ENERGY FUNCTIONS FOR MORSE SMALE SYSTEMS

by

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I. Introduction.

In [1] Smale introduced a class of vector fields on a manifold that are similar to gradient fields generated by Morse functions and have since been called Morse-Smale systems. Morse-Smale systems are allowed to have a finite number of closed orbits and singular points but they share with gradient fields the property that the α and ω limit sets of every trajectory can only be a singular point or a closed orbit. Hence there is no complicated recurrent motion. A Morse-Smale system without closed orbits is called gradient like. In [2] it is shown that for every gradient like system there exists a Morse function that is decreasing along trajectories. In this paper a larger class of functions is considered, called $\mathcal{E}$-functions, and it is shown in Theorem 1 that for every Morse-Smale system there exists an $\mathcal{E}$-function that is decreasing along the trajectories of the system. This reminds one of the energy function associated to a dissipative system in mechanics and hence the name $\mathcal{E}$-function.

The construction of the $\mathcal{E}$-function requires little more effort but the added generality has suggested new questions that are discussed here. It is natural to ask if the association of an $\mathcal{E}$-function to a Morse-Smale field is unique in some sense. Theorem 2 establishes that the functions corresponding to a particular field are topologically equivalent.

Several interesting special results are also obtained when the manifold is compact and two dimensional. In this case one has a necessary and sufficient condition for structural stability in terms of $\mathcal{E}$-functions and moreover there is a one-to-one correspondence between topological equivalence classes of structurally stable fields and $\mathcal{E}$-functions.

II. Definitions and Preliminaries.

In this paper smooth will always mean C^∞. Let M be a closed smooth manifold of dimension m with a distance function d inherited from some Riemannian metric. R^n will be Euclidean n-space, S^n the unit sphere in R^{n+1} and B^n the open unit ball in R^n. If X is a smooth vector field on M then φ_t will denote the 1-parameter group of diffeomorphisms generated by X. If $p \in M$ then $\gamma(p)$ will denote the trajectory of X through p, i.e. $\gamma(p) = \bigcup_t \varphi_t(p)$. If $p \in M$ then the α and ω limit sets of $\gamma(p)$ are defined in the usual manner by $\alpha(p) = \bigcap_{\tau \leq 0} \bigcup_{t \leq \tau} \varphi_t(p)^-$ and $\omega(p) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \varphi_t(y)^-$.

If A is a subset of M then A^o will denote the topological interior of A and A^- the topological closure of A.

Definition: A smooth vector field X will be called a Morse-Smale system (or field) provided

1) X has a finite number of singular points, say $\beta_1, \ldots, \beta_k$, each of generic type. A generic singular point is a singular point such that in local coordinates the matrix of partial derivatives of X has eigenvalues with non-zero real parts.

2) X has a finite number of closed orbits (i.e. periodic solutions), say $\beta_{k+1}, \ldots, \beta_n$, each of generic type. A generic orbit is a closed orbit such that all the characteristic multipliers, except the one corresponding to the orbit itself, have modulus different from one.

3) For any $p \in M$, $\alpha(p) = \beta_i$ and $\omega(p) = \beta_j$ for some i and j.

4) If β_i is a closed orbit then there is no $p \in M-\beta_i$ such that $\alpha(p) = \beta_i$ and $\omega(p) = \beta_i$.

5) The stable and unstable manifolds associated with the β_i have transversal intersection.

The sets $\beta_1,\ldots,\beta_n$ will be called the singular elements of the field X.

Let W_i and W_i^* denote the unstable and stable manifold associated to β_i. See [1] and [2] for a discussion of condition 5) and definition of W_i and W_i^*. Note that in [1] transversal intersection is called a normal intersection. A large number of the lemmas in [1] can be summarized by the following:

Lemma: Let X be a Morse-Smale system on M. Let $\beta_i \succ \beta_j$ mean that there is a trajectory not equal to β_i or β_j whose α-limit set is β_i and whose ω-limit set is β_j. Then $\succ$ satisfies:

1) it is never true that $\beta_i \succ \beta_i$

2) if $\beta_i \succ \beta_j$ and $\beta_j \succ \beta_\ell$ then $\beta_i \succ \beta_\ell$ (thus $\succ$ is a partial ordering of $\bigcup \beta_i$)

3) if $\beta_i \succ \beta_j$ then $\dim W_i \geqq \dim W_j$ and equality can only occur if β_j is a closed orbit.

Let E be a smooth function from M into R and let Δ denote the set of critical points of E. Let Δ_i denote the set of points in Δ where the Hessian of E has nullity i. It is well known (see [3]) that Δ_0 is a finite union of points, say

$\delta_1, \ldots, \delta_k$, and there exists a coordinate system (N_i, x_i) such that

$$E \circ x_i^{-1} = E(\delta_i) + Q(x)$$

where Q is a nonsingular quadratic form in x whose index is the same as the index of the Hessian of E at δ_i. For discussion and definitions relevent for these functions see [3].

Definition: A smooth function E from M into R will be called on $\mathcal{E}$-function for M provided

1) $\Delta = \Delta_0 \cup \Delta_1$
2) Δ_1 is the disjoint union of a finite number of circles, i.e. closed connected one dimensional submanifolds of M, such that the Hessian of E is constant on each circle. Denote these circles by $\delta_{k+1}, \ldots, \delta_n$.
3) For $i = k+1, \ldots, n$ there exists a neighborhood N_i of δ_i and a diffemorphism x_i such that x_i maps N_i into the product of B^{m-1} and S^1 if N_i is orientable or into the twisted product of B^{m-1} and S^1 if N_i is nonorientable with the property that $E \circ x_i^{-1} = E(\delta_i) + Q(x)$ where Q is a nonsingular quadratic form in $x_1, \ldots, x_{m-1}$, the coordinates in B^{m-1}, and is periodic of period 1 in x_m, the coordinate in S^1. Moreover, for each point in S^1 the quadratic form has index equal to the index of

E on δ_i.

In this paper the connection between Morse-Smale systems and $\mathcal{E}$-functions is investigated. In this respect the $\mathcal{E}$-function is closely related to the field when $\mathcal{E}$ is decreasing along trajectories. To formalize this we need:

Definition: Let X be a smooth vector field on M. Then an $\mathcal{E}$-function, E, for M will be called an $\mathcal{E}$-function for X provided

1) $XE(p) < 0$ for all $p \in M-\Delta$, i.e. E is decreasing along the trajectories of X or the trajectories of X are transversal to the level lines of E
2) if p is a singular point of X then $p \notin \Delta_1$
3) there exists a constant $\kappa > 0$ such that on each N_i

$$-XE(p) \geq \kappa d(p,\delta_i)^2 \quad \text{for} \quad p \in N_i$$

III. Existence of $\mathcal{E}$-functions.

The first result is that Morse-Smale systems admit $\mathcal{E}$-functions, that is

Theorem 1: If X is a Morse-Smale system then there exists an $\mathcal{E}$-function for X.

Proof: The first step is to define the $\mathcal{E}$-function on the β_i and since E must be decreasing along trajectories this must be done in a consistent way. The lemma shows that this can be done, that is, one can find n real numbers α_i such that if $\beta_i \succ \beta_j$

then $\alpha_i > \alpha_j$. Thus we define E on the β_i by $E(\beta_i) = \alpha_i$ and then construct E globally so that $\delta_i = \beta_i$ and E is decreasing along trajectories. Next E must be extended to a neighborhood of the β_i in such a way that the nondegenerating conditions are satisfied. If β_i is a singular point then in local coordinates X has the form $\dot{x} = Ax + f(x)$ where $f(0) = df(0) = 0$ and the eigenvalues of A have nonzero real parts. By Liapunov theory there exists symmetric matrices Q and C, C positive definite and Q nonsingular such that $A'Q + QA = -C$. Moreover, the index of Q is equal to the number of eigenvalues of A with positive real part. If we define $E(x) = \alpha_i + x'Qx$ then by standard Liapunov arguments there exists a neighborhood sufficiently small and a constant $\kappa_i > 0$ such that $-XE(p) \geq \kappa_i d(x,p)^2$ in this neighborhood. Take the N_i sufficiently small that the above holds and so that they do not overlap.

Now around a closed orbit β_i one can choose a neighborhood N_i' and a diffeomorphism x_i' mapping N_i' into $B^{m-1} X S^1$ or B^{m-1} twisted product with S' (if N_i' is non orientable) such that if y is the coordinates in B^{m-1} and θ is the coordinate in S^1 then X takes the form

$$\dot{\theta} = \omega + \Theta(\theta, y)$$
$$\dot{y} = A(\theta)y + Y(\theta, y)$$

where A is an $(m-1) \times (m-1)$ periodic matrix of period 1 i.e. A is a function on S^1. Θ and Y are periodic of period 1 in θ

$\Theta(\theta,0) = 0$ and $Y = o(\|y\|)$. By Floquet theory the fundamental matrix solution of $u' = -uA$ can be written in the form $e^{S\theta}P(\theta)$ where S is constant and P is either periodic or skew periodic of period 1 i.e. either $P(\theta) = P(\theta+1)$ or $P(\theta) = -P(\theta+1)$. By assumption S has no eigenvalue with zero real part and so by Liapunov theory there exists symmetric matrices Q and C, Q nonsingular and C positive definite such that $S^TQ + QS = C$. Define $E \circ x_i^{-1} = \alpha_i + y^TP(\theta)^TQP(\theta)y$ by direct computation then $XE = -y^TP^T(\theta)CP^T(\theta)y + \varepsilon(\theta,y)$ where $\varepsilon = o(\|y\|^2)$. We again restrict the neighborhood N_i so that they do not overlap and so that $-XE(p) \geq \kappa d(\delta_i,p)^2$ for $p \in N_i$.

Thus the $\mathcal{E}$-function is now defined in neighborhoods of the singular points and closed orbits of X. The extension of this function can now be accomplished by the same procedure as in [2]

As a partial converse of the above theorem we have

Proposition: Let X be a smooth vector field on M. If there exists an $\mathcal{E}$-function for X then X satisfies the conditions 1) 2) 3) and 4) in the definition of a Morse-Smale system. Moreover, the field X can be approximated arbitrarily closely in the C^r-topology for fields on M by a Morse-Smale system.

The first part follows by standard Liapunov arguments and the second part is established essentially the same way as proposition 2 in [4].

If M is compact and 2-dimensional the above result can be sharpened. In this case Morse-Smale systems are the same as structurally stable systems by a theorem of Peixoto [5]. If E is an $\mathcal{E}$-function for X such that all the sources of X lie in $E^{-1}(1)$; all saddle points of X lie in $E^{-1}(0)$ and all sinks of X lie in $E^{-1}(-1)$ then E will be called a special $\mathcal{E}$-function for X. It is clear from the above that if M is compact and two-dimensional then the construction of Theorem 2 could be made to yield a special $\mathcal{E}$-function for X.

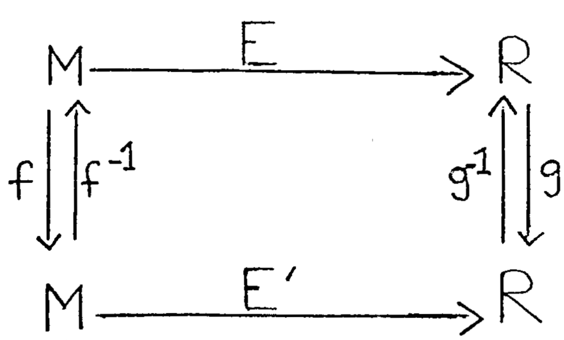
If E is a special $\mathcal{E}$-function for X then there can be no trajectory joining saddle points of X since E is decreasing along trajectories. Thus the stable and unstable manifolds have transversal intersection. Hence

Corollary: If M is compact and two dimensional then a necessary and sufficient condition for X to be structurally stable is the existence of a special $\mathcal{E}$-function for X.

IV. Uniqueness of $\mathcal{E}$-functions.

Clearly the $\mathcal{E}$-function constructed in Theorem 1 is not unique but if one introduces the concept of topological equivalence a form of uniqueness can be established.

Recall (see [6]) that two functions E and E' from M to R are said to be topologically equivalent if there exists homeomorphisms f and g, $f:M \to M$ and $g:R \to R$ such that the following diagram commutes



Recall that two vector fields X and X' on M are said to be topologically equivalent if there exists a homeomorphism $h: M \to M$ which sends the trajectories of X into the trajectories of X'.

In general two $\mathcal{E}$-functions for two topologically equivalent fields are not topologically equivalent since the $\mathcal{E}$-functions are defined quite arbitrarily on the singular points and closed orbits. To obtain uniqueness some regularity on the way the $\mathcal{E}$-functions are defined on the β_i's is necessary. This could be done by uniquely specifying the way the functions are defined on the β_i's as was done in the definition of the special $\mathcal{E}$-functions for two-dimensional fields. Instead of this we assume that the $\mathcal{E}$-functions are defined correctly on the critical elements.

Since hence forth we shall be considering two fields and two functions we shall use the same symbols as before and all unprimed symbols will refer to one system and all primed to the other.

<u>Theorem 2</u>. Let X and X' be two Morse-Smale systems on M that are topologically equivalent under the homeomorphism h. Let E and E' be $\mathcal{E}$-functions for X and X' respectively. Then if the two $\mathcal{E}$-functions are equivalent on the singular elements, i.e. on the singular points and closed orbits, then they are topologically equivalent. That is to say if there exists a homeomorphism $g: R \to R$ such that the diagram

$$\begin{array}{ccc} \bigcup_1^n \beta_i & \xrightarrow{\quad E \quad} & R \\ h \downarrow \uparrow h^{-1} & & g^{-1} \uparrow \downarrow g \\ \bigcup_1^n \beta_i' & \xrightarrow{\quad E' \quad} & R \end{array}$$

commutes then E and E' are topologically equivalent.

Proof: Let β_i and β_i' be so numbered that $h(\beta_i) = \beta_i'$. Observe that g is by assumption a homeomorphism of R into R that must satisfy a finite number of other requirements, namely $g \circ E(\beta_i) = E'(\beta_i')$. If such a g exists then a smooth $\tilde{g}$ exists satisfying the same conditions. Hence we can assume that E and E' agree the singular elements since otherwise we would consider E and $\tilde{g} \circ E'$.

We first define a special neighborhood of one singular element. Let β represent any one of the β_i or β_i' and N, x and E the corresponding N_i, N_i', x_i, x_i', E or E'. Then a $\mathcal{P}$-neighborhood, P, of β is defined as a closed neighborhood of β contained in N such that the boundary of P is the union of three sets: I a $m-1$ closed submanifold of M that lies in the level line $E^{-1}(E(\beta)+\varepsilon)$ for some $\varepsilon > 0$ or $I = \varphi$, O a $m-1$ closed submanifold of M that lies in the level line $E^{-1}(E(\beta)-\varepsilon)$ for some $\varepsilon > 0$ or $O = \varphi$ and U the union of trajectories that join the boundary of I to the boundary of O.

Such a neighborhood always exists as can be seen by the following. If β is a source take P to be the set of points in N where E is greater than $E(\beta)-\varepsilon$ with ε small and positive. If β is a sink P is defined similarly. Let β be a saddle point. Then $E(x) = E(\beta) + Q(x)$ in N where Q is a nonsingular quadratic form. Let T be the quadratic form that is equal to Q on the subspace of R^n where Q is negative definite and zero on the complement. For ε and δ sufficiently small the set I of points p where

$p \in E^{-1}(E(\beta) + \varepsilon)$ and $-T(p) \leq \delta$ is contained in the interior of N.

Moreover, if ε and δ are sufficiently small one can also fulfill the requirement that the set of all points p that lie on a trajectory through I and satisfy $E(\beta) - \varepsilon \leq E(p) \leq E(\beta) + \varepsilon$ is contained in N, let P be the closure of this set. It is easy to see that P is a closed neighborhood of β contained in N and that the boundary of P is composed of I as defined above, O and U where O and U satisfy the requirements of the definition of a $\mathscr{D}$-neighborhood. $\mathscr{D}$-neighborhoods for closed orbits are constructed in a similar way.

Let P_i be a $\mathscr{D}$-neighborhood for β_i, $i = k+1, k+2, \ldots, n$ and P_i^o its interior. We first construct f on $M - \bigcup_1^k \beta_i - \bigcup_{k+1}^n P_i^o$. Let $p \in M - \bigcup_1^k \beta_i - \bigcup_{k+1}^n P_i^o$ and define $f: p \to q$ where q is defined as the unique point on the X'-trajectory through $h(p)$ that satisfies $E'(q) = E(p)$. To make sure that this map is well defined observe that $E(\varphi_t(p))$ and $E'(\varphi_t'(h(p)))$ tend to the same limit as $t \to +\infty$ and the same limit as $t \to -\infty$ and moreover both are decreasing functions of t. Thus f is a homeomorphism taking level line into level line where it is defined.

Now f can be extended to the singular points by $f(\beta_i) = \beta_i'$ for $i = 1, \ldots, k$. To see that f is still a homeomorphism note that f maps $\mathscr{D}$-neighborhood of β_i onto $\mathscr{D}$-neighborhoods of β_i' and conversely. For closed orbits the extension is more difficult since the β_i no longer consist of single points.

The homeomorphism f is defined on the boundary of P_i, $i = k+1,\ldots,n$ and maps the boundary of P_i into the boundary of a $\mathcal{P}$-neighborhood, P_i', of β_i'. To see this observe that the image I_i' of I_i under f is contained in a level line of E' and similarly for the image of O. Moreover the image of U is the union of X' trajectories joining the boundary of I' to the boundary of O'. P_i' is defined once I' or O' are defined as can be seen by our construction of $\mathcal{P}$-neighborhoods.

We now show how to extend the definition of f to the interiors of P_i, $i = k+1,\ldots,n$. Since we shall be working locally we shall drop the subscripts. For definiteness let us consider the case when the neighborhood of β and the stable and unstable manifolds of β are orientiable. The nonorientable cases are similarly treated.

First let β be a source or a sink. Let N be a neighborhood of β containing P and x a diffeomorphism $x:N \to B^m \times S^1$, $x = (y,\theta)$, $y:N \to B^{m-1}$, $\theta:N \to S^1$ such that in N, $E(x) = E(\beta) + y^T y$. Let P', N', x', y', θ' be similarly defined. For simplicity let E be zero on β and 1 on the boundary of P. f is defined on the boundary of P and let $f = h$ on β. Let $p = (y,\theta) \in P^o - \beta$. p is on the curve $\tau(0,\theta) + (1-\tau)(\|y\|^{-1}y,\theta)$. Let $f(0,\theta) = (0,\theta_o')$ and $f((\|y\|^{-1}y,\theta)) = (y_1',\theta_1')$ and let q be the unique point on the curve $\tau(y_1',\theta_1') + (1-\tau)(0,\theta_o^1)$ that satisfies $E(p) = E'(q)$. By defining $f(p) = q$ we see that f has been extended to the criterior of P as a homeomorphism taking level line into level line.

Now let β be a saddle type closed orbit. Let N be a neighborhood of β containing P and $x = (y,\theta)$ a diffeomorphism

$y:N \to B^{n-1}$ and $\theta:N \to S^1$ such that in these local coordinates $E(x) = E(\beta) + y^T Qy$ where $Q = \text{diag}(1,\ldots,1,-1,\ldots,-1)$. Moreover, let N', x', y', θ' be similarly defined. Let Π be a $\mathscr{D}$-neighborhood of β completely interior to P. Define f on Π by $f:p \to q$ where $p \in \Pi$ and $q \in \Pi'$ and p and q have the same numerical coordinates in the unprimed and primed coordinates respectively.

Thus f must be extended to $P^o - \Pi$. This extension can be accomplished by dividing $P^o - \Pi$ into several parts each of which has a simple geometric type. Let a and b be the real numbers such that the region of the boundary of Π that is a region of ingress resp. egress is in the level line $E^{-1}(a)$ resp. $E^{-1}(b)$. Consider $K_1 = E^{-1}(\alpha) \cap (P-\Pi^o)$ and $K_2 = E^{-1}(b) \cap (P-\Pi^o)$. f is defined on the boundary of K_1 and K_2 and topologically K_1 and K_2 are just products of unit intervals and spheres. Let L_1 $(P-\Pi^o) \cap \{p \in M: E(I) \geq E(p) \geq a\}$, $L_2 = (P-\Pi^o) \cap \{p \in M: E(a) \geq E(p) \geq E(b)\}$ and $L_3 = (P-\Pi^o) \cap \{p \in M: E(b) \geq E(p) \geq E(0)\}$.

Topologically L_1, L_2 and L_3 are just the product of the unit interval and spheres. f is defined on the boundary of K_1 and K_2 and so we first extend f to K_1 and K_2. Now f is defined on the boundary of L_1, L_2 and L_3 and so f is then extended to their interiors.

Each extension is carried out in the same way as the extension was carried out for the source because in each case there is a set that acts as the center. That is if one of the sets is $I \times I \times S^1$ then $(0,0) \times S^1$ is the center.

The center is mapped homeomorphically on the center by fiat and then the extension is carried out by joining the center to the boundary by lines and carrying points proportionally.

Of course for special $\mathcal{E}$-functions the homeomorphism g may always be taken as the identity. In the case where M is compact and two dimensional the converse of Theorem 4 holds also. Namely

<u>Proposition:</u> Let M be a compact and two dimensional smooth manifold. Let X and X' be smooth vector field on M and let E and E' be special $\mathcal{E}$-functions for X and X' respectively. If E and E' are topologically equivalent then X and X' are topologically equivalent.

<u>Proof:</u> Let f be the homeomorphism of M that takes level lines of E into level lines of E' i.e. $E = E' \circ f$. f sets up a correspondence between the critical elements of E and E' let them be so numbered that $f(\delta_i) = \delta_i'$ and let the β_i and β_i' be numbered so that $\beta_i = \delta_i$ and $\beta_i' = \delta_i'$ as sets. Let $\Gamma = E^{-1}(0)$ and $\Gamma' = E'^{-1}(0)$. Then f is a homeomorphism of Γ onto Γ'. Define h to be equal to f on Γ.

The first thing to be established is that if $p \in \Gamma$ and $\alpha(p) = \beta_i$ and $\omega(p) = \beta_j$ then $\alpha'(f(p)) = \beta_i'$ and $\omega'(f(p)) = \beta_j'$. Let $p \in \Gamma$ and p not a saddle point and let $p^* = f(p)$. Consider the X'-trajectory through p^* and let it be reparameterized so that it is a map u from $(-1,1)$ into M where the new parameter is the value of E', this can be done since $E'(\varphi_t'(p^*))$ is a decreasing function of t. To be precise $u:(-1,1) \to M$ such that $u(\alpha)$, $\alpha \in (-1,1)$, is the unique point on the X' trajectory through p^* such that $E'(u(\alpha)) = \alpha$. In a similar manner let $v:(-1,1) \to M$ be the reparameterization of $f(\varphi_t(p))$ by values of E'. To be precise

$v(\alpha)$, $\alpha \in (-1,1)$, is the unique point on $f(\varphi_t(p))$ such that $E'(v(\alpha)) = \alpha$. We want to show that u and v are isotopic with an isotopy that moves points in a level line. That is we want to show that there exists a map $V:(-1,1) \quad [0,1] \to M$ such that $V(\cdot,0) = u$ and $V(\cdot,1) = v$ and moreover $E'(V(\alpha,t)) = \alpha$ for all $t \in [0,1]$. Clearly this will establish the fact that α and ω limit sets of trajectories correspond as described above.

Let A be a small disk about p' such that A contains no singular points of X'. For α different from zero the level lines $E'^{-1}(\alpha)$ is a smooth one manifold and so there is a unique arc a in A joining $u(\alpha)$ to $v(\alpha)$ of arc length $s(\alpha)$. Let $V(\alpha,t)$ be the unique point on the arc a such that the arc length from $u(\alpha)$ to $V(\alpha,t)$ is $ts(\alpha)$. Thus the isotopy V is defined so long as α is small but the extension is now obvious and so our claim is established.

The sets $\{p \in M:E'(p) \geq \frac{1}{2}\}$ and $\{p \in M:E' \leq -\frac{1}{2}\}$ are the disjoint union of $\mathscr{D}$-neighborhoods of all the sources and sinks respectively.

The homeomorphism h is now extended in the following way. Let p be a point of M not on a separatrix of X and such that $-\frac{1}{2} \leq E(p) \leq \frac{1}{2}$. The X trajectory through p meets at p^* let q be the unique point on the X'-trajectory through $h(p^*) = f(p^*)$ that satisfies $E(p) = E'(q)$. Now extend this map to all of $\{p \in E:-\frac{1}{2} \leq E(p) \leq \frac{1}{2}\}$ so that separatrix goes to separatrix and level line of E to level line of E'.

Thus the map f is defined on all but the interiors of $\mathscr{D}$-neighborhoods of the sources and sinks. The map f is defined on the boundaries of these $\mathscr{D}$-neighborhoods and takes the boundary of one particular $\mathscr{P}$-neighborhood of an X critical element into the boundary of a $\mathscr{P}$-neighborhood of an X' critical element of the same type.

But it is shown in [7] that if one is given two critical elements of the same type and an arbitrary homeomorphism of the boundaries of $\mathscr{P}$-neighborhoods for these two critical elements then the homeomorphism can be extended to the interior of the neighborhoods taking trajectories into trajectories. Thus f can be defined globally.

References

[1] Smale, S., Morse inequalities for dynamical systems, Bull. Ann. Math. Soc., 66 (1960), 43-9.

[2] Smale, S., On gradient dynamical systems, Ann. Math., 74 (1961), 199-206.

[3] Milnor, J., Morse Theory, Princeton University Press, (1963).

[4] Peixoto, M.M., On an approximation theorem of Kupka and Smale, (to appear).

[5] Peixoto, M. M., Structural stability on two-dimensional manifolds, Topology, 1, (1962), 101-120.

[6] Thom, R., La stabilite topologique des applications polynomiales, L'Enseignement Mathematique, VIII, (1962), 24-33.

[7] Peixoto, M. C., and Peixoto, M.M., Structural stability in the plane with enlarged boundary conditions, Ann. Bras. Acad. Sci., 31 (1959), 135-160.