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by Hans Sagan

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SUMMARY

L. S. Pontryagin et al. have shown ([1], pp. 248-256) that the maximum principle leads, as is to be expected, to the multiplier rule and the Weierstrass inequality for the problem of Lagrange.

In this paper, we will demonstrate how the transversality conditions for the Lagrange problem with a variable endpoint may be obtained from Pontryagin's maximum principle and transversality conditions for an optimal control problem with a variable endpoint ([1], pp. 45-50, 62-63).

We consider the Lagrange problem of finding a trajectory $(y_1(t),\dots,y_n(t)) \in C^1[a,b]$ (optimal trajectory) so that

$$\int_a^b f(t,y_1,\dots,y_n,y_1',\dots,y_n')dt \to \text{minimum}$$

under the constraints

$$\begin{aligned} y_1' &= \phi_1(t,y_1,\dots,y_n,y_{\mu+1}',\dots,y_\mu') \\ &\vdots \\ y_\mu' &= \phi_\mu(t,y_1,\dots,y_n,y_{\mu+1}',\dots,y_\mu'), \quad \mu < n \end{aligned}$$

which emanates from the given point

$$y_i(a) = y_i^a, \quad i = 1,2,\dots,n$$

and terminates for some b on the (smooth) $(n + 1 - k)$-dimensional manifold T

$$\begin{aligned} \chi_1(t,y_1,\dots,y_n) &= 0 \\ &\vdots \\ \chi_k(t,y_1,\dots,y_n) &= 0, \quad k < n. \end{aligned} \tag{1}$$

We will assume that f, ϕ_i, $\partial f/\partial t$, $\partial f/\partial y_j$, $\partial\phi_i/\partial t$, $\partial\phi_i/\partial y_j$ are continuous in an open set of the $(t,y_1,\dots,y_n)$-space, that contains the optimal trajectory, and that f, ϕ_i, are continuous for all $y_1',\dots,y_n'$. We will further assume that $\partial\chi_i/\partial t$, $\partial\chi_i/\partial y_j$ are continuous for all $t,y_1,\dots,y_n$ and that for every fixed t, grad χ_k are linearly independent for all $(y_1,y_2,\dots,y_n)$.

The transversality conditions as stated by G. A. Bliss ([2], p. 311) and as adopted to our specific problem and notation are as follows:

There have to exist $(k + 1)$ constants $(\mu_o,\mu_1,\ldots,\mu_k) \neq (0,0,\ldots,0)$ such that

$$(2) \qquad \sum_{i=1}^{k} \mu_i \left(\frac{\partial \chi_i}{\partial y_j}\right)_b = \left(\frac{\partial h}{\partial y_j'}\right)_b, \quad j = 1,2,\ldots,n$$

$$(3) \qquad \sum_{i=1}^{k} \mu_i \left(\frac{\partial \chi_i}{\partial t}\right)_b = - \mu_o(f)_b - \sum_{i=1}^{n} \left(\frac{\partial h}{\partial y_i'}\right)_b y_i'(b)$$

where

$$(4) \qquad h = - \lambda_o f + \sum_{i=1}^{\mu} \lambda_i (y_i' - \phi_i)$$

and where $(\lambda_o,\lambda_1,\ldots,\lambda_\mu) \neq (0,0,\ldots,0)$ are the Lagrange multipliers with $\lambda_o = \mu_o$.

In order to show how these conditions may be obtained from Pontryagin's maximum principle and transversality conditions, we first formulate the Lagrange problem as an optimal control problem, introducing

$$y_o(t) = \int_a^t f(s,\ y_1(s),\ldots,y_n(s),\ y_1'(s),\ldots,y_n'(s))ds,$$

$$u_1 = y_{\mu+1}',\ldots,u_m = y_n', \quad m = n - \mu,$$

$$\phi_o(t,y_1,\ldots,y_n,u_1,\ldots,u_m) = f(t,y_1,\ldots,y_n,\ \phi_1(t,y,u),\ldots,\phi_\mu(t,y,u),u_1,\ldots,u_m).$$

Then the problem will read as follows: To be found is a trajectory $(y_o(t),\ y_1(t),\ldots,y_n(t))$ and a control $(u_1(t),\ldots,u_m(t))$ so that

$$
(5) \qquad
\begin{aligned}
y_o' &= \phi_o(t,\ y_1,\dots,y_n,\ u_1,\dots,u_m)\\
&\vdots\\
y_\mu' &= \phi_\mu(t,\ y_1,\dots,y_n,\ u_1,\dots,u_m)\\
y_{\mu+1}' &= u_1\\
&\vdots\\
y_n' &= u_m
\end{aligned}
$$

where

$$y_o(a) = 0, \quad y_i(a) = y_i^a, \quad i = 1,2,\dots,n,$$

$$\chi_j(b,\ y_1(b),\dots,y_n(b)) = 0, \quad j = 1,2,\dots,k.$$

for some b, and

$$y_o(b) \to \text{minimum}.$$

In the spirit of the problem, as originally posed, the entire $(u_1,\dots,u_m)$-space is to be taken as the control region. Then the following relations between the Lagrange multipliers $\lambda_o,\lambda_1,\dots,\lambda_\mu$ and the solutions $\psi_o,\psi_1,\dots,\psi_n$ of the conjugate system to (5) follow from the maximum principle ([1], p. 59,251,252)

$$(6) \qquad \psi_i(t) = \lambda_i(t) - \frac{\partial f}{\partial y_i'}\,\psi_o, \quad i = 0,1,\dots,\mu$$

$$(7) \qquad \psi_{\mu+j}(t) = -\,\psi_o\,\frac{\partial f}{\partial y_{\mu+j}'} - \sum_{i=1}^{\mu} \frac{\partial \phi_i}{\partial y_{\mu+j}'}\,\lambda_i, \quad j = 1,2,\dots,m.$$

By (4)

$$
(8) \qquad \frac{\partial h}{\partial y_i'} =
\begin{cases}
-\,\lambda_o\,\dfrac{\partial f}{\partial y_i'} + \lambda_i, & i = 1,2,\dots,\mu\\[2ex]
-\,\lambda_o\,\dfrac{\partial f}{\partial y_i'} - \displaystyle\sum_{j=1}^{\mu} \lambda_j\,\frac{\partial \phi_j}{\partial y_i'}, & i = \mu+1,\ \mu+2,\dots,n.
\end{cases}
$$

Hence, in view of (6) and (7)

$$\frac{\partial h}{\partial y_i'} = \psi_i \tag{9}$$

along the optimal trajectory.

By Pontryagin's transversality condition, ([1], p. 63), it is necessary that at the termination point on T

$$\psi_1(b)p_1 + \ldots + \psi_n(b)p_n = 0$$

for any vector $(p_1,\ldots,p_n)$ that lies in the tangent plane to the (n-k)-dimensional (smooth) manifold T*

$$\chi_j(b,y_1,\ldots,y_n) = 0, \quad j = 1,2,\ldots,k$$

at the termination point. There are exactly k linearly independent vectors that are orthogonal to T* at the termination point, namely

$$(\text{grad } \chi_1)_b,\ldots,(\text{grad } \chi_k)_b.$$

Hence, by necessity

$$\psi_j(b) = \sum_{i=1}^{k} \mu_i \left(\frac{\partial \chi_k}{\partial y_j} \right)_b, \quad j = 1,2,\ldots,n \tag{10}$$

for some constants $\mu_1,\mu_2,\ldots,\mu_k$. Equations (9) and (10) yield the transversality conditions (2).

The remaining condition (3) is obtained as follows: Let $\mathcal{M}(\psi,y,t)$ denote the maximum in $u_1,\ldots,u_m$ of

$$\mathcal{H}(\psi,y,t,u) = \psi_o f + \psi_1\phi_1 + \ldots + \psi_\mu\phi_\mu + \psi_{\mu+1}u_1 + \ldots + \psi_n u_m$$

for fixed t,y,ψ. If $\hat{q} = (1,q_1,\ldots,q_n)$ is a tangent vector to T at $(b,y_1(b),\ldots,y_n(b))$, we have to have ([1], p. 62)

$$\mathcal{M}(\psi(b),y(b),b) = \sum_{i=1}^{n} q_i\psi_i(b). \tag{11}$$

Since $\hat{q}$ is a tangent vector to T, we have

$$\frac{\partial \chi_j}{\partial t} + \sum_{i=1}^{n} q_i \left(\frac{\partial \chi_j}{\partial y_i} \right)_b = 0, \; j = 1,2,\ldots,k.$$

We consider first the case where $(\mu_1,\mu_2,\ldots,\mu_k) \neq (0,0,\ldots,0)$, i.e., $(\psi_1(b), \psi_2(b),\ldots,\psi_n(b)) \neq (0,0,\ldots,0)$. Then we obtain after multiplication by μ_j, summation over j and observation of (10)

$$\sum_{j=1}^{k} \mu_j \left(\frac{\partial \chi_j}{\partial t} \right)_b = - \sum_{i=1}^{n} q_i \psi_i(b). \tag{12}$$

In view of (7), which is a consequence of $\partial \mathcal{H}/\partial u_i = 0$, $i = 1,2,\ldots,m$, we obtain along the optimal trajectory

$$\mathcal{M}(\psi,y,t) = \psi_o f + \psi_1\phi_1 + \ldots + \psi_\mu \phi_\mu + \sum_{i=1}^{m} (\psi_o \frac{\partial f}{\partial y'_{\mu+i}} + \sum_{j=1}^{\mu} \frac{\partial \phi_j}{\partial y'_{\mu+i}} \lambda_j) y'_{\mu+i},$$

which because of (8) yields after cumbersome manipulations

$$\mathcal{M}(\psi,y,t) = \psi_o f + \sum_{i=1}^{n} \frac{\partial h}{\partial y'_i} y'_i.$$

This together with (6), (11) and (12) leads directly to (3) with $(\mu_1,\mu_2,\ldots,\mu_k) \neq (0,0,\ldots,0)$ and hence $(\mu_o,\mu_1,\ldots,\mu_k) \neq (0,0,\ldots,0)$.

The case where $(\mu_1,\mu_2,\ldots,\mu_k) = (0,0,\ldots,0)$ is easily taken care of. If $(\mu_1,\mu_2,\ldots,\mu_n) = (0,0,\ldots,0)$, then $(\psi_1(b),\ldots,\psi_n(b)) = (0,0,\ldots,0)$ but $\psi_o < 0$ ([1], p. 18,19).

Then we have from (2) that $(\partial h/\partial y'_i)_b = 0$, $j = 1,\ldots,n$ and hence, $\mathcal{M}(\psi(b),y(b),b) = \psi_o(f)_b$. By (11) we obtain instead of (3)

$$- \psi_o(f)_b = 0$$

and since $\psi_o < 0$ and $\psi_o = \lambda_o = \mu_o$, we have again $(\mu_o,\mu_1,\ldots,\mu_n) \neq (0,0,\ldots,0)$.

Bibliography

[1] L. S. Pontryagin, et al. The Mathematical Theory of Optimal Processes, Interscience Publishers, New York, 1963.

[2] G. A. Bliss. "The problem of Mayer with variable endpoints", Trans. Am. Math. Soc., Vol. 19(1918), pp. 305-314.

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