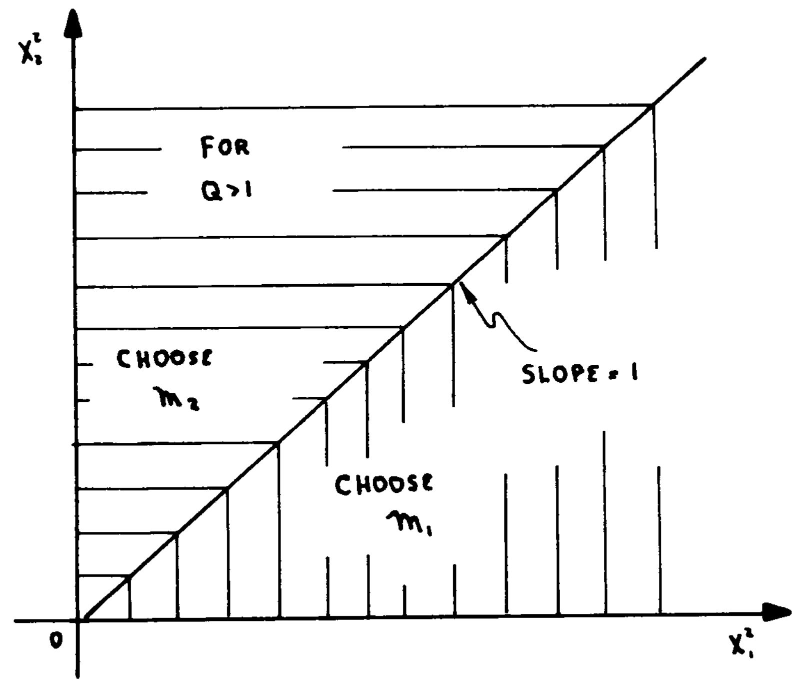


FIGURE 13. DECISION SPACE FOR THE CASE WHEN Q>1

When $Q > 1$ we have

$$\frac{P}{(1 - 2P)(2 \frac{E_T}{N_0} \gamma^2 + 1)} > 1$$

or

$$P > \frac{2 \frac{E_T}{N_0} \gamma^2 + 1}{4 \frac{E_T}{N_0} \gamma^2 + 3}$$

All the previous expressions can be simplified by observing that $\frac{E_T}{N_0} \gamma^2$ is actually the average received signal to noise ratio, which will hereafter be designated as $\frac{E_R}{N_0}$. Summarizing the results for case 1 we have, when

$$P \leq \frac{2 \frac{E_R}{N_0} + 1}{4 \frac{E_R}{N_0} + 3} \qquad (6.13)$$

$$P\ [E] = (1 - 2P)(2Q^D - Q^{2D}) + 2P \left(1 - Q^{D/c} + \frac{Q^{\frac{D(c+1)}{c}}}{c + 1}\right)$$

or when

$$P \geq \frac{2 \frac{E_R}{N_0} + 1}{4 \frac{E_R}{N_0} + 3}$$

$$P\ [E] = (1 - 2P) + 2P \left(\frac{1}{c + 1}\right)$$

where

$$Q = \frac{P}{(2 \frac{E_R}{N_0} + 1)(1 - 2P)}$$

$$D = 1 + \frac{1}{2 \frac{E_R}{N_0}}$$

$$c = 2 \frac{E_R}{N_0} R^2(\tau) + 1$$

6.7 Modified Binary Decision Rule - Case 2

By removing the line separating the decision spaces of m_1 and m_2 in Figure 10 we can develop a binary decision rule. (See Figure 14). We know that the optimum receiver will do as well as or better than this case since an optimum receiver, as the name implies, is the best one can do. The reason, however, for examining this case is twofold; first, it allows a check on the considerably more complicated optimum case and second, it presents a contrast to help decide what factors are most important for optimum performance.

We will follow the same general procedure for evaluating the performance of case 2 as we did for case 1.

For $Q \leq 1$ it is easily seen that $P[E/n]$ is exactly the same for this case as for case 1 and is given by (6.2)

$$P\,[E/n] = 2Q^D - Q^{2D} \quad \text{for} \quad Q < 1$$

When m_1 is sent the receiver makes an error if the point (x_1^2, x_2^2) lies inside the square of Figure 12. Then

$$P\,[E/m_1] = \overline{\int_{z_1=0}^{-\frac{\ln Q}{B}} P_{x_1^2}(z_1)\, dz_1 \int_0^{-\frac{\ln Q}{B}} P_{x_2^2}(z_2)\, dz_2}^{\,a} \qquad Q \leq 1 \tag{6.14}$$

where $P_{x_1^2}(z)$ and $P_{x_2^2}(z)$ are given by (6.3) and (6.4).

Rearranging (6.14) gives

$$P\,[E/m_1] = \int_0^{-\frac{\ln Q}{B}} \overline{P_{x_1^2}(z_1)}\, dz_1 \int_0^{-\frac{\ln Q}{B}} P_{x_2}(z_2)\, dz_2 \qquad Q \leq 1$$

$$= \int_0^{-\frac{\ln Q}{B}} \frac{1}{N_0 c}\, e^{-\frac{z_1}{N_0 c}}\, dz_1 \int_0^{-\frac{\ln Q}{B}} \frac{1}{N_0}\, e^{-\frac{z_2}{N_0}}\, dz_2$$

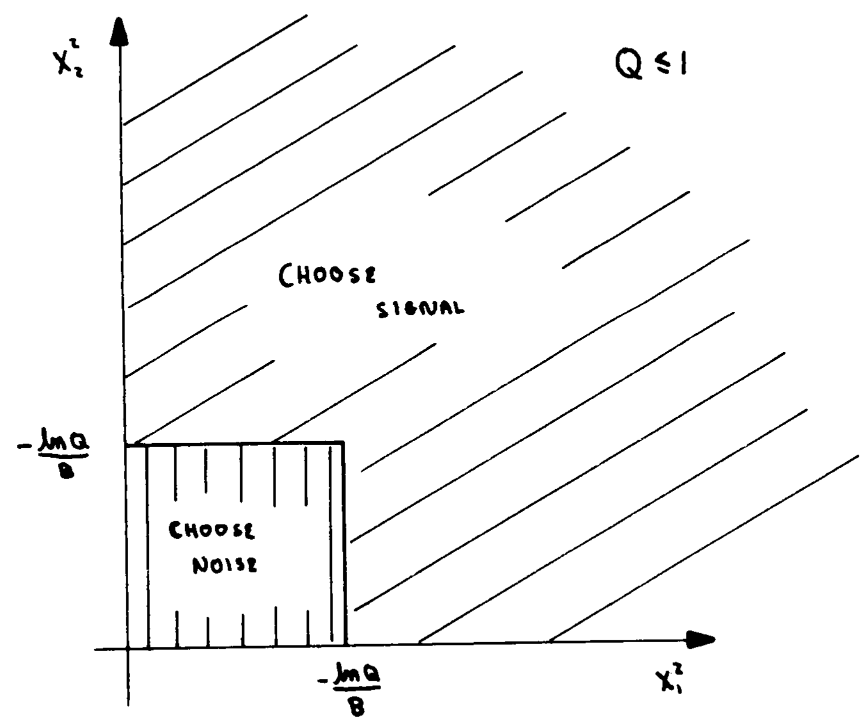


FIGURE 14. MODIFIED BINARY DECISION RULE

$$= (1 - Q^{D/c})(1 - Q^{D})$$

Again it can be seen by symmetry that $P[E/_{m_2}]$ will be the same as $P[E/_{m_1}]$.

$$P\ [E/_{m_2}] = (1 - Q^{D/c})(1 - Q^{D})$$

Finally when $Q > 1$ then the result is fairly obvious, since the receiver only guesses signal and therefore will be wrong anytime only noise is present. Therefore,

$$P\ [E/_{n}] = 1 \quad \text{for} \quad Q > 1$$

Using the fact that $P\ [E] = \sum_{i=1}^{3} P[m_i]\ P[E/_{m_i}]$ and collecting results we have, when

$$P \leq \frac{2\ (\frac{E_R}{N_0}) + 1}{4\ (\frac{E_R}{N_0}) + 3}$$

$$P[E] = (1 - 2P)(2Q^{D} - Q^{2D}) + 2P(1 - Q^{D})(1 - Q^{D/c})$$

or if

$$P \geq \frac{2\ (\frac{E_R}{N_0}) + 1}{2\ (\frac{E_R}{N_0}) + 3}$$

$$P[E] = 1 - 2P$$

where

$$Q = \frac{P}{(2\ (\frac{E_R}{N_0}) + 1)(1 - 2P)}$$

$$D = 1 + \frac{1}{2\ (\frac{E_R}{N_0})}$$

$$c = 2\ (\frac{E_R}{N_0})\ R^2(\tau) + 1$$

6.8 Optimum Binary Decision Rule - Case 3

In this section we design the optimum receiver to decide whether only noise is present or whether either message m_1 or m_2 was transmitted.

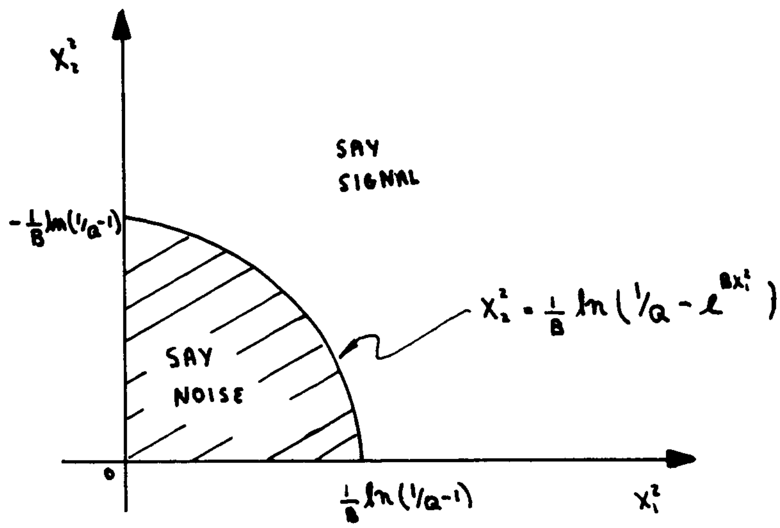


FIGURE 15. DECISION SPACE FOR OPTIMUM BINARY CASE

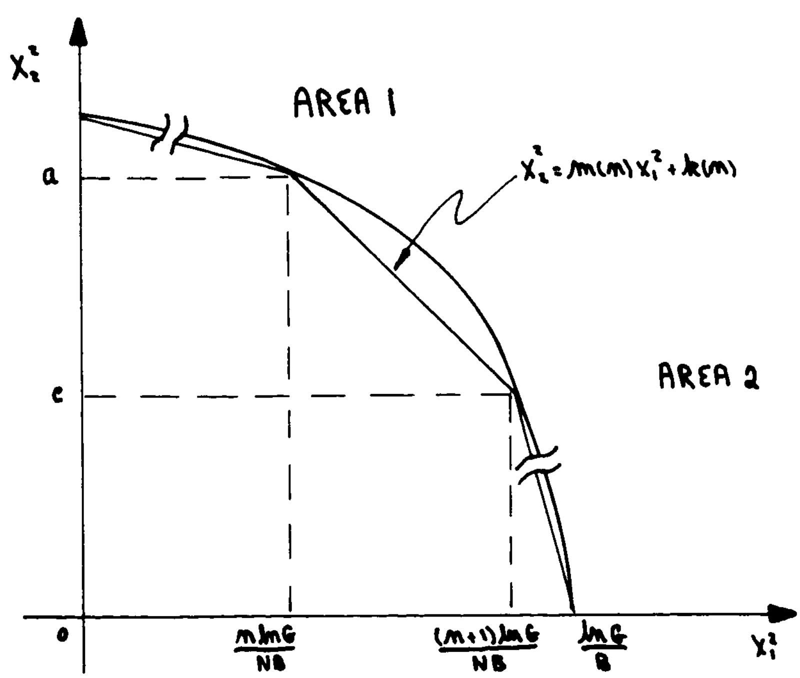


FIGURE 16. PIECEWISE LINEAR APPROXIMATION FOR CASE 3

The first step involves calculating the decision rule and space using the results of Chapter 4.

The decision rule is

$$A\ P\ e^{B\ x_1^2} + A\ P\ e^{B\ x_2^2} \underset{N}{\overset{s}{\gtrless}} 1 - 2P$$

Where the notation indicates to choose signal when the greater than equality holds, and noise when the less than does.

Solving this equation for x_2^2 gives

$$x_2^2 \underset{N}{\overset{s}{\gtrless}} \frac{1}{B} \ln \left(\frac{1 - 2P}{PA} - e^{B\ x_1^2}\right) \tag{6.14A}$$

As before we let $Q = \frac{PA}{1-2P}$

and so the decision rule becomes

$$x_2^2 \underset{N}{\overset{s}{\gtrless}} \frac{1}{B} \ln \left(\frac{1}{Q} - e^{B\ x_2^2}\right)$$

Since this function has a positive second derivative it is concave downwards. The decision space is shown in Figure 15. For convenience we define the intercept of the curve and the x_1^2 axis as G which makes

$$G = \frac{1}{Q} - 1$$

Thus the optimum receiver for the binary receiver "guesses" when $Q \leq \frac{1}{2}$, rather than when $Q \leq 1$, as was the case for the previous receiver.

Unfortunately, applying directly methods of solution used in the last two cases leads to integrals which cannot be evaluated in closed form. One apprpach is to "linearize" the problem by making an N-segment precise-linear approximation to the curve and then allowing $N \to \infty$. The piecewise linear approximation is shown in Figure 16. Next let's focus more closely on the n^{th} segment and determine its slope and intercept. Referring again to Figure 16, we see

$$a = \frac{1}{B} \ln \left(\frac{1}{Q} - G^{\frac{n}{N}}\right)$$

$$c = \frac{1}{B} \ln \left(\frac{1}{Q} - G^{\frac{n+1}{N}}\right)$$

Using the standard slope-intercept formulas the n^{th} segment's equation is of the form $x_2^2 = m(n)\, x_1^2 + k(n)$ where

$$m(n) = -\frac{N}{\ln G} \ln \left(\frac{\frac{1}{Q} - G^{\frac{n}{N}}}{\frac{1}{Q} - G^{\frac{n+1}{N}}} \right) \tag{6.15}$$

and

$$k(n) = \frac{1}{B} \ln \frac{\left(\frac{1}{Q} - G^{\frac{n}{N}}\right)^{n+1}}{\left(\frac{1}{Q} - G^{\frac{n+1}{N}}\right)^{n}} \tag{6.16}$$

6.9 Case 3 - P [E/noise] $Q \leq 1/2$

Referring to Figure 15 we see that the receiver is incorrect whenever noise only is present and the point (x_1^2, x_2^2) lies outside the shaded area. The statistics for x_1^2 and x_2^2 are the same as the noise only case in the last section.

$$P_{x_1^2}(z_1) = \frac{1}{N_0} e^{-\frac{z_1}{N_0}} \qquad P_{x_2^2}(z_2) = \frac{1}{N_0} e^{-\frac{z_2}{N_0}} \tag{6.17}$$

The expression for P[E/noise] can be broken up into two terms, area 1 (see Figure 16) which is the sum of all the contributions lying above the linear segments; and area 2 which is the region above the x_1^2-axis extending from $x_1^2 = \frac{1}{B} \ln G$ to ∞. With P[E/noise, N] denoting the N-segment approximation to P[E/noise] we have

$$P\left[E/_{\text{noise},N}\right] = \int_{\frac{\ln G}{B}}^{\infty} P_{\chi_1^2}(z_1)\,dz_1 \int_0^{\infty} P_{\chi_2^2}(z_2)\,dz_2$$

$$+ \sum_{m=0}^{N-1} \int_{\frac{m}{NB}\ln G}^{\frac{m+1}{NB}\ln G} P_{\chi_1^2}(z_1)\left[\int_{z_2=m(n)z_1+k(m)}^{\infty} P_{\chi_2^2}(z_2)\,dz_2\right]dz_1$$

(6.18)

Substituting (6.17) into (6.18) we have, for (6.18)

$$P\left[E/_{\text{noise},N}\right] = \frac{1}{N_0^2}\int_{\frac{\ln G}{B}}^{\infty}\int_0^{\infty} e^{-1/N_0(z_1+z_2)}\,dz_1\,dz_2$$

$$+ \sum_{m=0}^{N-1} \frac{1}{N_0^2}\int_{\frac{m}{NB}\ln G}^{\frac{m+1}{NB}\ln G}\int_{m(n)z_1+k(m)}^{\infty} e^{-1/N_0(z_1+z_2)}\,dz_1\,dz_2$$

(6.19)

Evaluating the first double integral, and the inner integral of the second, equation (6.19) reduces to

$$P\left[E/_{\text{noise},N}\right] = G^{-2} + \sum_{m=0}^{N-1} \frac{e^{-\frac{k(m)}{N_0}}}{N_0}\int_{\frac{m}{NB}\ln G}^{\frac{m+1}{NB}\ln G} e^{-z_1/N_0[1+m(n)]}\,dz_1$$

(6.20)

The integral in (6.20) can be evaluated straightforwardly noticing that we have two cases, when $m(n) + 1 = 0$ and when $m(n) + 1 \neq 0$

$$\int_{\frac{m}{NB}\ln G}^{\frac{m+1}{NB}\ln G} e^{-z_1/N_0(1+m(n))}\,dz_1 = \begin{cases} \dfrac{DN_0}{N}\ln G & m(n)+1=0 \\[2ex] \dfrac{N_0}{m(n)+1}\left[G^{-[m(n)+1]Dn/N} - G^{-[m(n)+1]D(n+1)/N}\right] & m(n)+1\neq 0 \end{cases}$$

Defining
$$T(n) = \frac{\left(\frac{1}{Q} - G^{\frac{n+1}{N}}\right)^{Dn}}{\left(\frac{1}{Q} - G^{\frac{n}{N}}\right)^{D(n+1)}} \tag{6.21}$$

Then (6.20) becomes

$$P[E/_{noise,N}] = G^{-D} + \sum_{m=0}^{N-1} \begin{cases} \dfrac{DT(m)\ln G}{N} & m(n)+1=0 \\[2ex] \dfrac{T(m)}{m(n)+1}\left(G^{-[m(n)+1]Dm/N} - G^{[m(n)+1]D(m+1)/N}\right) & m(n)+1\neq 0 \end{cases}$$

where D and G are defined as before.

6.10 Case 3 - $P[E/m_i]$, $Q \leq 1/2$

Since $P[E/m_1] = P[E/m_2]$ by symmetry arguments only $P[E/m_1]$ will be derived. It is somewhat easier to evaluate the integrals for this case if we find, instead of $P[E/m_1]$,

$$P[c/m_1] = P[\text{correct decision}/m_1\text{sent}] = 1 - P[E/m_1]$$

Since we are finding $P[c/m\ ]$ the space over which we integrate is the same as the last section. The distributions for x_1^2 and x_2^2 are

$$\overline{P_{x_1^2}(z_1)}^{\,a} = \frac{1}{N_0 c}\, e^{-\frac{z_1}{N_0 c}} \qquad z_1 > 0$$

$$P_{x_2^2}(z_2) = \frac{1}{N_0}\, e^{-\frac{z_2}{N_0}} \qquad z_2 > 0 \tag{6.22}$$

The expression for $\overline{P_{x_1^2}(z_1)^a}$ is given instead of $P_{x_1^2}(z_1)$ since it can be shown that any derivation for $P(E)$ uses $P_{x_1^2}(z_1)$ only in the form $\overline{P_{x_1^2}(z_1)^a}$. Breaking the integration into two regions to take into account the contributions of area above the curve and above the x_1^2 axis, and then averaging with respect to the random amplitude we have,

$$P\left[C/m_1, N\right] = \int_{A=0}^{\infty} \int_{z_1=\frac{\ln G}{B}}^{\infty} P_{x_1^2}(z_1)\,dz_1 \int_{z_2=0}^{\infty} P_{x_2^2}(z_2)\,dz_2\, P_a(A)\,dA$$

$$+ \sum_{m=0}^{N-1} \int_{A=0}^{\infty} \int_{z_1=\frac{m\ln G}{B}}^{\frac{(m+1)\ln G}{B}} P_{x_1^2}(z_1) \left[\int_{z_2=m(A)z_1+k(A)}^{\infty} P_{x_2^2}(z_2)\,dz_2 \right] dz_1\, P_a(A)\,dA$$

Rearranging the integrals and using the definition of $\overline{P_{x_1}(z_1)^a}$ gives

$$P\left[C/m_{1}, N\right] = \int_{z_1=\frac{\ln G}{B}}^{\infty} \overline{P_{x_1^2}(z_1)}^{a} \int_{z_2=0}^{\infty} P_{x_2^2}(z_2)\,dz_2$$

$$+ \sum_{m=0}^{N-1} \int_{z_1=\frac{m\ln G}{NB}}^{\frac{(m+1)\ln G}{NB}} \overline{P_{x_1^2}(z_1)}^{a} \left[\int_{z_2=m(m)+k(m)}^{\infty} P_{x_2^2}(z_2)\,dz_2\right] dz_1$$

(6.23)

Substituting equations (6.22) into (6.23) and collecting terms

$$P\left[C/m_{1}, N\right] = \int_{\frac{\ln G}{B}}^{\infty} \int_{0}^{\infty} \frac{1}{N_0^2 C}\, e^{-1/N_0\left(\frac{z_1}{C}+z_2\right)}\, dz_1\, dz_2$$

$$+ \sum_{m=0}^{N-1} \int_{\frac{m\ln G}{B}}^{\frac{(m+1)\ln G}{B}} \int_{m(m)z_1+k(m)}^{\infty} \frac{1}{N_0^2 C}\, e^{-\frac{1}{N_0}\left(\frac{z_1}{C}+z_2\right)}\, dz_2\, dz_1$$

Evaluating the first double integral and the inner integral of the second leaves us with

$$P\left[C/m_{1}, N\right] = G^{-D/C} + \sum_{m=0}^{N-1} e^{-\frac{k(m)}{N_0}} \int_{\frac{m\ln G}{NB}}^{\frac{m+1}{NB}\ln G} \frac{1}{N_0 C}\, e^{-\frac{z_1}{N_0}\left(1/C + m(m)\right)}\, dz_1$$

(6.24)

The remaining integral in (6.24) has two forms depending on the value of m(n) + 1/c.

$$\int_{\frac{m \ln G}{NB}}^{\frac{m+1}{NB}\ln G} \frac{1}{N_o C}\, e^{-\frac{z_1}{N_o}\left[1/c + m(m)\right]}\, dz_1 = \begin{cases} \dfrac{DN_o}{CN}\ln G & m(m) + 1/c = 0 \\[2ex] \dfrac{1}{m(m)c+1}\left[G^{-\frac{[1+m(m)c]Dm}{Nc}} - G^{-\frac{[1+m(m)c]D(m+1)}{Nc}} \right] & m(m) + 1/c \neq 0 \end{cases}$$

Using the above result and equations (6.16), (6.21), and (6.24) gives

$$P\left[c/m_1, N\right] = G^{-D/c} + \sum_{m=0}^{N-1} \begin{cases} \dfrac{T(m)\, D \ln G}{N} & m(m) + 1/c = 0 \\[2ex] \dfrac{T(m)}{m(m)c+1}\left[G^{-[1+m(m)c]Dm/Nc} - G^{-[1+m(m)c]D(m+1)/Nc} \right] \end{cases} \tag{6.25}$$

Note that the expression in equation (6.25) also represents $P[c/m_2, N]$ as mentioned earlier.

Now,

$$P[E/N] = \sum_{i=1}^{3} P[E/m_i, N]P[m_i]$$

$$= (1 - 2P)\, P\,[E/_{noise}, N] + 2P\,(1 - P\,[c/m_1, N])$$

As remarked before, these equations hold only when $Q \leq \frac{1}{2}$ or

$$P \leq \frac{2\left(\frac{E_R}{N_0}\right) + 1}{4\left(\frac{E_R}{N_0}\right) + 4}$$

6.11 P[E] for $Q > 1/2$ and Summary of Case 3 Results

When $Q > \frac{1}{2}$ the receiver will always say signal. However, with probability $1 - 2P$ only noise is present at the receiver input. Thus, the probability of error is $1 - 2P$.

$$P[E[= 1 - 2P$$

Summarizing the case 3 results we have;

for

$$P \leq \frac{2\left(\frac{E_R}{N_0}\right) + 1}{4\left(\frac{E_R}{N_0}\right) + 4}$$

$$P[E] = \lim_{N \to \infty} (1-2P)G^{-D} + 2P(1-G^{-D/c}) +$$

$$+ 2P \sum_{m=0}^{N-1} \begin{cases} \dfrac{DT(m)\ln G}{N} & m(m)+1 = 0 \\ \dfrac{T(m)}{m(m)+1}\left(G^{-[m(m)+1]Dm/N} - G^{-[m(m)+1]D(m+1)/N}\right) & m(m)+1 \neq 0 \end{cases}$$

$$+ 2P \sum_{m=0}^{N-1} \begin{cases} \dfrac{DT(m)\ln G}{NC} & m(m) + 1/c = 0 \\ \dfrac{T(m)}{m(m)c+1}\left(G^{-[m(m)c+1]Dm/NC} - G^{-[m(m)c+1]D(m+1)/NC}\right) & m(m)+\frac{1}{c} \neq 0 \end{cases}$$

while for

$$P > \frac{2\left(\frac{E_R}{N_0}\right) + 1}{4\left(\frac{E_R}{N_0}\right) + 4}$$

$$P[E] = 1 - 2P$$

where

$$Q = \frac{P}{(1 - 2P)(2\frac{E_R}{N_0} + 1)}$$

$$G = \frac{1}{Q} - 1$$

$$D = 1 + \frac{1}{2\left(\frac{E_R}{N_0}\right)}$$

$$T(n) = \left(\frac{\left(\frac{1}{Q} - G^{\frac{n+1}{N}}\right)^{Dn}}{\left(\frac{1}{Q} - G^{\frac{n}{N}}\right)^{D(n+1)}} \right)$$

$$m(n) = -\frac{N}{\ln G} \ln \frac{\frac{1}{Q} - G^{\frac{n}{N}}}{\frac{1}{Q} - G^{\frac{n+1}{N}}}$$

$$c = 2\frac{E_R}{N_0} R^2(\tau) + 1$$

In the next cahpter these rather involved equations are graphed and analyzed.

CHAPTER 7.

ANALYSIS OF RECEIVER PERFORMANCE VS. SIGNAL TO NOISE AND TIME DELAY

After considerable mathematical analysis we are now in a position to answer questions about the development of a receiver "philosophy". A comparison of P(E) for the different cases will reveal for which values of E_R/N_0 a particular receiver performs the best. We also have noticed the fact that there are times when a decision region receiver simply "throws in the towel" and guesses. A closer look at the reasons for this behavior and its effect on receiver performance will be made. Finally, the results of our bit-by-bit analysis will be extended by an example, to the more practical case of an error-correcting block code. The expressions developed in the previous chapter were quite complicated involving logarithms and non-integer values raised to non-integer powers. Therefore, in order to evaluate them for many points a computer program was written. The program also contains a graphing routine which displays the data both singly and five graphs to a page.

The different cases are studied in the next sections in the order in which they were derived.

7.1 Case 1, The 3-Ary Receiver

Observing the graphs of Figures 17-20, we notice that the expression for P[E] decreases monotonically with increasing signal to noise ratio (E_R/N_0). It is interesting to note that the complicated expression for P[E] for this case can be fairly accurately modeled over a range of E_R/N_0 from .1 to 100 by a function of the form

$P_{(e)}$ vs. E_b/N_o

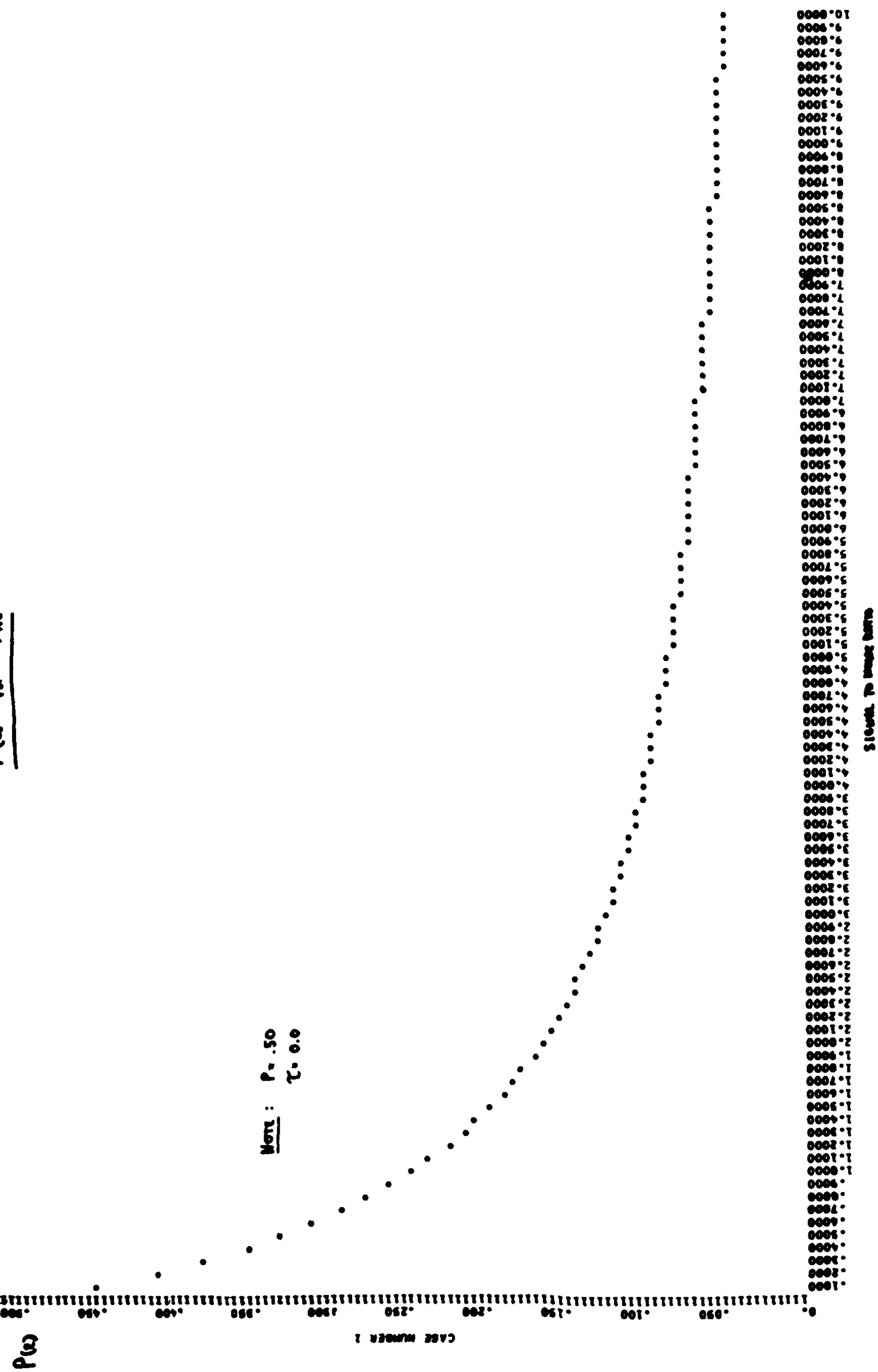


FIGURE 17

$P_{(e)}$ vs. E_b/N_o

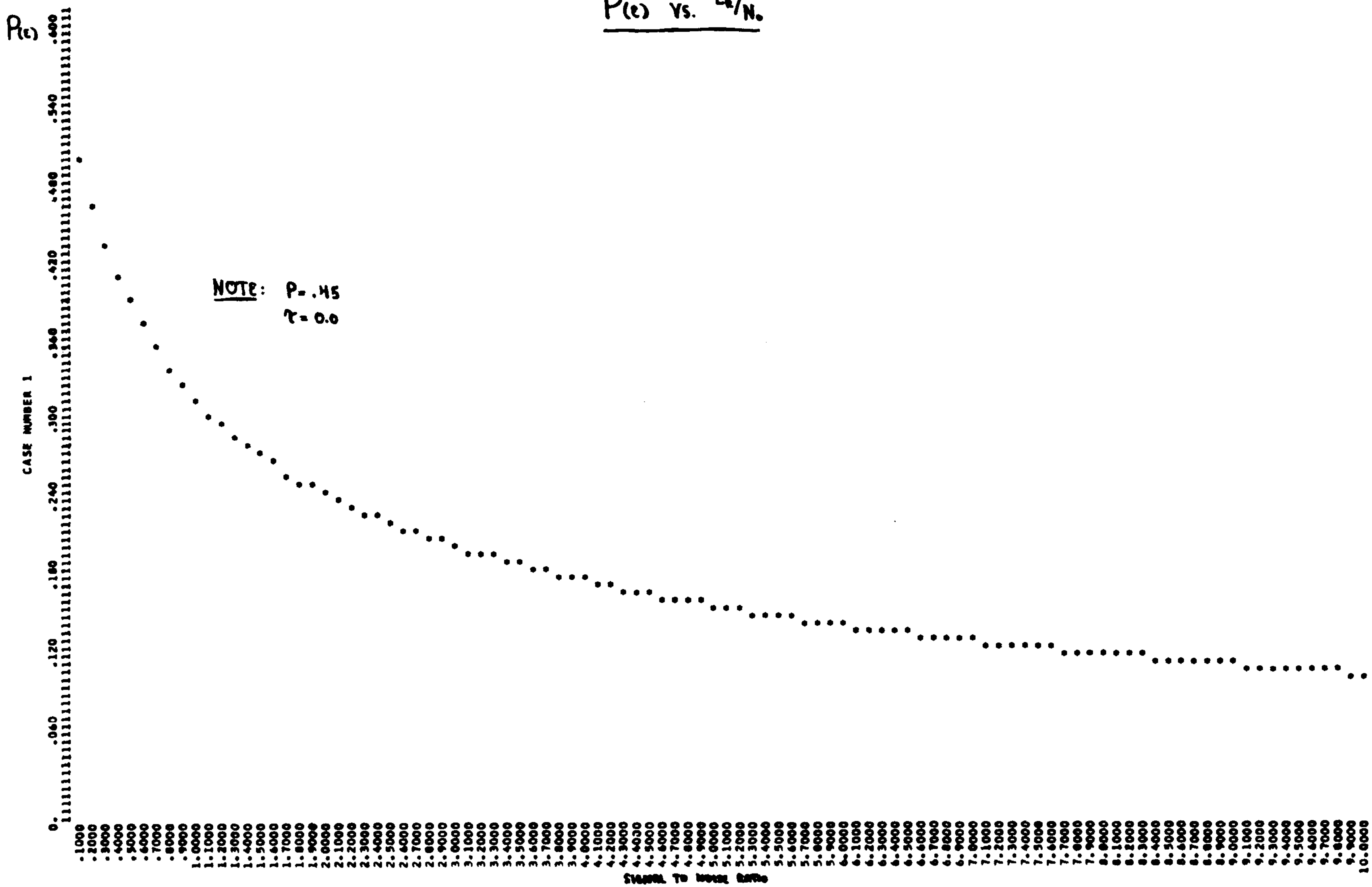


FIGURE 18

$P_{(E)}$ vs. E_b/N_0

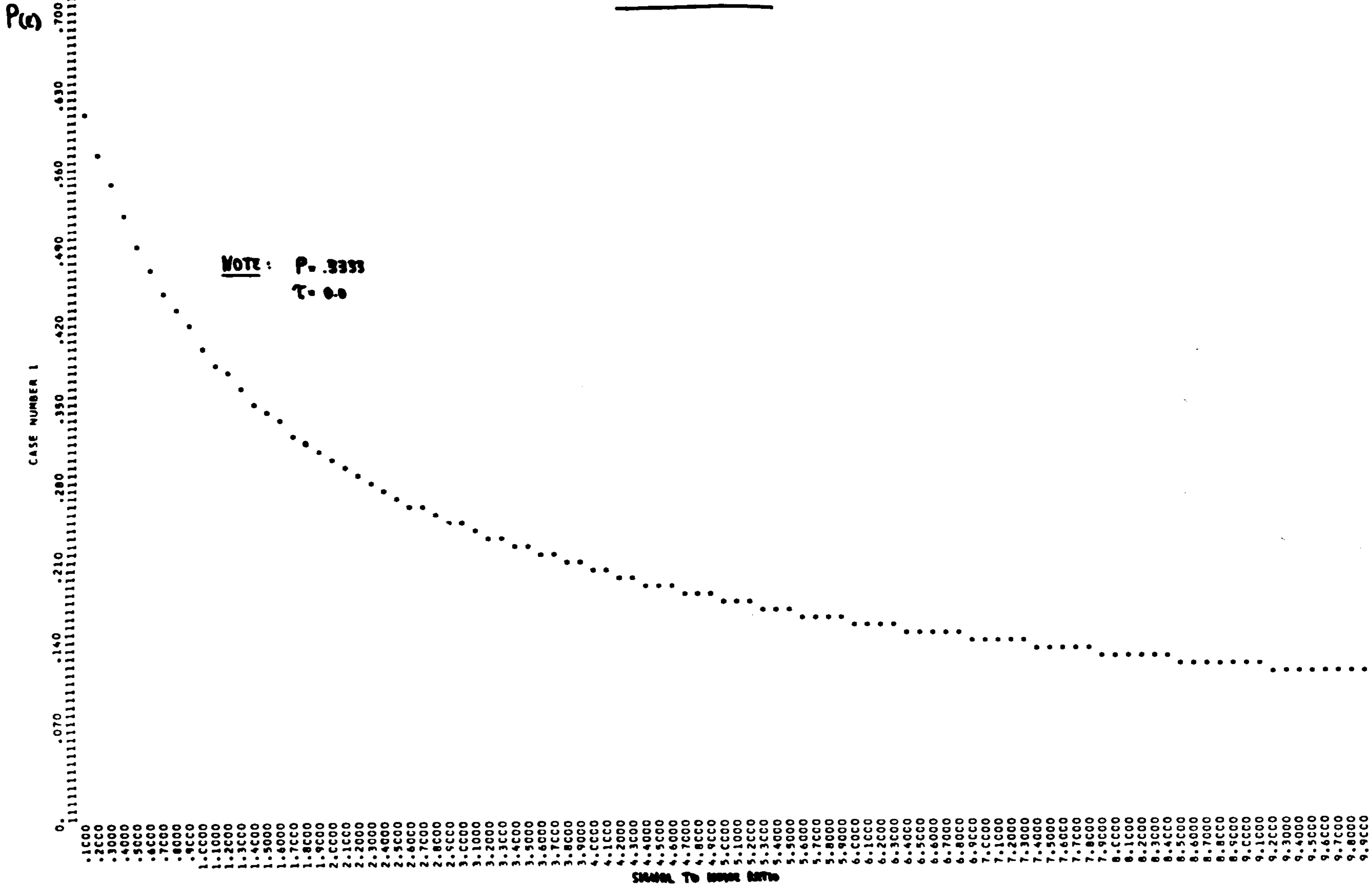


FIGURE 19

$P(\varepsilon)$ vs. E_b/N_o

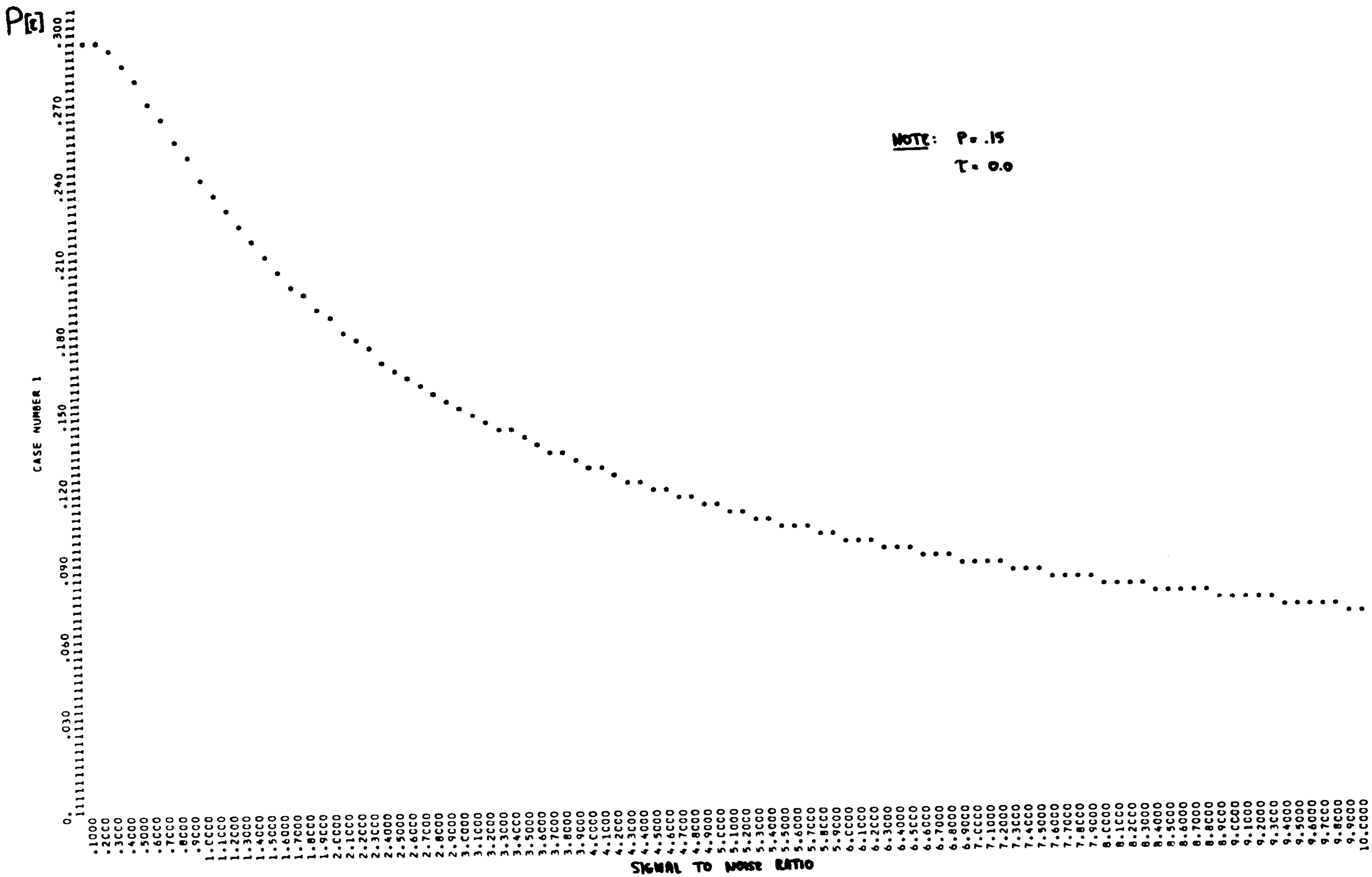


FIGURE 20

$a + \frac{b}{1 + \frac{E_R}{N_0} R(\tau)}$. The fact that the error probability decreases only inversely with the signal to noise ratio even for large values of E_R/N_0, is due primarily to the assumption of a scattering model in the channel. For a scattering model gives rise to a Rayleigh distribution for the received signal amplitude; and this means that on any given transmission, no matter how large the transmitted power, we may receive no signal.

Although it is not obvious from the graph, the case 1 receiver, for the $P = .45$ case, has begun to guess that noise is never sent once E_R/N_0 falls below about 1.8. This is actually a logical step for the receiver to take since saying noise is never received will add .1 to P[E], whereas using the poor (low E_R/N_0) received data causes a mistake to be made between the weak signals and noise more often than a tenth of the time. Actually, the receiver's "guessing" for small values of E_R/N_0, in the graphs of case 1 is obscured by the fact that the major portion of the receiver error occurs not from mistaking signal and noise, but rather from confusing signals one and two. A much more profound effect of receiver "guessing" on P[E] will be seen later when we eliminate the penalty for confusing s_1 and s_2.

It is worth noting that setting $P = .50$ leads to the simple case of choosing between one of two equal energy, equally likely signals; expression 6.13 for the case one P[E] then reduces to the standard result[1] for this problem.

---

[1] Wozencraft & Jacobs, p. 533.

7.2 Case 2, The Modified Binary Case

As mentioned earlier, this is a suboptimum decision rule involving the choice of either signal or noise. Of primary interest will be to verify that this case does indeed give an expression for P[E] which is always greater than or equal to the optimum binary case (case 3).

The first interesting phenomena we notice is that for the P = .5 case (Figure 21) the receiver has P[E] identically equal to zero for all E_R/N_0. This is due to the receiver using only the apriori knowledge that a signal will be transmitted with probability one. Since P = 1/2 is always greater than

$$\frac{2\frac{E_R}{N_0}+1}{4\left(\frac{E_R}{N_0}\right)+3}$$

the receiver "guesses" signal regardless of the values of x_i^2 and since a signal is always transmitted, it never makes a mistake[1].

Moving on to a more interesting case we notice that the graph for the P = .45 case not only shows guessing for E_R/N_0 less than 1.9 but also is not monotone decreasing. It is obvious that this is not an optimum receiver since one could do better by simply guessing signal at every point where P[E] is greater than .1. It is interesting to observe that the graphs of P[E] for cases 1 and 2 start to converge for large signal to noise ratios (for example, see Figure 25, curves labeled

---

[1] This is actually an uninteresting case because if we knew beforehand that a signal is broadcast with probability equal 1, we hardly need to design a receiver to tell us whether or not a signal was transmitted.

$P_{(E)}$ vs. E_b/N_0

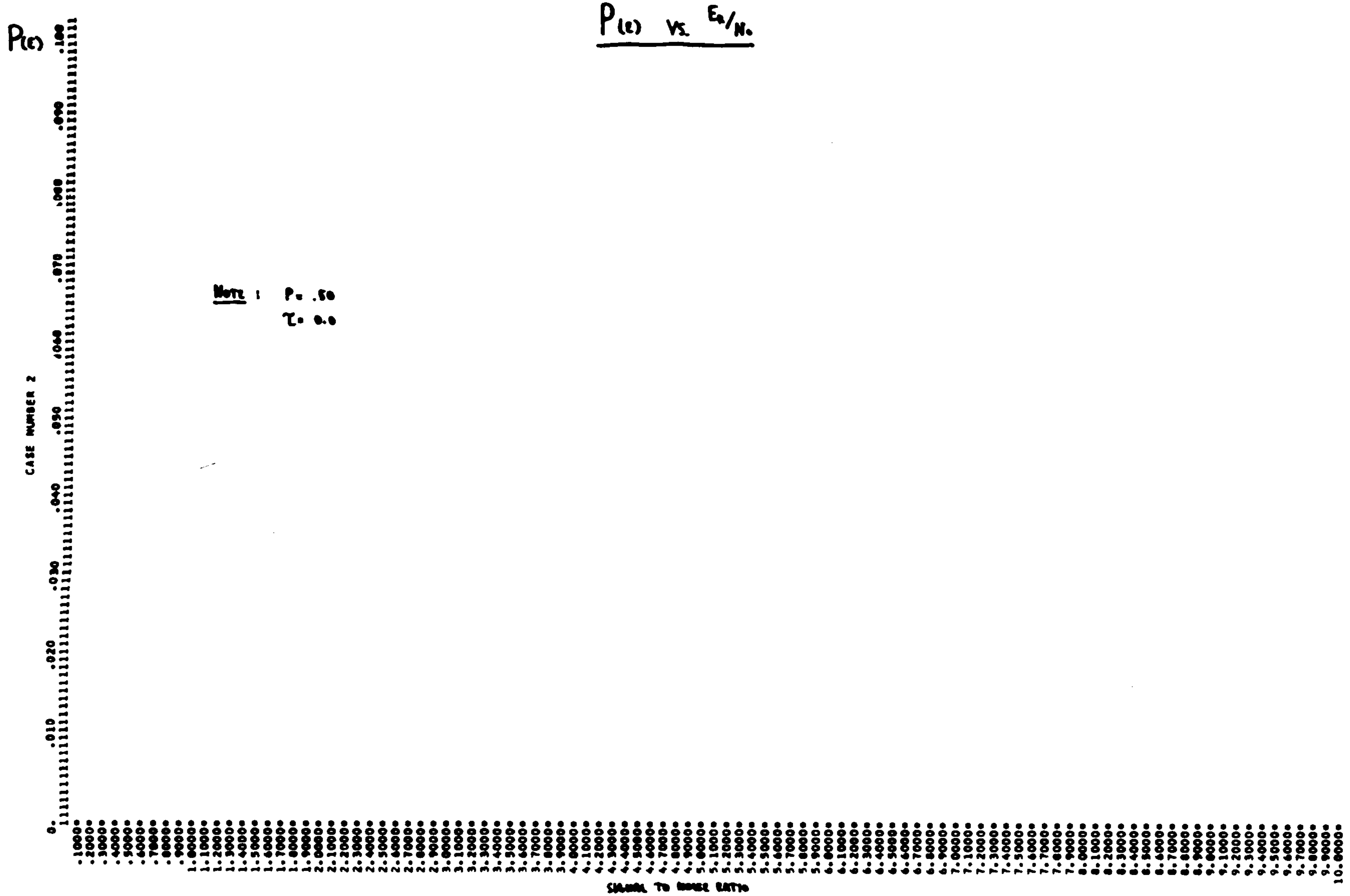


FIGURE 21

P[ε] vs E_R/N_0

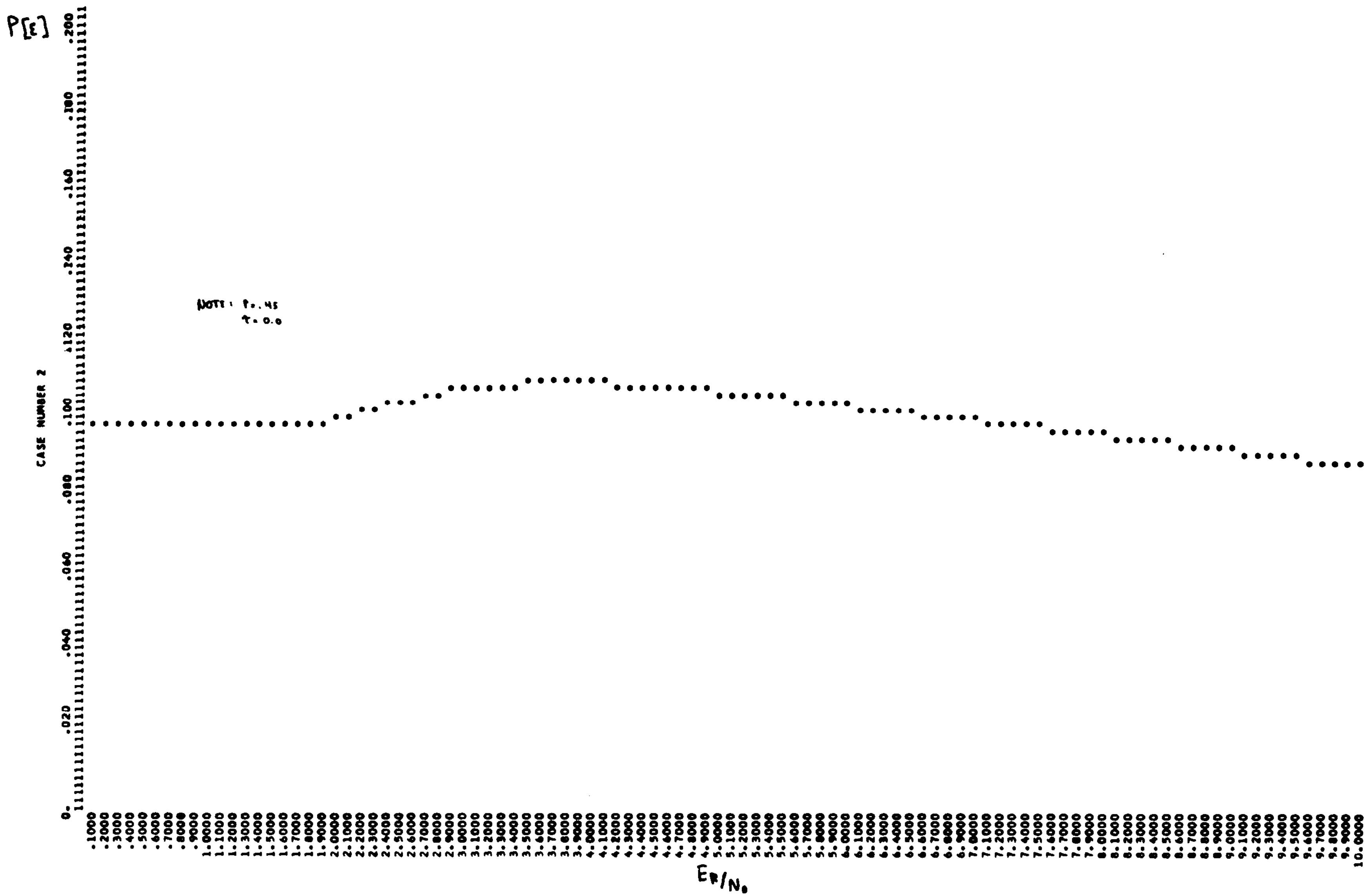


FIGURE 22

P[E] vs. E_R/N_o

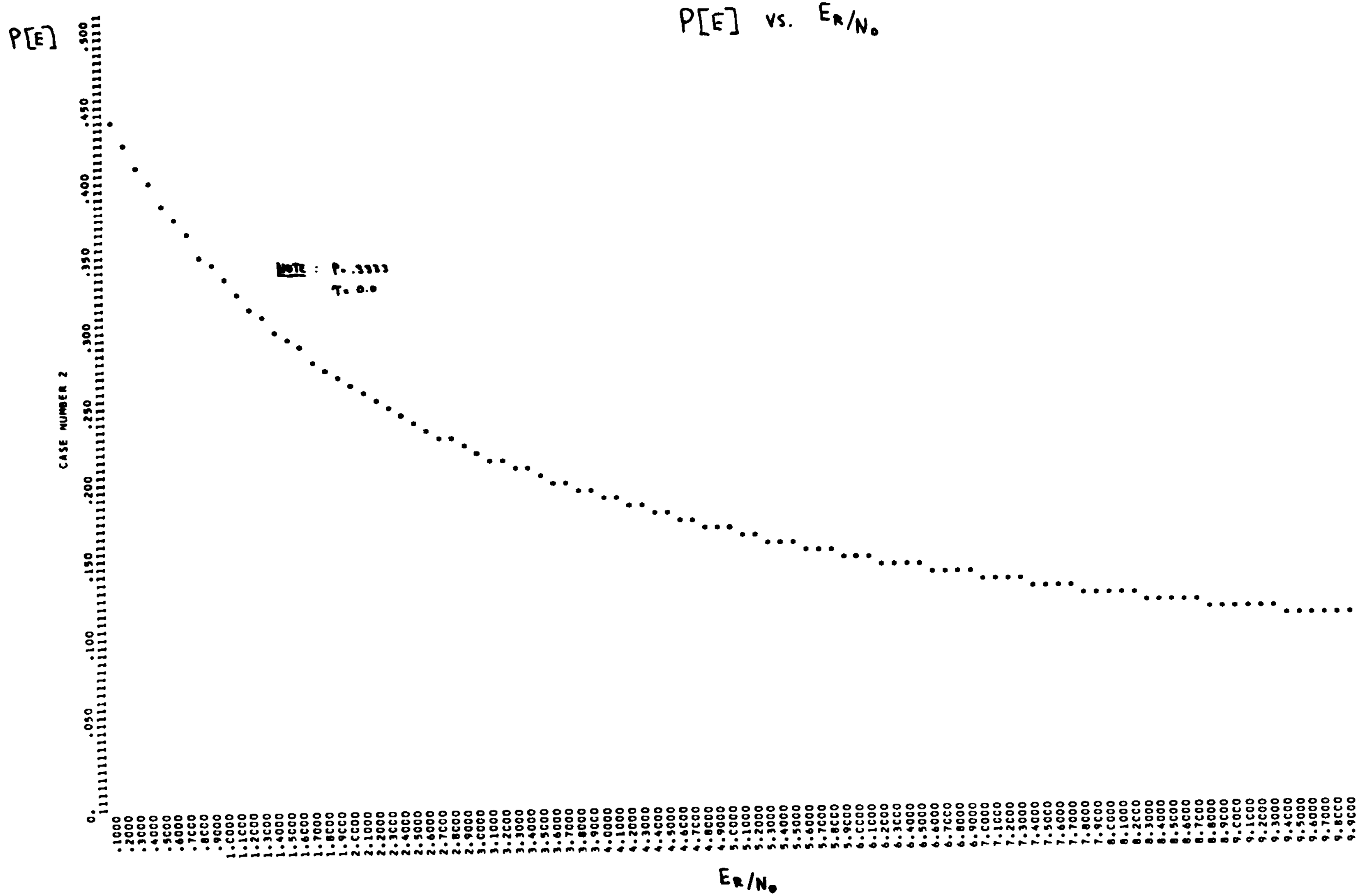


FIGURE 23

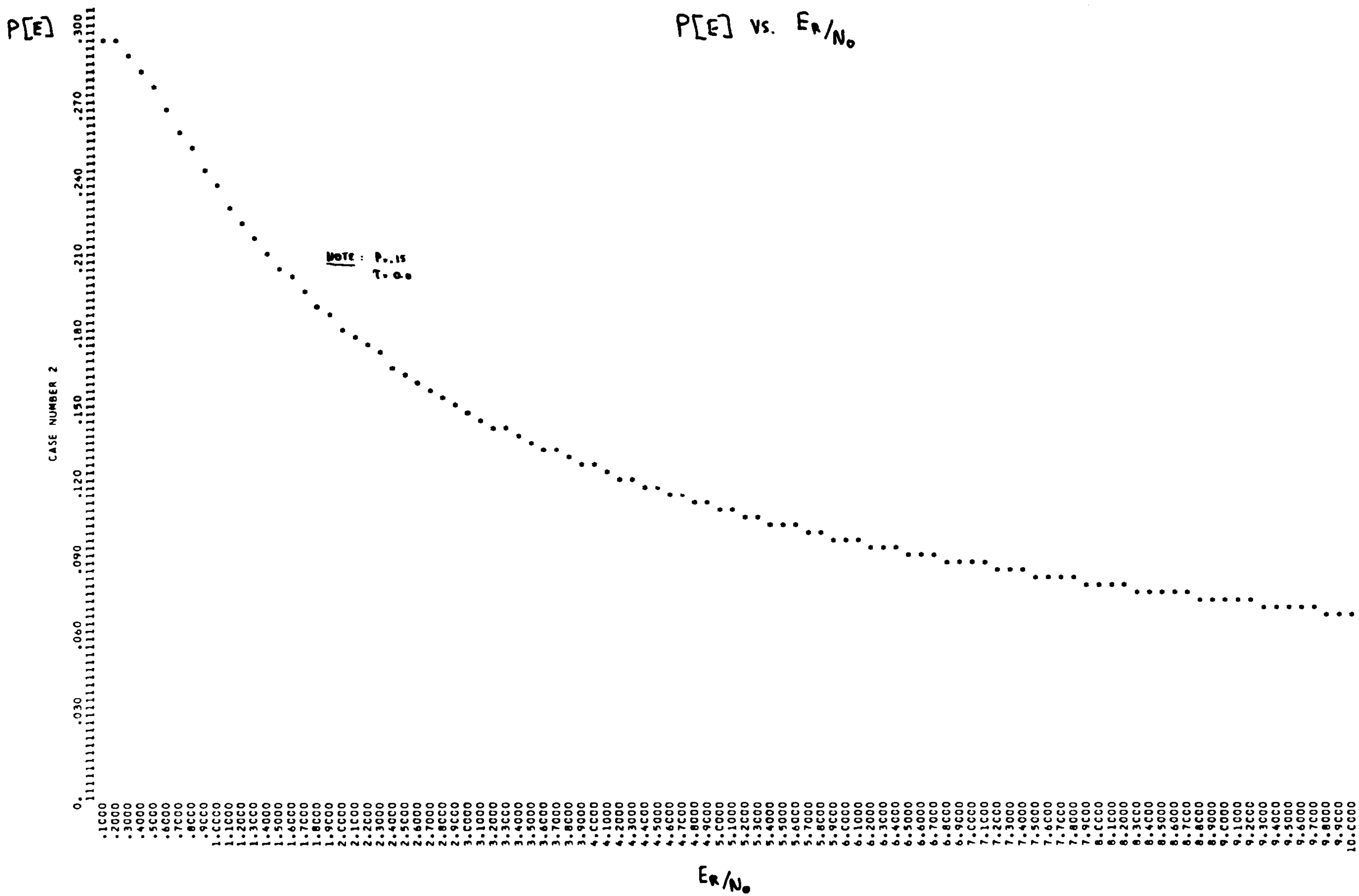
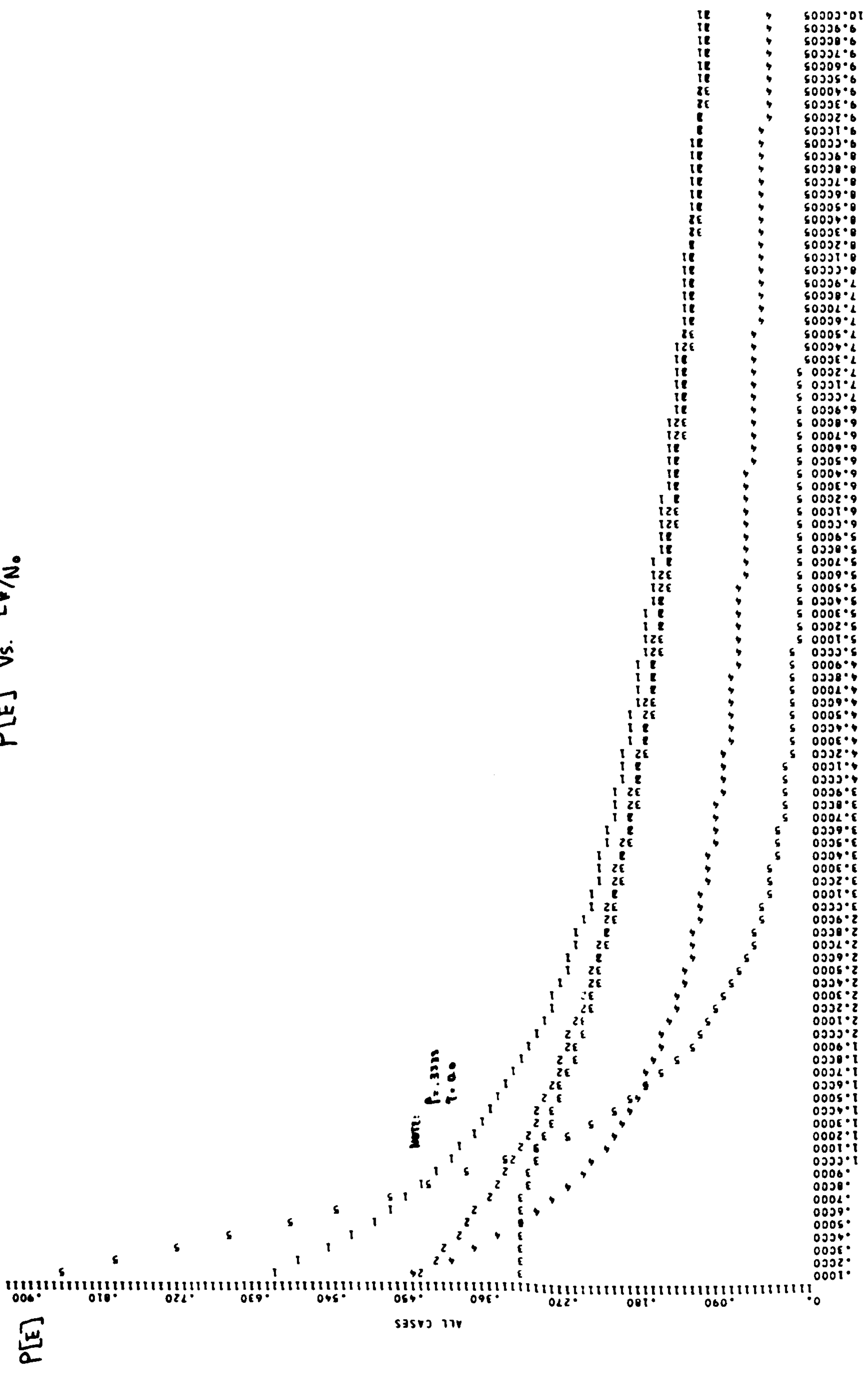


FIGURE 24

P[E] vs. E_R/N_o



E_R/N_o

FIGURE 25

1 and 2). This would indicate that for small signal to noise ratios the principal error is to mistake signals 1 and 2, but that as the signals get stronger this mistake is made less often and the limiting factor is the confusion of signals with noise. It is also worthwhile to notice that for large $^{E}R/_{N_0}$ the converging graphs actually lie almost on top of each other (to within .00001 for $^{E}R/_{N_0} = 100$). That is to say, a receiver operating in this range gains nothing by using the simpler decision rule. We will see that this result holds true also for the optimum binary case of the next section.

7.3 Case 3, The Optimum Binary Case

An evaluation of the performance of the case 3 receiver is further complicated over the previous cases by the need for linearizing the decision rule. A computer subroutine was written which evaluated P[E] while varying the number of segments from 1 to 50. A very surprising result was obtained; for any number of segments greater than 2 the P[E] was within one place in 10^5 of the 50-segment case! A closer examination of the decision rule reveals that for large signal to noise ratios the decision rule equation (6.14A) approaches very closely a square (see Figure 26). Since a two-segment approximation will fit a square as perfectly as a fifty, and since the decision rule is practically a square, it is then reasonable to expect for large $^{E}R/_{N_0}$ that the decision rule will be independent of the number of segments (above 2). For the small signal to noise ratios we find that the main contribution to the value of P[E] comes from integrating over area 2 of Figure 27. Since the contributions from area 1 are small, the exact shape of it, be it curved (Figure 27A), quadralateral (Figure 27B), or otherwise is relatively

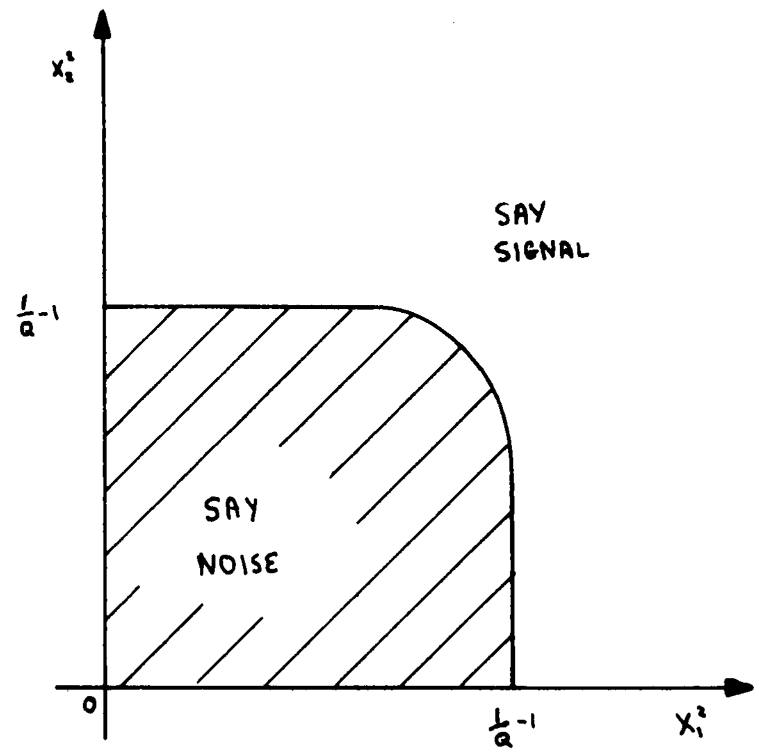
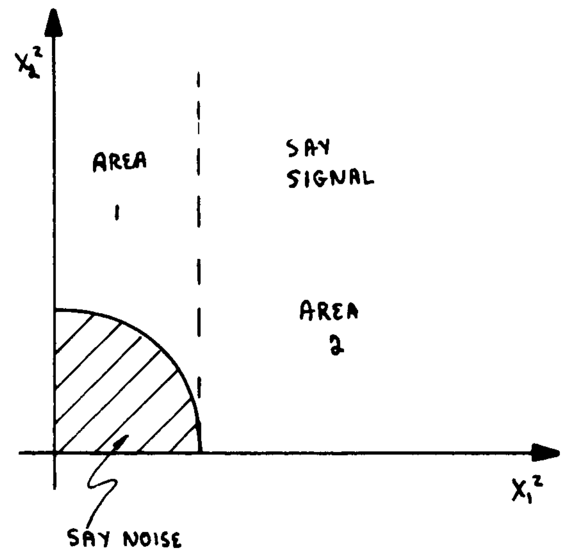
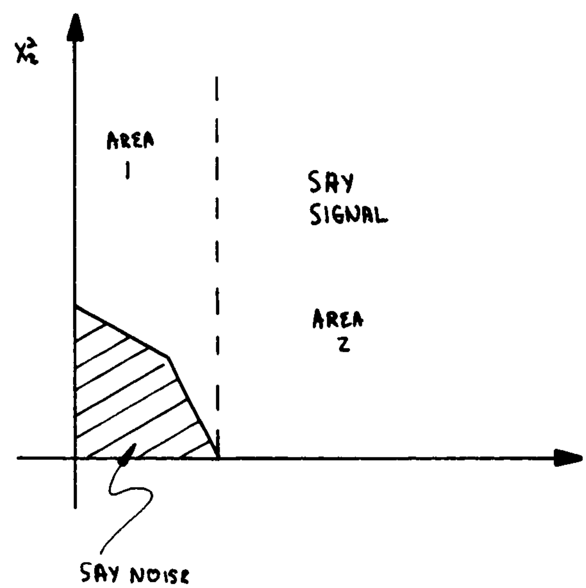


FIGURE 26. DECISION CURVE FOR LARGE E_R/N_0 FOR CASE 3



a)



b)

FIGURE 27 - CASE 3 DECISION RULE AND TWO TERM APPROXIMATION

unimportant. A combination of these two factors then helps to explain the surprising results of the computer program.

This result raises an interesting question. Why, if case 2 is a square, and the optimum case can be closely approximated by a square, is case 2 (as we have seen) not optimum? The answer to this question is that the case 2 decision rule has an x_1^2-intercept at $1/Q$ while the case 3 rule has the intercept at $1/Q - 1$. It would then appear that the critical factor in the design of an optimum binary receiver is not the exact shape of the decision curve but rather the intercept on the x_1^2 axes.

Case 3 with $P = 1/3$ (see Figure 30) illustrates a clear example of receiver guessing. The value of the probability of error expression starts to rise for decreasing signal to noise ratio. Finally, when P[E] based upon the input data is about to exceed the error probability for guessing, the receiver does guess and the P[E] curve levels off. From Chapter 6

$$\frac{1}{Q} = \frac{(1 - 2P)\left(2\frac{E_R}{N_0} R(\tau) + 1\right)}{P}$$

and, as E_R/N_0 becomes large, $1/Q - 1 \approx 1/Q$. Thus, for these signal to noise ratios the case 1 and 2 x_1^2-intercepts are about the same. From our previous discussion it was shown that equal intercepts give rise to approximately equal expressions for P[E]. Checking Figure 25 (curves 2, 3) we see that for E_R/N_0 greater than 4.7 the curves differ by a vanishingly small amount. This graph also points out the interesting result that for large signal to noise ratios, cases 1, 2, and 3 all behave approximately the same, and therefore we should use the case 1 receiver which extracts the most information.

P[E] vs. E/N_0

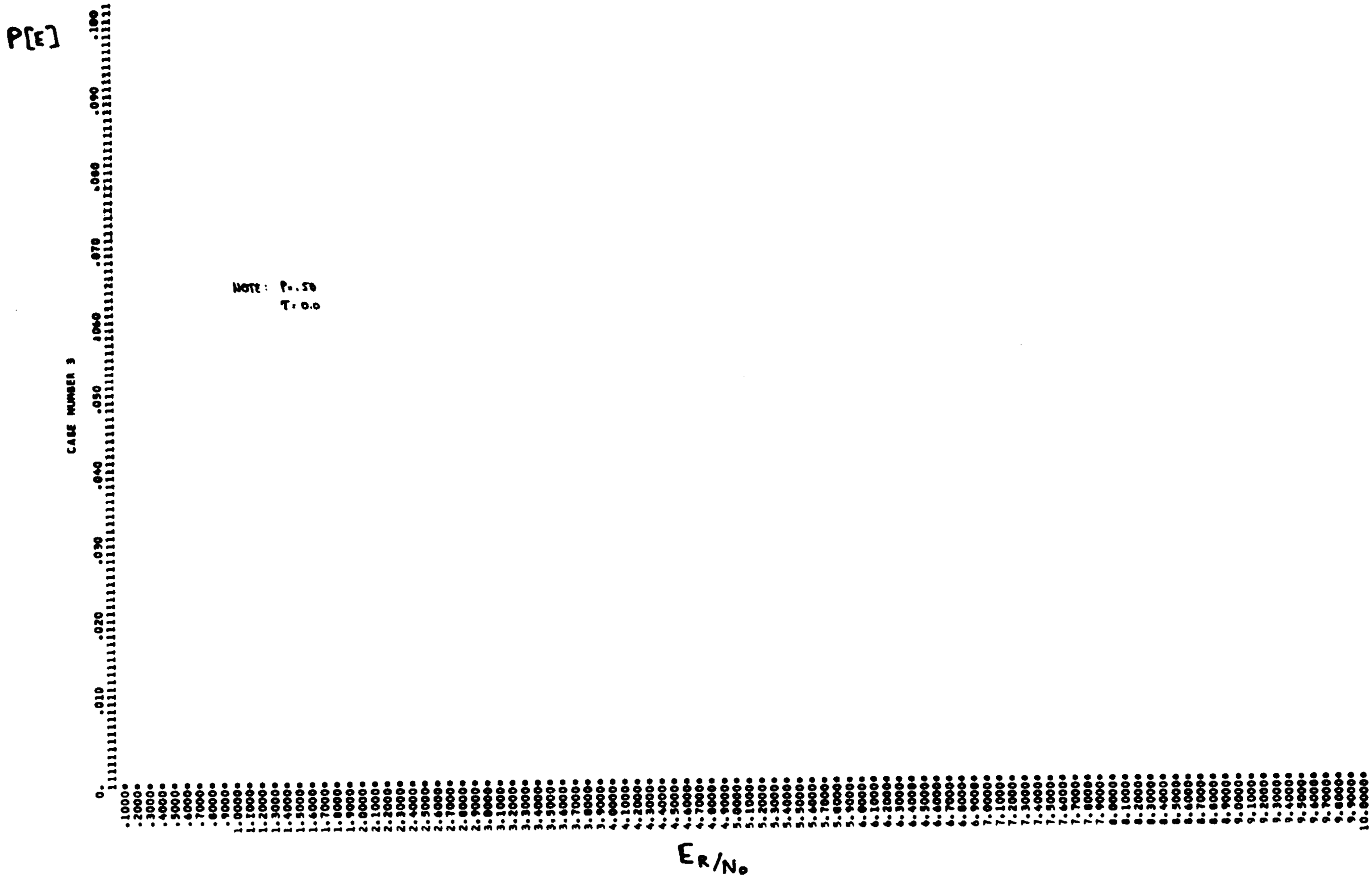


FIGURE 28

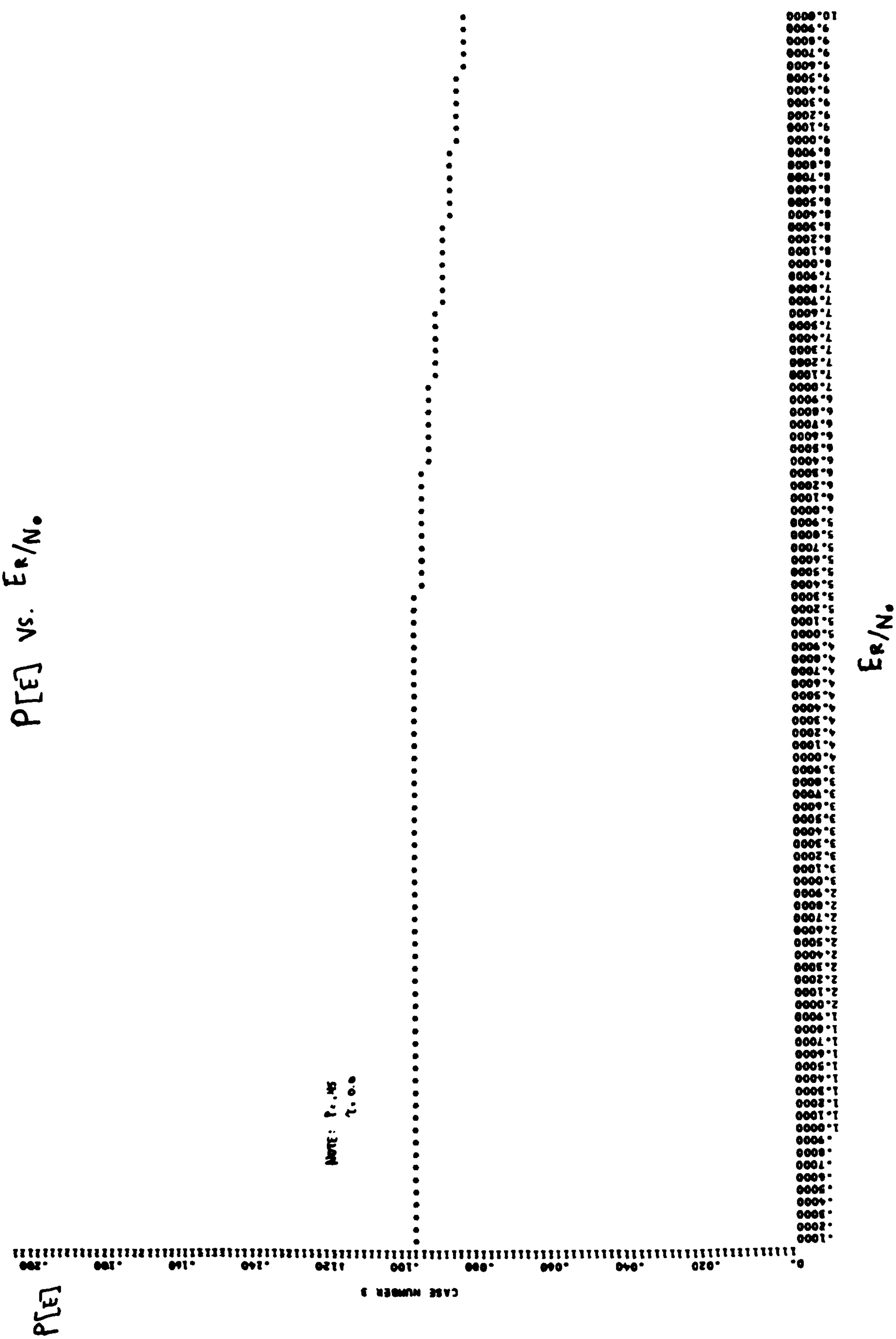


FIGURE 29

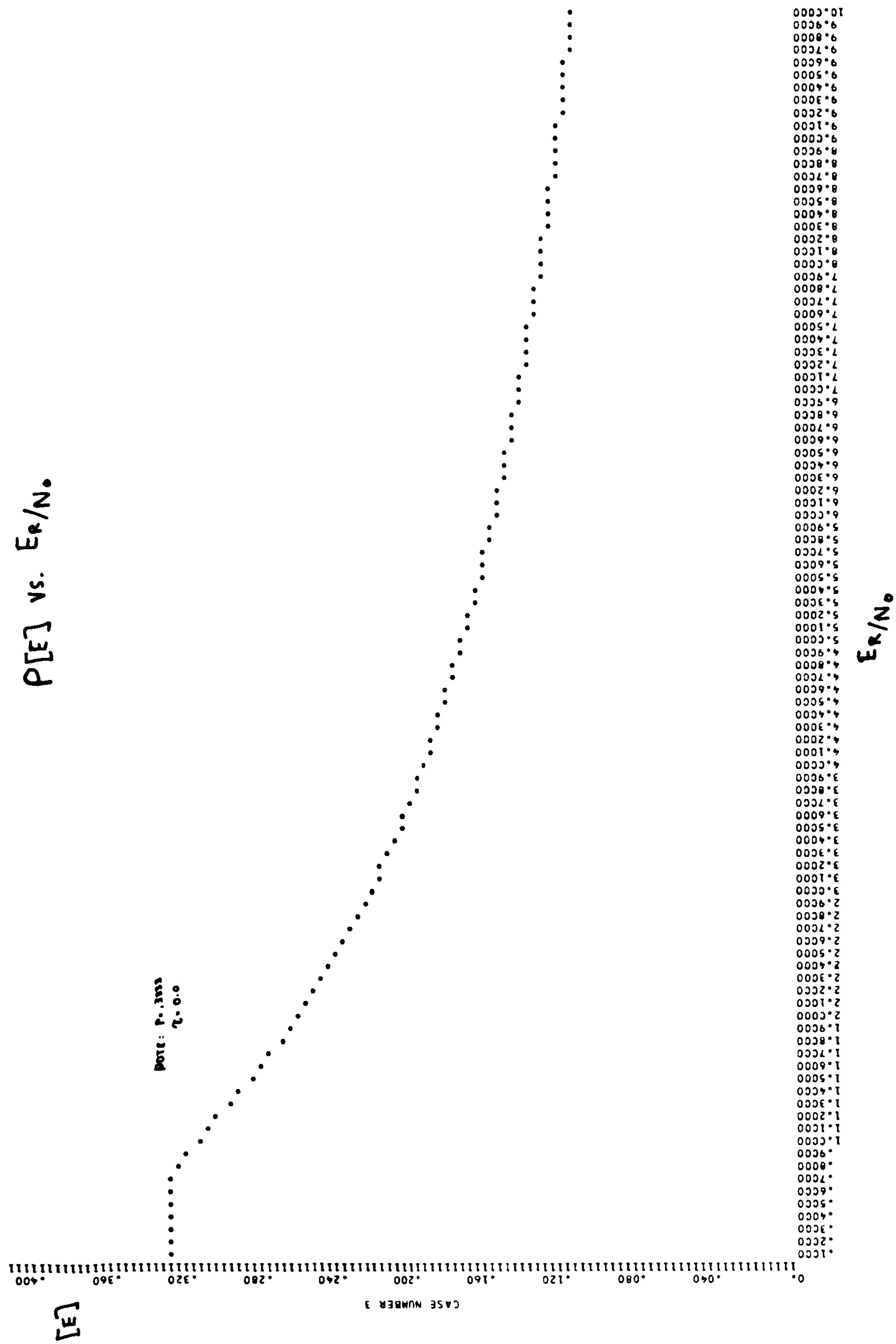

P[E] vs. E_R/N_0
P[E]
E_R/N_0
CASE NUMBER 3
.400
.360
.320
.280
.240
.200
.160
.120
.080
.040
0.

FIGURE 30

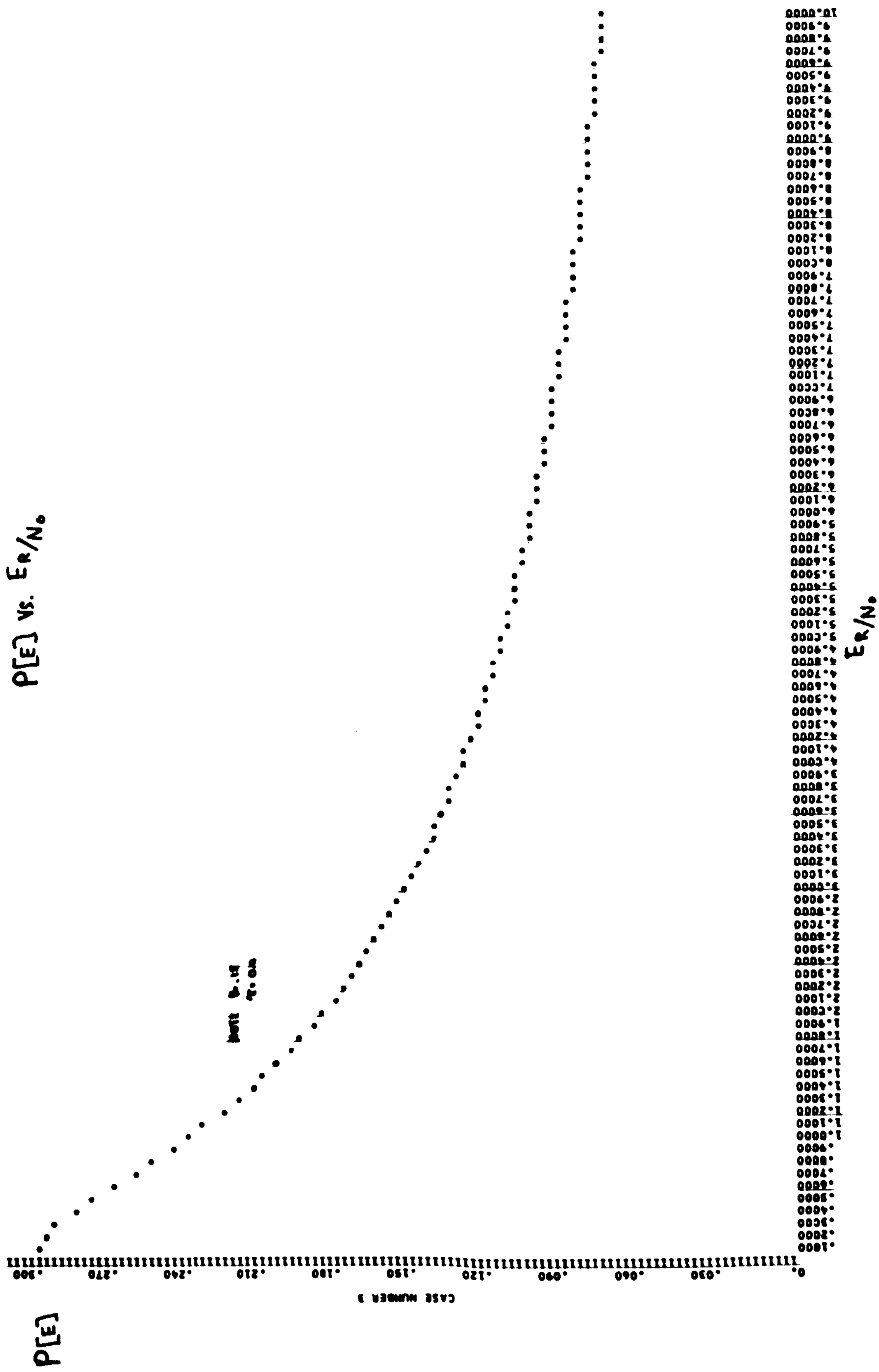


FIGURE 31

The last point to observe from Figure 25 is that case 3 not only begins to guess earlier than case 2 ($^1/_Q - 1$ as compared to $^1/Q$), but also decreases more rapidly for small signal to noise ratios.

Finally, the 2-segment case 3 expression was compared with case 2, for all values of $^E R/_{N_0}$ from .01 to 100, for values of P from .01 to .50, and for all values of τ between 0 and 75 usec. At every point the case 3 P[E] was less than or equal to the case 2 P[E], which indeed it should be if the previous theories were consistent.

7.4 Some Further Comments on Decision Region Receiver P[E] as a Function of $^E R/_{N_0}$

As mentioned previously the receiver guesses when it finds that the input data is of such poor quality that better results can be obtained by simply choosing signal. It should be noted that when the receiver guesses, it always chooses signal - never noise. This implies that a necessary condition for optimum receiver guessing is for the sum of the signal probabilities to be greater than $^1/_2$. Other conditions for guessing are placed upon the a priori probabilities by the receiver structure. For example, case 3 requires

$$\frac{1}{4} \leq P \leq \frac{1}{2}$$

while case 2 requires

$$\frac{1}{3} \leq P \leq \frac{1}{2}$$

Finally, it appears that, except for those times when the receiver is guessing, an optimum receiver has a P[E] function which is monotone decreasing with signal to noise ratio[1].

---

[1] It should be noted that this is not always true for non-optimum receivers. See case 2 graphs for an example.

This concludes the examination of the effects of varying signal to noise ratios on receiver performance. In the next section we shall study the results of having only a finite number of correlators available for use in the receiver.

7.5 P[E] Vs. Time Delay Error

From Chapter 4 we remember that the estimation rule is to wait until x_i^2 is a maximum and then choose this instant as the arrival time of the signals, i.e., set τ, the relative signal delay equal to zero. However, since we have only a finite number of correlators we are likely to choose a maximum for which $\tau \neq 0$. The effect of this nonzero τ, is shown in the following Figures, 32 through 35. For a low signal to noise ratio (.1) it actually makes little difference what τ is - the performance of the receiver is about equally bad for all values of time delay. For signal to noise ratio of 1 the error increases linearly with τ reaching a maximum at the baud length of the Barker code. As the signal to noise ratio further increases the curves for P[E] start becoming more concave. For E_R/N_0 equal to 100 we can even miss by 2/5 of a band length and still have error probabilities of less than 10^{-3}.[1] It also may be noticed that for any τ, P[E] for case 3 is always less than P[E] for case 2, as expected.

---

[1] By using coding, error probabilities of less than 10^{-4} can be obtained even after missing by 3/5 a band length. (See Figure 35, curve 5).

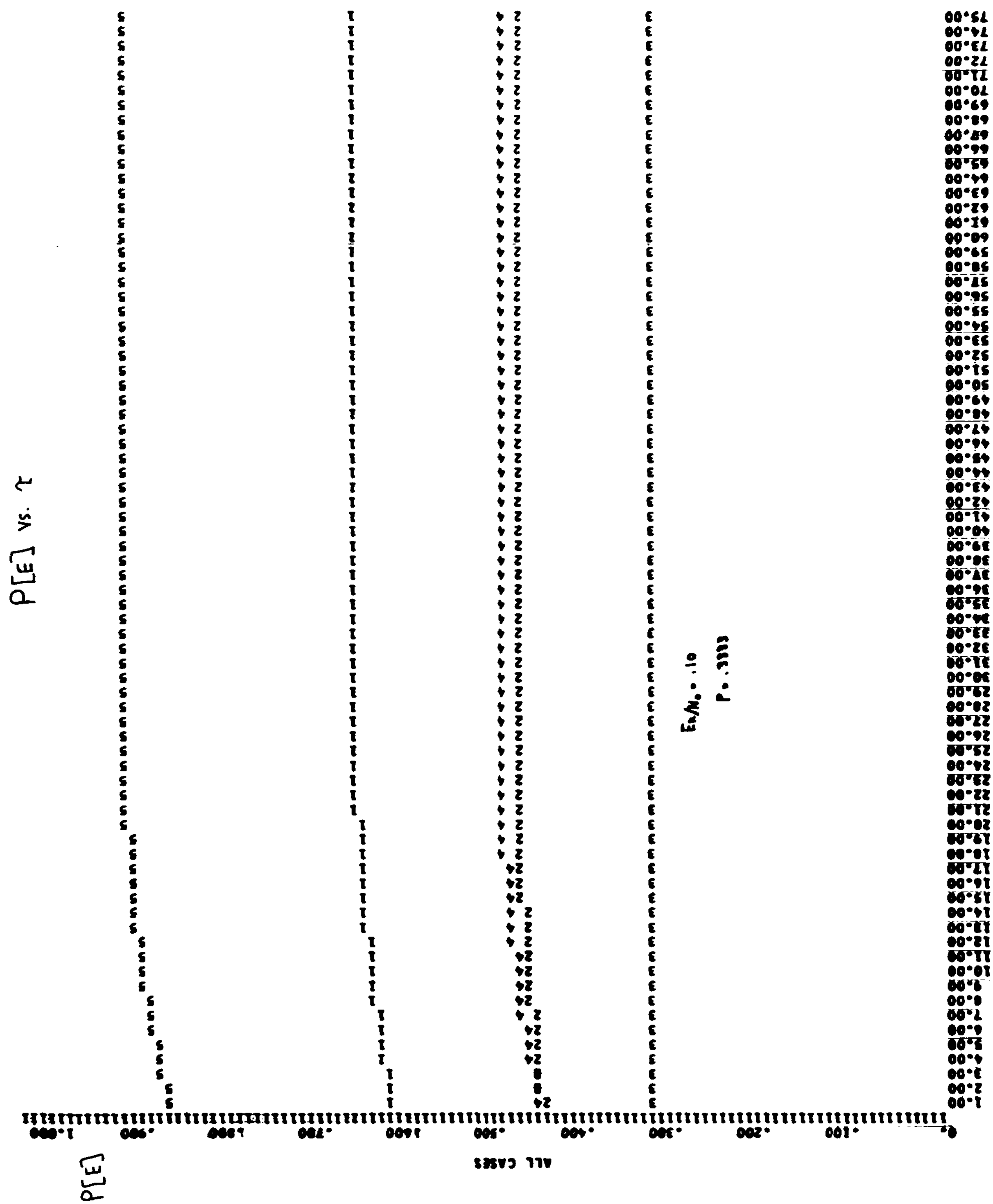


FIGURE 32

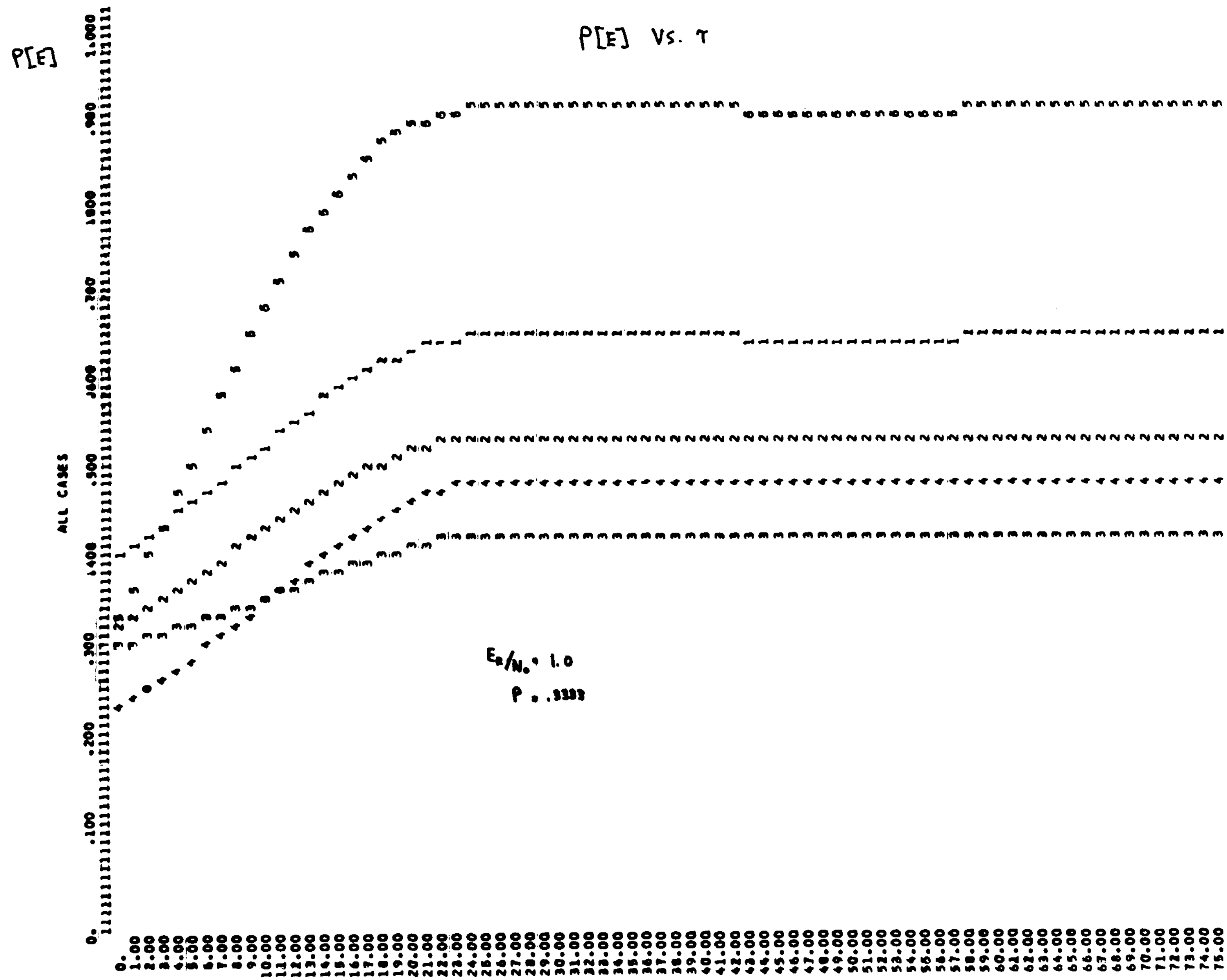


FIGURE 33

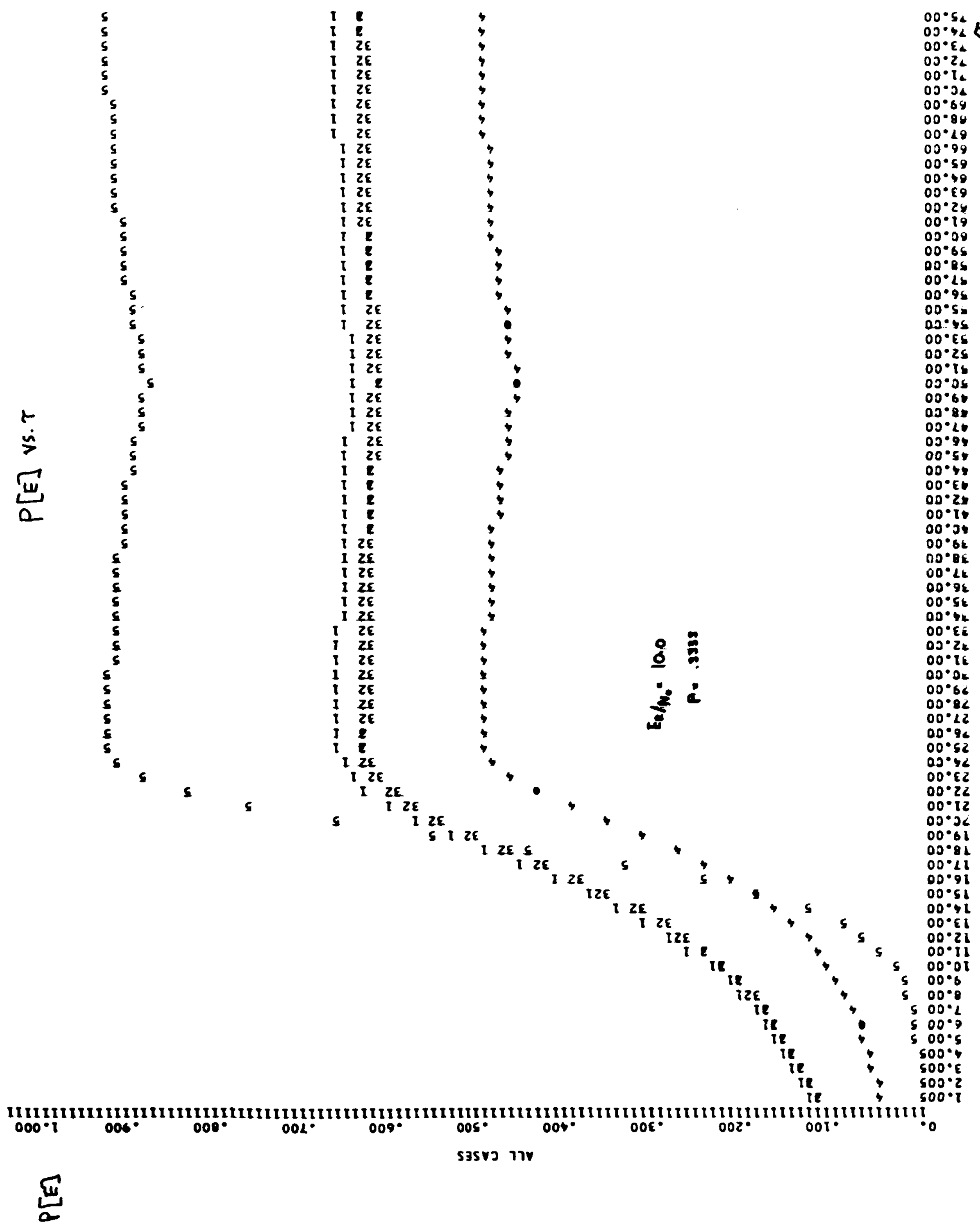


Figure 34

P[E] VS. τ

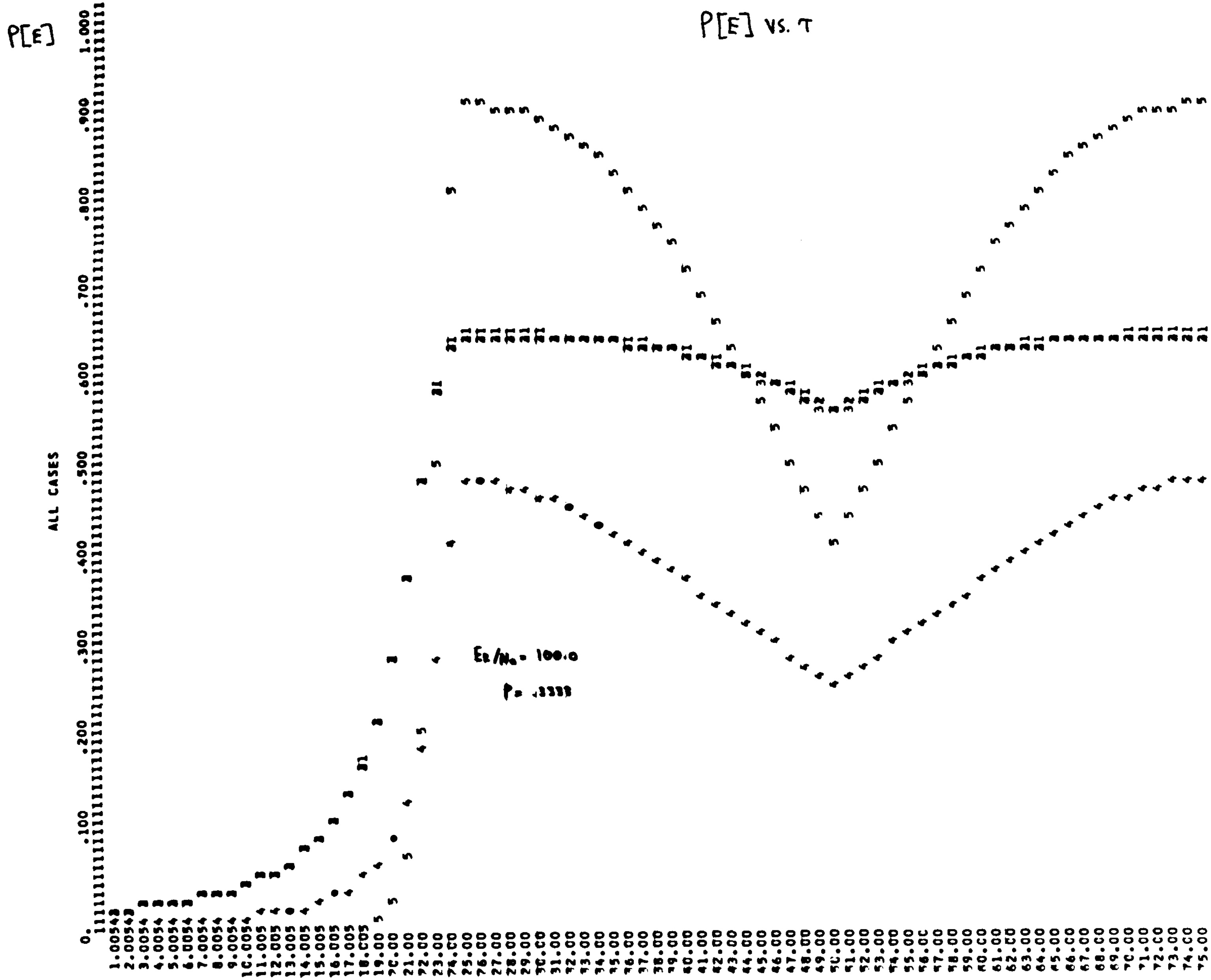


FIGURE 35

7.6 A Simple Coding Scheme

By using a coding scheme[1] it is possible to decrease the message errors over the bit by bit cases just studied. For example, curve 5 on the previous figures represents a coding scheme having eight code words of length 12 bits and minimum hamming distance 7. A 7-distance code allows for correction of three or less errors. The individual bit by bit errors used in the calculations for case 5 are given by case 4 (curve 4 on the graphs), an equal energy, equal probability optimum receiver. Note that the block probability of error is tiny in comparison to the bit by bit case until the per bit errors become moderate. Once this happens P[E] for the block signaling case rises extremely rapidly (exponentially) to a much greater value than the bit by bit error. If P [error/bit] = P_B then the block error probability can be expressed as

$$P[E_{BLK}] = \sum_{i=4}^{12} \binom{12}{i} P_B^i (1 - P_B)^{12-i}$$

This coding can supply an exponential decrease in receiver probability of error. However, this is at the expense of reduced transmission rates since we must transmit 12 bits to represent the same messages we could represent with 3 bits were it not for error correction.

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[1] Abramson, Information Theory and Coding

CHAPTER 8.

SUMMARY AND FINAL REMARKS

In the preceding chapters we have looked in detail at the problem of communicating through a noisy, scattering channel with time delay. It was observed that this channel rotated the "transmitted vector" in signal space, scaled it in amplitude and added to it a noise vector. Several decision rules, using the received vector as a basis, were developed. The two most important cases were the 3-ary decision rule (case 1) for choosing between message 1, message 2, or noise, and the optimum binary rule (case 3) for choosing between any message and noise. Which case should be used for the Sunblazer receiver was found to be determined primarily by the signal to noise ratio - for large $^{E}R/_{N_0}$ use case 1, while for small $^{E}R/_{N_0}$ use case 3.

It was also noticed that both the case 1 and case 2 receivers resorted to "guessing" under certain conditions of low signal to noise ratio. The optimum binary case was also examined in more detail and the critical factors for optimum reception were explored.

Finally, a simple coding scheme was used to illustrate the advantages which coding can realize over simple bit by bit signalling systems.

8.1 Future Ideas and Extensions

There are numerous extensions which can be made to this problem. These include the effect of dispersion on the signal, non-orthogonal signal sets, and different channel models. A particularly interesting channel model would be the Rician, since it is probably a more realistic model of the space environment through which Sunblazer must transmit. Further work might also be done on receiver "guessing" and the effect of the shape of the decision region on the expressions for $P[E]$. Finally, it might be interesting to examine other coding schemes besides the forward and backward Barker codes, particularly those with non-zero off diagonal terms in $\underline{R}(\tau)$.

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