A Note on Contact Transformations

by

K. R. Meyer*

Abstract

Not all contact transformations are of the form $p = W_q(q,P)$ and $Q = W_P(q,P)$ but this note shows that after a linear change of variables any contact transformation can be written in this form.

*This Research was supported by the National Aeronautics and Space Administration, Huntsville, under Contract No. NAS 8-11264 & NGR-40-002-015

It is well known that a transformation of the form

$$(1) \qquad p_i = \frac{\partial W}{\partial q_i}(q,P), \quad Q_i = \frac{\partial W}{\partial P_i}(q,P)$$

defines a contact transformation from the q,p variables to the Q,P variables where q,p,Q,P are n-vectors, W is a scalar function with continuous second partial derivatives with respect to all arguments and subscripts denote components of the vectors. See for example [1], and [2]. It is not always true that any contact transformation can be written in the form 1) or even in one of the other three common variations of 1). This fact is pointed out in [2] and [3] and the author recommends [2] as a careful and readable source on contact transformations (see in particular, page 69-70 of [2]). This note will show, however, that <u>any contact transformation can be written as a composition of a linear orthogonal contact transformation and a contact transformation of the form 1)</u>. That is to say, given any contact transformation one can first make a change of variables that is linear, orthogonal and preserves Hamiltonian form and then write the transformation in the form 1). The above is to be taken as a local statement, that is, the above statement holds only in a sufficiently small neighborhood of a point. Also we shall assume that all functions are sufficiently differentiable that the indicated derivatives are continuous and that the implicit function theorem can be applied. The assumption that all functions considered have continuous second partial derivatives with respect to all arguments will suffice.

To avoid confusion a contact transformation will be taken in the sense of Whittaker [1], page 293. That is:

Definition: A transformation

$$(2) \qquad \mathcal{F}: \quad Q = \varphi(q,p), \quad P = \psi(q,p)$$

where q,p,Q,P are n-vectors and ψ and φ are n-vector valued used functions of q and p will be called a contact transformation if there exists a scalar valued function $S(q,p)$ such that

$$(3) \qquad dS(q,p) = \sum_{i=1}^{n} \{p_i dq_i + \varphi_i(q,p) d\psi_i(q,p)\}.$$

Observe that 3) is often written $dS = \sum_{i=1}^{n} \{p_i dq_i + Q_i dP_i\}$ and that this short notation is the cause of some of the confusion in the literature. The equality 3) states that S must be considered as a function of p and q only. Indeed the whole question of when a contact transformation 2) can be written in the form 1) rests on the question of when can S be written as a function of q,P. If the second equation in 2) can be solved for p in terms of P and q and the result substituted into S we would have the desired function W. But when can we solve the second equation in 2) for p in terms of q, and P? If the sub-Jacobian $\det\{\frac{\partial\psi_i}{\partial p_j}\}$ is non zero then we can solve this equation, but there is no reason to suppose that it is nonzero. At this point a result in [3] can be used to straighten things out.

The formal proof is as follows. Let 2) or $\mathcal{F}$ be a given contact transformation. Without loss of generality we can assume that $\mathcal{F}$ takes the origin into the origin since otherwise we would shift the origin by a translation. Let T be the Jacobian matrix of $\mathcal{F}$ evaluated at the origin, i.e.

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = \{\frac{\partial\varphi_i}{\partial q_j}(0,0)\}$, $B = \{\frac{\partial\varphi_i}{\partial p_j}(0,0)\}$, $C = \{\frac{\partial\psi_i}{\partial q_j}(0,0)\}$ and $D = \{\frac{\partial\psi_i}{\partial p_j}(0,0)\}$.

Now by a result in [3], page 44 there exists nonsingular contact matrices O and R where O is orthogonal and R is positive definite symmetric such that T = RO. This result for contact matrices is the analog of the well known result in 3-dimensions that says that any matrix of a linear transformation is the product of a pure rotation (or rotation and reflection) and a pure dilation. It should be remarked that in [3] as in many other references a contact matrix is called symplectic and is sometimes given a different but equivalent definition (see [2]).

Let $\mathcal{O}$ be the transformation whose representation is the matrix O. Define a new transformation $\mathcal{G}$ by $\mathcal{G} = \mathcal{F} \circ \mathcal{O}^{-1}$

and so $\mathcal{F} = \mathcal{G} \circ \Theta$. Observe that we have "factored" the transformation $\mathcal{F}$ into two operations: first apply Θ and then $\mathcal{G} = \mathcal{F} \circ \Theta^{-1}$. Another way of looking at $\mathcal{G}$ is that we have changed coordinates by the linear transformation Θ and now $\mathcal{F}$ has the form $\mathcal{G}$ in the new coordinates. We now want to show that $\mathcal{G}$ can be written in the form 1).

$\mathcal{G}$ is a contact transformation since it is the composition of two contact transformations and moreover its Jacobian matrix at the origin is $TO^{-1} = (RO)O^{-1} = R$. Thus if $\mathcal{G}$ is given by $Q = a(q',p')$, $P = b(q',p')$ and $R = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ where $A' = \{\frac{\partial a_i}{\partial q'_j}(0,0)\}$ etc.

Now R is positive definite and symmetric and so by Sylvester's criterion [4], page 306, or [5] page 94 each principal subdeterminant of R is positive and so in particular $D' = \{\frac{\partial b_i}{\partial p'_i}(0,0)\}$ is nonsingular.

Thus, we can solve the equation $P = b(q',p')$ for p' to obtain $p' = h(q',P)$.

Since $\mathcal{G}$ is a contact transformation there exists a generating function $S'(q',p')$ such that

$$(4) \qquad dS'(q',p') = \sum_{i=1}^{n} \{p'_i dq'_i + b_i(q',p')da_i(q',p')\}.$$

Let $W(q',P) = S'(q',h(q',P))$ now

$$(5) \qquad dW(q',P) = \sum_{i=1}^{n} \{\frac{\partial W}{\partial q'} dq'_i + \frac{\partial W}{\partial P_i} dP_i\}$$

but $dW = dS$ at corresponding points and so

$$(6) \qquad dW(q',P) = \sum_{i=1}^{n} \{p_i' dq_i' + b_i(q',p')dP_i\}$$

where in 6) $p' = h(q',P)$.

Now since $dP_i = \sum_{j=1}^{n} \{\frac{\partial b_i}{\partial q_j'} dq_j' + \frac{\partial b_i}{\partial p_j'} dp_j'\}$ and since $\{\frac{\partial b_i}{\partial p_j}\}$ is nonsingular the differentials $dq_1', \ldots, dq_n', dP_1, \ldots, dP_n$ are linearly independent and so we can equate coefficients in 5) and 6) to obtain

$$(7) \qquad p_i' = \frac{\partial W}{\partial q_i'}(q',P) \quad \text{and} \quad Q_i = \frac{\partial W}{\partial P_i}(q',P).$$

Therefore $\mathcal{G}$ is of the form 1).

Observe that we can obtain one of the other common variations of 1) when any one of the other sub-Jacobian matrices is nonsingular. The procedure we have used gives that A' is nonsingular so this gives one variant. By changing variables again with the linear orthogonal contact matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ then the Jacobian of the new $\mathcal{G}$ is of the form $\begin{pmatrix} -B' & A' \\ -D' & C' \end{pmatrix}$ and so now the upper right and lower left sub-Jacobian matrices are nonsingular and by the same procedure you get the other two variants.

References

[1] Whittaker, E. T., A Treatise on the Analytic Dynamics of Particles and Rigid Bodies, Cambridge University Press, 4th edition, 1964.

[2] Pollard, H., Mathematical Introduction to Celestial Mechanics, Prentice Hall, 1966.

[3] Wintner, A., The Analytic Foundations of Celestial Mechanics, Princeton University Press, 1947.

[4] Gantmacher, F. R., The Theory of Matrices, Vol. 1, Chelsea Publishing Company, 1959, p. 306.

[5] Perlis, Theory of Matrices, Addison-Wesley Publishing Company, 1958, p. 94.