COVERED SPACES

by

Joel E. Edelman

TECHNICAL REPORT NO. 26

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION

DEPARTMENT OF MATHEMATICS

KENT STATE UNIVERSITY

KENT, OHIO

August 1967

EDELMAN, JOEL EDWARD, M.A., August 1967   MATHEMATICS
COVERED SPACES (26 pp.)
Director of Thesis: T.N. Bhargava

The object of this thesis is to investigate the extent to which the elementary properties of the basic concepts of topology remain valid in the more general concept of covered spaces. It is shown that virtually all of the properties of the topological operations of closure, interior and derivative are retained despite the weak hypotheses. In contrast to the case for topological spaces, however, it is found that the derivative and closure operations do not distribute over union of sets, and that the interior operation does not distribute over intersection of sets. An additional key difference found is that while complements of open sets are closed, the complement of a closed set in general is not open.

Several of the basic properties of set functions and of the separation axioms are shown to hold true under these more general conditions.

Next we investigate a particular form of disconnectedness for arbitrary subsets in a space. It is found that extreme degrees of connectedness are accompanied by specific topological properties, and in one case the complete topological structure is imposed.

Finally we define a limit point digraph corresponding to a covered space in terms of singleton set limit points. It is shown that a connected digraph can represent only a space which is connected in the particular sense previously mentioned, but that the converse relationship may not necessarily hold.

We also show a one-to-one correspondence between simplicial maps of these digraphs and a class of set functions which, on topological spaces, are homeomorphisms.

Acknowledgments

With sincere appreciation I thank Dr. T. N. Bhargava for his encouragement and guidance throughout the development of this thesis.

I am deeply grateful to Professors P. H. Doyle and Byron H. McCandless for many helpful suggestions which have resulted in an improvement of the thesis.

For financial support, I am indebted to the National Aeronautics and Space Administration through NASA Research Grant Number NsG-568.

Table of Contents

Chapter 0

Preliminaries

Introduction

In this thesis we are mainly concerned with investigating the elementary but basic topological concepts of neighborhoods, limit points, and closure, as applied to a space with a more general structure than that of a topology. Some work has been done in determining the nature of these concepts under general hypotheses, such as by Day [4], Barbălat [2] and Sierpinski [9]. But in all cases known to us, except that of Sierpinski, the authors have retained the properties that provide a space with topological structure. Our purpose here is to conduct our investigation with the bare minimum of hypotheses; such work is essentially due to Doyle [5]. In addition we investigate the nature of the open sets through a generalized definition of connectedness, and the nature of limit points through a directed graph.

Definitions and Notations

We use the usual set theoretic operations and notations as in Dugundji [6]. The complement of a set A relative to a set X is denoted by X-A; if A is a singleton set, {a}, we write its complement as X-a. The more elementary results from set theory are assumed.

Definition 0.1: By a (Doyle) covered space $(X,\mathcal{H})$ is meant an arbitrary set X and a collection $\mathcal{H}$ of subsets of X such that $\bigcup\{H\varepsilon\mathcal{H}\}=X$.

Clearly every topological space $(X,\mathcal{J})$ is a covered space, for the collection $\mathcal{J}$ contains the set X as a member. But obviously there are many covered spaces which are not topological spaces; for example, we may take a non-degenerate set X and take $\mathcal{H}$ to be the collection of singleton sets of elements from X.

A more interesting example of a non-topological covered space is the following space, which henceforth we refer to as the "three point covered space": Let $X = \{a,b,c\}$ and $\mathcal{H} = \{\{a,c\}, \{b,c\}\}$; clearly the union of all sets in $\mathcal{H}$ equals X, and we do not have any of the three topological axioms satisfied.

We note that either the set X or the collection $\mathcal{H}$ may be empty. However, this situation is not permitted in a topological structure, and as such we assume, without any loss of generality, that neither X nor $\mathcal{H}$ is empty.

At this point, we wish to point out a rather close connection between Sierpinski's (V) spaces, and Doyle's covered spaces. Both of these generalize "topology on a set". A covered space $(X,\mathcal{H})$ in general gives rise to many (V) spaces since we may assign to each point x in X a sub-collection of $\mathcal{H}$; this is not uniquely assignable. The natural one is to take all the elements in the covering that contain a fixed element to be the neighborhood system of that element. With this definition each covered space uniquely determines a (V) space. What is natural in the other direction is not necessarily straightforward, and there seem to be several ways of going about it. We only remark that between the two approaches (of Sierpinski and Doyle) any theories based on either that would be a generalization of topological notions should coincide. Probably a (V) space gives rise to more trivia than a covered space, but these vanish through the expedient of equivalence classes.

Definition 0.2: A subset H of X is said to be open if it belongs to the collection $\mathcal{H}$.

Remark: In a (V) space, H is open iff its complement is closed.

In the three point covered space, the set {a,c} is open, but the set {a,b} is not.

Definition 0.3: An open set H is said to be a neighborhood (open neighborhood) of a point x in X if $x \in H$.

Definition 0.4: A point x is said to be a limit point of a subset A of X if every neighborhood of x intersects A-x.

Definition 0.5: The derived set A' of a subset A of X is the set of all limit points of A.

In the three point covered space it can be seen that {a,b} is the derived set of the set A = {a,c}; the open set {a,c} is a neighborhood of x=a, and clearly it intersects the set {{a,c}-a} at the point c. Since this is the only neighborhood of a, it follows that a is a limit point of A. But c is not, for $\{b,c\}$ is a neighborhood of c, but $\{b,c\} \cap \{\{a,c\}-c\}=\emptyset$.

Definition 0.6: A subset F of X is said to be closed if it contains all of its limit points.

Definition 0.7: The closure of an arbitrary subset A, denoted $\overline{A}$, is $A \cup A'$.

Definition 0.8: The interior of an arbitrary subset A, denoted Int A, is $X-\overline{X-A}$.

Definition 0.9: The boundary of an arbitrary subset A, denoted Bd A, is $\overline{A} \cap \overline{X-A}$.

For conciseness we state that the separation axioms, T_0 through T_5, are taken as in Dugundji [6], and so are the definitions for sequences and convergence of sequences.

Chapter I

The Topological Operations

Let $(X,\mathcal{H})$ be an arbitrary covered space. The purpose of this chapter is to investigate these general spaces with respect to some of the topological operations, viz. closure, derived set, interior. We also investigate some of the more elementary properties of continuity and separation axioms as applied to these covered spaces.

All sets considered in this chapter are assumed to be subsets of X, in the context of the covered space $(X,\mathcal{H})$.

1.1 Derived Sets

Lemma 1.1.1: Let $A \subset B$. Then $A' \subset B'$.

Proof: If $x \in A'$, then every neighborhood of x intersects $A-x$. Since $A \subset B$, $A-x \subset B-x$. Thus every set which intersects $A-x$ must intersect $B-x$, so every neighborhood of x intersects $B-x$. Hence $x \in B'$. ∎

Lemma 1.1.2: For any set A, $x \in A'$ iff $x \in \overline{A-x}$.

Proof: If $x \in \overline{A-x}$, then $x \in (A-x) \cup (A-x)'$, or $x \in (A-x)'$. Thus by Lemma 1.1.1, $x \in A'$. If $x \in A'$, every neighborhood of x intersects $A-x$, so every neighborhood of x intersects $(A-x)-x$. Thus $x \in (A-x)'$, and hence $x \in (A-x) \cup (A-x)' = \overline{A-x}$. ∎

Lemma 1.1.3: The closure of every set is closed.

Proof: Let x be a limit point of $\overline{A}$ such that $x \notin \overline{A}$. Let H be any neighborhood of x. Since H intersects $\overline{A}-x$, it must intersect $A'-x$ or $A-x$. Suppose H does not intersect $A-x$. Then H is a neighborhood of some point $y \in A'$, $y \neq x$. But since y is a limit point of A, H must intersect $A-y$, and thus also A itself. Since $x \notin \overline{A}$, $x \notin A$, and thus H intersects $A-x$. This is a contradiction. ∎

Lemma 1.1.4: A set A is closed iff $A = \overline{A}$.

Proof: A is closed iff $A \supset A'$. But $A \supset A'$ iff $A = A \cup A'$, or equivalently $A = \overline{A}$. ∎

Lemma 1.1.5: $\emptyset' = \emptyset$.

Proof: If $x \in \emptyset'$, then every neighborhood of x intersects $\emptyset$. But no set can intersect $\emptyset$. ∎

Lemma 1.1.6: $\overline{\emptyset} = \emptyset$.

Proof: $\overline{\emptyset} = \emptyset \cup \emptyset' = \emptyset \cup \emptyset = \emptyset$. ∎

We now state and prove our main theorem regarding the derived sets in $(X, \mathcal{N})$.

Theorem 1.1.7: The derived set operation $A \to A'$ on subsets of X has the following properties:

(1) for any point $x \in X$, $x \notin \{x\}'$;

(2) $A' \cup B' \subset (A \cup B)'$;

(3) $A'' \subset A \cup A'$;

(4) $\emptyset' = \emptyset$.

Proof: (1) If $x \in \{x\}'$, then $x \in \overline{\{x\}-x}$, or $x \in \overline{\emptyset} = \emptyset$.

(2) Since $A \subset A \cup B$, $A' \subset (A \cup B)'$. Similarly we have $B' \subset (A \cup B)'$. Combining we obtain $A' \cup B' \subset (A \cup B)'$.

(3) $\overline{A} = A \cup A'$, so $(\overline{A})' = (A \cup A')' \supset A' \cup A''$. Now $\overline{A}$ is closed, so $\overline{A} \supset (\overline{A})'$, or $A \cup A' \supset (\overline{A})'$. Combining we have $A \cup A' \supset A'' \cup A'$, and since $A'' \cup A' \supset A''$, we find $A \cup A' \supset A''$.

(4) was proven in lemma 1.1.5. ∎

For covered spaces whose open sets form a topology, the second property of theorem 1.1.7 becomes an equality, and the theorem becomes the following familiar result: any operation on subsets of an arbitrary set X satisfying the four properties of the theorem, with (2) as an equality, determines a unique topology on X, and, for the topology, that operation is the derived set operation. That we do not have equality for covered spaces is illustrated by the three point covered space. Consider the subsets $A = \{a\}$, $B = \{b\}$. Clearly $A' = \emptyset$ and $B' = \emptyset$, so $A' \cup B' = \emptyset$. But $(A \cup B)' = \{a,b\}' = \{c\}$. Hence for covered spaces we cannot expect to have equality.

1.2 Closure

We now proceed similarly to examine the closure operator.

Lemma 1.2.1: Let A be a subset of B. Then $\overline{A} \subset \overline{B}$.

Proof: Since $A \subset B$, $A' \subset B'$, and so $A \cup A' \subset A \cup B'$. Similarly $A \subset B$ implies $A \cup B' \subset B \cup B'$. Thus $A \cup A' \subset B \cup B'$, or $\overline{A} \subset \overline{B}$. ∎

We now state and prove our main theorem regarding the closure operation in $(X, \mathcal{H})$.

Theorem 1.2.2: The closure operation $A \to \overline{A}$ on subsets of X has the following properties:

(1) $A \subset \overline{A}$;

(2) $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$;

(3) $\overline{\emptyset} = \emptyset$.

(4) $\overline{\overline{A}} = \overline{A}$.

Proof: (1) Clearly $A \subset A \cup A' = \overline{A}$.

(2) Since $A \subset A \cup B$, $\overline{A} \subset \overline{A \cup B}$. Similarly we find $\overline{B} \subset \overline{A \cup B}$. Combining we have $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

(3) was proven in lemma 1.1.6.

(4) follows immediately from the fact that $\overline{A}$ is a closed set and lemma 1.1.4. ∎

For covered spaces whose open sets form a topology, the second property of theorem 1.2.2 becomes an equality, with the result being Kuratowski's closure axioms and a uniqueness theorem similar to the one previously mentioned. That we do not have equality for covered spaces is illustrated again by the three point covered space, using the same subsets A and B. For both A and B, the set equals its closure, but the closure of $A \cup B$ is the whole set, X.

1.3 Interior Operation

Finally we consider the properties of the interior operation.

Lemma 1.3.1: Let A be a subset of B. Then Int A is a subset of Int B.

Proof: If $A \subset B$, then $X-B \subset X-A$. So $\overline{X-B} \subset \overline{X-A}$. Hence $X-\overline{X-A} \subset X-\overline{X-B}$, or Int $A \subset$ Int B. ∎

We now state and prove our main theorem regarding the interior operation in $(X,\mathcal{H})$.

Theorem 1.3.2: The interior operation $A \to \text{Int } A$ on subsets of X has the following properties:

(1) $\text{Int } A \subset A$;

(2) $\text{Int } A \cap \text{Int } B \supset \text{Int } A \cap B$;

(3) $\text{Int } X = X$;

(4) $\text{Int } (\text{Int } A) = \text{Int } A$;

(5) $X-A$ is closed iff $\text{Int } A = A$.

Proof: (1) $X-A \subset \overline{X-A}$, so $A \supset X-\overline{X-A} = \text{Int } A$.

(2) Now $A \cap B \subset A$, so $\text{Int } A \cap B \subset \text{Int } A$. Similarly $\text{Int } A \cap B \subset \text{Int } B$. Combining we have $\text{Int } A \cap B \subset \text{Int } A \cap \text{Int } B$.

(3) $\text{Int } X = X-\overline{X-X} = X-\overline{\emptyset} = X-\emptyset = X$.

(4) $\text{Int } (\text{Int } A) = X-\overline{X-\text{Int } A} = X-\overline{X-(X-\overline{X-A})} = X-(\overline{\overline{X-A}}) = X-\overline{X-A}$ $= \text{Int } A$, since $\overline{\overline{X-A}} = \overline{X-A}$ by theorem 1.2.2(4).

(5) $X-A$ is closed iff $X-A = \overline{X-A}$, or equivalently $A = X-\overline{X-A}$. But this means $A = \text{Int } A$. ∎

Provided the second property above is an equality, any operation with these properties would, as before, determine a unique topology for which the given operation would be the interior operation. And although equality can be proved for covered spaces whose open sets form a topology, inclusion is the best result possible without that structure. As an example we again mention the three point covered space used previously, taking $A = \{a,c\}$ and $B = \{b,c\}$. Clearly Int A

= A and Int B = B, so Int $A \cap$ Int B is not empty. But $\text{Int}(A \cap B)$ = Int {c} which is empty. Hence we cannot generally expect to have equality.

1.4 Some Additional Results

Before examining these operations in greater depth, we present two further results of interest.

Theorem 1.4.1: The closure of any set is the union of the interior and the boundary of that set.

Proof: $\overline{A} = (\overline{A} \cap (X-\overline{X-A})) \cup (\overline{A} \cap (\overline{X-A})) = \text{Bd } A \cup (\overline{A} \cap \text{Int } A) = \text{Bd } A \cup \text{Int } A.$ ▮

Theorem 1.4.2: The complement of the interior of a set is closed.

Proof: $X-\text{Int } A = \overline{X-A}$, which is closed. ▮

1.5 The Open and Closed Sets

We now state and prove for closure and interior two equivalent formulations, which are known for topological spaces. Following usual procedures, the union of an empty collection of sets is empty, and the intersection of an empty collection of sets is the whole set, X.

Theorem 1.5.1: Let A be a subset of X. Then

(1) $\overline{A} = \bigcap \{F \mid F \text{ closed} \supset A\}$;

(2) $\text{Int } A = \bigcup \{H \mid H \text{ open} \subset A\}$.

Proof: (1) Let $S = \bigcap \{F \mid F \text{ closed} \supset A\}$. Since $\overline{A}$ is closed, $\overline{A} \supset \bigcap F$, or $\overline{A} \supset S$. Now let F be a closed set containing A.

Then $\overline{F} \supset \overline{A}$. But F is closed, so $\overline{F} = F$, and $F \supset \overline{A}$. Since this is true for every closed set F containing A, we have $S \supset \overline{A}$. Thus $S = \overline{A}$.

(2) Now let $S = \bigcup\{H|H \text{ open} \subset A\}$. Let $x \in S$. Then $x \in H$ for some open $H \subset A$. So $x \notin X-A$. And since H is a neighborhood of x which does not intersect X-A, we have $x \notin (X-A)'$. Thus $x \notin \overline{X-A}$, and hence $x \in X-\overline{X-A}$, or $x \in \text{Int } A$. Now let $x \in \text{Int } A$. Then $x \notin \overline{X-A}$. So $x \notin X-A$ and $x \notin (X-A)'$. Since x is not a limit point of X-A, there exists some neighborhood H such that $H \cap ((X-A)-x) = \emptyset$. But since $x \notin X-A$, we may write $H \cap (X-A) = \emptyset$. This means $H \subset A$. Thus $x \in S$. Hence $S = \text{Int } A$. ▮

<u>Theorem 1.5.2</u>: The intersection of any collection of closed sets is closed.

<u>Proof</u>: Let S be the intersection of an arbitrary collection of closed sets. If the collection is empty, S = X, which is a closed set. Also if $S = \emptyset$, then it is closed. Now let x be a limit point of S. Every neighborhood of x intersects S. Thus if F is any member of the collection of closed sets, every neighborhood of x also intersects F, since $F \supset S$. Thus $x \in F'$, and so $x \in F$, since F is closed. But $x \in F$ for every closed set of the collection, by the above reasoning. Hence x belongs to the intersection, and thus to S. Hence S is closed. ▮

Thus we may characterize the closure of a set as the smallest closed set containing it.

Theorem 1.5.3: Int A is the largest set contained in A whose complement is closed.

Proof: Let S be a set such that $S \subset A$, and X-S is closed. Since $S \subset A$, we have $X-S \supset X-A$. Therefore $\overline{X-S} \supset \overline{X-A}$. But since X-S is closed, we know $\overline{X-S} = X-S$, and thus $X-S \supset \overline{X-A}$. Taking complements we find that $S \subset X-\overline{X-A}$, or equivalently, $S \subset \text{Int } A$. ∎

Theorem 1.5.4: A set is closed iff its complement is a union of open sets.

Proof: Let F be any set for which X-F is a union of open sets. Let $x \in F'$ and suppose $x \notin F$. Then $x \in X-F$. Now x must be in some open set H. But then H is a neighborhood of x which does not intersect F. So x cannot belong to F'. This is a contradiction.

Now suppose F is closed. Let $x \in X-F$. Since $x \notin F$, x is not a limit point of F, so there exists a neighborhood H of x which does not intersect F. Let S be the union of all such neighborhoods for the points of X-F. Clearly $S \supset X-F$. But since none of the neighborhoods intersects F, $S \cap F = \emptyset$. Thus $S \subset X-F$. Hence we conclude X-F = S, which is a union of open sets. ∎

Corollary 1.5.5: The complement of every open set is closed.

Corollary 1.5.6: Let H be an open set, and A any subset of X. Then if $H \cap A$ is empty, $H \cap \overline{A}$ is empty.

Proof: If $A \cap H = \emptyset$, then $A \subset X-H$. Thus $\overline{A} \subset \overline{X-H}$. But since H is open X-H is closed, and $\overline{X-H} = X-H$. Thus $\overline{A} \subset X-H$, and so $H \cap \overline{A} = \emptyset$. ∎

1.6 Continuity and the Separation Axioms

We now examine a notion of continuity, and the separation axioms. The usual properties of set functions are assumed.

Definition 1.6.1: A function $f:X\to Y$ is continuous if B open in Y implies $X-f^{-1}(B) = f^{-1}(Y-B)$ is closed in X.

Theorem 1.6.1: The following conditions on a transformation $f:X\to Y$ are equivalent, where X and Y are covered spaces:

(1) f is continuous;

(2) If B is closed in Y, then $f^{-1}(B)$ is closed in X;

(3) $\overline{f(A)}\supset f(\overline{A})$ for all $A\subset X$;

(4) $f^{-1}(\overline{B})\supset \overline{f^{-1}(B)}$ for all $B\subset Y$;

(5) If $x \in X$ and V is a neighborhood of $f(x)$ in Y, then there is a neighborhood U of x in X such that $f(U)\subset V$.

Proof: (1)$\Leftrightarrow$(2): If B is closed in Y, then Y-B is a union of open sets, $Y-B = \bigcup H_\alpha$. For each $H_\alpha, f^{-1}(Y-H_\alpha) = X-f^{-1}(H_\alpha)$ is closed since f is continuous. Thus $\bigcap(X-f^{-1}(H_\alpha))$ is closed. Hence $X-\bigcup f^{-1}(H_\alpha) = X-f^{-1}(\bigcup H_\alpha)$ is closed, or $X-f^{-1}(Y-B) = f^{-1}(B)$ is closed.

Now if B is open in Y, Y-B is closed in Y. Thus $f^{-1}(Y-B) = X-f^{-1}(B)$ is closed in X.

(2)$\Rightarrow$(3): Let A be any subset of X. Since $\overline{f(A)}$ is closed in Y, $f^{-1}(\overline{f(A)})$ is closed in X, by our hypothesis. Now $f^{-1}(\overline{f(A)})\supset f^{-1}(f(A))\supset A$, so $f^{-1}(\overline{f(A)})\supset\overline{A}$. Hence $\overline{f(A)}\supset f(\overline{A})$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $f^{-1}(B) \subset X$, so $\overline{ff^{-1}(B)} \supset f(\overline{f^{-1}(B)})$ by hypothesis. Now $B \supset ff^{-1}(B)$, and thus $\overline{B} \supset \overline{ff^{-1}(B)}$, which implies that $\overline{B} \supset f(\overline{f^{-1}(B)})$ and thus that $f^{-1}(\overline{B}) \supset \overline{f^{-1}(B)}$.

(4) $\Rightarrow$ (5): If $x \in X$ and V is a neighborhood of f(x) in Y, then Y-V is closed. Thus $\overline{Y-V} = Y-V$, so $f^{-1}(Y-V)$ $= f^{-1}(\overline{Y-V}) \supset \overline{f^{-1}(Y-V)}$. Hence $f^{-1}(Y-V)$ is closed, or equivalently $X-f^{-1}(V)$ is closed. Thus $f^{-1}(V)$ is the union of open sets of $(X,\mathcal{H})$. Now $f(x) \in V$ implies $x \in f^{-1}(V)$, so there must be some particular open set H which contains x. Now $H \subset f^{-1}(V)$, so $f(H) \subset V$.

(5) $\Rightarrow$ (2): Let B be closed in Y. Then Y-B is a union of open sets of Y. Now for all $x \in f^{-1}(Y-B)$, $f(x) \in Y-B$, and hence f(x) belongs to some open set H from the union. Therefore $f(x) \in H \subset Y-B$. H is a neighborhood of f(x), so there exists a neighborhood A of x in X such that $f(A) \subset H$. Thus $A \subset f^{-1}(H) \subset f^{-1}(Y-B)$. Hence for all $x \in f^{-1}(Y-B)$ there exists a neighborhood A of x in X such that $A \subset f^{-1}(Y-B)$. Let S be the union of all such A's for every $x \in f^{-1}(Y-B)$. Since each A is contained in $f^{-1}(Y-B)$, $S \subset f^{-1}(Y-B)$. But since every x is covered, $S \supset f^{-1}(Y-B)$. Thus $f^{-1}(Y-B)$ is a union of open sets of X. Hence $X-f^{-1}(Y-B)$ is closed, or equivalently, $X-(X-f^{-1}(B))$ is closed. Thus $f^{-1}(B)$ is a closed set in X. ∎

<u>Theorem 1.6.2:</u> Suppose B open in Y implies $f^{-1}(B)$ is

open in X. Then if B is closed in Y, $f^{-1}(B)$ is closed in X.

Proof: Let B be closed in Y. Then Y-B is a union of open sets, and by hypothesis the inverse of each of these is open in X. So the inverse of their union is a union of open sets: $f^{-1}(Y-B) = X-f^{-1}(B)$ is a union of open sets. Hence $f^{-1}(B)$ is closed in X. ▮

In other words, preservation of "openness" is sufficient to insure that we have all of the five equivalent conditions of theorem 1.6.1, but none of the five conditions guarantees preservation of"openness". An attempt to prove this for, say, the second condition of theorem 1.6.1 leads directly to the situation we must construct to provide a counterexample. For if we let B be open in Y, Y-B would be closed, and thus $f^{-1}(Y-B) = X-f^{-1}(B)$ is closed. But this only assures us that $f^{-1}(B)$ is a union of open sets and not necessarily itself an open set. Construction of such a situation is straightforward.

Theorem 1.6.3: A covered space $(X,\mathcal{H})$ is T_1 iff every point of X is closed.

Proof: Let x be a point of X which is not closed. Then there is some $y \neq x$ which is a limit point of x. Hence every neighborhood of y contains x, so $(X,\mathcal{H})$ is not T_1.

If $(X,\mathcal{H})$ is not T_1, there is a pair of distinct points, x and y, such that, say, every neighborhood of y contains x. Thus y is a limit point of x, and hence $\{x\}$ is not closed. ▮

<u>Theorem 1.6.4</u>: Let $(X,\mathcal{N})$ be a T_1 covered space. Then $T_5 \Rightarrow T_4$, $T_4 \Rightarrow T_3$, $T_3 \Rightarrow T_2$, $T_2 \Rightarrow T_1$, and $T_1 \Rightarrow T_0$.

<u>Proof</u>: $T_5 \Rightarrow T_4$: Let A, B be closed and disjoint. $\overline{A} \cap B = A \cap B = \phi$, and $A \cap \overline{B} = A \cap B = \phi$. Hence $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \phi$, and T_4 follows immediately.

$T_4 \Rightarrow T_3$: Let F be closed, $x \notin F$. Since $(X,\mathcal{N})$ is T_1, $\{x\}$ is closed, and T_3 follows using T_4.

$T_3 \Rightarrow T_2$: Let x and y be distinct points in X. Since $(X,\mathcal{N})$ is T_1, $\{x\}$ is closed, and since $y \notin \{x\}$, T_2 follows using T_3.

$T_2 \Rightarrow T_1 \Rightarrow T_0$: The proof of these is trivial. ▮

<u>Theorem 1.6.5</u>: In a Hausdorff covered space convergence is unique.

<u>Proof</u>: Let $x_n \to x$ and $x_n \to y$, $y \neq x$. There are neighborhoods U and V of x and y respectively such that $U \cap V = \phi$. Now $x_n \to x$ implies almost all the x_n's are in U and thus not in V. But $x_n \to y$ means almost all the x_n's are in V. This is a contradiction. ▮

Chapter II

Connectedness and Limit Points

In this chapter we investigate the nature of $\mathcal{H}$-disconnected sets and some properties of the limit point digraph, particularly with regard to generalized homeomorphisms. The usual definitions and elementary properties from graph theory are assumed as in Ore [8].

2.1 $\mathcal{H}$-connected Sets

Definition 2.1.1: A nonempty subset A of X is said to be $\mathcal{H}$-disconnected if there exist two open sets U and V such that $A \subset U \cup V$ and $U \cap A \neq \phi$, $V \cap A \neq \phi$, but $(U \cap A) \cap (V \cap A) = \phi$. Otherwise A is said to be $\mathcal{H}$-connected.

Theorem 2.1.1: A nonempty subset A is $\mathcal{H}$-disconnected iff it is of the form $S_1 \cup S_2$ where $S_1 \subset U-V$, $S_2 \subset V-U$, U and V are open sets of the space, and S_1 and S_2 are nonempty.

Proof: Given the conditions it is clear that A is $\mathcal{H}$-disconnected. Now let A be any $\mathcal{H}$-disconnected set. Then there exist open sets U and V such that $A \subset U \cup V$, $U \cap A \neq \phi$, $V \cap A \neq \phi$, and $(U \cap A) \cap (V \cap A) = \phi$. Since $A \subset U \cup V$, we can write $A \subset (U-V) \cup (V-U) \cup (U \cap V)$ and thus $A = (A \cap (U-V)) \cup (A \cap (V-U)) \cup (A \cap (U \cap V))$. But $A \cap (U \cap V) = \phi$. Thus $A = S_1 \cup S_2$, where

$S_1 = A \cap (U-V)$ and $S_2 = A \cap (V-U)$, and clearly the conditions above are satisfied. ∎

Theorem 2.1.2: Every subset of X is $\mathcal{H}$-connected iff the open sets of $(X,\mathcal{H})$ are linearly ordered by inclusion.

Proof: It is clear that all subsets in a covered space whose open sets are linearly ordered by inclusion are $\mathcal{H}$-connected, for if $A \subset U \cup V$, then either $A \subset U$ or $A \subset V$, since $U \cup V$ must be equal to either U or V. Now suppose we do not have linear order by inclusion. Then there exist two open sets U and V neither of which is a subset of the other. Thus U-V and V-U are not empty. Select x from U-V and y from V-U. Then $\{x,y\}$ is an $\mathcal{H}$-disconnected subset using the open sets U and V. ∎

We note that if in such a covered space the empty set is open, then the open sets of the space form a topology.

Theorem 2.1.3: X has an $\mathcal{H}$-disconnected subset iff it contains an $\mathcal{H}$-disconnected point pair.

Proof: An $\mathcal{H}$-disconnected point pair is of course an $\mathcal{H}$-disconnected subset. Suppose A is an $\mathcal{H}$-disconnected set. Let U and V be two open sets which satisfy the requirements. Since neither of $U \cap A$ or $V \cap A$ is empty we may select, say, x from $U \cap A$ and y from $V \cap A$. Then clearly x and y form an $\mathcal{H}$-disconnected point pair. ∎

Theorem 2.1.4: If a subset A is $\mathcal{H}$-connected, and $A \subset B \subset \overline{A}$, then B is $\mathcal{H}$-connected.

Proof: Suppose B is $\mathcal{H}$-disconnected: $B \subset U \cup V$ where U,V are open, $B \cap U \neq \phi$, $B \cap V \neq \phi$, and $(U \cap B) \cap (V \cap B) = \phi$. Then since $A \subset B$, we have $(U \cap A) \cap (V \cap A) = \phi$, which is a contradiction unless either A is not contained in $U \cup V$, or $U \cap A = \phi$ $(V \cap A = \phi)$. But $A \subset B \subset U \cup V$. Now suppose $U \cap A = \phi$. Then $U \cap \overline{A} = \phi$ by corollary 1.5.6, and since $B \subset \overline{A}$, $U \cap B = \phi$. But by hypothesis this is not so. Hence $U \cap A$ cannot be empty. Similarly $V \cap A \neq \phi$. Thus we have the contradiction. ▮

Corollary 2.1.5: If A is $\mathcal{H}$-connected, then $\overline{A}$ is $\mathcal{H}$-connected.

Corollary 2.1.6: If y is a limit point of x, then $\{x,y\}$ is $\mathcal{H}$-connected.

Corollary 2.1.7: If the point pair $\{x,y\}$ is $\mathcal{H}$-disconnected, then x (y) cannot be a limit point of $\{x,y\}$, and thus x (y) cannot be a limit point of the singleton set $\{y\}$ ($\{x\}$).

Theorem 2.1.8: All point pairs of a covered space are $\mathcal{H}$-disconnected iff the space is T_1.

Proof: If $(X,\mathcal{H})$ is T , for distinct points x and y there are open neighborhoods U and V respectively such that $y \notin U$ and $x \notin V$. Thus U and V $\mathcal{H}$-disconnect x and y.

Now let x and y be any two distinct points. If they are $\mathcal{H}$-disconnected, there exist open sets U and V such that $\{x,y\} \subset U \cup V$, $\{x,y\} \cap U \neq \phi$, $\{x,y\} \cap V \neq \phi$, and $(U \cap \{x,y\}) \cap (V \cap \{x,y\}) = \phi$. Now $U \cap \{x,y\} \neq \phi$. Suppose $x \in U$. If $y \in U$, we have $(U \cap \{x,y\}) \cap (V \cap \{x,y\}) = \{x,y\} \cap V \cap \{x,y\} = \{x,y\} \cap V \neq \phi$. This is a contradiction, so it must be that

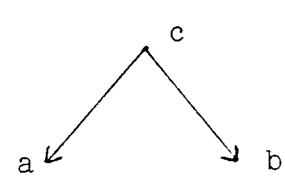
$y \notin U$. Therefore $y \in V$, and similarly we find that $x \notin V$. Thus $(X,\mathcal{H})$ is T_1. ▮

Clearly the covered space $(X,\mathcal{H})$, where $X = \{a,b\}$ and $\mathcal{H} = \{\{a\}, \{a,b\}\}$, is T_0. But we know this covered space has no $\mathcal{H}$-disconnected subsets. Hence there exists a T_0 covered space with no $\mathcal{H}$-disconnected subsets. Now every T_1 covered space is also T_0, so there exist T_0 covered spaces for which every point pair is $\mathcal{H}$-disconnected. Thus under the T_0 condition both extremes can occur.

2.2 The Limit Point Digraph

Definition 2.2.1: The limit point digraph $L(X)$ corresponding to the covered space $(X,\mathcal{H})$ is the ordered pair (X,E), where E is the subset of $X \times X$ determined as follows: the ordered pair $(x,y) \in E$ iff, in the context of the covered space, the point y is a limit point of the singleton set $\{x\}$.

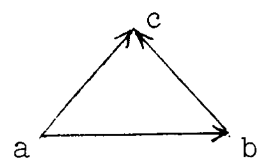
For example the digraph corresponding to the three point covered space from Chapter I contains the diedges (c,a) and (c,b).



A space covered entirely by singleton sets is represented by a digraph which contains no diedges, and the space

(X, {X}) on a nonempty set of n points is represented by the complete digraph on n points.

As a further example, the digraph below represents the covering of three points by the collection $\mathcal{H}$ = {{a}, {a,b}, {a,b,c}}.



Theorem 2.2.1: If (x,y) and (y,z) both belong to E, then (x,z) also belongs to E.

Proof: Let (x,y) and (y,z) belong to E. Then every neighbor hood of y contains x. But every neighborhood of z contains y. Thus every neighborhood of z also contains x, and hence (x,z) belongs to E.

We note that since $x \notin \{x\}'$, L(X) contains no loops. It now follows that every minimal closed path of L(X) contains either three diedges or an even number of diedges. ▮

Theorem 2.2.2: If {x} is open, then in L(X) the vertex x has no adjacent incoming diedges. Furthermore {x} is closed iff the vertex x has no outgoing diedges.

Proof: Suppose $(x,y) \in E$. Then every neighborhood of y contains x. Thus {y} is not open, for it would be a neighborhood of y not containing x.

Now suppose x is a closed point. Then {x} contains all its limit points. But x is the only point in {x}, and x is not a limit point of {x}, so $\{x\}' = \phi$. Thus x has no outgoing diedges.

If x has no outgoing diedges, {x} has no limit points, and thus it is closed. ▌

That the above is the best possible result is illustrated by the three point covered space, for the vertex c has no incoming diedges, but the set {c}is not open.

If the open sets of a covered space form a topology, then the converse of the statement for open sets holds, for the intersection of all open sets containing such a point is open since X is finite, and since every other point is excluded by at least one open set, the intersection must be the point itself.

Corollary 2.2.3: A covered space $(X,\mathcal{H})$ is T_1 iff L(X) contains no diedges.

Definition 2.2.2: A digraph is (weakly) connected if for every pair of vertices there exists a path between them, disregarding the directions of the diedges.

Theorem 2.2.4: If L(X) is connected, then X is $\mathcal{H}$-connected.

Proof: Suppose X is not $\mathcal{H}$-connected. Let $X = U \cup V$, $U \neq \phi$ and $V \neq \phi$, with $(X \cap U) \cap (X \cap V) = \phi$. Let x be any point in $X \cap U = U$, and y any point of $X \cap V = V$. Then clearly $x \notin \{y\}'$ and $y \notin \{x\}'$. Thus there cannot be a diedge between x and y in L(X). But this means none of the vertices associated with the points of $X \cap U = U$ can be connected to any of the vertices associated with the points of $X \cap V = V$. Thus L(X) cannot be connected. ∎

If a finite covered space is a topological space, we can conclude that if X is $\mathcal{H}$-connected then L(X) is connected. This can be seen as an immediate corollary to theorem 2.2.5, which is stated and proved subsequently. That this is not generally true is illustrated by any covered space of at least three points for which $\mathcal{H}$ consists only of singleton sets.

We remark that some results similar to ours, but in reference to topology on digraphs, have been obtained by Bhargava and Ahlborn [3].

Definition 2.2.3: By a homeomorphism of a space $(X,\mathcal{H})$ is meant a biunique mapping f:X onto X such that both f and f^{-1} satisfy the properties of theorem 1.6.1.

We now observe a connection between homeomorphisms of a space $(X,\mathcal{H})$ and simplicial maps of the digraph L(X). The class of covered spaces with which we are concerned has the following property, which we denote henceforth as property *: a point of X is a limit point of a subset of X iff it is a limit point of some point of the subset. As an example of such a space, let X = {a,b,c,d} and $\mathcal{H}$ = {{a,b},{c,d}}. We now mention a more interesting collection of spaces with this property.

Theorem 2.2.5: Every finite topological space has property *.

Proof: Let x be a limit point of a subset A. Let S be the intersection of all neighborhoods of x. Since X is finite, S is open, and hence S intersects A-x. Since every neighborhood of x contains S, x is a limit point of every point in $S \cap (A-x)$. ∎

We now state and prove our main result concerning the limit point digraph.

Theorem 2.2.6: Let (X,) be a covered space with property *. To every simplicial map s:L(X) onto L(X) there corresponds a unique homeomorphism of $(X,\mathcal{H})$, and conversely.

Proof: Let s be a simplicial map of L(X) onto L(X). This function naturally induces a function h of X onto X by h(x) = s(x) for all x ε X. We show that both h and h^{-1} satisfy the properties of theorem 1.6.1.

Let $A \subset X$. We show that $h(\overline{A}) \subset \overline{h(A)}$. Let x ε A'. Then there is some point a ε A such that x ε {a}'. This means (a,x) ε L(X), so then (s(a),s(x)) ε s(L(X)). Thus h(x) ε {h(a)}', and hence h(x) ε h(A)'. Thus h(x) ε $\overline{h(A)}$. So $h(\overline{A}) \subset \overline{h(A)}$.

By a similar argument we conclude $h^{-1}(\overline{A}) \subset \overline{h^{-1}(A)}$.

To prove the converse we show that homeomorphisms preserve limit points in the covered space. Suppose x is not a limit point of y. Then there exists an open set H such that x ε H and y ∉ H. Thus y ε X-H. Hence we have h(x) ε h(H) and h(y) ε h(X-H) = X-h(H), which is a closed set since h is a homeomorphism. Thus h(H) is a union of open sets, one of which must contain h(x). And since the union of these open sets does not contain h(y), no single one can, and thus h(x) is not a limit point of h(y).

By a similar argument we conclude that if h(x) is not a limit point of h(y), then $h^{-1}h(x) = x$ is not a limit point of $h^{-1}h(y) = y$. ∎

References

[1] Aull, C. E. and Thron, W. J., Separation Axioms between T_0 and T_1, Indagationes Math. 24, 26-37 (1962).

[2] Barbălat, B., Sur les Espaces Topologiques les plus Généraux, Bull. Math. Soc. Roumaine Sci. 45, 103-112 (1943).

[3] Bhargava, T. N. and Ahlborn, T. J., On Topological Spaces Associated with Digraphs, To appear in Acta Mathematica.

[4] Day, Mahlon M., Convergence, Closure, and Neighborhoods, Duke Math. J. 11, 181-199 (1944).

[5] Doyle, P. H., Lecture notes, Unpublished.

[6] Dugundji, James, Topology, Allyn and Bacon, Boston, 1966.

[7] Kelly, J. L., General Topology, Van Nostrand, Princeton, N. J., 1955.

[8] Ore, O., Theory of Graphs, Amer. Math. Soc. Coll. Publ., 38, Providence, 1962.

[9] Sierpinski, W., General Topology, University of Toronto Press, Toronto, 1956.