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# DYNAMIC PROGRAMMING AND PONTRYAGIN'S MAXIMUM PRINCIPLE

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*Prepared by*
NORTH CAROLINA STATE UNIVERSITY
Raleigh, N. C.

*for*

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# DYNAMIC PROGRAMMING AND PONTRYAGIN'S MAXIMUM PRINCIPLE

Hans Sagan

## SUMMARY

For an autonomous terminal control problem of not predetermined duration, an _admissible set of inception_ is defined as a simply connected domain such that every point in that domain represents an initial state from which a given terminal state can be reached by an optimal trajectory.

On such an admissible set of inception, _Hamilton's characteristic function_ S is defined as the minimum terminal value of one state variable ($y_o$), as a function of the initial state, that can be achieved by an optimal trajectory. If $\Omega$ is an admissible set of inception, then S is defined for all points in $\Omega$.

It is shown that if there exists an admissible set of inception $\Omega$ and if S satisfies certain differentiability assumptions on $\Omega$, then Bellman's functional equation is valid and Pontryagin's maximum principle follows from Bellman's functional equation.

Two simple examples are discussed where one or the other of these assumptions on $\Omega$ and S are not met and hence Bellman's functional equation and the maximum principle as derived from this functional equation are not applicable, while the maximum principle in Pontryagin's general version is still valid and leads in both cases to optimal controls and optimal trajectories.

## 1. STATEMENT OF PROBLEM

We consider the terminal control problem of not predetermined duration of finding a control $\hat{u} = \hat{u}(t) = (u_1(t),\dots,u_n(t)) \varepsilon C_s[t_o,t_1]$ and a corresponding trajectory $\hat{y} = \hat{y}(t) = (y_o(t),\dots,y_n(t)) \varepsilon C_s^1[t_o,t_1]$, where $t_o$ is given and where $t_1$ is unspecified, such that

$$\begin{aligned} y_o' &= f_o(y_1,\dots,y_n, u_1,\dots,u_m) \\ &\vdots \\ y_n' &= f_n(y_1,\dots,y_n, u_1,\dots,u_m) \end{aligned} \tag{1}$$

under observation of the initial conditions

$$y_o(t_o) = 0, \quad y_1(t_o) = y_1^o,\dots,y_n(t_o) = y_n^o, \tag{2}$$

and the terminal conditions

$$y_1(t_1) = y_1^1,\dots,y_n(t_1) = y_n^1 \tag{3}$$

and such that

$$y_o(t_1) \to \text{minimum.}$$

We require hereby, that for all $t \varepsilon [t_o,t_1]$, $\hat{u} \varepsilon U$, where $U$ denotes some given subset of the $(u_1,\dots,u_m)$-space. We assume that

$$f_k, \frac{\partial f_k}{\partial y_i} \varepsilon C(U \times Y) \tag{4}$$

where $Y$ denotes the $(y_1,\dots,y_n)$-space.

The solution $\hat{u} = \hat{u}(t) \varepsilon C_s[t_o,t_1]$ and with $\hat{u} \varepsilon U$ we call the <u>optimal control</u>, and the corresponding trajectory $\hat{y} = \hat{y}(t)$ we call the <u>optimal trajectory</u>.

Geometrically, this means that a control $\hat{u} = \hat{u}(t)$ has to be found such that the corresponding trajectory $\hat{y} = \hat{y}(t)$ which is a solution of (1) emanates from the point $P_o(0, y_1^o, \ldots, y_n^o)$ and terminates on the line L that is given by $y_1 = y_1^1, \ldots, y_n = y_n^1$, with the smallest possible $y_o$-coordinate. (See Fig. 1.)

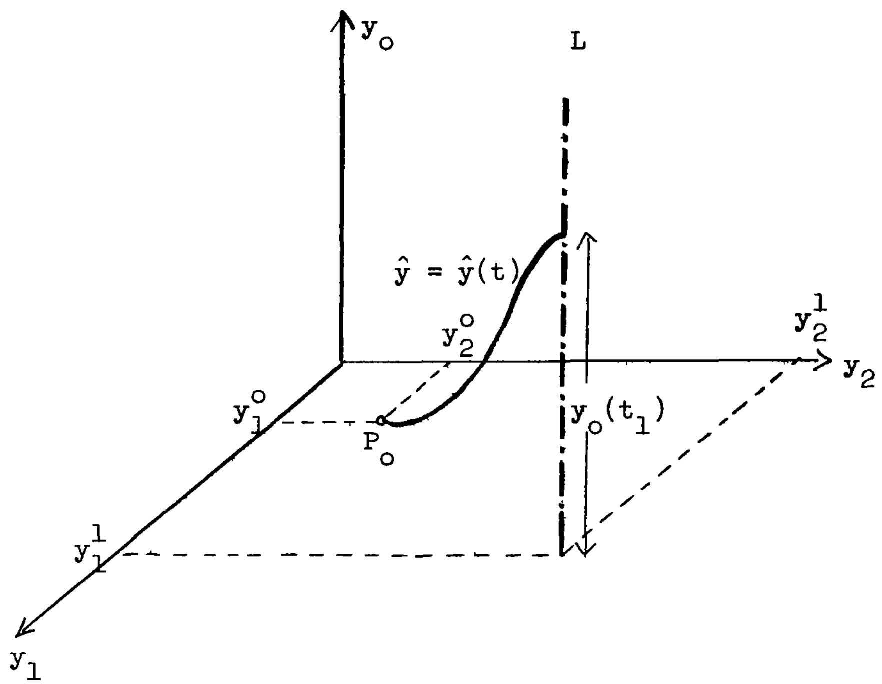


Figure 1.

In the following discussion, we will make use of two assumptions labeled (I) and (II) which we will now proceed to formulate.

If there exists a simply connected domain $\Omega$ in $(y_o, \ldots, y_n)$-space such that from every point $(y_o, \ldots, y_n) \in \Omega$ there emanates at some $t = \tau_o$ an optimal trajectory $\hat{y} = \hat{y}(t)$ which terminates on L for some $t = \tau_1$ and is such that $\hat{y}(t)$ remains in $\Omega$ for all $t \in [\tau_o, \tau_1)$, then we call $\Omega$ an <u>admissible set of inception</u>.

(I) There exists an admissible set of inception for the terminal control problem [(1),(2),(3)].

If $S(y_o,\ldots,y_n)$ denotes the minimum of $y_o$ that is attainable by an optimal trajectory that emanates from $(y_o,\ldots,y_n)$ and terminates on $L$, then, $S(y_o,\ldots,y_n)$ which we call <u>Hamilton's characteristic function</u>, is defined for all $(y_o,\ldots,y_n) \in \Omega$ if $\Omega$ is an admissible set of inception.

(IIa) $$S(y_o,\ldots,y_n) \in C^1(\Omega).$$

(IIb) $S(y_o,\ldots,y_n) \in C^1(\Omega)$, $\dfrac{\partial^2 S}{\partial y_i \partial y_k}$ exist for all $(y_o,\ldots,y_n) \in \Omega$.

## 2. DYNAMIC PROGRAMMING

We will now derive <u>Bellman's functional equation</u> ([1], p. 135) which constitutes a <u>necessary condition</u> for a control and trajectory to be optimal, under the condition that (I) and (IIa) are satisfied.

We assume that $\hat{u} = \hat{u}(t)$ is an optimal control and that $\hat{y} = \hat{y}(t)$ is the corresponding optimal trajectory of problem [(1), (2), (3)] and that $t_1$ is the terminal value of the independent variable.

Let $t \in (t_o,t_1)$ be an arbitrary, but fixed value of the independent variable. We replace $\hat{u} = \hat{u}(t)$ in $[t, t + \Delta t)$, where we assume that $\Delta t$ is sufficiently small so that $t + \Delta t \in (t_o,t_1)$, by a constant control $\hat{v} = (v_1,\ldots,v_n) \in U$. (See Figure 2.)

Such a <u>variation</u> of the original control is called a <u>needle shaped</u> variation.

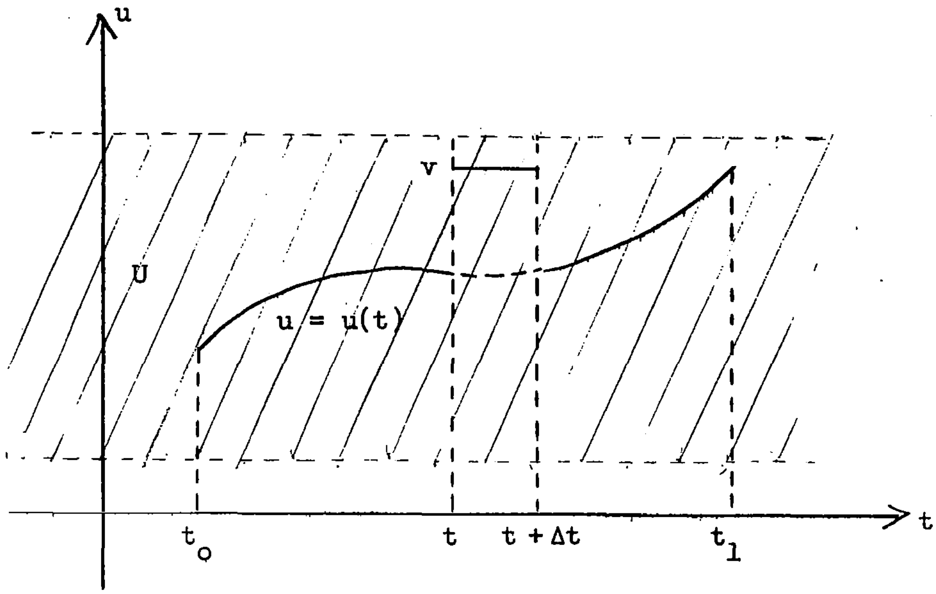


Figure 2.

This new control will, in general, lead away from the point $\hat{y} = \hat{y}(t)$ on the optimal trajectory, to a point, the coordinates of which are to be found from (1) by integration:

$$\bar{y}_k(t + \Delta t) = y_k(t) + \int_t^{t+\Delta t} f_k(\bar{y}_1(s),\ldots,\bar{y}_n(s),v_1,\ldots,v_m)ds$$

$$= y_k(t) + f_o(y_1(t),\ldots,y_n(t),v_1,\ldots,v_m)\Delta t + o(\Delta t),\ k = 0,1,\ldots,n, \tag{5}$$

where $\lim\limits_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$. (Note that this approximation is permissible in view of (4).) If $\Delta t$ is sufficiently small, which we will assume, then, $(\bar{y}_o(t + \Delta t),\ldots,\bar{y}_n(t + \Delta t)) \in \Omega$ and hence, $S(\bar{y}_o(t + \Delta t),\ldots,\bar{y}_n(t + \Delta t))$ is defined.

In particular, there exists an optimal control $\hat{u} = \hat{u}(t)$ and a corresponding optimal trajectory $\hat{y} = \hat{y}(t)$ that emanates from

$(\bar{y}_o(t + \Delta t),\ldots,\bar{y}_n(t + \Delta t))$ and terminates on $L$ for some $t = \tilde{t}_1$. Clearly,

$$S(y_o,\ldots,y_n) \leq S(\bar{y}_o(t + \Delta t),\ldots,\bar{y}_n(t + \Delta t)). \tag{6}$$

Otherwise, $\hat{y} = \hat{y}(t)$ would not be the optimal trajectory because the control

$$\hat{u} = \begin{cases} \hat{u}(t) & \text{for } t \in [t_o, t) \\ \hat{v} & \text{for } t \in [t, t + \Delta t) \\ \hat{u}(t) & \text{for } t \in [t + \Delta t, \tilde{t}_1] \end{cases}$$

would yield a trajectory that terminates on $L$ with a smaller value than $y_o(t_1)$, namely, $\tilde{y}_o(\tilde{t}_1) = S(\bar{y}_o(t + \Delta t),\ldots,\bar{y}_n(t + \Delta t))$.

We obtain from (6) by application of the mean-value theorem, observation of (5) and (IIa) that

$$- \frac{\partial S(y_o,\ldots,y_n)}{\partial y_o} f_o(y_1,\ldots,y_n, v_1,\ldots,v_m)\Delta t - \ldots$$

$$\ldots - \frac{\partial S(y_o,\ldots,y_n)}{\partial y_n} f_n(y_1,\ldots,y_n, v_1,\ldots,v_m)\Delta t + o(\Delta t) \leq 0.$$

Division by $\Delta t$ and $\Delta t \to 0$ yields

$$\begin{aligned} Q(y_o,\ldots,y_n, v_1,\ldots,v_m) \equiv & - \frac{\partial S(y_o,\ldots,y_n)}{\partial y_o} f_o(y_1,\ldots,y_n, v_1,\ldots,v_m) - \ldots \\ & - \frac{\partial S(y_o,\ldots,y_n)}{\partial y_n} f_n(y_1,\ldots,y_n, v_1,\ldots,v_m) \leq 0 \end{aligned} \tag{7}$$

for all points $(y_o,\ldots,y_n)$ on the optimal trajectory $\hat{y} = \hat{y}(t)$ and for all $\hat{v} \in U$.

On the other hand, we have for any point on the optimal trajectory $\hat{y} = \hat{y}(t)$

$$S(y_o(t),\ldots,y_n(t)) = S(0,y_1^o,\ldots,y_n^o)$$

for all $t \in (t_o,t_1)$ and hence,

$$\left(\frac{dS}{dt}\right)_{\hat{y}=\hat{y}(t)} = \frac{\partial S}{\partial y_o} y_o' + \ldots + \frac{\partial S}{\partial y_n} y_n'$$

$$= \frac{\partial S(y_o,\ldots,y_n)}{\partial y_o} f_o(y_1,\ldots,y_n,u_1,\ldots,u_m) + \ldots,$$

$$\ldots + \frac{\partial S(y_o,\ldots,y_n)}{\partial y_n} f_n(y_1,\ldots,y_n,u_1,\ldots,u_m) = 0,$$

i.e.,

(8) $$Q(y_o,\ldots,y_n,u_1,\ldots,u_m) = 0.$$

From (7) and (8)

$$\max_{\hat{v} \in U} Q(y_o,\ldots,y_n,v_1,\ldots,v_m) = Q(y_o,\ldots,y_n,u_1,\ldots,u_m) = 0 \qquad (9)$$

for every point $(y_o,\ldots,y_n)$ on the optimal trajectory $\hat{y} = \hat{y}(t)$ and for the values $(u_1,\ldots,u_m)$ of the optimal control $\hat{u} = \hat{u}(t)$ that correspond to this point.

Condition (9), when written in detail as

$$\max_{\hat{v} \in U}\left[- \frac{\partial S(y_o,\ldots,y_n)}{\partial y_o} f_o(y_1,\ldots,y_n,v_1,\ldots,v_m) - \ldots \right.$$

$$\left. - \frac{\partial S(y_o,\ldots,y_n)}{\partial y_n} f_n(y_1,\ldots,y_n,v_1,\ldots,v_m)\right] = 0,$$

or, in the more familiar form

$$\min_{\hat{v}\,\varepsilon\,U}\left[\frac{\partial S(y_o,\dots,y_n)}{\partial y_o} f_o(y_1,\dots,y_n,v_1,\dots,v_m) + \dots \right.$$
$$\left. + \frac{\partial S(y_o,\dots,y_n)}{\partial y_n} f_n(y_1,\dots,y_n,v_1,\dots,v_m)\right] = 0 \tag{10}$$

is easily recognized as Bellman's functional equation ([1], p. 135).

This condition has to be satisfied, by necessity, at every point $(y_o,\dots,y_n)$ of the optimal trajectory $\hat{y} = \hat{y}(t)$. The values $v_1 = u_1,\dots,v_m = u_m$, for which the minimum is assumed are the values of the optimal control $\hat{u} = \hat{u}(t)$ at that point.

Formula (10) assumes the well-known form of Bellman's functional equation ([2], p. 191) when applied to the simple variational problem

$$\int_a^b f(t,y,y')dt \rightarrow \text{minimum}, \quad y(a) = y_a, \quad y(b) = y_b. \tag{11}$$

We introduce the new variables

$$y_o = \int_a^t f(s,y,y')ds, \quad y_1 = y, \quad y_2 = t, \quad u = y'$$

and formulate this problem as a terminal control problem:

$$y_o' = f(y_2,y_1,u)$$
$$y_1' = u$$
$$y_2' = 1$$

with the boundary conditions $y_o(a) = 0$, $y_1(a) = y_a$, $y_2(a) = a$, $y_1(t_1) = y_b$, $y_2(t_1) = b$. For $\int_a^b f(t,y,y')dt \rightarrow$ minimum we have now $y_o(t_1) \rightarrow$ minimum. (Note that $y_2' = 1$ together with $y_2(a) = a$ and $y_2(t_1) = b$ forces $t_1 = b$.)

Since

$$S(y_o,y_1,y_2) = y_o + S(0,y_1,y_2) \tag{12}$$

where $S(0,y_1,y_2)$ is simply <u>Hamilton's characteristic function</u> (<u>Bellman's optimal value function</u> ([1], p. 71) $P(t,y) = P(y_2,y_1) = S(0,y_1,y_2)$ of the variational problem (11), we have

$$\frac{\partial S}{\partial y_o} = 1.$$

Then, replacing $y_1,y_2,u$ by $y,t,y'$, (10) will assume the form

$$\min_{(y')} [f(t,y,y') + \frac{\partial P}{\partial y} y' + \frac{\partial P}{\partial t} ] = 0. \tag{13}$$

<u>Bellman's recursion formula</u> ([3], p. 85) for the approximate computation of the optimal control and the optimal trajectory for the variational problem (11) together with a constraint of the type $y' \varepsilon U$ is obtained from (6) as follows:

If the right side of (6) is considered for all possible values of $\hat{v} \varepsilon U$ and $\Delta t$ sufficiently small, then,

$$\begin{aligned} S(y_o,y_1,y_2) &= \min_{v \varepsilon U} S(\bar{y}_o(t + \Delta t), \bar{y}_1(t + \Delta t), \bar{y}_2(t + \Delta t)) \\ &= \min_{v \varepsilon U} S(y_o + f(y_2,y_1,v)\Delta t + o(\Delta t), y_1 + v\Delta t, y_2 + \Delta t). \end{aligned}$$

If we neglect $o(\Delta t)$, which is small of higher than first order, and observe (12), we obtain

$$y_o + S(0,y_1,y_2) = \min_{v \varepsilon U} [y_o + f(y_2,y_1,v)\Delta t + S(0, y_1 + v\Delta t, y_2 + \Delta t)]$$

which reads in terms of the optimal value function $P(t,y)$ and after cancellation of $y_o$ as

$$P(t,y) = \min_{y' \varepsilon U} [f(t,y,y')\Delta t + P(t + \Delta t, y + y'\Delta t)].$$

(This formula leads again to (13) if one applies the mean-value theorem to $P(t + \Delta t, y + y'\Delta t)$, divides by $\Delta t$ and lets $\Delta t \to 0$.)

Summarizing, we may state:

Theorem 1.

If (I) and (IIa) hold and if $\hat{u} = \hat{u}(t)$ is an optimal control and if $\hat{y} = \hat{y}(t)$ is the corresponding optimal trajectory of [(1),(2),(3)], then it is necessary that

$$\frac{\partial S(y_o,\dots,y_n)}{\partial y_o} f_o(y_1,\dots,y_n,u_1,\dots,u_m) + \dots$$

$$\dots + \frac{\partial S(y_o,\dots,y_n)}{\partial y_n} f_n(y_1,\dots,y_n,u_1,\dots,u_m)$$

$$= \min_{\hat{v} \varepsilon U} \left[ \frac{\partial S(y_o,\dots,y_n)}{\partial y_o} f_o(y_1,\dots,y_n,v_1,\dots,v_m) + \dots \right.$$

$$\left. \dots + \frac{\partial S(y_o,\dots,y_n)}{\partial y_n} f_n(y_1,\dots,y_n,v_1,\dots,v_n) \right] = 0$$

for every point of the optimal trajectory.

When applied to the variational problem (11) with $y' \varepsilon U$, the condition becomes

$$f(t,y,y') + \frac{\partial P}{\partial y} y' + \frac{\partial P}{\partial t} = \min_{v \varepsilon U} [f(t,y,v) + \frac{\partial P}{\partial y} v + \frac{\partial P}{\partial t}] = 0$$

where $P(t,y)$ is the optimal value function of the problem (11).

## 3. PONTRYAGIN'S MAXIMUM PRINCIPLE

We use condition (9) as a point of departure to derive Pontryagin's maximum principle for the case where conditions (I) and (IIb) are satisfied.

Since $(y_o,\dots,y_n) \in \Omega$ and since $\Omega$ is open, $(y_o + \Delta y_o,\dots,y_n + \Delta y_n) \in \Omega$ provided $(\Delta y_o)^2 + \dots + (\Delta y_n)^2$ is sufficiently small. Since $\Omega$ is an admissible set of inception, there exists an optimal trajectory $\hat{y} = \hat{\bar{y}}(t)$ that emanates from $(y_o + \Delta y_o,\dots,y_n + \Delta y_n)$ for some $t = \tau_o$ and terminates on L for some $t = \tau_1$. This optimal trajectory corresponds to an optimal control $\hat{u} = \hat{\bar{u}}(t)$. By (9)

$$\max_{\hat{v} \in U} Q(\bar{y}_o,\dots,\bar{y}_n,v_1,\dots,v_m) = Q(\bar{y}_o,\dots,\bar{y}_n,\bar{u}_1,\dots,\bar{u}_m) = 0$$

for every point $(\bar{y}_o,\dots,\bar{y}_n)$ on the optimal trajectory $\hat{y} = \hat{\bar{y}}(t)$. Hence

$$Q(\bar{y}_o,\dots,\bar{y}_n, u_1,\dots,u_m) \leq 0$$

for any $(u_1,\dots,u_m) \in U$, and, in particular, for the value of the optimal control $\hat{u} = \hat{u}(t)$ that corresponds to the point $(y_o,\dots,y_n)$ of the optimal trajectory $\hat{y} = \hat{y}(t)$. This, together with $Q(y_o,\dots,y_n,u_1,\dots,u_m)=0$ yields

$$Q(y_o,\dots,y_n, u_1,\dots.u_m) = \max_{(\hat{\bar{y}})} Q(\bar{y}_o,\dots,\bar{y}_n, u_1,\dots,u_m), \qquad (14)$$

where $\hat{\bar{y}} = (\bar{y}_o,\dots,\bar{y}_n) \in \Omega$. Since $\Omega$ is open, and since (IIb) is assumed to hold, we obtain the necessary condition

$$\frac{\partial Q(y_o,\dots,y_n,u_1,\dots,u_m)}{\partial y_i} = 0, \quad i = 0,1,\dots,n. \qquad (15)$$

Since

$$\frac{\partial Q}{\partial y_i} = - \sum_{k=0}^{n} \frac{\partial^2 S}{\partial y_k \partial y_i} f_k - \sum_{k=0}^{n} \frac{\partial S}{\partial y_k} \frac{\partial f_k}{\partial y_i}$$

and since $f_k = y_k'$ on the optimal trajectory, we have

$$\sum_{k=0}^{n} \frac{\partial^2 S}{\partial y_k \partial y_i} f_k = \sum_{k=0}^{n} \frac{\partial^2 S}{\partial y_k \partial y_i} y_k' = \frac{d}{dt} \left(\frac{\partial S}{\partial y_i}\right)$$

and we obtain for (15)

$$\frac{d}{dt} \left(\frac{\partial S}{\partial y_i}\right) = - \sum_{k=0}^{n} \frac{\partial S}{\partial y_k} \cdot \frac{\partial f_k}{\partial y_i}$$

on the optimal trajectory. This relation enables us to find $\left(\frac{\partial S}{\partial y_i}\right) = -\psi_i(t)$ as functions of $t$ along the optimal trajectory without knowledge of S, as the solutions of a system of linear first order differential equations, namely

$$\psi_i'(t) = - \sum_{k=0}^{n} \frac{\partial f_k(y_1,\ldots,y_n,u_1,\ldots,u_m)}{\partial y_i} \psi_k(t), \; i = 0,1,\ldots,n. \quad (16)$$

This is called the <u>conjugate system</u> to (1). Thus we see that for every point on the optimal trajectory and a suitable solution $\hat{\psi} = (\psi_o,\ldots,\psi_n)$ of (16)

$$\sum_{k=0}^{n} \psi_k(t) f_k(y_1,\ldots,y_n,v_1,\ldots,v_m) \le 0$$

for all $\hat{v} \in U$ and

$$\sum_{k=0}^{n} \psi_k(t) f_u(y_1,\ldots,y_n,u_1,\ldots,u_m) = 0$$

where $\hat{u} = (u_1,\ldots,u_m)$ are the values of the optimal control $\hat{u} = \hat{u}(t)$ at the point $(y_1,\ldots,y_n)$. In terms of

$$\mathcal{H}(\hat{\psi},y,\hat{v}) = \sum_{k=0}^{n} \psi_k(t) f_k(y_1,\ldots,y_n,v_1,\ldots,v_m)$$

we can state that for the optimal control $\hat{u} = \hat{u}(t)$

$$\mathcal{H}(\hat{\psi},y,\hat{u}) = \max_{\hat{v}\,\varepsilon\,U} \mathcal{H}(\hat{\psi},y,\hat{v}) = 0$$

for every point $y = (y_1,\ldots,y_n)$ of the optimal trajectory. Since

$$S(y_o,y_1,\ldots,y_n) = y_o + S(0,y_1,\ldots,y_n),$$

we have

$$\frac{\partial S}{\partial y_o} = 1$$

and hence

$$\psi_o = -1.$$

We summarize our result in

Theorem 2.

If (I) and (IIb) are satisfied and if $\hat{u} = \hat{u}(t)$ is the optimal control and $\hat{y} = \hat{y}(t)$ the corresponding optimal trajectory of the terminal control problem [(1),(2),(3)], then there exists a solution $\hat{\psi} = (\psi_o,\ldots,\psi_n)$ with $\psi_o = -1$ of the conjugate system (16) such that

$$\mathcal{H}(\hat{\psi},y,\hat{u}) = \max_{\hat{v}\,\varepsilon\,U} \mathcal{H}(\hat{\psi},y,\hat{v}) = 0$$

for every $t\ \varepsilon\ (t_o,t_1)$.

This is essentially the maximum principle of Pontryagin ([4], p. 19). This principle holds (with $\psi_o \leq 0$ instead of $\psi_o = -1$) even if (I) and (IIb) are not satisfied as Pontryagin has shown ([4], p. 75-108).

It may be of interest to note that Bellman's functional equation (10) is only one step removed from the maximum principle within the limits imposed by (I) and (IIb).

This one step is the recognition of (14) and subsequent differentiation of $Q$ with respect to $y_i$ and taking $\frac{\partial Q}{\partial y_i} = 0$.

## 4. EXAMPLES

The assumptions (I) and (IIb), while permitting a simple and compelling derivation of the maximum principle, severely limit its scope of applicability.

Assumption (I) excludes the case where the optimal trajectory, or portions thereof, may lie on the boundary of $\Omega$. The assumption that $\Omega$ be open and that the optimal trajectory remain inside $\Omega$ proved essential in two instances. First, in the establishment of the inequality (6) and then again in the derivation of condition (15). Without either, the entire proof technique - and dynamic programming - would have to be abandoned.

Assumption (IIa) entered in the derivation of (7) from (6) and (IIb) was required for (15). Again we have to state that without either, the argumentation which we carried out would not work.

We will now discuss two simple examples that will clearly demonstrate the shortcomings of dynamic programming and the derivation of the maximum principle that is based on dynamic programming.

First, we consider the problem

$$\int_0^1 (1 - y'^2)^2 dt \to \text{minimum}, \quad y(0) = 0, \quad y(1) = 0, \quad |y'| \leq \frac{1}{2},$$

which we formulate as a terminal control problem by introduction of the new variables

$$y_0 = \int_0^t (1 - y'^2)^2 dt, \quad y_1 = y, \quad y_2 = t, \quad u = y'.$$

Then,

$$y_o' = (1 - u^2)^2$$

$$y_1' = u$$

$$y_2' = 1$$

with the boundary conditions

$$y_o(0) = 0, \quad y_1(0) = 0, \quad y_2(0) = 0$$

$$y_1(t_1) = 0, \quad y_2(t_1) = 1$$

and the minimum condition

$$y_o(t_1) \to \text{minimum}.$$

We obtain for the conjugate system

$$\psi_o' = 0, \quad \psi_1' = 0, \quad \psi_2' = 0$$

of which $\hat{\psi} = (-1, 0, 9/16)$ is a nontrivial solution. Then

$$\max_{|v| \leq \frac{1}{2}} \mathcal{H}(\hat{\psi}, y, v) = \max_{|v| \leq \frac{1}{2}} [-(1 - v^2)^2 + \tfrac{9}{16}] = 0$$

yields for the optimal control $u = \pm \frac{1}{2}$. (That this control is indeed optimal can be seen directly from the original formulation of the problem.)

One optimal trajectory that corresponds to the optimal control

$$u = \begin{cases} \frac{1}{2} & \text{for} \quad 0 \leq t \leq \frac{1}{2} \\ -\frac{1}{2} & \text{for} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

is indicated in Fig. 3 by a bold line and other optimal trajectories are indicated by dotted lines. While any point P in the shaded region R can

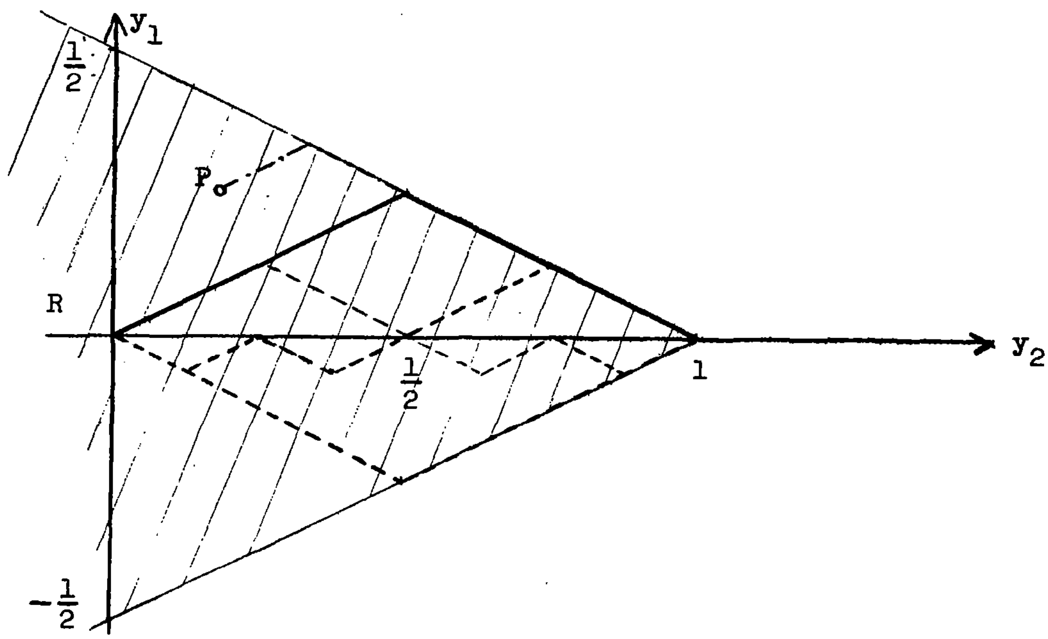


Figure 3.

be joined to (1,0) by an optimal trajectory (see stroke-dotted line), at least a portion of every optimal trajectory lies on the boundary

$$y_1 = \begin{cases} -\frac{1}{2} y_2 + \frac{1}{2} \\ \frac{1}{2} y_2 - \frac{1}{2} \end{cases}$$

of R and hence, an admissible set of inception $\Omega$ as postulated in (I) cannot exist. We obtain for $S(y_o, y_1, y_2)$ for $(y_1, y_2) \in R$, $-\infty < y_o < \infty$,

$$S(y_o, y_1, y_2) = y_o + \int_{y_2}^{1} \frac{9}{16}\, dt = y_o + \frac{9}{16}(1 - y_2)$$

and we see that (IIb) is met, at least for all $(y_1, y_2)$ in the interior of R.

Next, we consider the problems where the duration of the process of transferring a moving masspoint from the state $(y(0),y'(0))$ to the state $(0,0)$ is to be minimized by a proper choice of a control $u = u(t)$ in the equation of motion

$$\frac{d^2y}{dt^2} = u(t)$$

where we require that $|u(t)| \leq 1$. Letting

$$y_1 = y, \quad y_2 = y',$$

we may formulate this problem as a terminal control problem as follows:

$$y_o' = 1$$

$$y_1' = y_2$$

$$y_2' = u$$

with the boundary conditions

$$y_o(0) = 0, \quad y_1(0) = y_1^o, \quad y_2(0) = y_2^o$$

$$y_1(t_1) = 0, \quad y_2(t_1) = 0$$

and the minimum condition

$$y_o(t_1) = t_1 \to \text{minimum.}$$

Application of the maximum principle yields $u = \pm 1$ ([4], p. 23-27) and trajectories that are depicted in Fig. 4. Assuming that these trajectories are indeed optimal and depending on whether the initial state $(y_1^o,y_2^o)$ is in $R_1$ or in $R_2$, one obtains by elementary, though cumbersome manipulations

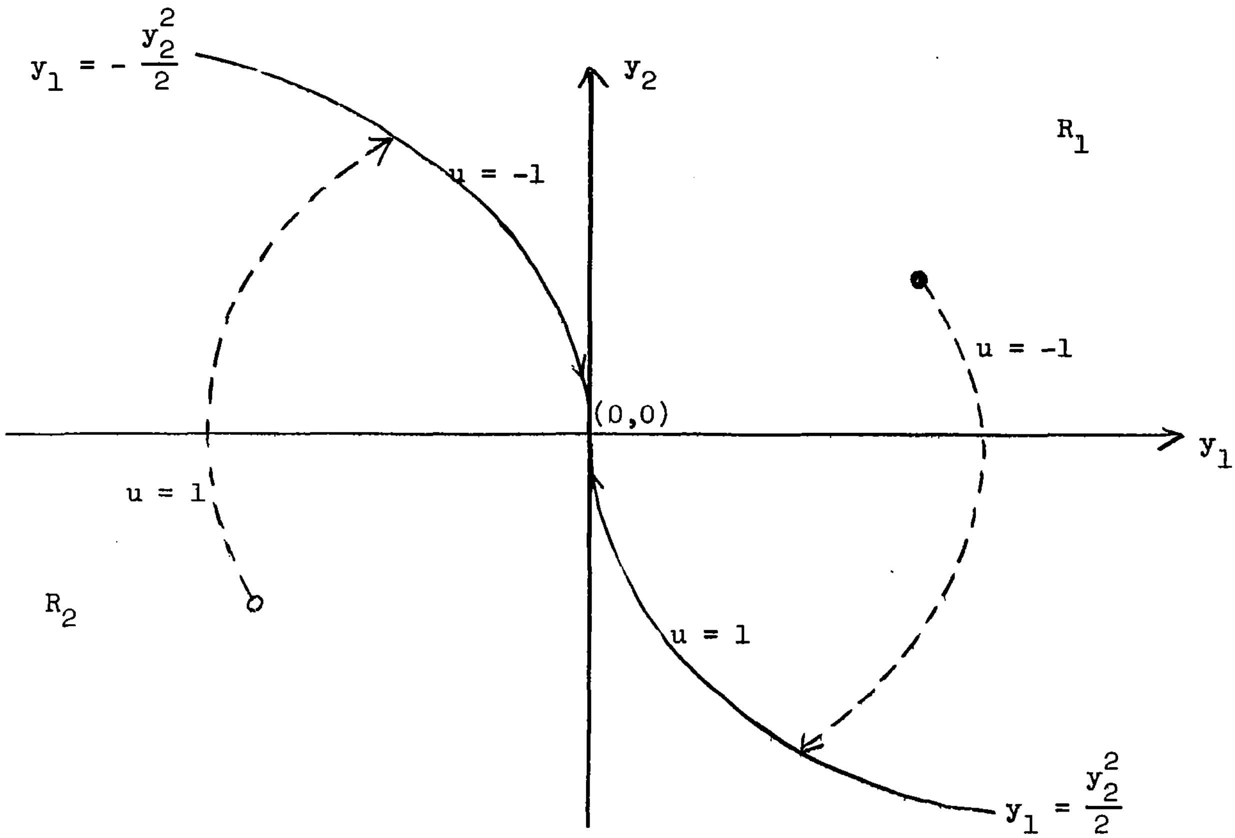


Figure 4.

$$S(y_o,y_1,y_2)_{(y_1,y_2)\ \varepsilon\ R_1} = y_o + y_2 + 2\sqrt{y_2^2/2 + y_1}$$

$$S(y_1,y_1,y_2)_{(y_1,y_2)\ \varepsilon\ R_2} = y_o - y_2 + 2\sqrt{y_2^2/2 - y_1}$$

and we see that S is not even continuous on $y_1 = y_2^2/2$ for $y_1 > 0$. Hence, (IIb) cannot possibly be satisfied. On the other hand, $\Omega$ is the entire $(y_o,y_1,y_2)$-space and condition (I) is met.

We note that, although Theorem 2 of section 3 does not apply to these two examples, the maximum principle of Pontryagin in its general formulation _is_ applicable. However, the application of dynamic programming in its various manifestations is not justified.

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