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A THEORETICAL STUDY OF ANTENNAS IN MOVING IONIZED MEDIA

PART II

THE COMPLEX DOPPLER EFFECT

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ABSTRACT

A theory of the complex Doppler effect of an oscillating electromagnetic source moving uniformly through a homogeneous dispersive medium is presented. The analysis is based on a method of asymptotic evaluation of a certain threefold Fourier integral. The theory presented and the results obtained in this paper generally differ from those already reported in the literature. To illustrate the effect a few special cases are worked out in detail.

CONTENTS

I. INTRODUCTION

It is well known that the color of radiation emitted by a harmonic light source moving relative to an observer varies with time. This is the familiar Doppler effect. When such a source is moving through a dispersive medium, however, a new feature arises. The radiation incident upon an observer may have several spectral components although the source is oscillating at a single frequency. This phenomenon of frequency splitting is called the complex Doppler effect and has been studied by Frank[1] in connection with the problem of calculating the radiation field of an oscillating dipole moving uniformly through a homogeneous, isotropic, dispersive medium. Barsukov[2], Barsukov and Kolomenskii[3], and others[4] have extended Frank's analysis to homogeneous, anisotropic, dispersive media, such as the magneto-ionic medium.

Frank's method of analysis is basically as follows. The moving oscillating dipole is first spectrally analyzed into a harmonic line source. Then, the time-harmonic Maxwell equations are solved and the spectrum of the far field is obtained. Frank notes that for a dispersive medium the spectrum may peak at several frequencies, whereas in a non-dispersive medium it peaks only at one frequency which is the ordinary Doppler-shifted frequency. The frequencies at which the spectrum peaks are determined by solving for ω the Doppler equation

$$\omega = \gamma^{-1}\omega_0' + \beta\omega n(\omega) \cos \theta, \qquad (1)$$

where ω_0' = proper frequency of the source, ω = wave frequency measured by a distant observer at rest in the medium, n = refractive index, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$, $\beta = v/c$, θ = angle between the velocity $\underline{v}$ of the source and the wave vector $\underline{k}$. Equation (1) in general admits several roots for ω since the refractive index can be a complicated function of ω. Thus Frank and his followers conclude that at a given direction θ a distant observer may find several distinct frequencies in the radiation field, i.e., several wave packets whose main frequencies are determined by eq. (1). However, they can not ascribe any physical interpretation to their results and the origin of the complex Doppler effect remains a mystery.

Actually, the Doppler eq. (1) just connects the frequencies of a homogeneous plane wave observed in two different inertial frames K and K′, with K′ moving relative to K at a constant velocity $\underline{v}$. If the plane wave measured in K consists of several spectral components, then in K′ the wave appears to be a harmonic wave made up of a bundle of plane waves with the same frequency ω_0' but with different propagation directions. This can be seen from the aberration formula[5]

$$\tan\theta' = \frac{1}{\gamma}\,\frac{\tan\theta}{1 - \frac{\beta}{n(\omega)}\sec\theta}\,, \qquad (2)$$

θ' being the angle between $\underline{v}$ and $\underline{k}'$. Here and henceforth, the primed quantities are always referred to K′ while the unprimed quantities are always referred to K. Thus, according to Frank's analysis the far field in K′, the reference frame in which the oscillating source is at rest, is a monochromatic wave oscillating at the source frequency and consisting of a bundle of plane waves, whereas in K this bundle of plane waves becomes a polychromatic plane wave with a unique direction of propagation; that is to say, the radiation incident upon a distant observer at a given direction consists of several wave packets. From a physical viewpoint this is impossible since the group velocity takes different values at different frequencies for a given direction of propagation. Therefore, at any time the distant observer cannot receive more than one wave packet at a given direction. From a mathematical viewpoint the analysis of

Frank and his followers is equally unacceptable since they identify the spectrum of the far field of an oscillating line source with the spectrum of the radiation field of a uniformly moving oscillating point source. The identification is permissible only for the spectrum of the full field but is not permissible for that of the far field alone. Moreover, in a complicated medium such as the magneto-ionic medium, the "outgoing-wave condition" in the frequency-domain is not necessarily the counterpart of the "retardation condition" in the time-domain, both being the commonly used radiation condition.

Although the analysis of Frank and his followers is of dubious validity, their prediction of the existence of the complex Doppler effect in a dispersive medium is correct. In the following we shall present a method, which is mathematically rigorous, for calculating the radiation field of an oscillating source moving uniformly through a homogeneous, lossless, dispersive medium. The method is to express the field as a superposition of plane waves by a threefold Fourier integral with the radiation condition built in the integrand and then to estimate the integral asymptotically to get the radiation field in space and time. Our mathematical results prove the existence of the complex Doppler effect in a dispersive medium. The origin of the effect is found in examining the group velocity of each individual wave packet arriving at a distant observer. The wave packets, each of which has its own characteristic spectrum, were actually emitted by the moving source at different time, propagate to the observer at different directions and at different group speeds, and finally reach the observer simultaneously. From the viewpoint of the observer the wave packets were emitted by different sources at different locations.

The remainder of this report is divided into two parts. The first part contains the mathematical theory of the method for solving the general problem of finding the radiation field of an oscillating source traversing a homogeneous dispersive medium. The theory is then applied in the second part to a few special cases where the mathematics remains tractable: an oscillating source travels through (i) a vacuum, (ii) an isotropic cold plasma, and (iii) a cold plasma biased by a strong magnetostatic field such that the gyro-frequency is much larger than the plasma and the wave frequency. Only in case (iii) does the complex Doppler effect arise. According to Frank's analysis, however, the effect should also occur in case (ii), whereas according to our analysis there can be no frequency splitting.

II. THEORY

A. FORMULATION OF THE PROBLEM

In order to avoid the unnecessary mathematical complications which arise from Lorentz contraction for bodies of finite size, we shall consider only an electromagnetic point source. To fix ideas we shall assume the point source to be an oscillating electric dipole moving at a constant velocity $\underline{v}$ through a homogeneous medium of infinite extent. Although we are considering a very particular source, no essential features will be lost regarding the phenomenon of complex Doppler effect.

Let us consider two inertial frames of reference K and K′, K being the rest frame of the medium with respect to which K′ is moving at the constant velocity $\underline{v}$. Thus, our oscillating electric dipole is at rest in K′ and we have for the external electric and magnetic polarization vectors $\underline{P}_e'$ and $\underline{M}_e'$ the following expressions:

$$\underline{P}_e' = \underline{p}' \, \delta(\underline{r}') \, e^{-i\omega_o' t'} \tag{3}$$

$$\underline{M}_e' = 0,$$

where $\underline{p}'$ is the electric dipole moment and ω_o' is its proper frequency. With respect to K we then have[6]

$$\underline{P}_e = \left[\gamma\underline{p}' + \frac{1-\gamma}{v^2}\,\underline{v}\underline{v}\cdot\underline{p}'\right] \delta(\underline{r}') e^{-i\omega_o' t'} \tag{4}$$

$$\underline{M}_e = \gamma\underline{p}' \times \underline{v} \; \delta(\underline{r}') e^{-i\omega_o' t'} ,$$

We now transform $(\underline{r}',t')$ to $(\underline{r},t)$ in eq. (4) by means of the Lorentz transformations. For convenience $\underline{v}$ is taken to be along the z and z′ axes. Thus,

$$x' = x,\ y' = y,\ z' = \gamma(z-vt),\ t' = \gamma(t-vz/c^2),$$

by means of which we have

$$\delta(x')\delta(y')\delta(z')e^{-i\omega_o' t'} = \delta(x)\delta(y)\delta[\gamma(z-vt)]e^{-i\gamma\omega_o'(t-vz/c^2)}$$

$$= \gamma^{-1}\delta(x)\delta(y)\delta(z-vt)e^{-i\gamma^{-1}\omega_o' t} .$$

The last expression follows by setting $z=vt$ in the exponent of the second expression. This is permissible because of the presence of $\delta(z-vt)$. Generalizing the above expression for arbitrary direction of $\underline{v}$, we have

$$\delta(\underline{r}')e^{-i\omega_o' t'} = \gamma^{-1}\delta(\underline{r}-\underline{v}t)e^{-i\omega_o t} , \qquad (5)$$

where

$$\omega_o = \gamma^{-1}\omega_o' .$$

Substitution of eq. (5) into (4) gives

$$\underline{P}_e = \left[\underline{p}' + \frac{\gamma^{-1}-1}{v^2}\,\underline{v}\underline{v}\cdot\underline{p}'\right]\delta(\underline{r}-\underline{v}t)e^{-i\omega_o t} \qquad (6)$$

$$\underline{M}_e = \underline{p}' \times \underline{v}\ \delta(\underline{r}-\underline{v}t)e^{-i\omega_o t} .$$

With $\underline{P}_e$ and $\underline{M}_e$ as external source functions the Maxwell equations for non-magnetic, homogeneous, anisotropic and dispersive media are

$$\nabla \times \underline{E} = -\mu_o \frac{\partial}{\partial t} \underline{H} \tag{7}$$

$$\nabla \times \underline{H} = \frac{\partial}{\partial t} \hat{\underline{\underline{\varepsilon}}} \cdot \underline{E} + \frac{\partial}{\partial t} \underline{P}_e + \nabla \times \underline{M}_e \ ,$$

whence

$$\nabla \times \nabla \times \underline{E} + \mu_o \frac{\partial^2}{\partial t^2} \hat{\underline{\underline{\varepsilon}}} \cdot \underline{E} = -\mu_o \frac{\partial^2}{\partial t^2} \underline{P}_e - \mu_o \frac{\partial}{\partial t} \nabla \times \underline{M}_e, \tag{8}$$

where $\underline{\underline{\varepsilon}}$ is a dyadic integral operator and is defined as

$$\hat{\varepsilon}_{ij} E_j = \int_{-\infty}^{t} \varepsilon_{ij}(t-\tau) E_j(\tau) d\tau,$$

repeated indices being summed.

Writing

$$\underline{E}(\underline{r},t) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \underline{E}(\underline{k},t) e^{i \underline{k}\cdot\underline{r}} d^3k, \tag{9}$$

and similar integral representations for $\underline{P}_e$, $\underline{M}_e$ and substituting them into eq. (8) we have

$$\left(k_i k_j - k^2 \delta_{ij} - \mu_o \frac{d^2}{dt^2} \hat{\varepsilon}_{ij}\right) E_j(\underline{k},t) = F_i(\underline{k}) e^{-i\omega t}, \tag{10}$$

where $\omega = \omega_o + \underline{k}\cdot\underline{v}$ and δ_{ij} = Kronecker's delta. The operator $\hat{\varepsilon}_{ij}$ when acting on the function $\exp(-i\omega t)$ simply multiplies the function by $\varepsilon_{ij}(\omega)$. Hence, eq. (10) gives

$$E_i(\underline{k},t) = \frac{(\text{adj } V_{ij}) F_j}{\det V_{ij}} e^{-i\omega t} \ , \tag{11}$$

where adj V_{ij} and det V_{ij} denote respectively the adjoint and determinant of the matrix with elements given by

$$V_{ij} = k_i k_j - k^2 \delta_{ij} + \omega^2 \mu_0 \, \epsilon_{ij}(\omega) \tag{12}$$

Inserting (11) in (9) we have

$$E_i(\underline{r},t) = e^{-i\omega_0 t} \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \frac{N_i(\omega,\underline{k})}{D(\omega,\underline{k})} e^{i\,\underline{k}\cdot(\underline{r}-\underline{v}t)} d^3k, \tag{13}$$

where $N_i = (\text{adj } V_{ij})F_j$, $D = \det V_{ij}$ and $\omega = \omega_0 + \underline{k}\cdot\underline{v}$.

Our problem now is to evaluate the three-dimensional Fourier integral (13) for $|\underline{r}-\underline{v}t| \to \infty$, i.e., for observation points very far away from the trajectory. Before attempting to do so we note that for lossless media ϵ_{ij} is either real symmetric or hermitian. In this case D will be a real function of $\underline{k}$ and will have zeros on the paths of integration. Hence, the integral (13) has no meaning. However, when the radiation condition is introduced either the zeros of D will have to be removed off the paths of integration or the paths of integration will have to be modified to avoid the zeros of D. The simplest way of introducing this radiation condition is to replace the source frequency ω_0' by $\omega_0' + i\epsilon$ $(\epsilon > 0)$, and afterwards to let ϵ go to zero. Physically, this means that the source is slowly turned on as $\exp(\epsilon t - i\omega_0 t)$ and we seek solutions of Maxwell's equations of the order $\exp(\epsilon t)$. The solutions obtained in this way will be called the outgoing solutions. Thus, we shall have to evaluate, instead of the integral (13), the following integral:

$$E_i^{out}(\underline{r},t) = e^{-i\omega_0 t} \lim_{\epsilon\to 0} \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \frac{N_i\,(\omega,\underline{k})}{D_\epsilon(\omega+i\epsilon,\ \underline{k})} e^{i\,\underline{k}\cdot(\underline{r}-\underline{v}t)} d^3k \tag{14}$$

for $|\underline{r}-\underline{v}t| \to \infty$. The subscript ϵ in D is used to distinguish $D(\omega+i\epsilon)$ from $D(\omega)$ when the argument is not written out explicitly.

So far we have assumed that the singularities of the integrand in (14) come entirely from the zeros of D_ϵ, neglecting any possible poles arising from N_i. This is justified since N_i can be shown to have no poles.

B. ASYMPTOTIC EVALUATION

Our objective now is to obtain an asymptotic value for $E_i^{out}(\underline{r},t)$ by evaluating the integral in (14) for $|\underline{r}-\underline{v}t| \to \infty$. We anticipate this asymptotic value to come from the zeros of D which are the only singularities of the integrand. The method of evaluating the integral (14) asymptotically is very familiar in radiation problems where the far-zone fields from a localized source are sought. The only difference now is that we shall speak in general terms to achieve maximum generality, since we do not wish to specify D, or equivalently the dielectric tensor ϵ_{ij}, explicitly.

An asymptotic estimate of an integral of the form (14) has been obtained by Lighthill[7] who uses the theory of generalized functions. Here we shall re-derive the results in a straightforward and customary way, void of the theory of generalized functions and, moreover, we shall supply in some detail part of the discussions touched upon in Lighthill's paper.

The method we shall adopt in evaluating the integral in (14) consists of the calculus of residues and the method of stationary phase. First, let us choose a coordinate system (ξ, η, ζ) where the ξ-axis is along $\underline{r}-\underline{v}t$ whose magnitude is denoted by x Hence, the integral in (14) becomes

$$I = \lim_{\epsilon\to 0} \iint_{-\infty}^{\infty} d\eta d\zeta \int_{-\infty}^{\infty} \frac{N_i(\xi,\eta,\zeta,\omega)}{D_\epsilon(\xi,\eta,\zeta,\omega+i\epsilon)} e^{i\xi x} d\xi \tag{15}$$

for $x \to \infty$. The inner integral can be evaluated by the method of residues in the complex ξ-plane. Let ξ_m' be one of the zeros of D_ϵ and ξ_m the corresponding one of D. Then, expanding ξ_m' around ξ_m and assuming for the moment $\partial D/\partial\xi$ at $\xi = \xi_m$ is non-zero we have, by recalling that $\omega = \omega_o + \underline{k}\cdot\underline{v}$,

$$D_\epsilon(\xi_m',\eta,\zeta,\omega+i\epsilon) = D(\xi_m,\eta,\zeta,\omega) + (\xi_m'-\xi_m)\left(\frac{\partial D}{\partial\xi}\right)_{\xi_m} + i\epsilon\left(\frac{\partial D}{\partial\omega_o}\right)_{\xi_m} \ldots = 0 \tag{16}$$

whence approximately,

$$\xi_m' = \xi_m - i\epsilon \left(\frac{\partial D/\partial\omega_o}{\partial D/\partial\xi}\right)_{\xi_m} = \xi_m + i\epsilon \left(\frac{\partial\omega_o}{\partial\xi}\right)^{-1}_{\xi_m} . \tag{17}$$

Here we have used $\partial\omega_o/\partial\xi = -(\partial D/\partial\xi)(\partial D/\partial\omega_o)^{-1}$ which is obtained by differentiating with respect to ξ and ω_o the equation

$$D(\xi,\eta,\zeta,\omega_o + \underline{k}\cdot\underline{v}) = 0 \tag{18}$$

Equation (18) describes a surface in the (ξ,η,ζ) space and this surface is often called the "wave-number surface."

The inner integral in (15) is now evaluated by closing the contour in the upper ξ-plane and, therefore, only zeros of D_ϵ having positive imaginary part contribute. By the method of residues we have

$$\int_{-\infty}^{\infty} \frac{N_i}{D_\epsilon} e^{i\xi x} d\xi = 2\pi i \sum_m \left(\frac{N_i}{\partial D_\epsilon/\partial\xi}\right)_{\xi_m'} e^{i\xi_m' x} , \ \operatorname{Im} \xi_m' > 0 \tag{19}$$

As is evident from (17), the condition Im $\xi_m' > 0$ implies that

(a) if ξ_m is purely real, $\dfrac{\partial \omega_o}{\partial \xi} > 0$ at $\xi = \xi_m$;

(b) if ξ_m is complex, Im $\xi_m > 0$.

Passing to the limit $\epsilon \to o$ in (19) and substituting the result into (15) we then have

$$I = 2\pi i \sum_m \iint_{-\infty}^{\infty} \left(\frac{N_i}{\partial D/\partial \xi} \right)_{\xi_m} e^{i\xi_m x} \, d\eta d\zeta \tag{20}$$

To estimate (20) for $x\to\infty$, we divide the infinite domain into S and R (Figure 1). S is the area of the surface D = 0 projected onto the $\eta\zeta$-plane, R the remaining area in the infinite $\eta\zeta$-plane. Clearly, $\xi_m(\eta, \zeta, \omega_o)$ is real when the point (η, ζ) lies in S; ξ_m ceases to be real if the point (η, ζ) lies outside S. Since $x\to\infty$ and Im $\xi_m > 0$, the integrand in (20) is vanishingly small for points in R and, consequently, this part of integration can be neglected from integral (20). Thus the integral we shall have to estimate for $x\to\infty$ is

$$I = 2\pi i \sum_m \iint_S \left(\frac{N_i}{\partial D/\partial \xi} \right)_{\xi_m} e^{i\xi_m x} \, d\eta d\zeta, \tag{21}$$

where ξ_m is real and therefore lies on the surface D = 0, and at $\xi = \xi_m$, $\partial\omega_o/\partial\xi > 0$ must be satisfied.

To get the term proportional to x^{-1} from (21), we can use the method of stationary phase. The stationary points are determined from the equations

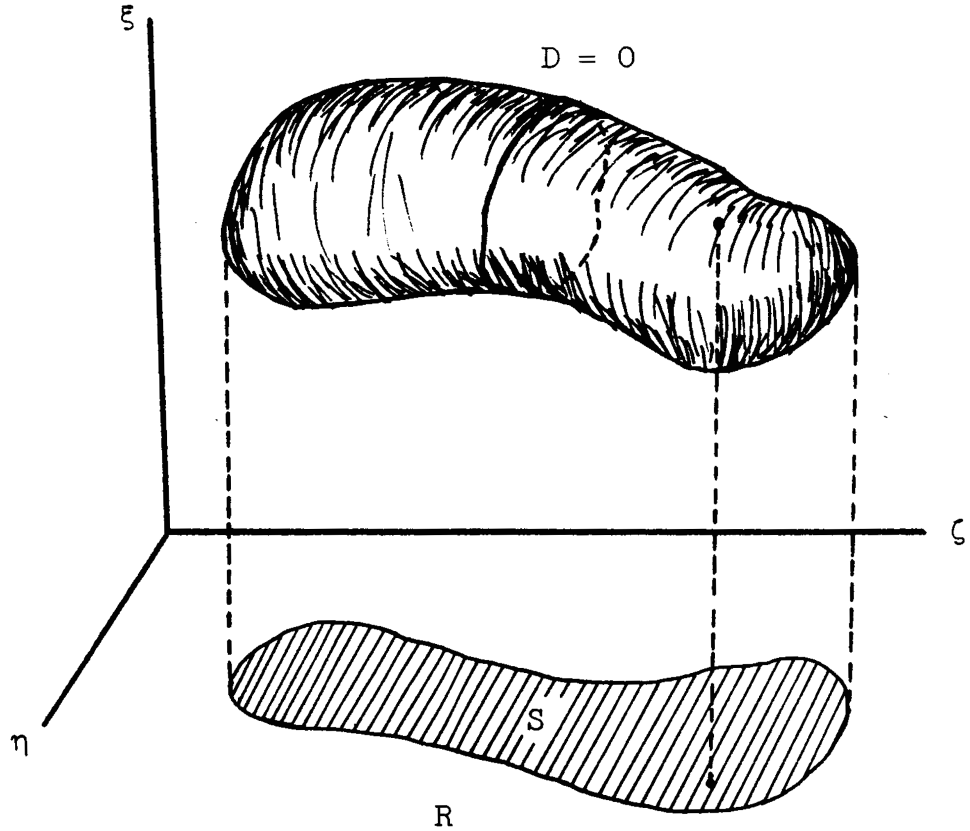


Figure 1. S is the projected area of the wave-number surface, D = 0, onto the ηζ-plane, R the remaining area in the infinite ηζ-plane.

$$\frac{\partial \xi_m}{\partial \eta} = \frac{\partial \xi_m}{\partial \zeta} = 0 \tag{22}$$

which simply mean that the tangent planes to the surface D = 0 at the stationary points are parallel to the $\eta\zeta$-plane. Thus the asymptotic value of (21) is given by

$$I \sim 4\pi^2 i \sum \frac{N_i}{\partial D/\partial \xi} \frac{e^{\frac{1}{4}\pi i(\operatorname{sgn} \kappa_\eta + \operatorname{sgn} \kappa_\zeta)}}{\sqrt{|\kappa_\eta \kappa_\zeta|}} \frac{e^{i\xi x}}{x} .$$

Here $\kappa_\eta, \kappa_\zeta$ are the two principal curvatures, and $\sum$ is the sum over the points on the surface D = 0 where $\partial \omega_o/\partial \xi$ is positive and where the normal is parallel to the x-direction.

We are now in a position to write down the asymptotic value of E_i^{out} given in (14). To do this we simply make the following substitutions in (23): $\xi_m x$ by $\underline{k}_m \cdot (\underline{r}-\underline{v}t)$, x by $|\underline{r}-\underline{v}t|$, $\kappa_\eta \kappa_\zeta$ by the Gaussian curvature K, and $\partial D/\partial \xi$ by $\partial D/\partial \underline{k}$. Then we have

$$E_i^{out} \sim \frac{1}{|\underline{r}-\underline{v}t|} \sum_m A_i(\underline{k}_m)\, e^{i\underline{k}_m \cdot \underline{r}}\, e^{-i(\omega_o + \underline{k}_m \cdot \underline{v})t} . \tag{24}$$

Here $\sum$ is summing over the set of points $\underline{k}_m$ on the surface D = 0 where the normal is parallel to $\underline{r}-\underline{v}t$ (the asymptotic condition) and the component of $\partial \omega_o/\partial \underline{k}$ along $\underline{r}-\underline{v}t$ is positive (the radiation condition). The latter condition implies the the group velocity $\partial \omega/\partial \underline{k}$ minus $\underline{v}$ has a positive component along $\underline{r}-\underline{v}t$.

The amplitude A_i is given by

$$A_i = \frac{C\, N_i}{2\pi |K|^{\frac{1}{2}} |\partial D/\partial \underline{k}|} , \tag{25}$$

where C is

(a) $\pm i$ when $K < 0$ and $\partial D/\partial \underline{k}$ is in the direction of $\pm(\underline{r}-\underline{v}t)$

(b) ± 1 when $K > 0$ and the surface is convex to the direction of $\pm(\partial D/\partial \underline{k})$.

In what follows our primary concern will be to examine the frequencies of the far field and no attention will be paid to the amplitude A_i. Although, to be sure, A_i may vanish at some $\underline{k}_m$'s, that is to say, the modes corresponding to that set of $\underline{k}_m$'s cannot be excited for a source function of given distribution, we shall not consider this point, but instead we shall study the frequencies of all the possible modes that can be excited.

At this point it may be asked if it is at all meaningful to talk about discrete frequency components of the far field since the field radiated by a moving oscillating source actually has a continuous spectrum. Strictly speaking, the far field (24) has a continuous frequency spectrum because of the presence of the factor $|\underline{r}-\underline{v}t|^{-1}$ and the fact that the locations of the points $\underline{k}_m$ on the wave-number surface vary from time to time. However, both of these factors hardly change over a period of oscillation and, therefore, most of the radiated energy is contained in the discrete frequencies ω_m equal to $\omega_o + \underline{k}_m \cdot \underline{v}$.

Before concluding this section two points should be mentioned. In the foregoing analysis we have assumed that there were only simple poles in the complex ξ-plane and that the Gaussian curvature was non-zero. At a multiple pole, i.e., a pole of order greater than one, our previous expression (19) has to be modified according to the theory of residues, and in this case the radiation condition can no longer be interpreted in terms of the group velocity, a physically meaningful quantity.

In the situtation where the Gaussian curvature vanishes, an asymptotic estimate of the two-dimensional integral (21) will lead to the Airy function which decays slower than the inverse of a distance. Here we shall not write out the cumbersome general formula for the case of multiple poles and for the case of vanishing Gaussian curvature. However, a detailed discussion will be given when such situations arise in treating a specific problem.

III. APPLICATIONS

The general theory set out in the last chapter applies to the problem of calculating the radiation field of an oscillating source of any distribution stationary in or moving through a linear, homogeneous, lossless medium of infinite extent. The medium can be dispersive as well as anisotropic. We shall now apply the general theory to a few cases where the analyses are mathematically tractable. We shall limit, however, our consideration only to the finding of the principal frequencies of the radiation field and, therefore, the construction of the wave-number surface for each particular case will suffice, as was pointed out in section II B.

A. CASE OF A VACUUM

Let us begin with the simplest and yet important case where a harmonically oscillating source travels at constant speed in a vacuum. The dispersion equation, D = 0, is well known as

$$\xi^2 + \eta^2 + \zeta^2 - \frac{1}{c^2}(\omega_o + \underline{k}\cdot\underline{v})^2 = 0 \tag{26}$$

Here, as before, (ξ, η, ζ) are the Cartesian components of $\underline{k}$. Without loss of generality we choose the ξ-axis along $\underline{v}$. Equation (26) then becomes

$$(1-\beta^2)\xi^2 + \eta^2 + \zeta^2 - 2\beta k_o\xi - k_o^2 = 0, \tag{27}$$

where $k_o = \omega_o/c$. In contrast to the stationary case where the wave-number surface is spherical, eq. (27) describes the surface of a prolate spheroid whose projection onto the $\xi\eta$-plane is an ellipse:

$$(1-\beta^2)\xi^2 + \eta^2 - 2\beta k_o\xi - k_o^2 = 0,$$

or

$$\frac{(\xi-\beta\gamma^2k_o)^2}{\gamma^4k_o^2} + \frac{\eta^2}{\gamma^2k_o^2} = 1,$$

or

$$k = \frac{k_o}{1-\beta\cos\chi} \tag{28}$$

Equation (28) is plotted in Figure 2a. There are only two points A and B where the normals to the ellipse given by $D(\xi,\eta,\omega_o) = 0$ are parallel to $\underline{r}-\underline{v}t$. To see if both A and B satisfy the radiation condition, we construct the surface $D(\xi,\eta,\omega_o + \Delta\omega_o) = 0$, $\Delta\omega_o$ being small and positive. From Figure 2a we can easily see that the direction from A to A′ is along $\underline{r}-\underline{v}t$, whereas the direction from B to B′ is opposite to $\underline{r}-\underline{v}t$. Thus at A, $(\underline{r}-\underline{v}t)\cdot\partial\omega_o/\partial\underline{k} > 0$, while at B, $(\underline{r}-\underline{v}t)\cdot\partial\omega_o/\partial\underline{k} < 0$. That is to say (see section II.B), at B the radiation condition is violated and only A contributes to the radiation field.

The group velocity $\underline{u}$ of the wave is given by

$$u_\xi = \frac{\partial\omega}{\partial\xi} = \frac{\partial\omega_o}{\partial\xi} + v$$

$$u_\eta = \frac{\partial\omega}{\partial\eta} = \frac{\partial\omega_o}{\partial\eta} .$$

Evaluating $\partial\omega_o/\partial\xi$ and $\partial\omega_o/\partial\eta$ from (28) we get

$$u_\xi = c\,\frac{\xi}{\beta\xi + k_o} = c\,\frac{\xi}{k} = c\cos\chi,$$

$$u_\eta = c\,\frac{\eta}{\beta\xi + k_o} = c\,\frac{\eta}{k} = c\sin\chi. \tag{29}$$

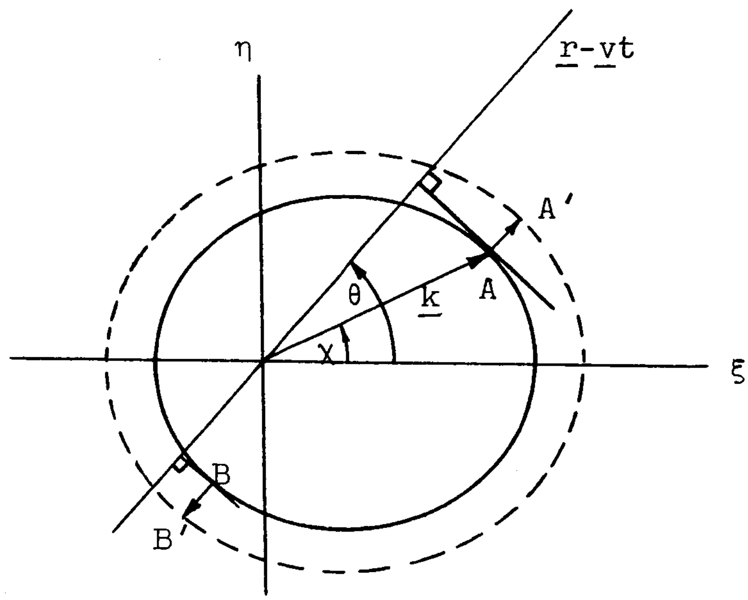


Figure 2a. Cross section (the ellipse in solid line) of the wave-number surface in any axial plane for the vacuum case. The ellipse in broken line is obtained when ω_0 is replaced by $\omega_0 + \Delta\omega_0$ in the dispersion equation; $\Delta\omega_0$ being small and positive.

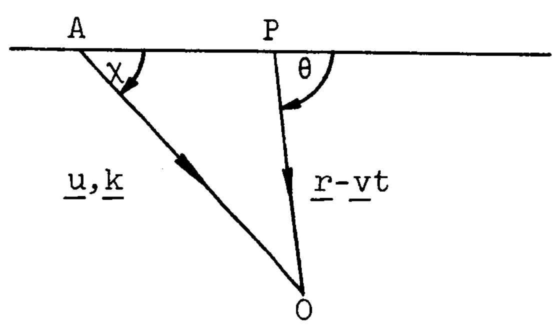


Figure 2b. The corresponding picture of Figure 2a in ordinary space. P and A are respectively the present and retarded positions of the source. O is the distant observer. The angles θ and χ correspond to those in Figure 2a. AO = ct. AP = vt.

Thus, as expected, $\underline{u}$ is in the direction of $\underline{k}$ (see Figure 2b) and has magnitude equal to c, the vacuum speed of light.

Figure 2b plots the relative positions of the observer and the source at the present and retarded moments. It is a simple matter to show that the direction from A to O in Figure 2b corresponds to the direction of $\underline{k}$ in Figure 2a. Thus, by means of the wave-number surface one can determine with comparative ease the relation between the present and retarded positions of the radiating source. The relation between θ and χ in Figure 2a or Figure 2b can be obtained from eq. (28) and is found to be

$$\tan\theta = \frac{\sin\chi}{\cos\chi - \beta} \qquad (30)$$

The variation of the wave frequency ω with the angle χ can also be obtained from the wave-number surface Figure 2a. Since

$$\omega = \omega_o + \underline{k}\cdot\underline{v} = \omega_o + \xi v,$$

and since from eq. (28)

$$\xi = \frac{k_o \cos\chi}{1-\beta\cos\chi},$$

we find that

$$\omega = \frac{\omega_o}{1-\beta\cos\chi} = \frac{\omega_o'}{\gamma(1-\beta\cos\chi)}, \qquad (31)$$

where, as before, ω_o' is the proper frequency of the source. A plot of (31) is given in Figure 4c by the curve $X = 0$.

If one wishes, he can also find the amplitude of the radiation field from eq. (25) for any given source function. However, this point will not be pursued here.

By the simple example of a vacuum we have shown how the characteristics of the radiation field can be determined solely from a knowledge of the wave-number surface.

B. CASE OF AN ISOTROPIC COLD PLASMA

A second simple illustration of the general theory is the case in which a harmonic source travels through an ionized gas of vacuum permeability and of permittivity given by

$$\epsilon = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2}\right),$$

where ω_p is the plasma frequency. In this case the dispersion equation, D = 0, is

$$k^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} = 0$$

which becomes

$$\xi^2 + \eta^2 + \zeta^2 - (k_o + \beta\xi)^2 + k_p^2 = 0 . \qquad (32)$$

After suppressing the variable ζ, we have from eq. (32)

$$\frac{\left(\xi - \beta\gamma^2 k_o\right)^2}{\gamma^2\left(\gamma^2 k_o^2 - k_p^2\right)} + \frac{\eta^2}{\gamma^2 k_o^2 - k_p^2} = 1. \qquad (33)$$

Assume for the moment $k_o > k_p$, i.e., $X = k_p^2/k_o^2 < 1$. The ellipse described by (33) is plotted in Figure 3a, which shows that there is only one point on the ellipse where the normal is parallel to $\underline{r}-\underline{v}t$ and $(\underline{r}-\underline{v}t)\cdot\partial\omega_o/\partial\underline{k} > 0$. Hence, it is concluded that only one wave will propagate to the distant observer.

The rectangular components u_ξ and u_η of the group velocity $\underline{u}$ are found, with the aid of (33), to satisfy the equation of an ellipse:

$$(1 + \beta^2 X)u_\xi^2 + u_\eta^2 - 2\beta c X u_\xi = c^2(1-X). \qquad (34)$$

The group-velocity surface is obtained by rotating this ellipse about the u_ξ-axis. Figure 3b is a plot of eq. (34), showing that the wave has a greater speed when the source is approaching.

The relative positions of the observer and the source are shown in Figure 3c. The angle χ in both Figures 3a and 3c can be shown to be the same. From Figure 3a the relation between the present and retarded positions of the source is found to be given by

$$\tan\theta = \frac{\sin\chi}{\cos\chi - v/u}\,. \qquad (35)$$

We now proceed to find the wave frequency ω as a function of χ from Figure 3a. First, we substitute $\eta = \xi\tan\chi$ into eq. (33) and solve for ξ. Then from $\omega = \omega_o + v\xi$ we get

$$\frac{\omega}{\omega_o} = \frac{1 + \beta\cos\chi\sqrt{1-X(1-\beta^2\cos^2\chi)}}{1-\beta^2\cos^2\chi}, \qquad (36)$$

which is given in Figure 4c by the curve $X < 1$.

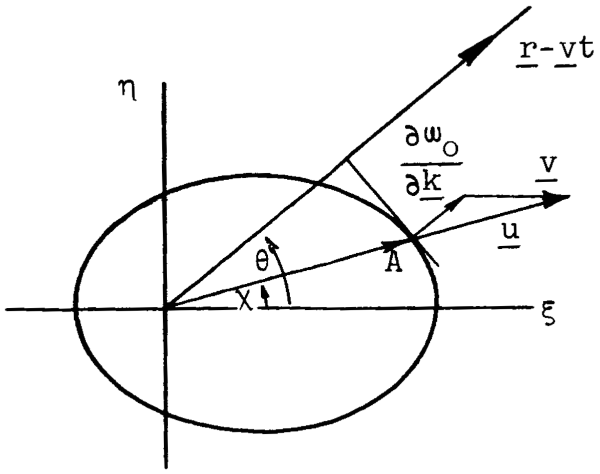


Figure 3a. Cross section of the wave-number surface in any axial plane for the case of an isotropic cold plasma. $\underline{u}$ is parallel to $\underline{k}$ and equal to the vectorial sum of $\partial\omega_o/\partial\underline{k}$ and $\underline{v}$.

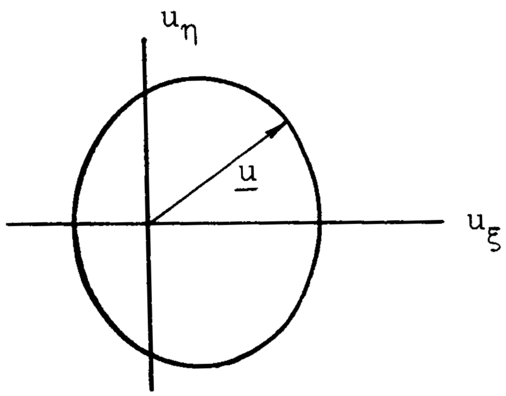


Figure 3b. Cross section of the group-velocity surface in any axial plane for the case of an isotropic cold plasma.

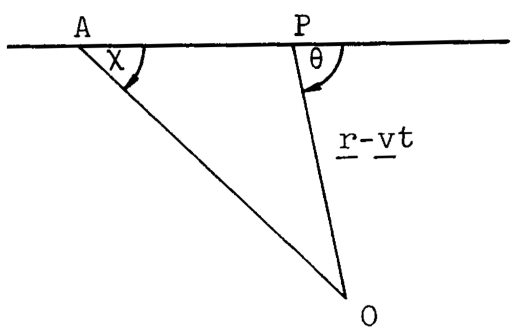


Figure 3c. Refer to Figure 2b for notation. AO = ut. AP = vt.

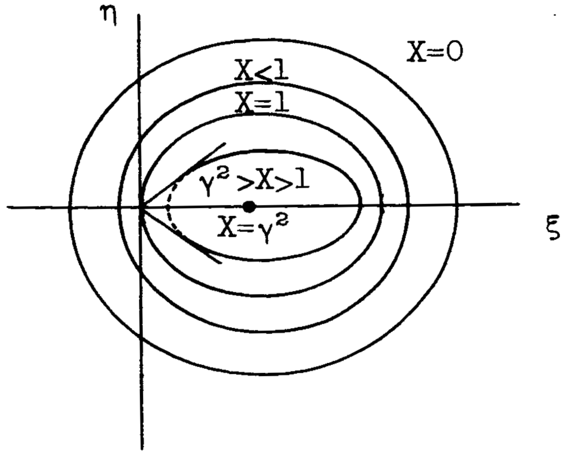
Let us now consider briefly the variation of Figures 3a and 3c with X, keeping v constant. As X increases, the wave-number surface and the group-velocity surface start shifting to the right and at the same time they are decreasing in size. Finally, as X reaches the value γ^2, they shrink to a point and thereafter disappear. The situations are depicted in Figure 4a and 4b. Figure 4c illustrates the variation of the wave frequency with the angle χ and also with X. It shows that the greater X becomes the smaller the cutoff angle χ_c is. At $X = \gamma^2$, i.e., $\omega_o' = \omega_p$, no radiation occurs at any direction. It is interesting to note that in the stationary case the radiation field is also cut off at the source frequency equal to the plasma frequency.

C. CASE OF A DISPERSIVE UNIAXIAL MEDIUM

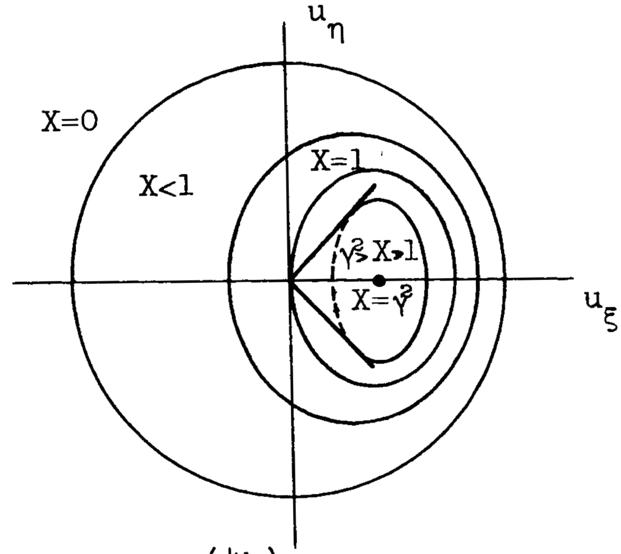
In the two cases discussed above, the Doppler effect is normal in the sense that there is only one peak in the frequency spectrum of the radiation field. The simplest example which will exhibit the phenomenon of complex Doppler effect is when a harmonically oscillating source travels through an ionized gas permeated by a magnetostatic field which is so strong that the gyro-frequency is much larger than the wave frequency and the plasma frequency. Under such a condition the ionized gas behaves as a uniaxial medium and the permittivity tensor is given by

$$\epsilon_{ij} = \epsilon_o \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\omega_p^2/\omega^2 \end{pmatrix} , \qquad (37)$$

where the z-axis has been chosen along the magnetostatic field. Substituting (37) into (12) we obtain the dispersion equation, by setting the determinant V_{ij} equal to zero,



(4a)



(4b)

Figure 4a,b. Variations of cross sections of the wave-number surface and the group-velocity surface with X. The radiation condition is not satisfied by points on that part of curves in broken line.

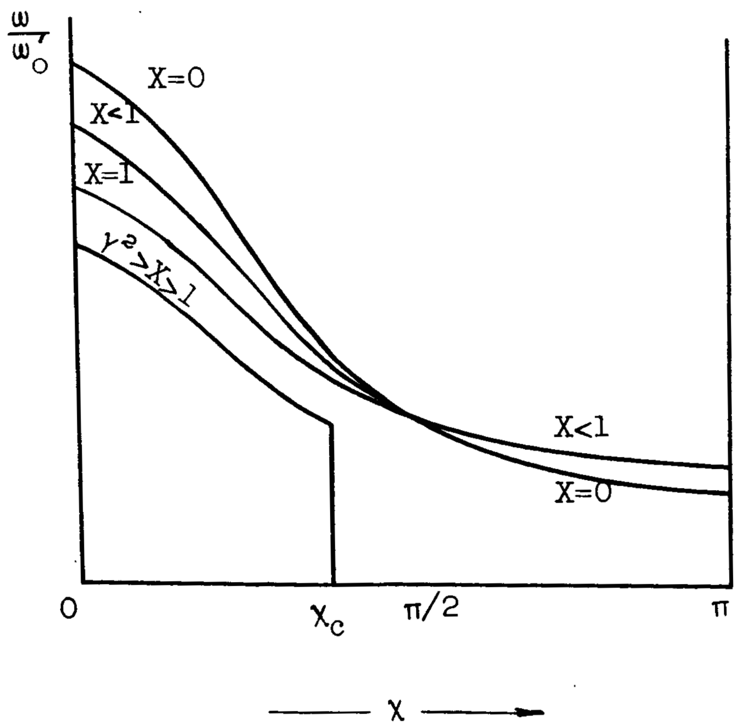


Figure 4c. Doppler-shifted frequency (or the wave frequency) versus χ for different values of X. χ_c is the cutoff angle.

$$\frac{\omega^2}{c^2}\left(a_4\, k^2 - \frac{\omega^2}{c^2}\, a_o\right)\left(k^2 - \frac{\omega^2}{c^2}\right) = 0, \qquad (38)$$

where

$$\omega = \omega_o + \underline{k}\cdot\underline{v}$$

$$a_4 = 1 - \frac{\omega_p^2}{\omega^2}\cos^2\Phi$$

$$a_o = 1 - \frac{\omega_p^2}{\omega^2}$$

Φ = angle between $\underline{k}$ and the z-axis.

Two cases will be studied separately: in case (1) the magnetostatic field is perpendicular to the plane containing the trajectory of the source and the observation point, and in case (2) the field is along the motion of the source.

-1. Field Perpendicular to the Source Motion

Equation (38) is simplified further if $\Phi = 90^o$ and $\underline{v}$ is perpendicular to the z-axis. In this case eq. (38) reduces to

$$\frac{\omega^4}{c^4}\left(k^2 - \frac{\omega^2}{c^2} + k_p^2\right)\left(k^2 - \frac{\omega^2}{c^2}\right) = 0. \qquad (39)$$

The case where $\omega = 0$ corresponds to non-radiated field and thus is of no concern to us. This equation, apart from the factor ω^4/c^4, describes surfaces of two concentric prolate spheroids: one is the wave-number surface of the vacuum case while the other is the wave-number surface of the case of an isotropic cold plasma. Hence, the results in III.A and III.B apply simultaneously to the present case. With the aid of

Figures 5a, 5b, 5c, and 5d we can describe the picture at the site of a distant observer. When the source is approaching, two waves, one being denoted as ordinary and the other extraordinary, arrive at the observer with the ordinary wave at a higher frequency. When the source comes directly above, these two waves coalesce and thereafter they split again into two with the extraordinary wave at a higher frequency. This phenomenon of frequency splitting is called the complex Doppler effect.

To see physically how the phenomenon of complex Doppler effect comes about, we resort to the group-velocity surface (Figure 5c). Since the ordinary wave travels at a group velocity c and the extraordinary wave at a slower speed u ($c > u > v$), these two waves which simultaneously reach the observer have actually been emitted at different time from the moving source in its course of passage through the medium. To state it another way, the moving source emits at any point along its trajectory two types of waves, ordinary and extraordinary. The ordinary wave and the extraordinary one emitted earlier are propagating at different speeds and at different directions, and they arrive at the observer at the same time. The situation is clearly depicted in Figure 5b.

2. Field Along the Source Motion

The phenomenon of complex Doppler effect becomes more pronounced when the oscillating source travels along the direction of the magnetostatic field. Equation (38), apart from the first multiplicative factor ω^2/c^2, now takes the form

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\left[\left(\frac{\omega^2}{c^2} - k_p^2 \cos^2 \Phi\right)k^2 - \frac{\omega^2}{c^2}\left(\frac{\omega^2}{c^2} - k_p^2\right)\right] = 0. \qquad (40)$$

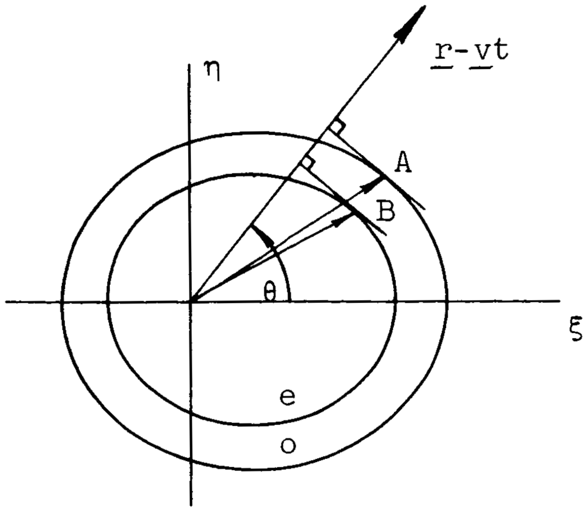


Figure 5a. Cross sections of wave-number surfaces for the perpendicular case of the uniaxial medium with X<1. o denotes the ordinary wave and e the extraordinary wave.

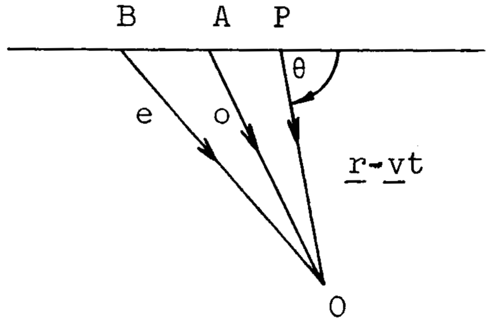


Figure 5b. The present position P and the retarded positions A, B are shown with respect to the distant observer O.

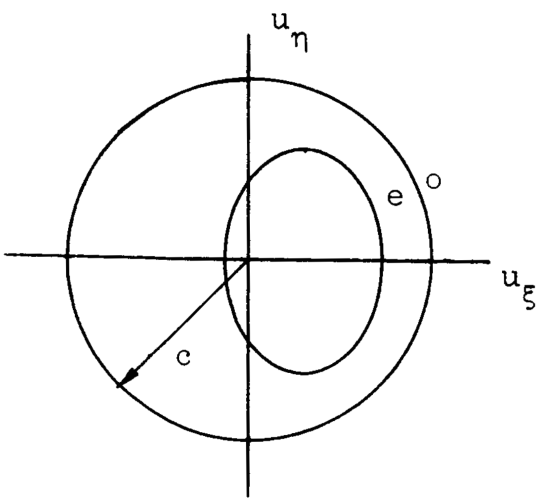


Figure 5c. Cross sections of group-velocity surfaces.

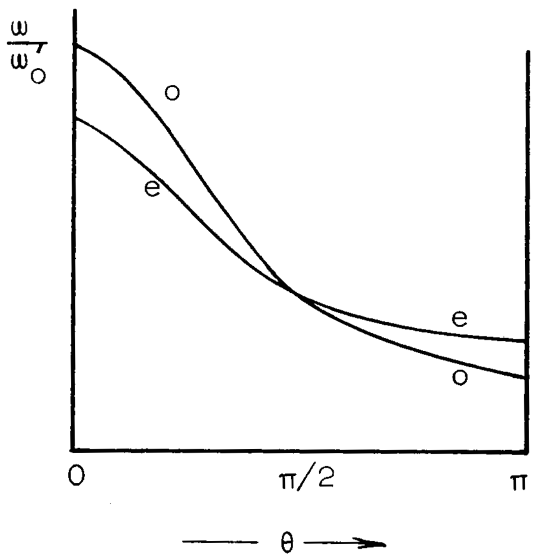


Figure 5d. Variations of wave frequencies with θ.

One of the branches of (40) is given by

$$k^2 - \frac{\omega^2}{c^2} = (1-\beta^2)\xi^2 + \eta^2 + \zeta^2 - 2\beta k_o\xi - k_o^2 = 0 \tag{41}$$

which is just the dispersion equation of the vacuum case. The propagating wave associated with this branch is therefore unaffected by the medium. The other branch of (40) is

$$\left(\frac{\omega^2}{c^2} - k_p^2 \cos^2 \Phi\right)k^2 - \frac{\omega^2}{c^2}\left(\frac{\omega^2}{c'^2} - k_p^2\right)$$

$$= \frac{\omega^2}{c^2}\left(\eta^2 + \zeta^2\right) - \left(\frac{\omega^2}{c'^2} - \xi^2\right)\left(\frac{\omega^2}{c'^2} - k_p^2\right) = 0. \tag{42}$$

This is a fourth order equation in ξ, since $\omega = \omega_o + v\xi$. Solving (42) for $\eta^2 + \zeta^2$ we have

$$\eta^2 + \zeta^2 = \frac{(\omega^2/c^2 - \xi^2)(\omega^2/c'^2 - k_p^2)}{\omega^2/c'^2} . \tag{43}$$

The oval and the open branch in Figure 6a correspond to eq. (43), while the ellipse corresponds to eq. (41). At A, B, C, and D the normals are parallel to $\underline{r}-\underline{v}t$ and $(\underline{r}-\underline{v}t)\cdot\partial\omega_o/\partial\underline{k} > 0$, that is to say, the asymptotic condition and the radiation condition are satisfied at these points. This can be easily seen by constructing the wave-number surface corresponding to $\omega_o + \Delta\omega_o$, as was done in the vacuum case. Thus, for the situation shown in Figure 6a, all these four points contribute to the far field. These four waves, having distinct frequencies ω_A, ω_B, ω_C, and ω_D, propagate to the distant observer at different directions as shown in Figure 6b.

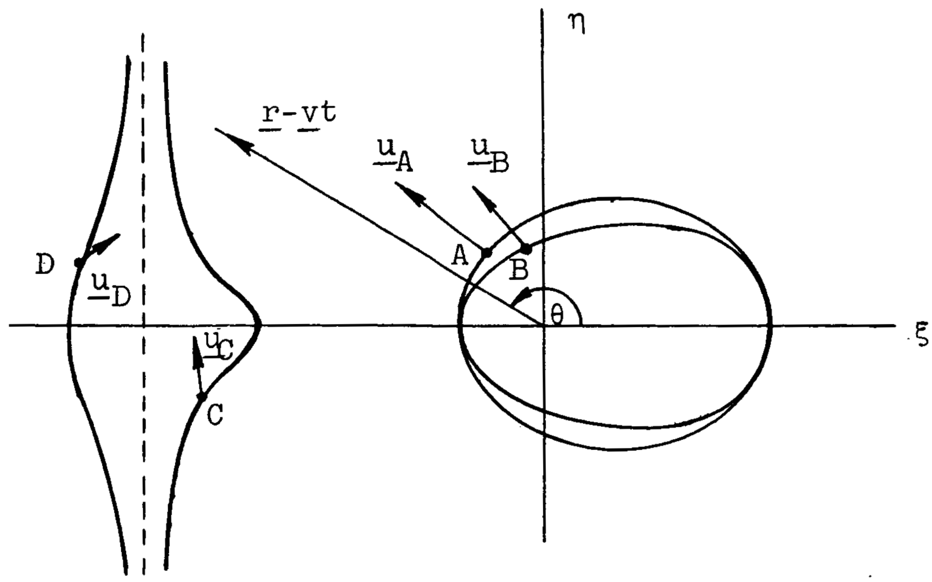


Figure 6a. Cross sections of wave-number surfaces in any axial plane for the parallel case of the uniaxial medium with X<1. A, B, C, D are the only points satisfying the radiation condition and the asymptotic condition as described in section II.B. $\underline{u}_A$, $\underline{u}_B$, $\underline{u}_C$, $\underline{u}_D$ are the corresponding group velocities of the waves.

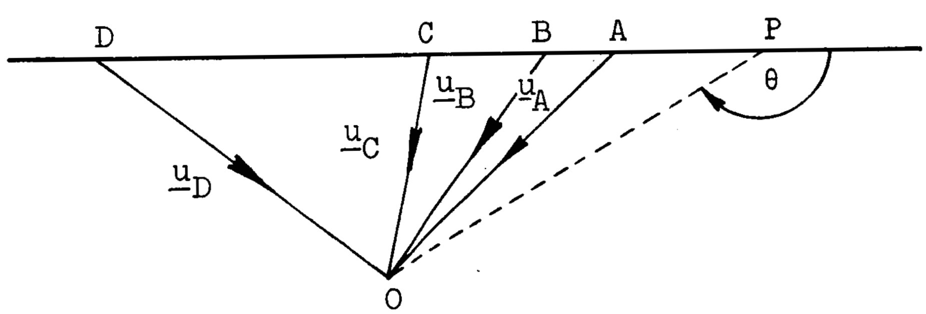


Figure 6b. The corresponding picture of Figure 6a in ordinary space. The present position P of the source and its retarded positions A, B, C, D are shown with the distant observer O.

ω_A is identical to the Doppler-shifted frequency in the vacuum case (section III.A), while ω_B is approximately given by the Doppler-shifted frequency in the case of an isotropic cold plasma (section III.B). The two new wave frequencies ω_C and ω_D can be shown to be always smaller than ω_p, and hence they belong to low frequency components compared to ω_A and ω_B.

In order to gain an understanding of this complex Doppler effect we shall now find the group velocity of each individual wave. The components of the group velocity are found from (43) to be

$$u_\xi = \frac{\partial\omega}{\partial\xi} = \frac{c^2\xi}{\omega}\left(\frac{\omega^2}{c^2} - k_p^2\right)\left(2\frac{\omega^2}{c^2} - \xi^2 - \eta^2 - k_p^2\right)^{-1}$$

$$u_\eta = \frac{\partial\omega}{\partial\eta} = \eta\omega\left(2\frac{\omega^2}{c^2} - \xi^2 - \eta^2 - k_p^2\right)^{-1} \qquad (44)$$

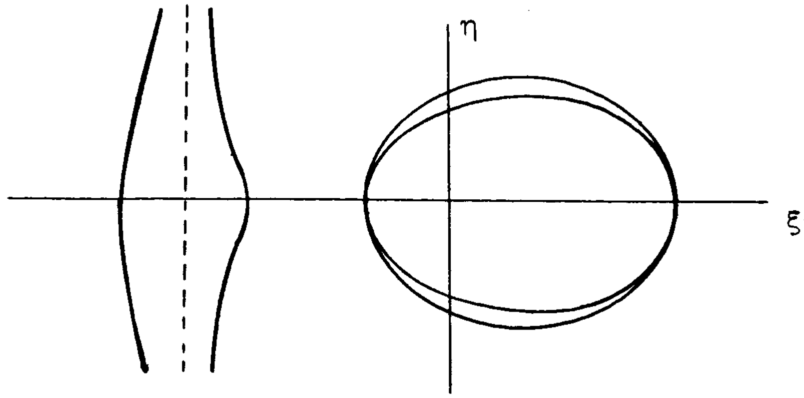
$$u^2 = u_\xi^2 + u_\eta^2 = \omega^2\left(\frac{\omega^2}{c^2} - k_p^2\right)\left(\frac{\omega^4}{c^4} - k_p^2\xi^2\right)^{-1}$$

Here we have suppressed u_ζ and the coordinate ζ because of rotational symmetry. From (44) and (43) we can deduce the following (Figures 6a, b):

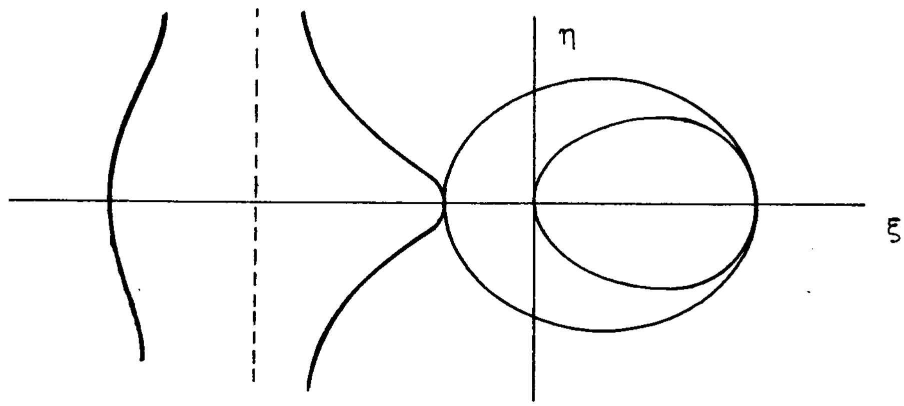
1. at D, $u_\xi>0$, $u_\eta>0$ and $u_D<v$;
2. at C, $u_\xi<0$, $u_\eta>0$ and $u_C \lesseqgtr v$ depending on the relative magnitude of X and β;
3. at B, $u_\xi<0$, $u_\eta>0$ and $v<u_B<c$;
4. at A, $u_\xi<0$, $u_\eta>0$ and $u_A = c$;
5. $u_A>u_B>u_C>u_D$.

With this information on the group velocities of the four waves, we are led to the following explanation for the phenomenon of complex Doppler effect. In traversing an ionized gas permeated by a very strong magnetostatic field, an oscillating source whose motion is along the direction of the field emits at any direction three waves of different group velocities. The "fast" wave and the "medium" wave emitted earlier arrive at the distant observer simultaneously and so will the "medium" wave and the "fast" wave emitted at a later time, while the "slow" wave is still on its long journey to the distant observer. Thus, in the early time when the source is approaching, only two waves appear at the observation point. After some time the "slow" wave will catch up with three other faster waves emitted when the source is receding, and reach the observer at the same time. From the viewpoint of the observer the four waves appear to originate from four different sources located at four different places. This situation is depicted in Figure 6b.

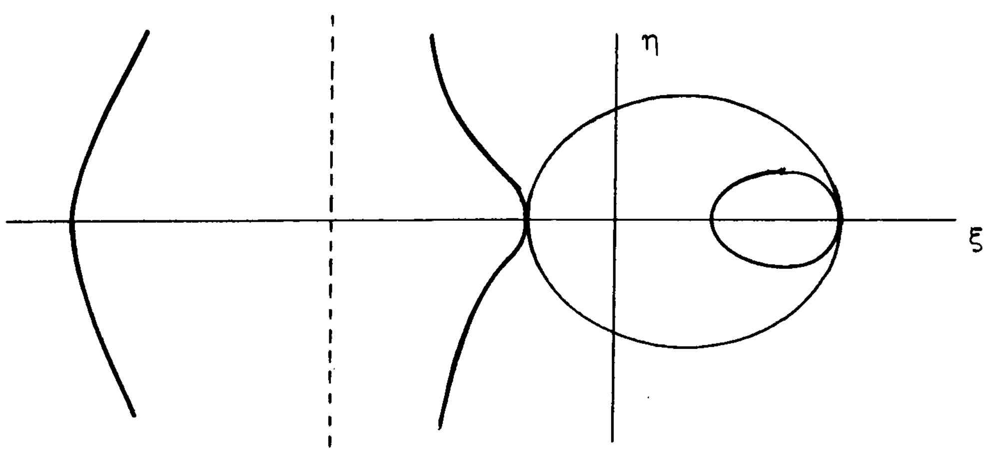
Up to now our discussions have been restricted to the case where $X < 1$. The variation of the wave-number surface with X is sketched in Figure 7 from which one can readily deduce how the complex Doppler effect changes with X.



(a) X < 1



(b) X = 1



(c) X > 1

Figure 7. Variations of Figure 6a with X for fixed k_o and β.

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