TG #23

DISCUSSION OF RADIO WAVE PROPAGATION EXPERIMENTS TO EXPLORE THE EARTH'S MAGNETOSPHERE

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August, 1966

Contract No: NSR-24-005-047

Prepared by

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Minneapolis, Minnesota

For

HEADQUARTERS, NATIONAL AERONAUTICS & SPACE ADMINISTRATION
Washington, D. C. 20546

DISCUSSION OF RADIO WAVE PROPAGATION EXPERIMENTS TO EXPLORE THE EARTH'S MAGNETOSPHERE

ABSTRACT

A discussion is given of several radio wave propagation experiments designed to study the mean electron density between the Earth and the Moon, the distribution of this electron density within certain regions and electron density fluctuations associated with the magnetospheric boundary. The experiments require a low power multi-frequency transmitter on the Moon. The emission from this transmitter is monitored for phase and delay information either at a ground-based station or in a satellite in orbit round the Earth. The choice of frequencies, the power requirements, as well as possible sources or error in the experiments are considered in some detail. It is concluded that mean electron densities less than 10 cm^{-3} can be measured using a lunar based transmitter with an output power of a few watts.

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INTRODUCTION

It is the purpose of these notes to outline some experiments which will provide a vastly improved picture of the properties of the earth's magnetosphere which in the tail region extends to at least some 50 Earth's radii. The experiments are based on placing a radio transmitting station on the side of the Moon which is facing the Earth and to monitor the signals on the Earth as well as from a satellite in orbit around the Earth. The monitoring of the transmitters on the ground will be a fairly straightforward task not requiring highly sophisticated equipment and could well be carried out by a number of universities on a routine basis. The monitoring of the Moon-based transmitters in the Earth's orbiting satellite, on the other hand, will probably be more demanding as far as data transmission equipment and receiving antennas are concerned.

In what follows, a very brief outline is given of the recently available model of the magnetosphere, the basic philosophy of the experiment is then described (ref. 1), the expected power requirements for the Moon-based transmitters are worked out, and finally, a discussion is given of various considerations regarding modulation schemes, orbit parameters, choice of frequencies, etc.

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* Operated with support from the U. S. National Aeronautics and Space Administration.

MODEL OF THE EARTH'S MAGNETOSPHERE

The Earth is surrounded by an ionosphere with a maximum electron density of about 10^6 cm^{-3} at a height of some 300 km. Beyond this height the electron density declines relatively monotonically out to a distance of about four Earth radii where the density is about 100 cm^{-3} (ref. 2). At this point, whistler data appear to indicate that the electron density drops abruptly to 1 cm^{-3}, at least in the solar direction. On the other hand, radar experiments relying on the measurement of phase delay and group delay appear to indicate that the electron density beyond the plasma sphere may be as high as 200 cm^{-3} (ref. 1). Arguments against this high density have been advanced by Dessler and Michel (ref. 3) on the grounds that the supply of ionized particles is much too low to maintain such a high density. Scarf (ref. 4) has criticized the Stanford experiment on the basis of a possible presence of coherent oscillations in the plasma density somewhere along the propagation path. It is concluded tentatively from calculations on more realistic models that density fluctuations are unlikely to influence the results of the Stanford experiments appreciably, see Appendix. We are, therefore, faced with the problem of exploring a region with the following electron density characteristics:

Height	Density	Plasma Freq.
300 km	10^6 cm^{-3}	9×10^6 Hz
1500 km	10^4 cm^{-3}	9×10^5 Hz
4-5 Earth radii	10^2 cm^{-3}	9×10^4 Hz
Beyond	{ 1 cm^{-3} or 2×10^2	{ 9×10^3 Hz or 1.3×10^5 Hz
Solar wind	10 cm^{-3}	2.8×10^4 Hz

PRINCIPLES OF THE EXPERIMENTS

The propagation vector k of a non-magnetized plasma is given by:

$$k^2 = k_o^2 \cdot \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right] = k_o^2 \, (1 - X); \; X = \frac{N \cdot e^2}{m \, \epsilon_o \, \omega^2} \text{ (MKSA units)}$$

where ω_p is the angular plasma frequency, where ω is the angular frequency of the wave and where k_o is the propagation constant in vacuo. In the remainder of this section we deal separately with experiments measuring the phase and the group delays of the waves.

Phase Delay Method

The phase delay in radians over a path between $\vec{r}_1$ and $\vec{r}_2$ is given by:

$$\Phi\,(\omega) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{k} \cdot d\vec{s} = k_o \, S\,(\vec{r}_2)$$

where $S(\vec{r}_2)$ is the eikonal of the wave-field. Suppose $\vec{r}_2$ is the position of the orbiting satellite and that the position is changing according to $\vec{r}_2 = \vec{R}(t)$. The Doppler frequency observed at the receiver is given by:

$$\Delta\omega = - \frac{d\phi}{dt} = - k_o \frac{dS}{dt} = - k_o \frac{d\vec{R}}{dt} \cdot \text{grad } S = - k_o \vec{v} \cdot \text{grad } S =$$

$$= -k \, \vec{v} \cdot \vec{n} = - \vec{v} \cdot \vec{n} \, k_o \sqrt{1 - X(\vec{R})}$$

where $\vec{n}$ is the unit wave normal at the point of observation and $\vec{v}$ is the velocity of the observer. Implicit in this derivation is the assumption that the rate of change of the medium is small.

For high frequencies $X \ll 1$ and we may write

$$\Delta\omega \approx -\vec{n}\cdot\vec{v}\,k_o\left(1-\frac{X}{2}\right) = -\vec{n}\cdot\vec{v}\,\frac{\omega}{c}\left[1-\frac{1}{2}\,\frac{N\cdot e^2}{m\,\epsilon_o\,\omega^2}\right]$$

Now, suppose doppler frequencies are recorded at two harmonically related frequencies ω_1 and ω_2 such that ($\mu > 1$).

$$\omega_2 = \mu\cdot\omega_1$$

A comparison of the two signals observed in the orbiting satellite after multiplying the frequency of the low frequency signal with the harmonic ration μ gives a difference frequency:

$$\delta(\Delta\omega) = \Delta\omega_2 - \mu\,\Delta\omega_1 = \frac{1}{2}\,\frac{N\cdot e^2}{m\,\epsilon_o\,\omega_2}\,(1-\mu^2)\,(\vec{n}\cdot\vec{v})$$

$$\approx -\frac{1}{2}\,\Delta\omega_2\,\frac{N\cdot e^2}{m\,\epsilon_o\,\omega_2^2}\cdot c\,(1-\mu^2) =$$

$$= \frac{1}{2}\,\Delta\omega_2\,X_2\,(\mu^2-1) = \frac{1}{2}\,\Delta\omega_1\,X_1\left(1-\frac{1}{\mu^2}\right)$$

Since both $\delta(\Delta\omega)$ and $\Delta\omega_2$ are measurable and μ is predetermined, the local electron density at the satellite can be determined in principle. Complications arise due to the motion of the Moon-based transmitter which makes $\vec{r}_1$ also vary with time. Considerations must also be given to the change with time of the path between $\vec{r}_1$ and $\vec{r}_2$. Simplest conditions obviously prevail when $\vec{v}$ and $\vec{n}$ are parallel, indicating that an eccentric orbit with apogee pointing in the general direction of the Moon is to be preferred.

Group Delay Method

The group delay in units of time between the two points $\vec{r}_1$ and $\vec{r}_2$ is given by:

$$D = \int_{\vec{r}_1}^{\vec{r}_2} \frac{\partial k}{\partial \omega} \cdot d\vec{s} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{\partial k}{\partial \omega} (\vec{n} \cdot d\vec{s}) = \int_{\vec{r}_1}^{\vec{r}_2} (\vec{n} \cdot d\vec{s}) \cdot \frac{1}{c} \frac{1}{\sqrt{1 - X}}$$

$$\cong \frac{1}{c} \int_{\vec{r}_1}^{\vec{r}_2} (\vec{n} \cdot d\vec{s}) \ (1 + \frac{1}{2} X)$$

If the group delay is measured at frequencies ω_1 and ω_2 corresponding respectively to X_1 and X_2 , the difference in group delays becomes:

$$\delta D = D_1 - D_2 = \frac{1}{2c} \int_{\vec{r}_1}^{\vec{r}_2} (\vec{n} \cdot d\vec{s}) \ (X_1 - X_2) = \frac{1}{2c} \int_{\vec{r}_1}^{\vec{r}_2} ds \ X_1 (1 - \frac{1}{\mu^2})$$

Since $d\vec{s} \parallel \vec{n}$ when $X \ll 1$.

In this experiment μ need not be an integer. The measurement of δD, therefore, provides direct information on the <u>integrated</u> electron density over the transmission path.

Some Numerical Examples

Consider the time delay experiment first. Suppose we specify the difference in time delay to be 100 μsec which should be rather easily measurable. For a range of $3.8 \cdot 10^8$ m between transmitter and receiver and for $\mu \gg 1$, the required lower frequency f_1 is found to be:

$N(cm^{-3})$	f_1
100	$7.2 \cdot 10^6$ Hz
10	$2.1 \cdot 10^6$ Hz
1	$7.2 \cdot 10^5$ Hz

Consider next the measurement of differential doppler frequency. For the same frequencies as above and again for $\mu << 1$ one obtains, assuming a typical velocity to be $5 \cdot 10^3$ m/sec:

$N(cm^{-3})$	f_1	$\delta \Delta f$
100	$7.2 \cdot 10^6$ Hz	$\sim 10^{-2}$ Hz
10	$2.1 \cdot 10^6$ Hz	$\sim 10^{-3}$ Hz
1	$7.2 \cdot 10^5$ Hz	$\sim 10^{-4}$ Hz

The same $\delta\Delta f$'s would have been obtained with a constant f_1 also. At least the two higher ones of these numbers should be readily measurable.

SOME NOTES ON POWER REQUIREMENTS

The construction of an efficient satellite born antenna at these low frequencies is fairly difficult. We shall, however, assume that an antenna can be built with unity directivity, i.e., with $g \approx 1$. The collecting area of such an antenna is approximately equal to $\lambda^2/12$ $(A = g\lambda^2/4\pi)$. The received power is determined from the equation:

$$P_r = P_t A_r g_t / 4\pi r^2$$

with the distance r of $3.8 \cdot 10^8$ m, $g_t = 1$ and with $\lambda = 300$ m (1 MHz) one obtains:

$$P_r = P_t \cdot 0.43 . 10^{-14} \text{ watts.}$$

The noise background temperature at 1 MHz is equal to $2.3 \cdot 10^7$ °K (ref. 5) and the noise power in the receiver will be:

$$P_n = 1.38 . 10^{-23}\, 2.3 . 10^7 \cdot B = 3.2 . 10^{-16} \text{ B watts}$$

where B is the noise bandwidth in Hz. The predetection signal-to-noise power ratio therefore becomes:

$$\frac{P_r}{P_n} = \frac{20\, P_t}{B}$$

Since the effective bandwidth of the transmitted signal may be only a few cycles per second for the measuring principles described above it seems that a signal-to-noise ratio in excess of unity could be achieved for a transmitted power of only one watt on the Moon. In view of the possible availability of as much as 50 watts on the Moon in the near future the power requirements seem to be quite reasonable.

DISCUSSION

The Moon-based transmitters should operate simultaneously at at least two carrier frequencies at a time. The lowest frequency suitable for reception in an Earth orbiting satellite should be within the range 1 - 10 MNz, the lower frequency limit being determined principally by the availability of suitable transmitter and receiver antenna designs, whereas the upper limit is determined by the detectability criteria. The choice of the higher frequencies required for comparison purposes is quite uncritical.

As far as the phase or the Doppler frequency measurement is concerned the transmission might well be unmodulated. The group delay may be carried out as originally suggested by Dr. V. R. Eshleman for a similar radar experiment by transmitting a pair of frequencies near f_1 with a separation Δf and simultaneously another pair near f_2 with exactly the same frequency separation Δf. The phase difference of the "beats" of the two frequency pairs will be directly proportional to the difference in group delay of f_1 and f_2. This phase difference should preferably be measured in the satellite -- but if this requires too much data processing in the satellite the raw-unprocessed data may be transmitted to the ground on a telemetry link for further processing. The tuning of the receivers to accommodate the Doppler shift should be made by means of a tracking receiver at one of the frequencies, the local oscillator of the other receiver should be locked to that of the former.

The orbit parameters of the satellite should be chosen in such a way that the satellite goes through the magnetospheric "knee", and, if at all possible, the satellite should be made to precess in such a way that apogee is in the general direction of the Moon most of the time.

A number of other experiments can also be done by this simple setup. When the propagation path passes obliquely through the boundary layer between the magnetospheric tail and the solar wind region one should expect the signal to exhibit fluctuations in phase and direction of arrival. With a well designed point source of radiation on the Moon it should not be difficult to derive a considerable amount of information from these fluctuations about the nature of the boundary layer.

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APPENDIX

COMMENTS ON PAPER BY F. L. SCARF

Scarf (ref. 4) has recently advanced some criticism of the methods of measuring electron densities described in this report. The argument is that electron density fluctuations, particularly coherent ones, could very substantially modify the effective mean propagation constants in the mediu. As Scarf's conclusions in his coherent density fluctuation case leads to a highly resonant effect and also appears to rest on invalid solutions of a transcendental equation, we shall reexamine his ideas very briefly in this appendix.

Let the wave propagation constant in the unperturbed plasma be k_o, the electron density N and the scalar wave field in the unperturbed medium be $\varphi_o(\vec{r})$. In the perturbed medium the field at a point $(\vec{r})$ can then be thought of as a linear superposition of the unperturbed wave $\varphi_o(\vec{r})$ and a scattered wave:

$$\varphi(\vec{r}) = \varphi_o(\vec{r}) + \frac{X}{1 - X} k_o^2 \frac{\delta N}{N} \int G(\vec{r}; \vec{r}') f(\vec{r}') \cdot \varphi(\vec{r}') d(\vec{r}') \qquad \text{(A-1)}$$

where $X = N \cdot e^2 / m\epsilon_o \omega^2 = \omega_p^2/\omega^2$

$\delta N \cdot f(\vec{r})$ = perturbation in electron density at $\vec{r}$

$G(\vec{r}; \vec{r}')$ = Green's function for wave equation of unperturbed medium, i.e.,

$$(\nabla^2 + k_o^2) G(\vec{r};\vec{r}') = \delta(\vec{r}-\vec{r}') \quad \text{or:}$$

$$G(r, r') = - \frac{1}{4\pi} e^{+ i k_o |\vec{r}-\vec{r}'|} / |\vec{r}-\vec{r}'| \qquad \text{(A-2)}$$

For the mean field $\langle \varphi(\vec{r}) \rangle$ we must have:

$$\langle f(\vec{r}) \varphi(\vec{r}) \rangle = \frac{X}{1-X} k_o^2 \frac{\delta N}{N} \int G(\vec{r};\vec{r}') \langle f(\vec{r}) f(\vec{r}) \varphi(\vec{r}') \rangle d(\vec{r}') \qquad \text{(A-4)}$$

We now assume that

$$\varphi(\vec{r}) = \langle \varphi(\vec{r}) \rangle + \delta\, \varphi(\vec{r}) \tag{A-5}$$

where $\delta\,\varphi(\vec{r})$ is a zero mean variable which is also assumed to be at least approximately gaussian. Substitution of (A-4) into (A-3) gives:

$$\langle \varphi(\vec{r}) \rangle = \varphi_o(\vec{r}) + \frac{X^2}{(1-X)^2}\, k_o^4 \left(\frac{\delta N}{N}\right)^2 \iint d(\vec{r})\, d(\vec{r}'')\, \langle \varphi(\vec{r}'') \rangle \langle f(\vec{r}'')\, f(\vec{r}') \rangle \cdot$$

$$\cdot\ G(\vec{r};\vec{r}')\, G(\vec{r};\vec{r}'') \tag{A-6}$$

Applying the operator $(\nabla^2 + k_o^2)$ to both sides of (A-6) one obtains the equation for $\langle \varphi(\vec{r}) \rangle$:

$$(\nabla^2 + k_o^2) \langle \varphi(\vec{r}) \rangle = \frac{X^2}{(1-X)^2}\, k_o^4 \left(\frac{\delta N}{N}\right)^2 \int d(\vec{r}') \langle f(\vec{r}')\, f(\vec{r}) \rangle \langle \varphi(\vec{r}') \rangle\, G(\vec{r};\vec{r}') \tag{A-7}$$

The dispersion equation is obtained by substituting a plane wave of the form $\exp(\vec{f}\,\vec{k} \cdot \vec{r})$ into (A-7) and by changing the variable of integration in (A-7) to $\vec{\rho} = \vec{r} - \vec{r}$, and by introducing the spatial correlation function defined by:

$$R(\vec{\rho}) = \langle f(r')\, f(r) \rangle \tag{A-8}$$

one obtains:

$$k^2 - k_o^2 = \left(\frac{X}{1-X}\right)^2 k_o^4\, \frac{\delta N}{N}^2 \cdot \frac{1}{4\pi} \underbrace{\int d(\vec{\rho})\, e^{i\,\vec{k}\cdot\vec{\rho}}\, \frac{e^{+i\,k_o |\vec{\rho}|}}{|\vec{\rho}|}\, R(\vec{\rho})}_{I} \tag{A-9}$$

which is identical to Scarf's equation (ref. 8) except for occurrence of a factor $\exp(i\,\vec{k}\cdot\vec{\rho})$ in (A-9) rather than the factor $\exp(i\,\vec{k}_o \cdot \vec{\rho})$ in Scarf's equation. We now show that this discrepancy leads to an erroneous result in Scarf's paper, particularly for the coherent oscillation case.

Consider Scarf's first case, i.e., that with

$$R(\rho) = \exp(-\rho/\ell) \tag{A-10}$$

Solving the dispersion equation on the assumption that $k \approx k_o$ one obtains:

$$k^2 = k_o^2 + (i\, k_o/\ell)\left(1 - \sqrt{1 - \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2 \frac{2(k_o \ell)^3}{2\, k_o \ell + i}}\right)$$

$$\approx k_o^2 \left(1 + \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2 \frac{(k_o \ell)^2}{1 + (2k_o \ell)^2} (1 + 2\, i\, k_o \ell)\right) \tag{A-11}$$

This is in agreement with Scarf's result except for the factor of 2 multiplying the correction term under the square root sign. We are therefore in agreement with Scarf's conclusions in this case.

In the second case, namely that of coherent density fluctuations, as considered by Scarf the correlation function was isotropic and of the form:

$$R(\rho) = \sin \varkappa \rho / \varkappa \rho \tag{A-12}$$

Substitution of this into equation (A-9) gives:

$$k^2 = k_o^2 \left[1 + \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2 \frac{k_o^2}{4\, \varkappa\, k} \left\{ \log n \left|\frac{\varkappa + k + k_o}{\varkappa - (k + k_o)}\right| + \log n \left|\frac{\varkappa + k - k_o}{\varkappa - k + k_o}\right| + \right.\right.$$

$$\left.\left. + \, i \begin{Bmatrix} 0 \\ \pi \\ 0 \end{Bmatrix} \right\}\right] \quad \begin{matrix} k + k_o > \varkappa \\ |k - k_o| < \varkappa < k + k_o \\ |k - k_o| > \varkappa \end{matrix}$$

This only reduces to Scarf's result when $k = k_o$ is substituted on the right hand side of the equation. With this substitution $k^2 \to \infty$ when $\varkappa \to 2\, k_o$. The result that coherent resonant fluctuations have a profound effect on the wave propagation is closely linked with this artificial infinity, and Scarf's arguments cannot be immediately accepted.

If we assume the attenuation to be negligible and make the following substitutions in (A-13)

$$y = \frac{k}{k_o} \qquad a = \frac{\varkappa}{k_o} \qquad B = \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2$$

we obtain:

$$y^2 = 1 + \frac{B}{8\,ay} \left\{ \log n \left(\frac{1 - (a+y)^2}{1 - (a-y)^2} \right)^2 \right\} \tag{A-14}$$

or

$$8(y^3 - y) = \frac{B}{a} \log n \left(\frac{1 - (a+y)^2}{1 - (a-y)^2} \right)^2$$

Let us consider an extreme example, viz. B = 0.01. This might for instance correspond to $\delta N = 0.1N$ and $X = \frac{1}{2}$, or $\omega_p = 0.707\,\omega$. Let us solve equation (A-14)numerically for a few values of a. Note that in this case the increase in phase velocity over that in vacuum caused by the unperturbed plasma ($\delta N = 0$) corresponds to the ratio

$$\frac{C_{plasma}}{C_{vacuum}} = 1.41 \text{ or } 41\% \tag{A-15}$$

The following table gives some values of k/k_o for different values of a:

$\varkappa/k_o = a$	$k/k_o = y$
1	1.0014
2	1.0042
2.1	1.0021
2.2	1.0017
4	1.0003

Hence, we see that even in this case, which Scarf considers extreme, the effect of the irregularities on the propagation constant is only on the order of 1% of the effect of the smeared out plasma background. For smaller values of X the relative importance of the coherent density fluctuations becomes even smaller since $y \to 1$ as X^2 whereas the refractive index of the smooth background plasma goes to unity as X.

The resonance condition introduced by Scarf in order to cause the density fluctuations to affect the propagation properties of the medium is also logically unsatisfactory since it would be a strange coincidence to have the condition $\kappa = 2\,k_o$ fulfilled for the particular wavelength selected for the experiment. For this reason it might be of interest to explore some other possibilities which would affect the propagation characteristics in a non-resonant fashion. In order for the irregularities to become important one must have an appreciable second order scattering in the medium. The first order scattering component is ineffective in modifying the mean propagation properties. The second order scattering can be effective in that waves are scattered out of the main beam and then back into the main beam with a definite phase relationship with respect to the main wave beam. It was therefore thought that density waves travelling perpendicularly to the direction of propagation of the wave might be effective in modifying the propagation properties of the wave because this would seem to enhance the second order forward scattering required to modify the unperturbed wave. Such density irregularities might well be expected to exist particularly when the direction of propagation lies along magnetic field lines. The density fluctuations may be thought of as being associated with so-called longitudinal hydromagnetic waves (longitudinal refers to the direction of particle motion with respect to the direction of wave motion (ref. 6).

Consider therefore the case:

$$\vec{k} \parallel \text{z-axis} \qquad \text{and} \qquad R(\vec{\rho}\,) = e^{-\sqrt{x^2+y^2}/\ell} \tag{A-16}$$

Substitution into (A-9) and integration then gives:

$$k^2 = k_o^2\left[1 + \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2 \left\{ \frac{1}{1-(^k/k_o)^2 + (1/k_o\ell)^2} - \frac{i/k_o\ell}{\left(1-(k/k_o)^2+(1/k_o\ell)^2\right)^{3/2}} \cdot \right.\right.$$

$$\left.\left. \cdot\left(\frac{\pi}{2} - \text{Arctan}\left(\frac{i/k_o\ell}{\left(1-(k/k_o)^2+(1/k_o\ell)^2\right)^{1/2}}\right)\right)\right\}\right] \tag{A-17}$$

We distinguish between two main cases, viz

1) $\left|(k^2-k_o^2)\ell^2\right| \gg 1$ 2) $\left|(k^2-k_o^2)\ell^2\right| \ll 1$

so that we may expand the Arctan function conveniently.

<u>Case 1</u>

$$(k^2-k_o^2) = \left(\frac{\delta N}{N}\right)^2 \left(\frac{X}{1-X}\right)^2 k_o^2 \frac{(k_o\ell)^2}{1+(k^2-k_o^2)\ell^2} + \text{imaginary part} \tag{A-18}$$

neglecting the imaginary part and solving the second degree equation one obains:

$$(k^2-k_o^2)\ell^2 + \frac{1}{2} \mp\sqrt{(k_o\ell)^4\,B + \frac{1}{4}} \tag{A-19}$$

where B has the same definition as in (A-9). Only the case $(k_o\ell)^4 B \gg 1$ is possible in view of the initial assumption in this case, and we obtain:

$$k^2 = k_o^2\left(1 + X \cdot \frac{\delta N}{N}\right) \approx k^2_{\text{vacuum}}\left(1 - X(1 - \frac{\delta N}{N})\right) \tag{A-20}$$

where only that sign is used which appears to be physically reasonable. Hence, the correction can only amount to a few per cent, depending on ($\delta N/N$).

<u>Case 2</u>

Solving this in the same manner as above, one obtains:

$$k^2 = k^2_{vacuum} \left\{1 - X\left(1 - 0.707\frac{\delta N}{N}\right)\right\} \qquad \text{(A-21)}$$

and again the correction to the propagation constant indtroduced by the density fluctuations can only be rather slight.

Let us finally consider the case where we have density fluctuations corresponding to a collection of plane, monchromatic density waves which all travel perpendicularly to the direction of propagation of the electromagnetic wave. If they are undamped and all of the allowed directions are equally likely then:

$$R(\vec{\rho}) = J_o\left(\kappa\sqrt{x^2 + y^2}\right) \qquad \text{(A-22)}$$

Substituting this into (A-9) and carrying out the integration one obtains:

$$k^2 - k_o^2 = -\frac{\kappa^2}{2} + \sqrt{B \cdot k_o^4 + \frac{\kappa^4}{4}}$$

Again one can show that the correction to the non-perturbed case is of the form:

$$k^2 = k^2_{vacuum} \left\{1 - X\left(1 + \alpha\frac{\delta N}{N}\right)\right\} \qquad \text{(A-23)}$$

where α is of the order of unity or less.

Hence it appears that the correction to the apparent electron density caused by the density fluctuations, be they coherent or not, can only amount to at most a fraction $\delta N/N$. No large discrepancy between observed and actual plasma density should arise due to the presence of irregularities. Also, the apparent density seems to be smaller — not larger than the true density as one would have to require to bring Yoh et al's experiement (ref. 1) into line with current physical ideas about the magnetosphere (ref. 3).