ON THE CONTROLLABILITY OF DELAY-DIFFERENTIAL SYSTEMS

by

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1. Introduction.

The importance of dealing effectively with the inevitable delays of signal transmission within a control system is attested to by the volume of literature devoted to this problem over the years [1]. The early textbooks on control generally treated the problem of time lags by ad hoc and approximation methods, some of which involved modelling a system with pure delay by a higher order system without pure delay. (See Repin [2] for a detailed discussion of this technique.)

For a wide class of systems, however, it is natural and important that the model show the delay explicitly (See [3,4]), which motivates the consideration of delay-differential equations as models and the study of their properties from a system-theoretic point of view.

One of the fundamental system-theoretic properties of a control system is that of "controllability", which can be viewed as pertaining to the question of whether a given (optimal) control problem is well-posed or not, and which therefore impinges on questions of existence of solutions to such problems. Exactly how one should define the concept of controllability depends on the nature of the problems one is considering. Even in the case of control systems with finite dimensional state spaces, there is more than one natural way of defining controllability [5]. In the case of infinite dimensional spaces and with possibly infinite dimensional target sets, the controllability concept of interest certainly depends on the precise nature of the target set.

In this paper we define and discuss a type of controllability which is likely to play an important role in a broad class of optimal control problems for systems described by delay-differential equations. One of our objectives is to illustrate that some techniques which have been found to be eminently useful in obtaining results for ordinary differential equations can also be profitably used when dealing with delay equations. In particular, the approach we take to the solution of the problem discussed in the sequel is analogous to that for ordinary differential equations given by Markus and Lee [6] as modified by Kalman [7]. The results subsume the controllability results given by Chyung and Lee [8] in their paper on optimal control of delay-differential systems with target sets in euclidean space.

2. Definition of Controllability and Some Preliminary Remarks.

Consider the system

$$\dot{x}(t) = f(t,x(t),x(t-h),u(t)), \quad t > t_o \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^p$ and u is measurable and bounded on every finite time interval,* h = positive constant (the delay),

*Such functions will be referred to as "admissible".

$f \in C^1$ in all its arguments and $f(t,0,0,0) \equiv 0$. Let $\mathcal{B}$ be the Banach space of real n-vector-valued continuous functions defined on the interval $[t_o-h,t_o]$ with the uniform norm, i.e., if $\varphi \in \mathcal{B}$, we have $\|\varphi\| = \max_{t\in[t_o-h,t_o]} |\varphi(t)|$. Then a solution of (1) exists and is unique for $t > t_o$ if one specifies an initial function $\varphi \in \mathcal{B}$ [9].

Remark: The assumption of a single constant delay is for convenience only. All the results in this paper can be easily generalized to the case of multiple delays and these delays can also be time-varying as long as they are appropriately bounded so that their values do not overlap.

Let $\mathcal{X}$ be an abstract normed linear space of functions defined on the interval $[t_o-h,t_o]$. Then we give the following:

DEFINITIONS: (1) A system (1) is controllable to a function $\psi(\cdot) \in \mathcal{X}$ with respect to the space of initial functions $\mathcal{B}$ if, for any given $\varphi \in \mathcal{B}$, there exists a time t_1, $t_o < t_1 < \infty$, and an admissible control segment* $u_{[t_o,t_1+h]}$ such that $x(t;t_o,\varphi,u) = \psi(t-t_1+t_o-h)$, $t \in [t_1,t_1+h]$, where $x(t;t_o,\varphi,u)$ is the

*A segment $g_{[a,b]}$ denotes a function g defined over the interval $[a,b]$.

solution of (1), starting at time t_o, with initial function φ and control u.

(2) If the system (1) is controllable to all functions in $\mathcal{H}$ it is controllable to the space $\mathcal{H}$.

(3) If $\psi(\cdot) \equiv 0$ in definition (1), then the system is controllable to the origin.

(4) If t_1 is constant with φ in any of the above definitions, the corresponding type of controllability is uniform.

In the sequel, we shall give sufficient conditions for (1) to be controllable to the origin as well as to a function with respect to the space $\mathcal{B}$. We shall also give sufficient conditions under which the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + C(t)u(t) \tag{2}$$

(where $x(t) \in R^n$, $u(t) \in R^p$, and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are continuous matrix functions) is controllable to the origin and to a function with respect to $\mathcal{B}$. The aforementioned conditions for (2) will be shown to be necessary if a certain assumption about the space of trajectories of the homogeneous equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) \tag{3}$$

is true.

It should be strongly emphasized that controllability to the origin for a delay-differential system does not imply, in general, controllability to a function or a space of functions. However, the techniques which are used in this paper to study controllability to the origin are completely applicable to the study of controllability to a function or function space. This fact is illustrated in Section 6, where some results along this line are given.

3. The Linear Problem.

Consider equation (2) with $\mathcal{B}$ the space of initial functions. The solution of (2) for time $t > t_o$, and corresponding to initial function $\varphi \in \mathcal{B}$, has the form [10]

$$x(t) \equiv x(t;\, t_o,\varphi,u) = M(t,t_o,\varphi) + \int_{t_o}^{t} K(s,t)C(s)u(s)ds \tag{4}$$

where $M(t,t_o,\varphi)$ is the solution of the homogeneous equation (3) corresponding to initial time t_o and initial function φ, i.e.

$$M(t,t_o,\varphi) = \varphi(t) \quad \text{for} \quad t \in [t_o-h,t_o]. \tag{5}$$

The kernel $K(s,t)$ is defined for $t \geq t_o$ and $t_o \leq s \leq t$ and is an $n \times n$ matrix solution of the equations

$$\frac{\partial K(s,t)}{\partial s} = -K(s,t)A(s) - K(s+h,t)B(s+h), \quad t_o \leq s \leq t - h \tag{6a}$$

$$\frac{\partial K(s,t)}{\partial s} = -K(s,t)A(s), \quad t - h \leq s \leq t \tag{6b}$$

with $K(t,t) = I$ (the identity matrix).

Equation (6b) shows the obvious fact that over one delay interval, the delay equation behaves similarly to an ordinary differential equation with $K(s,t)$ playing the role of a fundamental matrix solution of the homogeneous equation [11].

LEMMA 1: Given (2) with any $\varphi \in \mathcal{B}$. A sufficient condition for existence of an admissible control which results in the solution having a zero-crossing in finite time is that there exists $t_1 > t_o$ such that

$$\text{(7)} \qquad \text{rank} \int_{t_o}^{t_1} K(s,t_1)C(s)C'(s)K'(s,t_1)ds = n$$

where (') indicates transpose.

<u>Proof</u>: Let $\mathcal{B}(t_o,t_1) = \int_{t_o}^{t_1} K(s,t_1)C(s)C'(s)K'(s,t_1)ds.$

In equation (4), substitute

$$u(s) = -C'(s)K'(s,t_1)\mathcal{B}(t_o,t_1)^{-1}M(t_1,t_o,\varphi).$$

Then $x(t_1) = 0$.

<u>DEFINITIONS</u>: (5) <u>The Force-Free Attainable Set at time t</u> of a system (2) is the set of all points in R^n that can be reached at time t by the trajectories of (3) resulting from all initial functions contained in $\mathcal{B}$.

(6) A system (2) whose Force-Free Attainable Set at any time t is all of R^n is pointwise complete.
Since we have been unable to give an example to the contrary, we present for the reader's amusement, the following:

CONJECTURE: All constant coefficient systems of the form (2) are pointwise complete.

Remark: The conjecture is true if we consider the trajectories only on the interval $t_o - h \leq t \leq t_o + h$, since the elements of $\mathcal{B}$ span all of R^n at any time $t \in [t_o-h, t_o]$ and the system (3) behaves as an ordinary differential equation on the interval $[t_o, t_o+h]$.

LEMMA 2: If a system (2) is pointwise complete, then (7) is necessary as well as sufficient for existence of a control which results in a zero-crossing in finite time of the solution of (2) for any $\varphi \in \mathcal{B}$.

Proof: Given any $\varphi \in \mathcal{B}$, suppose there exists $t_1 > t_o$ and a control $u_{[t_o,t_1]}$ such that $x(t_1) = 0$, but (7) doesn't hold. The latter implies that there exists a nonzero vector $x_1 \in R^n$ such that $x_1'K(s,t_1)C(s) = 0$, $t_o \leq s \leq t_1$. Then, from (4), $x_1'M(t_1,t_o,\varphi) = 0$. By hypothesis, however, φ can be chosen so that $M(t_1,t_o,\varphi) = x_1$. Then $x_1'x_1 = 0$ which contradicts the assumption that $x_1 \neq 0$.

THEOREM 1: A pointwise complete system (2) is controllable to the origin with respect to $\mathcal{B}$ if and only if

(i) there exists $t_1 > t_o$ such that (7) holds

(ii) given $\varphi \in \mathcal{B}$, then with t_1 as in (7) and for some admissible $u_{[t_o,t_1]}$ such that $x(t_1,t_o,\varphi,u_{[t_o,t_1]}) = 0$ the equation

$$C(t)u(t) = -B(t)x(t-h;\ t_o,\varphi,u_{[t_o,t_1]}), \tag{8}$$

has an admissible solution $u(\cdot)$ on the interval (t_1,t_1+h).

Proof: By Lemma 1 we have that for any $\varphi \in \mathcal{B}$ there exists $u_{[t_o,t_1]}$ such that $x(t_1;t_o,\varphi,u_{[t_o,t_1]}) = 0$. If (8) holds, then over the interval (t_1,t_1+h), equation (2) becomes

$$\dot{x}(t) = A(t)x(t), \qquad x(t_1) = 0. \tag{9}$$

It follows by the uniqueness theorem for ordinary differential equations that $x(t) = 0$ for all $t \in [t_1,t_1+h]$.

Conversely if (2) is controllable to the origin with respect to $\mathcal{B}$, then for any $\varphi \in \mathcal{B}$, there exists $t_1 > t_o$ and an admissible control $u_{[t_o,t_1+h]}$ such that $x(t;t_o,\varphi,u_{[t_o,t_1+h]}) = 0$ $\forall t \in [t_1,t_1+h]$ which implies (8). Since $x(t_1,t_o,\varphi,u_{[t_o,t_1]}) = 0$ and the system is pointwise complete, then (7) must hold by Lemma 2. Q.E.D.

Remark: If the control $u_{[t_o,t_1+h]}$ transfers an initial function $\varphi \in \mathcal{B}$ of the system (2) to the origin (the zero function on the interval $[t_1,t_1+h]$), then if $u(t) = 0$ for all $t > t_1 + h$, the system will remain at the origin.

4. On the Solution of (8).

Consider the following facts.

(1) An admissible solution of (8) will exist on an interval (t_1,t_1+h) if and only if $-B(t)x(t-h;\ t_o,\varphi,u_{[t_o,t_1]}$ is in the range of $C(t)$ almost everywhere on (t_1,t_1+h). Standard techniques can then be employed to construct a solution [12].

(2) If "controllable" is replaced by "uniformly controllable" in Theorem 1, then the right side of (8) must be in the range of $C(t)$ for all $\varphi \in \mathcal{B}$ on (t_1,t_1+h) where t_1 is fixed.

(3) No solution of (8) can be unique since one can add to it any vector-valued function of time which is in the null space of $C(\cdot)$ almost everywhere on (t_1,t_1+h).

To obtain sharper results than the preceding, it is necessary to do some deep analysis of the attainable set for (2), as indicated by the results below.

Consider equation (8) over an interval (t_1,t_1+h), and let P be the set of initial functions in $\mathcal{B}$ which are controllable to the origin using admissible controls defined over $[t_o,t_1+h]$. ($P = \mathcal{B}$ for uniform controllability). For each $\varphi \in P$, let $K_\varphi = \{u_\lambda^\varphi,\ \lambda \in \Lambda(\varphi)\}$ = the set of admissible controls taking φ to the origin (the zero function defined over the fixed time interval $[t_1,t_1+h]$). Invoking the axiom of choice, define

$$Q = \{\psi;\ \psi:P \rightarrow \bigcup_{\varphi \in P} K_\varphi\}$$

(i.e., $\psi \in Q \Longrightarrow \psi(\varphi) = u_\lambda^\varphi$ for some $\lambda \in \Lambda(\varphi)$).

Now, let

$$S_\psi(t) = \{x(t;\ t_o,\varphi,\psi(\varphi));\quad \varphi \in P\}$$

where $x(t;\ t_o,\varphi,\psi(\varphi))$ denotes the value at time t of the trajectory of (2) generated by initial function φ and control $\psi(\varphi)$. We then have

LEMMA 3: If for each $\psi \in Q$ and each $t \in (t_1-h,t_1)$ the set $S_\psi(t)$ covers all directions in euclidean n-space, then a necessary and sufficient condition for (8) to have a solution independent of $u_{[t_o,t_1]}$ almost everywhere on (t_1,t_1+h) is that there exists a $p \times n$ matrix $D(t)$ with bounded measurable elements such that $B(t) = C(t)D(t)$ almost everywhere on (t_1,t_1+h).

Proof: Fix $t \in (t_1,t_1+h)$. The problem reduces to solving the algebraic equation

$$Cu = -Bx$$

where x is an n-vector which can take on values corresponding (except for a magnitude constraint) to any collection of n basis vectors. Then $-Bx \in$ range C if and only if the columns of B are linear combinations of those of C, i.e., there exists D such that $B = CD$. Continuity of $B(t)$ and $C(t)$ assure that this process can be repeated for each $t \in (t_1,t_1+h)$ with the matrix $D(t)$ having bounded measurable elements on that interval. Q.E.D.

Remark: Under the above conditions, the solution for $u(\cdot)$ has the form

$$(10)\quad u(t) = \sum \alpha_i e_i(t) + D(t)x(t-h;t_o,\varphi,u_{[t_o,t_1]}),\ t_1 < t < t_1 + h$$

where $e_i(t) \in$ null space of $C(t)$ and α_i = constant. The preceding facts plus Theorem 1 immediately imply

THEOREM 2: A pointwise complete system (2), which satisfies the hypothesis of Lemma 3 is uniformly controllable to the origin with respect to $\mathcal{B}$ if and only if

(i) There exists $t_1 > t_o$ such that (7) holds

(ii) There exists an $n \times p$ matrix, $D(t)$, with bounded measurable elements such that, with t_1 defined as above, $B(t) = C(t)D(t)$ a.e. on (t_1,t_1+h).

Since engineers have an aversion (and rightfully so!) to measurable solutions of control problems, we give the conditions under which one can find an absolutely continuous solution to (8) over the interval (t_1,t_1+h). The result emerges as an application of the next lemma which is due to Dǒlezal* [13].

LEMMA 4 (Dǒlezal): Let $G(t)$ be an $n \times p$ matrix defined on an interval $[a,b]$ and continuous, at least. Suppose there exists an integer $r \leq p$ such that rank $G(t) = r$ for all $t \in [a,b]$. Then there exists an $p \times p$ matrix $H(t)$, defined and nonsingular on

*This important Lemma has a variety of applications to problems in system theory [16,17].

$[a,b]$ and whose degree of smoothness matches that of $G(t)$, such that

$$G(t)H(t) = [F(t):0], \quad t \in [a,b],$$

where $F(t)$ is $n \times r$, rank $F(t) = r$ for all $t \in [a,b]$.

THEOREM 3: If equation (8) has an admissible solution and if rank $C(t) = r =$ constant for all $t \in [t_1, t_1+h]$, then that solution can be chosen to be absolutely continuous.

Proof: By Lemma 4, there exist real n-vector-valued continuous functions $c_1(t), \ldots, c_r(t)$ which span the range $C(t)$ at each $t \in [t_1, t_1+h]$. Then, if (8) has a solution almost everywhere on (t_1, t_1+h) we can write

$$(11) \qquad B(t)x(t-h;\, t_o, \varphi, u_{[t_o,t_1]}) = \sum_{i=1}^{r} \alpha_i(t) c_i(t), \quad \text{a.e. on} \quad (t_1, t_1+h).$$

But since the left side of (11) is absolutely continuous, then the α_i's can be chosen absolutely continuous. It then follows that an absolutely continuous solution of (8) exists. Q.E.D.

5. The Nonlinear Problem.

The problem will be solved in two steps. First, conditions are given under which one can control a system (1) to an arbitrarily small neighborhood of the origin in finite time, and then we give conditions under which the origin can be reached in finite time from a point in its neighborhood.

DEFINITION: (6) A system (1) is quasi-controllable to the origin with respect to $\mathcal{B}$ if for any $\varphi \in \mathcal{B}$ and any $\epsilon > 0$, there exists $t_1 > t_o$ and an admissible control $u_{[t_o,t_1+h]}$ such that $\|x(\cdot,t_o,\varphi,u)\|_{[t_1,t_1+h]} \equiv \max_{t_1 \leq t \leq t_1+h} |x(t;\ t_o,\varphi,u)| < \epsilon$.

Consider the system (1) with $f(t,0,0,0) \equiv 0$, $f \in C^1$ in $R \times R^n \times R^n \times R^p$, $u(t) \in R^p$, and $\varphi \in \mathcal{B}$.

Define the functions:

$\omega(\cdot)$ = continuous, real-valued nondecreasing function such that $\omega(s) > s$, $s > 0$;

$\mu(\cdot)$ and $\nu(\cdot)$ = continuous, real-valued functions of s defined for $s \geqq 0$, and positive and nondecreasing for $s \neq 0$.

$\beta(\cdot)$ = continuous, real-valued function of s defined for $s \geqq 0$, and positive for $s \neq 0$.

THEOREM 4: Given the system (1) and the above defined quantities. Suppose there exists a real-valued function $V(t,x)$, defined and continuous for $t \geqq t_o - h$, $x \in R^n$, and a real p-vector-valued function $U(x)$ which is C^1 in R^n such that

(i) $\mu(|x|) \leqq V(t,x) \leqq \nu(|x|)$, $t \geqq t_o - h$

$$\text{(ii)}\quad \left.\frac{\partial V(t,x)}{\partial t}\right|_{x=\rho(t)} + \left.\frac{\partial V(t,x)}{\partial x}\right|_{x=\rho(t)} \cdot f(t,\rho(t),\rho(t-h),U(\rho(t)))$$

$$\leq -\beta(|\rho(t)|)$$

for all $t \geq t_o$ and all continuous, real n-vector-valued function segments $\rho_{[t-h,t]}$ such that

$$\text{(iii)}\quad V(\xi,\rho(\xi)) < \omega(V(t,\rho(t))), \quad t - h \leq \xi \leq t.$$

Then the system (1) is quasi-controllable with respect to $\mathcal{B}$.

<u>Remark</u>: Theorem 4 is an easy generalization of a theorem originally due to Krasovskii [14] on uniform asymptotic stability of delay-differential equations. The proof follows precisely the novel but lengthy proof given by Driver [9] of the original theorem and will therefore not be reproduced here. Suffice it to say that if the conditions of the theorem are met, then for any initial-function $\varphi \in \mathcal{B}$, there exists an admissible control which has the effect of driving the system to an ϵ-neighborhood of the origin (in function space) in finite time.

Now, consider the following:

<u>DEFINITION</u>: (7) A system (1) is <u>locally controllable to the origin with respect to</u> $\mathcal{B}$ if it is controllable to the origin with respect to a neighborhood $N(0^{\mathcal{B}})$ of the origin in $\mathcal{B}$.

(8) The <u>first variation of</u> (1) <u>about the zero-solution</u> is

the system (2) where

$$A(t) = \frac{\partial f}{\partial x}(t,0,0,0)$$

$$B(t) = \frac{\partial f}{\partial x_d}(t,0,0,0) \quad (x_d(t) = x(t-h))$$

$$C(t) = \frac{\partial f}{\partial u}(t,0,0,0)$$

THEOREM 5: A system (1) is locally controllable to the origin with respect to $\mathcal{B}$ if its first variation about the zero-solution satisfies the conditions

(i) there exists $t_1 > t_o$ such that (7) holds

(ii) with t_1 defined as above, there exists an $n \times p$ matrix $D(t)$ with bounded, measurable elements such that $B(t) = C(t)D(t)$ a.e. on (t_1,t_1+h).

Proof: Following Kalman [7], we introduce a parameter ξ into the control u and define

$$(12) \qquad u^{\xi}(t) = u(t,\xi) = \begin{cases} C'(t)K'(t,t_1)\xi, & t_o \leq t \leq t_1 \\ \text{solution* of } C(t)u(t) = -B(t)x(t-h;\ t_o,0^{\mathcal{B}},u^{\xi}), & t_1 < t < t_1 + h \end{cases}$$

Notes:

(i) $u(t,0) = u^{o}(t) = 0$ for $t \in [t_o,t_1]$.

(ii) If $\varphi \equiv 0$, then $x(t;\ t_o,0^{\mathcal{B}},u^{o}) = 0$ on $[t_o-h,t_1]$.

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*An admissible solution exists by hypothesis (ii).

Let

$$J(t) = \frac{\partial x(t;\ t_o, 0^{\mathcal{B}}, u^{\xi})}{\partial \xi}\bigg|_{\xi=0} \tag{13}$$

Since $\varphi \equiv 0$, the solution of (1) is written as

$$x(t;t_o,0^{\mathcal{B}},u^{\xi}) \equiv x(t;\xi) = \int_{t_o}^{t} f(\tau,x(\tau),x_d(\tau),u^{\xi}(\tau))d\tau, \quad t_o \leq t \leq t_1 + h.$$

From (i) and (ii) above, it follows that

$$J(t) = \frac{\partial x}{\partial \xi}\bigg|_{\xi=0} = \int_{t_o}^{t} \left[\frac{\partial f}{\partial x}(\tau,0,0,0)\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial x_d}(\tau,0,0,0)\frac{\partial x_d}{\partial \xi} + \frac{\partial f}{\partial u}(\tau,0,0,0)\frac{\partial u}{\partial \xi}\right]\bigg|_{\xi=0} d\tau$$

$$= \int_{t_o}^{t} [A(\tau)J(\tau) + B(\tau)J(\tau-h) + C(\tau)\frac{\partial u}{\partial \xi}(\tau,0)]d\tau.$$

Differentiating,

$$\dot{J}(t) = A(t)J(t) + B(t)J(t-h) + C(t)\frac{\partial u}{\partial \xi}(t,0), \quad t_o \leq t \leq t_1 + h.$$

But from (12),

$$\frac{\partial u}{\partial \xi}(t,0) = C'(t)K'(t,t_1), \quad t_o \leq t \leq t_1$$

and

$$C(t)\frac{\partial u}{\partial \xi}(t,0) = -B(t)J(t-h), \quad t_1 < t < t_1 + h$$

Therefore

$$(14)\quad \dot{J}(t) = A(t)J(t) + B(t)J(t-h) + \begin{cases} C(t)C'(t)K'(t,t_1), & t_o \leq t \leq t_1 \\ -B(t)J(t-h), & t_1 < t < t_1 + h \end{cases} .$$

The solution of (14) over the interval $[t_o, t_1]$ is then

$$(15)\qquad J(t) = \int_{t_o}^{t} K(s,t)C(s)C'(s)K'(s,t_1)ds, \quad t_o \leq t \leq t_1.$$

By hypothesis, equation (15) implies that $\det J(t_1) \neq 0$. Moreover, on the interval (t_1, t_1+h), equation (14) is

$$(16)\qquad \dot{J}(t) = A(t)J(t)$$

so that $J(t)$ is a fundamental matrix solution for (16). It follows that $\det J(t) \neq 0$ for $t \in [t_1, t_1+h]$.

Since $J(t)$ is defined by (13), the above facts suggest that one may use an implicit function theorem to solve the equation

$$x(t; t_o, \varphi, \xi) = 0, \quad t_1 \leq t \leq t_1 + h$$

for ξ as a function of φ. More precisely, consider the following theorem from Dieudonné [15].

THEOREM 6: Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be Banach spaces; g a continuously differentiable map of an open subset S of $\mathcal{B}_1 \times \mathcal{B}_2$ into $\mathcal{B}_3$. Let $(x_o, y_o) \in S$ where $g(x_o, y_o) = 0$ and let the Frechet derivative of g with respect to y be a linear homeomorphism of $\mathcal{B}_2$ into

$\mathcal{B}_3$. Then there exists an open neighborhood, N_o, of x_o in $\mathcal{B}_1$ such that for every open connected neighborhood N of x_o contained in N_o, there exists a unique continuous map $\Pi: N \to \mathcal{B}_2$ such that $\Pi(x_o) = y_o$, $(x,\Pi(x)) \in S$ and $g(x,\Pi(x)) = 0$ for all $x \in N$. Furthermore Π is continuously differentiable in N.

Application: Let $\mathcal{B}_1$ = the space $\mathcal{B}$ of all continuous functions on $[t_o-h,t_o]$, $\mathcal{B}_2 = R^n$, $\mathcal{B}_3$ = the space of all continuous functions on $[t_1,t_1+h]$, g = a solution segment of (1) i.e. $g(\cdot,\cdot) = x_{[\cdot,\cdot]}(t_o,\cdot,\cdot)$.

Let $S = \mathcal{B} \times \Gamma$ where $\Gamma \subset R^n$ is an open neighborhood of the origin in R^n and represents the permissible range of ξ. (Thus $(0^{\mathcal{B}},0^{\Gamma})$ is an interior point of S). The Frechet derivative of g with respect to ξ is a map which takes R^n onto $\mathcal{B}_3$. The fact that the Jacobian matrix $J(t)$ is a homeomorphism of R^n onto R^n for each $t \in [t_1,t_1+h]$ implies that the Frechet derivative of g is a homeomorphism from R^n onto $\mathcal{B}_3$. Now, since $x_{[t_1,t_1+h]}(t_o,0^{\mathcal{B}},0^{\Gamma}) = 0$, then by Theorem 6 there exists a neighborhood $N(0^{\mathcal{B}})$ of $0^{\mathcal{B}}$ and a unique continuous map $\Pi: N(0^{\mathcal{B}}) \to R^n$ such that $\varphi \in N(0^{\mathcal{B}})$ implies $(\varphi,\Pi(\varphi)) \in S$ and $x_{[t_1,t_1+h]}(t_o,\varphi,\Pi(\varphi)) = 0$, that is if $\varphi \in N(0^{\mathcal{B}})$ then the equation

$$x(t;\ t_o,\varphi,\xi) = 0, \quad t_1 \leqq t \leqq t_1 + h$$

has an admissible solution $\xi = \Pi(\varphi)$. This completes the proof of Theorem 5.

Theorems 4 and 5 provide sufficient conditions for controllability to the origin with respect to the space $\mathcal{B}$ for the system (1).

6. Controllability to a Function.

To repeat our earlier assertion; controllability to the origin does not necessarily imply controllability to a function or to a function space. To illustrate this, and to show how the techniques presented thus far can be adapted to study controllability to a function, we present some results for controllability of (2) (and local controllability of (1)) to a function in the space $\mathcal{H}$ of real n-vector-valued C^1-functions defined on the interval $[t_o-h, t_o]$.

THEOREM 7: Consider a pointwise complete system (2) and let $\mathcal{L}_t(\cdot) = \frac{d}{dt}(\cdot) - A(t)(\cdot)$. Let $\alpha \in \mathcal{H}$. Then (2) is controllable to $\alpha \in \mathcal{H}$ with respect to $\mathcal{B}$ if and only if

(i) there exists $t_1 > t_o$ such that (7) holds

(ii) with t_1 defined as above, for any $\varphi \in \mathcal{B}$, and for some admissible $u_{[t_o,t_1]}$ such that $x(t_1; t_o,\varphi,u_{[t_o,t_1]}) = \alpha(t_o-h)$, there exists an admissible solution to the equation

$$(17)\qquad C(t)u(t) = (\mathcal{L}_t\alpha)(t-t_1+t_o-h) - B(t)x(t-h; t_o,\varphi,u_{[t_o,t_1]})$$

on the interval (t_1,t_1+h).

Proof: Essentially the same as for Theorem 1.

Now consider the following

DEFINITION: (9) A system (1) is locally controllable to a function $\alpha \in \mathcal{H}$ with respect to $\mathcal{B}$ if, given any initial time t_o, and a trajectory $x^o(\cdot;\ t_o,\varphi_\alpha,u_\alpha)$, $\varphi_\alpha \in \mathcal{B}$, u_α admissible, such that for some $t_1 > t_o$, $x^o(t;\ t_o,\varphi_\alpha,u_\alpha) = \alpha(t-t_1+t_o-h)$ for all $t \in [t_1,t_1+h]$, then there is a neighborhood $N(\varphi_\alpha)$ of the initial function φ_α such that for each $\varphi \in N(\varphi_\alpha)$ there exists an admissible control u^* defined on $[t_o,t_1+h]$ such that $x(t;\ t_o,\varphi,u^*) = \alpha(t-t_1+t_o-h)$ for all $t \in [t_1,t_1+h]$.

(10) The first variation of (1) about the trajectory $x^o(\cdot;\ t_o,\varphi_\alpha,u_\alpha)$ is given by (2) where

(18) $$A(t) = \frac{\partial f}{\partial x}(t,x^o(t;\ t_o,\varphi_\alpha,u_\alpha),\ x^o(t-h;\ t_o,\varphi_\alpha,u_\alpha),\ u_\alpha(t))$$

(19) $$B(t) = \frac{\partial f}{\partial x_d}(t,x^o(t;\ t_o,\varphi_\alpha,u_\alpha),\ x^o(t-h;\ t_o,\varphi_\alpha,u_\alpha),\ u_\alpha(t))$$

(20) $$C(t) = \frac{\partial f}{\partial u}(t,x^o(t;\ t_o,\varphi_\alpha,u_\alpha),\ x^o(t-h;\ t_o,\varphi_\alpha,u_\alpha),\ u_\alpha(t)).$$

We then have

THEOREM 8: A system (1) is locally controllable to a function $\alpha \in \mathcal{H}$ with respect to $\mathcal{B}$ if its first variation about the trajectory $x(\cdot,t_o,\varphi_\alpha,u_\alpha)$ as defined in Definition 9 satisfies the conditions

(i) (7) holds for t_1 as defined in Definition 9

(ii) with t_1 as above; $(\mathcal{L}_t\alpha)(t-t_1+t_o-h) \in$ range $C(t)$ almost everywhere on (t_1,t_1+h).

(iii) there exists an $n \times p$ matrix $D(t)$ with measurable bounded elements such that $B(t) = C(t)D(t)$ almost everywhere on (t_1, t_1+h).

Proof: Essentially the same as that for Theorem 5, but is outlined for illustrative purposes.

Let $x^o(t;\ t_o, \varphi_\alpha, u_\alpha) \equiv x^o(t)$ and perform the substitution in (1)

(21) $$x(t) = y(t) + x^o(t).$$

Then (1) can be written as

(22) $$\dot{y}(t) = -\dot{x}^o(t) + f(t, x(t), x(t-h), u(t)).$$

Solving for y assuming the zero initial function (corresponding to initial function $\varphi_\alpha \in \mathcal{B}$ for x) we obtain

(23) $$y(t) = -x^o(t) + \varphi_\alpha(t_o) + \int_{t_o}^{t} f(\tau, x(\tau), x(\tau-h), u(\tau))d\tau$$

Now introduce a parameter ξ into $u(t)$ and let

(24) $$u^\xi(t) = u(t,\xi) = \begin{cases} u_\alpha(t) + C'(t)K'(t,t_1)\xi, & t_o \leqq t \leqq t_1 \\ u_\alpha(t) + \text{solution}^* \text{ to } C(t)u(t) = \\ -B(t)y(t-h;\ t_o, 0, u^\xi_{[t_o,t_1]}), & t_1 < t < t_1 + h, \end{cases}$$

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*An admissible solution exists by hypotheses (ii) and (iii).

where K represents the kernel matrix in the solution of (3) with $A(\cdot)$, $B(\cdot)$ given by (18) and (19), and $C(t)$ is given by (20). Let the corresponding solution of (22) be $y(t;\ t_o,0,\xi)$ and define

$$J(t) = \left.\frac{\partial y(t;\ t_o,0,\xi)}{\partial \xi}\right|_{\xi=0}$$

Since $u^o(t) = u_\alpha(t)$ and $y(t;\ t_o,0,0) = 0$ we have, upon differentiating (23)

$$J(t) = \int_{t_o}^{t} [A(\tau)J(\tau) + B(\tau)J(\tau-h) + C(\tau)\left.\frac{\partial u^\xi(\tau)}{\partial \xi}\right|_{\xi=0}]d\tau$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are as in (18), (19), (20) respectively.

The remaining steps are now exactly as in the proof of Theorem 5, i.e., it follows from (7), (17), and (24) that $\det J(t) \neq 0$ for all $t \in [t_1,t_1+h]$. Hence we can apply Theorem 6 to show existence of a solution to the equation

(25) $$y(t;\ t_o,\varphi,\xi) = 0 \quad \text{for all} \quad t \in (t_1,t_1+h)$$

of the form $\xi = \pi(\varphi)$ for φ in some small neighborhood of the origin in y-space. (And the range of the control is contained in a neighborhood of the range of u_α.) But since, by definition,

$$\begin{aligned} y(t;\ t_o,\varphi,\xi) &= x(t;\ t_o,\varphi^*,\xi) - x^o(t;\ t_o,\varphi_\alpha,u_\alpha) \\ &= x(t;\ t_o,\varphi^*,\xi) - \alpha(t-t_1+t_o-h),\ t \in (t_1,t_1+h) \end{aligned}$$

where $\varphi^* = \varphi - \varphi_\alpha$, then the solution of (25) implies that the equation

$$x(t;\ t_o,\varphi^*,\xi) = \alpha(t-t_1+t_o-h), \quad t \in (t_1,t_1+h)$$

has a solution $\xi = \Pi^*(\varphi^*)$ for all φ^* in a small neighborhood of φ_α and with the range of the control contained in a neighborhood of the range of u_α.

Q.E.D.

To obtain sufficient conditions for controllability of (1) to $\alpha \in \mathcal{H}$ with respect to $\mathcal{B}$ we need merely complement Theorem 8 with a theorem which yields quasi-controllability of (1) to $\alpha \in \check{\mathcal{H}}$. Such a theorem is easily obtained by rewriting Theorem 4 so that it pertains to equation (22).

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