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INTERACTION OF A RING-REINFORCED SHELL AND A FLUID MEDIUM

by

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SUMMARY

This work is concerned with the transient dynamic response of a periodically ring-reinforced, infinitely long, circular cylindrical shell to a uniform pressure suddenly applied through the surrounding acoustic medium. The incident particle velocity is zero and the rings are assumed to be slightly flexible.

A classical theory of the Donnell type is used to analyze the shell while the fluid is described by the linear acoustic field equation.

The solution is obtained by assuming a power series expansion in the ring stiffness parameter and utilizing a technique which reduces the transient dynamic problem to an equivalent steady-state formulation.

Numerical results are presented for a steel shell immersed in salt water for different ring spacings.

For the case of rigid rings, a cylindrical and plane wave approximation was also used to represent the fluid field. It is shown that the cylindrical wave approximation yields reasonably accurate results.

Flexible ring results, although limited, indicate that undamped or nonradiating components of the shell vibration are activated.

LIST OF SYMBOLS

A	cross-sectional area of ring
a	radius of the median surface of the shell
c	acoustic velocity of the fluid
c_s	longitudinal wave velocity in the shell $[c_s = (E_s/\rho_s)^{1/2}]$
c_r	longitudinal wave velocity in the rings $[c_r = (E_r/\rho_r)^{1/2}]$
E_s, E_r	modulus of elasticity of shell and ring material, respectively
h	shell thickness
$H(\tau)$	Heaviside step function
$H_n^{(1)}$, $H_n^{(2)}$	n^{th} order Hankel functions of the first and second kind, respectively
K	shell parameter $[4K^4 = 12(\frac{a}{h})^2(1 - \nu^2)]$
K_n	n^{th} order modified Bessel function of the second kind
$2L^*$	distance between supports
$2L$	nondimensional support spacing $(2L = 2L^*/a)$
p_i	incident fluid pressure
p	total pressure acting on shell
r^*	radial coordinate
r	nondimensional radial coordinate $(r = r^*/a)$
s	Laplace transform parameter
t	time
$2T$	forcing function period
w^*	radial shell displacement
w	nondimensional radial shell displacement, $(w = w^*/a)$
x^*	axial coordinate

x	nondimensional axial coordinate
$X_j(x)$	eigenfunction
β	ring stiffness parameter $[\beta = (\frac{h^2}{A})(\frac{E_s}{E_r})\frac{1}{6(1-\nu^2)}]$
λ_j	eigenvalue
ν	Poisson's ratio
ρ	fluid mass density
ρ_r	ring mass density
ρ_s	shell mass density
τ	nondimensional time $(\tau = \frac{ct}{a})$
φ	fluid potential function
ω_j	free vibration frequency
$(\ )_{,i}$	$\frac{\partial}{\partial i}$
$(\breve{\ })$	complex conjugate
$(\dot{\ })$	$\frac{d}{d\tau}(\ )$
$(\ )'$	derivative with respect to argument
$(\bar{\ })$	transformed function

1. INTRODUCTION

Problems concerned with the vibrations of cylindrical shells in vacuo or immersed in a fluid medium have achieved a good deal of attention for more than a decade. The free vibration of ring-reinforced cylindrical shells in vacuo was considered in papers by Bleich[1], Gondikas[2], Galletly[3] and Garnet and Goldberg[4]. Bleich and Baron[5] investigated the free vibration of an unreinforced cylindrical shell immersed in a fluid. The steady-state forced vibrations of infinitely long shells with periodically spaced rigid septa and reinforcing rings, surrounded by an infinite acoustic fluid was analyzed by Junger[6].

The interaction of a plane shock wave and an infinitely long cylindrical shell immersed in an infinite fluid medium was studied in papers by Baron[7], Mindlin and Bleich[8], Carrier[9] and Payton[10]. In the first two articles, approximations of the acoustic field equation resulted in the assumption that each element radiates a plane wave into the surrounding fluid. For this approximation the error increases as time increases. Haywood[11] solves the same problem assuming a cylindrical radiated wave.

Herman and Klosner[12] found an exact solution to the transient problem of an infinitely long, cylindrical shell immersed in a fluid medium, subjected to an engulfing wave. In their solution the assumption that the pressure wave had a harmonic spatial variation along the shell axis enabled them to quickly eliminate the axial space coordinate from consideration and emphasize the time response. The solution was then obtained by means of a Fourier transformation on the time coordinate.

The present study investigates the transient response of an infinitely long, periodic ring reinforced, cylindrical shell subjected to a suddenly applied pressure through the surrounding acoustic medium. The pressure is assumed constant along the shell axis and the rings are assumed to be slightly flexible. The solution is obtained by means of a technique in which the transient problem is reduced to an equivalent steady-state problem. This method was recently demonstrated by Garnet and Crouzet-Pascal[13], Klosner and Berglund[14], and was previously suggested by Bisplinghoff, Isakson and Pian[15].

II. GOVERNING DIFFERENTIAL EQUATIONS

Consider an infinitely long, circular, cylindrical shell with flexible ring supports periodically spaced at distances of $2L^*$ (Fig. 1). The shell is immersed in a fluid and is initially at rest. At some time ($\tau = 0$) there is a sudden rise in the surrounding fluid pressure, given by

$$p_i = p_o H(\tau) \tag{1}$$

where $H(\tau)$ is the Heaviside step function; the incident particle velocity is assumed to be zero.

The structural response is described by the following Donnell-type shell equation

$$w_{,xxxx} + 4K^4 w + 4K^4 \left(\frac{c}{c_s}\right)^2 w_{,\tau\tau} = 4K^4 \left(\frac{a}{h}\right) \frac{p}{E_s} \tag{2}$$

for which the axial inertia term has been neglected and the longitudinal stress resultant is assumed to vanish; c is the acoustic velocity of the fluid and [$c_s = (E_s/\rho_s)^{1/2}$] is the longitudinal wave velocity of the shell. The nondimensional distances, displacements and time are $x = x^*/a$, $r = r^*/a$, $L = L^*/a$, $w = w^*/a$ and $\tau = ct/a$. The starred coordinates and displacements are indicated in Fig. 1; t = time.

The shell boundary conditions are

$$w_{,x}(0,\tau) = w_{,xxx}(0,\tau) = w_{,x}(L,\tau) = 0 \tag{3}$$

$$w_{,xxx}(L,\tau) = \left(\frac{c}{c_r}\right)^2 \frac{1}{\beta} w_{,\tau\tau}(L,\tau) + \frac{1}{\beta} w(L,\tau) \tag{3a}$$

where the ring stiffness parameter

$$\beta = \left(\frac{h^2}{A}\right)\left(\frac{E_s}{E_r}\right)\left(\frac{h}{a}\right)\frac{1}{6(1-\nu^2)} \tag{4}$$

and A is the cross-sectional area of the ring; E_s, E_r = Young's modulus of the shell and ring. For a rigid ring $\beta = 0$ and Eq. (3a) simplifies to

$$w(L,\tau) = 0$$

The shell is assumed to be at rest initially, i.e.,

$$w_{,\tau}(x,0) = w(x,0) = 0 \tag{5}$$

The fluid field is described by the axisymmetric acoustic potential field equation

$$\varphi_{,rr} + \frac{1}{r}\varphi_{,r} + \varphi_{,xx} = \varphi_{,\tau\tau} \tag{6}$$

where φ is the velocity potential function. At the shell-fluid interface (r = 1) the radial velocity of the shell and fluid must be equal, so that

$$w_{,\tau}(x,\tau) = \frac{1}{ac}\varphi_{,r}(1,x,\tau) \tag{7}$$

The vanishing of the radiated velocity and pressure at $\tau = 0$ requires that

$$\varphi(r,x,0) = \varphi_{,\tau}(r,x,0) = 0 \tag{8}$$

while the symmetry conditions of the problem lead to

$$\varphi_{,x}(r,0,\tau) = \varphi_{,x}(r,L,\tau) = 0 \tag{9}$$

The pressure p acting on the shell is the sum of that exerted by the incident and radiated acoustic waves, i.e.,

$$p = p_o H(\tau) + \frac{\rho c}{a} \varphi_{,\tau}(1,x,\tau) \tag{10}$$

Power Series Expansion

For an almost-rigid ring stiffened shell, $\beta << 1$, the solution can be expressed in terms of a power series expansion in β. Thus the radial shell displacement can be represented by the following series

$$\begin{aligned} w(x,\tau) &= w_o(x,\tau) + \beta w_1(x,\tau) + \beta^2 w_2(x,\tau) + \cdots + \beta^m w_m(x,\tau) + \cdots \\ &= \sum_{m=0}^{\infty} \beta^m w_m(x,\tau) \end{aligned} \tag{11}$$

For the acoustic potential we may similarly write

$$\begin{aligned} \varphi(r,x,\tau) &= \varphi_o(r,x,\tau) + \beta\varphi_1(r,x,\tau) + \beta^2\varphi_2(r,x,\tau) + \cdots + \beta^m\varphi_m(r,x,\tau) + \cdots \\ &= \sum_{m=0}^{\infty} \beta^m \varphi_m(x,\tau) \end{aligned} \tag{12}$$

If Eq. (11) is to be a valid representation of w, then it is necessary that $w_o(x,\tau)$ and $w_m(x,\tau)$ $(m = 1,2,3,\cdots\infty)$ satisfy the equations given in the preceeding section. This leads to the following system of equations

$$w_{o,xxxx} + 4K^4 w_o + 4K^4 (\frac{c}{c_s})^2 w_{o,\tau\tau} = 4K^4 (\frac{a}{h}) \frac{1}{E_s} [p_o H(\tau) + \frac{\rho c}{a} \varphi_{o,\tau}(1,x,\tau)] \quad (13)$$

$$w_{o,x}(0,\tau) = w_{o,x}(L,\tau) = w_{o,xxx}(0,\tau) = w_o(L,\tau) = 0 \quad (13a)$$

$$w_{o,\tau}(x,\tau) = \frac{1}{ac} \varphi_{o,r}(1,x,\tau) \quad (13b)$$

$$w_{o,\tau}(x,0) = w_o(x,0) = 0 \quad (13c)$$

and

$$w_{m,xxxx} + 4K^4 w_m + 4K^4 (\frac{c}{c_s})^2 w_{m,\tau\tau} = 4K^4 (\frac{a}{h}) \frac{1}{E_s} \frac{\rho c}{a} \varphi_{m,\tau}(1,x,\tau) \quad (14)$$

$$w_{m,x}(0,\tau) = w_{m,x}(L,\tau) = w_{m,xxx}(0,\tau) = 0 \quad (14a)$$

$$w_m(L,\tau) + (\frac{c}{c_r})^2 w_{m,\tau\tau}(L,\tau) = w_{m-1,xxx}(L,\tau) \quad (14b)$$

$$w_{m,\tau}(x,\tau) = \frac{1}{ac} \varphi_{m,r}(1,x,\tau) \quad (14c)$$

$$w_{m,\tau}(x,0) = w_m(x,0) = 0 \qquad (m = 1,2,3,\cdots\infty) \quad (14d)$$

The first set of equations (13) represents the formulation of the rigid ring problem ($\beta = 0$), for which the ring displacements vanish [$w_o(L,\tau) = 0$].

The second set of equations (14) are recurrence expressions [see Eq. (14b)] for the m^{th} term of the series expansion of w.

The nonhomogeneous boundary condition of Eq. (14b) can be removed by the following transformation

$$w_m(x,\tau) = u_m(x,\tau) + F_m(\tau) \tag{15}$$

where

$$u_m(L,\tau) = 0 \quad .$$

Eqs. (14) are thus recast as follows:

$$u_{m,xxxx} + 4K^4 u_m + 4K^4 (\frac{c}{c_s})^2 u_{m,\tau\tau} = 4K^4 (\frac{a}{h}) \frac{1}{E_s} \frac{\rho c}{a} \varphi_{m,\tau}(1,x,\tau) - 4K^4 \{F_m(\tau)[1 - (\frac{c_r}{c_s})^2] + w_{m-1,xxx}(L,\tau)\} \tag{16}$$

$$u_{m,x}(0,\tau) = u_{m,xxx}(0,\tau) = u_{m,x}(L,\tau) = u_m(L,\tau) = 0 \tag{16a}$$

$$u_{m,\tau} + F_{m,\tau} = \frac{1}{ac} \varphi_{m,r}(1,x,\tau) \tag{16b}$$

$$F_m + (\frac{c}{c_r})^2 F_{m,\tau\tau} = w_{m-1,xxx}(L,\tau) = u_{m-1,xxx}(L,\tau) \tag{16c}$$

$$u_m(x,0) = u_{m,\tau}(x,0) = F_m(0) = F_{m,\tau}(0) = 0 \qquad (m = 1,2,3,\cdots\infty) \tag{16d}$$

Finally we have that each coefficient of the series expansion of the acoustic potential must satisfy the following set of equations:

$$\varphi_{m,rr} + \frac{1}{r}\varphi_{m,r} + \varphi_{m,xx} = \varphi_{m,\tau\tau} \tag{17}$$

$$\varphi_{m,x}(r,0,\tau) = \varphi_{m,x}(r,L,\tau) = 0 \qquad (m = 0,1,2,\cdots\infty) \tag{17a}$$

III. STEADY-STATE FORMULATION

The reformulation of the transient dynamic problem to an equivalent steady-state problem is accomplished by replacing the Heaviside function in equation (13) by a periodic forcing function, $F(\tau)$ (Fig. 2), constructed by superimposing appropriately signed and time shifted Heaviside functions[14]. The period 2T must be chosen so that at $\tau = 2nT$ $(n = 0,1,2,3,\cdots)$ the initial conditions of the shell are satisfied, that is, the velocity and displacements vanish. For damped systems which are brought to a state of rest within the interval $\bar{\tau}_r$ the above conditions will be achieved if $T/2 \geq \bar{\tau}_r$. For systems which contain undamped components, that is for those that are brought to a state of periodic motion of period $\bar{\tau}$, the initial conditions will be satisfied if $(T/2) = n\bar{\tau}$ $(n = 1,2,3,\cdots)$, where $n\bar{\tau}$ must be at least as great as the time necessary for decay of the transient motion.

Thus the acoustic pressure of Eq. (10) is replaced by

$$p = p_o F(\tau) + \frac{\rho c}{a} \varphi_{,\tau}(1,x,\tau) \tag{18}$$

where the complex Fourier series expansion of $F(\tau)$ is

$$F(\tau) = \sum_{k=1,3,5}^{\infty} \frac{1}{k\pi} \{[\sin \frac{k\pi}{2} + \frac{i}{2}(1 - \cos k\pi)]e^{-\frac{ik\pi\tau}{T}} + [\sin \frac{k\pi}{2} - \frac{i}{2}(1-\cos k\pi)]e^{\frac{ik\pi\tau}{T}}\} \tag{19}$$

The form of the acoustic potential which is periodic in time and satisfies the field equation (17) and the boundary condition (17a) is

$$\varphi_m = \sum_{k=1,3,5}^{\infty} \left[\sum_{n=0,1,2}^{\infty} R_{nk}^{m}(r)\cos\frac{n\pi x}{L}\, e^{-\frac{ik\pi\tau}{T}} + \sum_{n=0,1,2}^{\infty} \bar{R}_{nk}^{m}(r)\cos\frac{n\pi x}{L}\, e^{\frac{ik\pi\tau}{T}} \right] \tag{20}$$

where

$$\frac{d^2R_{nk}^{m}}{dr^2} + \frac{1}{r}\frac{dR_{nk}^{m}}{dr} + \left[\left(\frac{k\pi}{T}\right)^2 - \left(\frac{n\pi}{L}\right)^2\right]R_{nk}^{m} = 0 \tag{21}$$

and

$$\frac{d^2\bar{R}_{nk}^{m}}{dr^2} + \frac{1}{r}\frac{d\bar{R}_{nk}^{m}}{dr} + \left[\left(\frac{k\pi}{T}\right)^2 - \left(\frac{n\pi}{L}\right)^2\right]\bar{R}_{nk}^{m} = 0 \tag{22}$$

$$(m = 0, 1, 2, \cdots \infty)$$

The forms of R_{nk}^{m} and $\bar{R}_{nk}^{m}$ vary in different regions as follows:

<u>Case A: $(k\pi/T)^2 - (n\pi/L)^2 = \gamma_{nk}^{2} > 0$</u>

For an outgoing wave,

$$R_{nk}^{m} = A_{nk}^{m} H_o^{(1)}(\gamma_{nk} r) \quad ; \quad \bar{R}_{nk}^{m} = B_{nk}^{m} H_o^{(2)}(\gamma_{nk} r)$$

where $H_j^{(1)}$ and $H_j^{(2)}$ are Hankel functions of order j of the first and second kind.

Case B: $(n\pi/L)^2 - (k\pi/T)^2 = \beta_{nk}^2 > 0$

For an exponentially decaying potential field

$$R_{nk}^m = A_{nk}^m H_o^{(1)}(i\beta_{nk} r) \quad ; \quad \bar{R}_{nk}^m = B_{nk}^m H_o^{(1)}(i\beta_{nk} r)$$

Case C: $(n\pi/L)^2 - (k\pi/T)^2 = 0$

For this case $(k = nT/L)$, $R_{nk}^m = A_{nk}^m \ell nr$; $\bar{R}_{nk}^m = B_{nk}^m \ell nr$. But this violates the far field condition and thus the chosen period 2T must be such that the ratio nT/L is not equal to an odd integer, i.e.,

$$\frac{nT}{L} \neq k \qquad (k = 1,3,5,\cdots\infty)$$

Thus the final form of the potential function can be expressed as

$$\varphi_m(r,x,\tau) = \sum_{k=1,3,5}^{\infty} \left\{ \sum_{n=0}^{\infty} A_{nk}^m \begin{Bmatrix} H_o^{(1)}(\gamma_{nk} r) \\ H_o^{(1)}(i\beta_{nk} r) \end{Bmatrix} \cos \frac{n\pi x}{L}\, e^{-\frac{ik\pi\tau}{T}} \right.$$

$$\left. + \sum_{n=0}^{\infty} B_{nk}^m \begin{Bmatrix} H_o^{(2)}(\gamma_{nk} r) \\ H_o^{(1)}(i\beta_{nk} r) \end{Bmatrix} \cos \frac{n\pi x}{L}\, e^{\frac{ik\pi\tau}{T}} \right\} ,$$

for

$$\begin{Bmatrix} (\frac{k\pi}{T})^2 - (\frac{n\pi}{L})^2 = \gamma_{nk}^2 > 0 \\ (\frac{n\pi}{L})^2 - (\frac{k\pi}{T})^2 = \beta_{nk}^2 > 0 \end{Bmatrix} \qquad (m = 0,1,2,\cdots\infty) \qquad (23)$$

IV. SOLUTION OF THE SHELL EQUATION

For the rigid ring problem, described by Equation (13), with boundary conditions (13a), we assume a solution of the form:

$$w_o(x,\tau) = \sum_{k=1,3,5}^{\infty} \left[\sum_{j=1}^{\infty} \xi_{jk}^{o} X_j(x) e^{-\frac{ik\pi\tau}{T}} + \sum_{j=1}^{\infty} \eta_{jk}^{o} X_j(x) e^{\frac{ik\pi\tau}{T}} \right] \tag{24}$$

where

$$X_j(x) = \cos\lambda_j x - \frac{\cos\lambda_j L}{\cosh\lambda_j L} \cosh\lambda_j x \tag{25}$$

are the mode shapes for the free vibration of a periodically (rigid) ring-reinforced infinite cylindrical shell in vacuo [see Appendix (A)] and are solutions of the differential equation

$$X_j^{IV}(x) + \lambda_j^4 X_j(x) = 0 \tag{26}$$

The corresponding eigenvalues λ_j are obtained from the following transcendental equation

$$\tan\lambda_j L + \tanh\lambda_j L = 0 \tag{27}$$

The substitution of Eqs. (19), (23) and (24) into Eq. (13) yields

$$\sum_{k=1,3,5}^{\infty} \{ \sum_{j=1}^{\infty} \xi_{jk} {}^{o}X_j^{IV}(x) e^{-\frac{ik\pi\tau}{T}} + \sum_{j=1}^{\infty} \eta_{jk} {}^{o}X_j^{IV}(x) e^{\frac{ik\pi\tau}{T}} \}$$

$$+ 4K^4 \sum_{k=1,3,5}^{\infty} \{ \sum_{j=1}^{\infty} \xi_{jk} {}^{o}X_j(x) e^{-\frac{ik\pi\tau}{T}} + \sum_{j=1}^{\infty} \eta_{jk} {}^{o}X_j(x) e^{\frac{ik\pi\tau}{T}} \}$$

$$- 4K^4 (\frac{c}{c_s})^2 \sum_{k=1,3,5}^{\infty} (\frac{k\pi}{T}) \{ \sum_{j=1}^{\infty} \xi_{jk} {}^{o}X_j(x) e^{-\frac{ik\pi\tau}{T}} + \sum_{j=1}^{\infty} \eta_{jk} {}^{o}X_j(x) e^{\frac{ik\pi\tau}{T}} \}$$

$$= 4K^4 (\frac{a}{h}) \frac{P_o}{E_s} \sum_{k=1,3,5}^{\infty} \frac{1}{k\pi} \{ [\sin \frac{k\pi}{2} + \frac{i}{2} (1-\cos k\pi)] e^{-\frac{ik\pi\tau}{T}}$$

$$+ [\sin \frac{k\pi}{2} - \frac{i}{2} (1-\cos k\pi)] e^{\frac{ik\pi\tau}{T}} \}$$

$$- \frac{\rho c}{a} \sum_{k=1,3,5}^{\infty} \frac{ik\pi}{T} \{ \sum_{n=0}^{\infty} A_{nk} \, {}^{o}\begin{Bmatrix} H_o^{(1)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix} \cos \frac{n\pi x}{L} e^{-\frac{ik\pi\tau}{T}}$$

$$- \sum_{n=0}^{\infty} B_{nk} \, {}^{o}\begin{Bmatrix} H_o^{(2)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix} \cos \frac{n\pi x}{L} e^{\frac{ik\pi\tau}{T}} \} \qquad (28)$$

where ( )' represents the derivative with respect to the argument. Expanding the terms of the right-hand side of Eq. (28) into a series of the eigenfunctions X_j, we obtain the following

$$4K^4\left(\frac{a}{h}\right)\frac{p_o}{E_s}\frac{1}{k\pi}\left[\sin\frac{k\pi}{2}+\frac{i}{2}(1-\cos k\pi)\right]=\sum_{j=1}^{\infty} I_{jk}^{o} X_j(x) \tag{29}$$

$$4K^4\left(\frac{a}{h}\right)\frac{p_o}{E_s}\frac{1}{k\pi}\left[\sin\frac{k\pi}{2}-\frac{i}{2}(1-\cos k\pi)\right]=\sum_{j=1}^{\infty} J_{jk}^{o} X_j(x) \tag{30}$$

and

$$\cos\frac{n\pi x}{L}=\sum_{j=1}^{\infty} C_{nj}X_j(x) \tag{31}$$

where

$$C_{nj}=\frac{4\lambda_j^{\,4}\sin\lambda_j L\cos n\pi}{\left[\lambda_j^{\,4}-\left(\frac{n\pi}{L}\right)^4\right]\lambda_j L\left[\left(\frac{\cos\lambda_j L}{\cosh\lambda_j L}\right)^2+1\right]} \tag{32}$$

$$\begin{Bmatrix} I_{jk}^{o} \\ J_{jk}^{o} \end{Bmatrix}=4K^4\left(\frac{a}{h}\right)\frac{p_o}{E}\frac{1}{k\pi}\left[\sin\frac{k\pi}{2}\pm\frac{i}{2}(1-\cos k\pi)\right]\frac{4\sin\lambda_j L}{\lambda_j L\left[\left(\frac{\cos\lambda_j L}{\cosh\lambda_j L}\right)^2+1\right]} \tag{33}$$

Upon substituting Eqs. (26, 29-33) into Eq. (28), we obtain the following system of equations

$$\left[\lambda_j^{\,4}+4K^4-4K^4\left(\frac{c}{c_s}\right)^2\left(\frac{k\pi}{T}\right)^2\right]\xi_{jk}^{o}-I_{jk}^{o}+4K^4\left(\frac{a}{h}\right)\frac{\rho c}{aE_s}\frac{ik\pi}{T}\sum_{n=0}^{\infty}A_{nk}^{o}C_{nj}\begin{Bmatrix} H_o^{(1)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix}=0 \tag{34}$$

$$[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\eta_{jk}^o - J_{jk}^o - 4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}\sum_{n=0}^{\infty} B_{nk}^o C_{nj}\begin{Bmatrix} H_o^{(2)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix} = 0 \tag{35}$$

where $(j=1,2,3,\cdots\infty;$

$k=1,3,5,\cdots\infty)$

In a similar manner the interface boundary condition (13b) may be recast as

$$\xi_{jk}^o = -\frac{T}{ik\pi ac}\sum_{n=0}^{\infty} A_{nk}^o C_{nj}\begin{Bmatrix} \gamma_{nk}H_o'^{(1)}(\gamma_{nk}) \\ i\beta_{nk}H_o'^{(1)}(i\beta_{nk}) \end{Bmatrix} \tag{36}$$

and

$$\eta_{jk}^o = \frac{T}{ik\pi ac}\sum_{n=0}^{\infty} B_{nk}^o C_{nj}\begin{Bmatrix} \gamma_{nk}H_o'^{(2)}(\gamma_{nk}) \\ i\beta_{nk}H_o'^{(1)}(i\beta_{nk}) \end{Bmatrix} \tag{37}$$

Utilizing the identity

$$\frac{\rho c^2}{E_s} = (\frac{\rho}{\rho_s})(\frac{\rho_s}{E_s})c^2 = (\frac{\rho}{\rho_s})(\frac{c}{c_s})^2 \tag{38}$$

and noting that [16]

$$H_o'^{(1)}(\gamma_{nk}) = -H_1^{(1)}(\gamma_{nk})$$

$$H_o'^{(1)}(i\beta_{nk}) = -H_1^{(1)}(i\beta_{nk})$$

(continued on next page)

$$H_o^{'(2)}(\gamma_{nk}) = - H_1^{(2)}(\gamma_{nk}) \tag{39}$$

Equations (34-37) can be combined to yield:

$$\sum_{n=0}^{\infty} \{[\lambda_j^4+4K^4-4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\begin{Bmatrix}\gamma_{nk}H_1^{(1)}(\gamma_{nk}) \\ i\beta_{nk}H_1^{(1)}(i\beta_{nk})\end{Bmatrix}$$

$$- 4K^4(\frac{a}{h})(\frac{\rho}{\rho_s})(\frac{c}{c_s})^2(\frac{k\pi}{T})^2\begin{Bmatrix}H_o^{(1)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk})\end{Bmatrix}\} \frac{C_{nj}T}{ik\pi ac} \; A_{nk}^o = I_{jk}^o \tag{40}$$

$$\sum_{n=0}^{\infty} \{[\lambda_j^4+4K^4-4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\begin{Bmatrix}\gamma_{nk}H_1^{(2)}(\gamma_{nk}) \\ i\beta_{nk}H_1^{(1)}(i\beta_{nk})\end{Bmatrix}$$

$$- 4K^4(\frac{a}{h})(\frac{\rho}{\rho_s})(\frac{c}{c_s})^2(\frac{k\pi}{T})^2\begin{Bmatrix}H_o^{(2)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk})\end{Bmatrix}\} \frac{C_{nj}T}{ik\pi ac} \; B_{nk}^o = - J_{jk}^o \tag{41}$$

$$(j=1,2,3,\cdots\infty \; ; \; k=1,3,5,\cdots\infty)$$

From Eq. (33) we note that I_{jk}^o is the complex conjugate of J_{jk}^o, or

$$I_{jk}^o = \tilde{J}_{jk}^o$$

where ($\tilde{\ }$) represents the complex conjugate. Also, by definition

$$H_m^{(1)}(\gamma_{nk}) = \tilde{H}_m^{(2)}(\gamma_{nk})$$

$$H_o^{(1)}(i\beta_{nk}) = -\frac{2}{\pi}\, iK_1(\beta_{nk})$$

$$H_1^{(1)}(i\beta_{nk}) = -\frac{2}{\pi}\, K_1(\beta_{nk})$$

Thus by observing Eqs. (40) and (41) we conclude that

$$B_{nk}^{o} = \mp \tilde{A}_{nk}^{o} \quad , \quad \text{for} \begin{cases} (\frac{k\pi}{T})^2 - (\frac{n\pi}{L})^2 = \gamma_{nk}^{\ 2} > 0 \\ (\frac{n\pi}{L})^2 - (\frac{k\pi}{T})^2 = \beta_{nk}^{\ 2} > o \end{cases} \tag{42}$$

and from Eqs. (36) and (37) we observe that

$$\eta_{jk}^{o} = \tilde{\xi}_{jk} \quad , \quad \text{for } \gamma_{nk}^{\ 2} > 0 \text{ and } \beta_{nk}^{\ 2} > 0 \tag{43}$$

that is, for both ranges we have shown that $\eta_{jk}^{\ o}$ is equal to the complex conjugate of $\xi_{jk}^{\ o}$.

As a consequence of equation (43), the expression for the deflection function (24) may be rewritten as

$$w_o(x,\tau) = 2 \sum_{k=1,3,5}^{\infty} [\sum_{j=1}^{\infty} \mathrm{Re}(\xi_{jk}^{\ o}\, e^{-\frac{ik\pi\tau}{T}}) X_j(x)] \tag{44}$$

where Re( ) represents the real part of the complex function.

The complex coefficients ξ_{jk}^{o} can be determined from equations (36) and (37).

It should be noted that Eq. (40) represents an infinite set of algebraic equations for the unknown coefficients A_{nk}^{o} $(n=0,1,2,\cdots,k=1,3,5,\cdots)$, which must be solved by truncating the series.

V. HIGHER ORDER TERMS

Higher order terms in the shell deflection series (11) are defined in equation (15) and are obtained by solving the set of equations (16). A comparison of the rigid ring equations (13) and (13a) to equations (16) and (16a) reveals a term for term correspondence, with differences appearing only on the right-hand sides of the differential equations (13) and (16), i.e., the incident pressure term is replaced by the term $-4K^4\{F_m(\tau)[1-(\frac{c_r}{c_s})^2] + w_{m-1,xxx}(L,\tau)\}$. Thus we see that the m^{th} term of the expansion of the radial deflection is determined from the shear in the shell at the ring support for the $(m-1)^{th}$ term. It follows therefore that a solution to equation (16) which satisfies the boundary conditions (16a) can also be written in the form

$$u_m(x,\tau) = \sum_{k=1,3,5}^{\infty} [\sum_{j=1}^{\infty} \xi_{jk}^{m} X_j(x) e^{-\frac{ik\pi\tau}{T}} + \sum_{j=1}^{\infty} \eta_{jk}^{m} X_j(x) e^{\frac{ik\pi\tau}{T}}]$$

$$(m=1,2,3,\cdots\infty) \qquad (45)$$

Equations (16) and (16c) both contain the third derivative of the $(m-1)^{th}$ term of the radial displacement evaluated at $x = L$. The determination of this derivative by direct differentiation yields a slowly converging series. To eliminate this numerical difficulty, the differential equation (16) is integrated. The result is

$$w_{m-1,xxx}(L,\tau) = \sum_{k=1,3,5}^{\infty}\Bigg[\{-4K^4[1-(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\sum_{j=1}^{\infty}\xi_{jk}^{m-1}\frac{2}{\lambda_j}\sin\lambda_j L$$

$$-4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}A_{ok}^{m-1}LH_o^{(1)}(\gamma_{ok})\}e^{-\frac{ik\pi\tau}{T}}$$

$$+\{-4K^4[1-(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\sum_{j=1}^{\infty}\eta_{jk}^{m-1}\frac{2}{\lambda_j}\sin\lambda_j L$$

$$+4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}B_{ok}^{m-1}\;LH_o^{(2)}(\gamma_{ok})\}e^{\frac{ik\pi\tau}{T}}\Bigg]$$

$$-4K^4L\{F_{m-1}[1-(\frac{c_r}{c_s})^2]+w_{m-2,xxx}(L,\tau)\} \qquad (m=2,3,\cdots\infty)$$

(46)

and for m = 1

$$w_{o,xxx}(L,\tau) = \sum_{k=1,3,5}^{\infty}\Bigg[\{-4K^4[1-(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\sum_{j=1}^{\infty}\xi_{jk}^{o}\frac{2}{\lambda_j}\sin\lambda_j L$$

$$-4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}A_{ok}^{o}\;LH_o^{(1)}(\gamma_{ok})\}e^{-\frac{ik\pi\tau}{T}}$$

$$+\{-4K^4[1-(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\sum_{j=1}^{\infty}\eta_{jk}^{o}\frac{2}{\lambda_j}\sin\lambda_j L$$

(continued on next page)

$$+ 4K^4 (\frac{a}{h}) \frac{\rho c}{aE_s} \frac{ik\pi}{T} B_{ok}^{o} LH_o^{(2)}(\gamma_{ok})\} e^{\frac{ik\pi\tau}{T}} \Big] + 4K^4 (\frac{a}{h}) \frac{p_o}{E_s} LF(\tau) \qquad (47)$$

The expansion of the preceding equations into a series of the eigenfunctions yields

$$F_m(\tau)[1-(\frac{c_r}{c_s})^2] + w_{m-1,xxx}(L,\tau) = -\frac{1}{4K^4} \sum_{k=1,3,5}^{\infty} [\sum_{j=1}^{\infty} I_{jk}^{m} X_j(x) e^{-\frac{ik\pi\tau}{T}}$$

$$+ \sum_{j=1}^{\infty} J_{jk}^{m} X_j(x) e^{\frac{ik\pi\tau}{T}}] \qquad (m=1,2,3,\cdots\infty) \qquad (48)$$

where

$$I_{jk}^{m+1} = -4K^4\{-4K^4[1+(\frac{c}{c_s})^2(\frac{k\pi}{T})^2] \sum_{j=1}^{\infty} \xi_{jk}^{m} \frac{2}{\lambda_j} \sin\lambda_j L$$

$$- 4K^4 (\frac{a}{h}) \frac{\rho c}{aE_s} \frac{ik\pi}{T} A_{ok}^{m} LH_o^{(1)}(\gamma_{ok})\} C_{oj} - 4K^4 L I_{jk}^{m}$$

$$J_{jk}^{m+1} = -4K^4\{-4K^4[1+(\frac{c}{c_s})^2(\frac{k\pi}{T})^2] \sum_{j=1}^{\infty} \eta_{jk}^{m} \frac{2}{\lambda_j} \sin\lambda_j L$$

$$+ 4K^4 (\frac{a}{h}) \frac{\rho c}{aE_s} \frac{ik\pi}{T} B_{ok}^{m} LH_o^{(2)}(\gamma_{ok})\} C_{oj} - 4K^4 L J_{jk}^{m} \qquad (49)$$

$$(m=0,1,2,\cdots\infty)$$

and from Eq. (32)

$$C_{oj} = \frac{4 \sin\lambda_j L}{\lambda_j L[(\frac{\cos\lambda_j L}{\cosh\lambda_j L})^2+1]} \tag{50}$$

(Note that expressions for I_{jk}^{o} and J_{jk}^{o} are found in equation (33)). The solution of Eq. (16c) which satisfies the initial conditions (16d) can be written as

$$F_m(\tau) = - \sum_{k=1,3,5}^{\infty} \frac{1}{4K^4[2-(\frac{c_r}{c_s})^2+(\frac{c}{c_r})^2(\frac{k\pi}{T})^2]} [\sum_{j=1}^{\infty} I_{jk}^{m} X_j(x) e^{-\frac{ik\pi\tau}{T}}$$

$$+ \sum_{j=1}^{\infty} J_{jk}^{m} X_j(x) e^{\frac{ik\pi\tau}{T}}] \qquad (m=1,2,3,\cdots\infty) \tag{51}$$

By substituting equations (20, 45, and 48) into equation (16) we obtain

$$[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\xi_{jk}^{m} - I_{jk}^{m} + 4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}\sum_{n=0}^{\infty} A_{nk}^{m} C_{nj} \begin{Bmatrix} H_o^{(1)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix} = 0$$

(52)

$$[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]\eta_{jk}^{m} - J_{jk}^{m} - 4K^4(\frac{a}{h})\frac{\rho c}{aE_s}\frac{ik\pi}{T}\sum_{n=0}^{\infty} B_{nk}^{m} C_{nj} \begin{Bmatrix} H_o^{(2)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix} = 0$$

(53)

The above equations are identical in form to their rigid ring counterparts, equations (34) and (35).

The interface boundary condition, equation (16b), which yields the required additional set of simultaneous equations, can be recast by using equations (20, 45 and 51). Thus

$$\xi_{jk}^{m} = \frac{T}{ik\pi ac} \sum_{n=0}^{\infty} C_{nj} A_{nk}^{m} \begin{Bmatrix} \gamma_{nk} H_1^{(1)}(\gamma_{nk}) \\ i\beta_{nk} H_1^{(1)}(i\beta_{nk}) \end{Bmatrix} + \frac{I_{jk}^{m}}{4K^4[2-(\frac{c_r}{c_s})^2+(\frac{c}{c_r})^2(\frac{k\pi}{T})^2]} \tag{54}$$

$$\eta_{jk}^{m} = -\frac{T}{ik\pi ac} \sum_{n=0}^{\infty} C_{nj} B_{nk}^{m} \begin{Bmatrix} \gamma_{nk} H_1^{(2)}(\gamma_{nk}) \\ i\beta_{nk} H_1^{(1)}(i\beta_{nk}) \end{Bmatrix} + \frac{J_{jk}^{m}}{4K^4[2-(\frac{c_r}{c_r})^2+(\frac{c}{c_r})^2(\frac{k\pi}{T})^2]} \tag{55}$$

The solution of the set of simultaneous equations (52-55) yields

$$\sum_{n=0}^{\infty} \{[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2] \begin{Bmatrix} \gamma_{nk} H_1^{(1)}(\gamma_{nk}) \\ i\beta_{nk} H_1^{(1)}(i\beta_{nk}) \end{Bmatrix}$$

$$- 4K^4(\frac{a}{h})(\frac{\rho}{\rho_s})(\frac{c}{c_s})^2(\frac{k\pi}{T})^2 \begin{Bmatrix} H_o^{(1)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix}\} \frac{C_{nj}T}{ik\pi ac} A_{nk}^{m}$$

(continued on next page)

$$= \{1 - \frac{[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]}{4K^4[2-(\frac{c_r}{c_s})^2+(\frac{c}{c_r})^2(\frac{k\pi}{T})^2]}\} I_{jk}^m \tag{56}$$

and

$$\sum_{n=0}^{\infty} \{[\lambda_j^4 + 4K^4 - 4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2] \begin{Bmatrix} \gamma_{nk} H_1^{(2)}(\gamma_{nk}) \\ i\beta_{nk} H_1^{(1)}(i\beta_{nk}) \end{Bmatrix}$$

$$- 4K^4(\frac{a}{h})(\frac{\rho}{\rho_s})(\frac{c}{c_s})^2(\frac{k\pi}{T})^2 \begin{Bmatrix} H_o^{(2)}(\gamma_{nk}) \\ H_o^{(1)}(i\beta_{nk}) \end{Bmatrix}\} \frac{C_{nj}T}{ik\pi ac} B_{nk}^m$$

$$= - \{1 - \frac{[\lambda_j^4+4K^4-4K^4(\frac{c}{c_s})^2(\frac{k\pi}{T})^2]}{4K^4[2-(\frac{c_r}{c_s})^2+(\frac{c}{c_r})^2(\frac{k\pi}{T})^2]}\} J_{jk}^m \tag{57}$$

Equations (52-55) correspond to the rigid ring equations (34-37). As in the rigid ring solution, equation (56) represents an infinite set of algebraic equations in the coefficients A_{nk}^m ($n=0,1,2,\cdots\infty$; $k=1,3,5,\cdots\infty$) which must be solved by truncation. Once having obtained these values they are substituted into equations (54) and (55) and the final expressions for the coefficients ξ_{jk}^m, η_{jk}^m are obtained. As in the case of the rigid ring it can be shown that since $I_{jk}^m = \tilde{J}_{jk}^m$ then

$$\xi_{jk}^m = \tilde{\eta}_{jk}^m$$

Thus the final expression of equation (45) is

$$u_m(x,\tau) = 2 \sum_{k=1,3,5}^{\infty} \left[\sum_{j=1}^{\infty} \mathrm{Re}\left(\xi_{jk}^{m} e^{-\frac{ik\pi\tau}{T}}\right) X_j(x) \right] \tag{58}$$

A similar form can be used for the expression of $F_m(\tau)$.

Finally, the solution for the m^{th} component of the asymptotic expansion of the radial displacement is

$$w_m(x,\tau) = 2 \sum_{k=1,3,5}^{\infty} \left[\sum_{j=1}^{\infty} \mathrm{Re}\left(\xi_{jk}^{m} e^{-\frac{ik\pi\tau}{T}}\right) X_j(x) \right]$$

$$- \sum_{k=1,3,5}^{\infty} \frac{2}{4K^4\left[2-\left(\frac{c_r}{c_s}\right)^2+\left(\frac{c}{c_r}\right)^2\left(\frac{k\pi}{T}\right)^2\right]} \left[\sum_{j=1}^{\infty} \mathrm{Re}\left(I_{jk}^{m} e^{-\frac{ik\pi\tau}{T}}\right) X_j(x) \right] \tag{59}$$

VI. APPROXIMATE SOLUTIONS

A. Plane Wave Approximation

By assuming that each element of the shell radiates a plane wave into the surrounding medium, an assumption accurate for small time, the acoustic field equation can be reduced to

$$\varphi_{,r}(r,x,\tau) = -\varphi_{,\tau}(r,x,\tau) \tag{60}$$

At the shell-fluid interface the boundary condition therefore becomes

$$w_{,\tau}(x,\tau) = -\frac{1}{ac}\varphi_{,\tau}(1,x,\tau) \tag{61}$$

Substituting equation (61) into the shell equation (2) we obtain the following uncoupled relation

$$w_{,xxxx} + 4K^4 w + 4K^4\left(\frac{c}{c_s}\right)^2 w_{,\tau\tau} = 4K^4\left(\frac{a}{h}\right)\frac{1}{E_s}\left[p_o H(\tau) - \rho c^2 w_{,\tau}\right] \tag{62}$$

Noting the identity (38), we may write equation (62) as

$$w_{,xxxx} + 4K^4 w + 4K^4\left(\frac{c}{c_s}\right)^2 w_{,\tau\tau} + 4K^4\left(\frac{c}{c_s}\right)^2\left(\frac{a}{h}\right)\left(\frac{\rho}{\rho_s}\right) w_{,\tau} = 4K^4\left(\frac{a}{h}\right)\frac{p_o}{E_s} H(\tau) \tag{63}$$

Assuming a solution of Eq. (63) in the form

$$w(x,\tau) = \sum_{i=1}^{\infty} T_i(\tau) X_i(x) \tag{64}$$

we have that

$$\ddot{T}_i + (\frac{a}{h})(\frac{\rho}{\rho_s})\dot{T}_i + \frac{(\lambda_i^{\ 4} + 4K^4)}{4K^4(\frac{c}{c_s})^2} T_i = \frac{C_i}{4K^4(\frac{c}{c_s})^2} \tag{65}$$

and

$$4K^4(\frac{a}{h})\frac{p_o}{E_s} = \sum_{i=1}^{\infty} C_i X_i (x) \tag{66}$$

$$C_i = 4K^4(\frac{a}{h})\frac{p_o}{E_s} \frac{4 \sin\lambda_i L}{\lambda_i L[(\frac{\cos\lambda_i L}{\cosh\lambda_i L})^2 + 1]} \tag{67}$$

$$(\dot{\ }) = \frac{d}{d\tau}(\ )$$

The solution of Eq. (64) which satisfies the stationary initial conditions (4) is

$$T_i = \frac{r_1 r_2}{(r_1 - r_2)} \frac{C_i}{(\lambda_i^{\ 4} + 4K^4)} [\frac{e^{r_1\tau}}{r_1} - \frac{e^{r_2\tau}}{r_2} + \frac{(r_1 - r_2)}{r_2 r_1}] \tag{68}$$

where

$$\left\{ \begin{matrix} r_1 \\ r_2 \end{matrix} \right\} = -\frac{1}{2}(\frac{a}{h})(\frac{\rho}{\rho_s}) \pm [\frac{1}{4}(\frac{a}{h})^2(\frac{\rho}{\rho_s})^2 - \frac{(\lambda_i^{\ 4} + 4K^4)}{4K^4(\frac{c}{c_s})^2}]^{1/2} \tag{69}$$

Finally, the complete solution is

$$w(x,\tau) = \sum_{i=1}^{\infty} \{\frac{r_1 r_2}{(r_1-r_2)} \frac{C_i}{(\lambda_i^4+4K^4)} [\frac{e^{r_1\tau}}{r_1} - \frac{e^{r_2\tau}}{r_2} + \frac{(r_1-r_2)}{r_2 r_1}](\cos\lambda_i x - \frac{\cos\lambda_i L}{\cosh\lambda_i L} \cosh\lambda_i x)\} \tag{70}$$

B. Cylindrical Wave Approximation

If it is assumed that each element of the shell radiates a cylindrical wave into the surrounding fluid, the acoustic field equation (5) can be replaced by Haywood's approximate relation[11]

$$\varphi_{,r}(r,x,\tau) = - \varphi_{,\tau}(r,x,\tau) - \frac{\bar{g}_o}{r} \varphi(r,x,\tau); \; \bar{g}_o = 0.363 \tag{71}$$

The interface boundary condition (7) then can be written

$$w_{,\tau}(x,\tau) = - \frac{1}{ac} [\varphi_{,\tau}(1,x,\tau) + \bar{g}_o \varphi(1,x,\tau)] \tag{72}$$

We denote the Laplace transform of a function $h(\tau)$ as

$$\bar{h}(s) = \int_0^{\infty} e^{-s\tau} h(\tau) d\tau \tag{73}$$

and thus express the transform of the shell equation (2) and of the interface boundary condition (72) as

$$\bar{w}_{,xxxx} + 4K^4 \bar{w} + 4K^4 (\frac{c}{c_s})^2 s^2 \bar{w} = 4K^4 (\frac{a}{h}) \frac{p_o}{E_s} [\frac{1}{s} + \frac{\rho c}{a} s \bar{\varphi}(1,x,s)] \tag{74}$$

and

$$s\bar{w} = -\frac{1}{ac}(s + \bar{g}_o)\bar{\varphi}(1,x,s) \tag{75}$$

The elimination of $\varphi(1,x,s)$ between equations (74) and (75) yields

$$\bar{w}_{,xxxx} + 4K^4\bar{w} + 4K^4\left(\frac{c}{c_s}\right)^2 s^2\bar{w} = 4K^4\left(\frac{a}{h}\right)\frac{p_o}{E_s}\left[\frac{1}{s} - \rho c^2 \frac{s^2\bar{w}}{(s+\bar{g}_o)}\right] \tag{76}$$

Upon re-arranging terms and taking note of the identity (38), we may rewrite the above as

$$\bar{w}_{,xxxx} + 4K^4\left[1 + s^2\left(\frac{c}{c_s}\right)^2 + \left(\frac{a}{h}\right)\left(\frac{\rho}{\rho_s}\right)\left(\frac{c}{c_s}\right)^2 \frac{s^2}{(s^2+\bar{g}_o)}\right]\bar{w} = 4K^4\left(\frac{a}{h}\right)\left(\frac{p_o}{E_s}\right)\left(\frac{1}{s}\right) \tag{77}$$

The solution to equation (77) is

$$\bar{w} = \sum_{i=1}^{\infty} \bar{W}_i(s)X_i(x) \tag{78}$$

where

$$\bar{W}_i(s) = \frac{\Lambda_i\left(\frac{a}{h}\right)\left(\frac{p_o}{E_s}\right)\left(\frac{\bar{g}_o}{s} + 1\right)\left(\frac{c_s}{c}\right)^2}{s^3 + s^2\left[\bar{g}_o + \left(\frac{\rho}{\rho_s}\right)\left(\frac{a}{h}\right)\right] + s\left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right] + \bar{g}_o\left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right]} \tag{79}$$

and

$$\Lambda_i = \frac{4\sin\lambda_i L}{\lambda_i L\left[\left(\frac{\cos\lambda_i L}{\cosh\lambda_i L}\right)^2 + 1\right]}$$

The partial fraction expansion of equation (79) is

$$\overline{W}_i(s) = \Lambda_i\left(\frac{a}{h}\right)\left(\frac{p_o}{E_s}\right)\left(\frac{c_s}{c}\right)^2\left[-\frac{1}{R_1R_2R_3}\frac{\overline{g}_o}{s} + \frac{(\overline{g}_o + R_1)}{R_1(R_1-R_2)(R_1-R_3)(s-R_1)}\right.$$

$$\left. + \frac{(\overline{g}_o + R_1)}{R_2(R_2-R_1)(R_2-R_3)(s-R_2)} + \frac{(\overline{g}_o + R_3)}{R_3(R_3-R_1)(R_3-R_2)(s-R_3)}\right] \tag{80}$$

where the R_k's $(k = 1,2,3,)$ are the roots of

$$s^3 + s^2\left[\overline{g}_o + \left(\frac{\rho}{\rho_s}\right)\left(\frac{a}{h}\right)\right] + s\left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right] + \overline{g}_o\left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right] = 0 \tag{81}$$

Upon noting that

$$R_1 + R_2 + R_3 = -\left[\overline{g}_o + \left(\frac{\rho}{\rho_s}\right)\left(\frac{a}{h}\right)\right]$$

$$R_1R_2 + R_1R_3 + R_2R_3 = \left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right]$$

$$R_1R_2R_3 = -\overline{g}_o\left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right] \tag{82}$$

we may express the inverse of Eq. (80) as

$$W_i(\tau) = \Lambda_i\left(\frac{a}{h}\right)\left(\frac{p_o}{E_s}\right)\left(\frac{c_s}{c}\right)^2\left[\frac{1}{\Gamma_i} - \sum_{n=1}^{3}\frac{(\overline{g}_o + R_n)e^{R_n\tau}}{GR_n^2 + 2\Gamma_iR_n + 3\overline{g}_o\Gamma_i}\right] \tag{83}$$

where

$$\Gamma_i = \left(\frac{c_s}{c}\right)^2\left[1 + \frac{\lambda_i^4}{4K^4}\right] \quad ; \quad G = \left[\overline{g}_o + \left(\frac{\rho}{\rho_s}\right)\left(\frac{a}{h}\right)\right] \tag{84}$$

The final solution can therefore be written as follows

$$w(x,\tau) = \sum_{i=1}^{\infty} \{\Lambda_i (\frac{a}{h}) (\frac{p_o}{E_s}) (\frac{c_s}{c})^2 [\frac{1}{\Gamma_i} - \sum_{n=1}^{3} \frac{(\bar{g}_o + R_n) e^{R_n \tau}}{GR_n^2 + 2\Gamma_i R_n + 3\bar{g}_o \Gamma_i}] (\cos\lambda_i x - \frac{\cos\lambda_i L}{\cosh\lambda_i L} \cosh\lambda_i x)\} \quad (85)$$

VII. NUMERICAL CALCULATIONS

All calculations were performed on the IBM 7040 computer located at the Polytechnic Institute of Brooklyn. The physical properties used were

$$(\frac{a}{h}) = 87\frac{1}{3}\ ,\ \nu = 0.3,\ (\frac{c_r}{c_s})^2 = 1\ ,\ (\frac{\rho}{\rho_s}) = 0.13054\ ,\ (\frac{c}{c_s})^2 = 0.08815$$ (steel shell immersed in salt water),

while L was varied.

The infinite set of linear algebraic equations (40) in the unknowns A^o_{nk} were solved by assuming a forcing function period 2T, truncating the system and then solving the resulting finite number of equations. The deflection w was found by substituting the A^o_{nk}'s into equation (36) and then using equation (44). A similar procedure was used to solve equation (56), (54) and (59) for the higher order terms. For all cases 6 terms in space and 41 in time was found adequate to ensure accuracy.

As previously noted, if the system under investigation has a damping mechanism which brings it to a state of rest in a period of time $\bar{\tau}_r$ then T must be chosen so that $T/2 \geq \bar{\tau}_r$. Conditions of zero velocity and displacement will therefore be obtained at times $\tau_o = nT$ (n = integer). The amplitude of the displacement immediately following the times at which zero velocity and displacements occur will then represent the desired "transient" dynamic response, and will be valid for a period of time equal to $\tau_o + T/2$. If the vibration of the system contains undamped components, that is if the response degenerates into a periodic motion having a period $\bar{\tau}$, the initial conditions of zero displacement and velocity will be satisfied if $T/2 \geq n\bar{\tau}$, (n = integer) where $n\bar{\tau}$ is at least as great as the time necessary for decay of the transient motion[14].

For the rigid ring problem ($\beta = 0$) an assumed forcing function half period $T \geq 12$ was found to result in a solution which satisfied initial conditions. For the second term in the power series, ($\beta \neq 0$) a half period of $T = 14.928$ was found to be sufficient to ensure zero initial velocity and displacement. This value corresponds to $8\bar{\tau}$ where $\bar{\tau}$ is the period of undamped motion in the second term. Since any value for τ above 12 was adequate for the rigid ring solution $T = 14.928$ was used for all cases.

VIII. DISCUSSION OF RESULTS

The time history of the mid span deflections of rigid ring reinforced cylindrical steel shells immersed in sea water [$(c/c_s)^2$=0.08815] are presented in Figures 3 and 4 for ring spacings 2L = 0.3321 and 6.0. The solutions exhibit damped responses thereby indicating that energy is radiated into the fluid far field. The exact solutions over a wide range of bay lengths are presented in Fig. 5.

In a previous investigation by Herman and Klosner[12], in which a reinforced shell was suddenly subjected to an increased pressure (having harmonic spatial variation in the axial direction) of the surrounding fluid, it was shown that the shell response could be decomposed into radiating and nonradiating components. When the nonradiating component was large, it was found that the characteristic response obtained from the cylindrical (C) and plane wave (P) approximations were quite different from those obtained from the exact solutions.

It has also been shown[5] that no far field radiation occurs when the shell wavelength is smaller than the fluid wavelength, while energy is radiated to the far field when the shell wavelength is the larger of the two. For the nonradiation case, an axial flow from the adjacent compressed and rarified regions of the pressure field takes place. At high frequencies (short fluid wavelengths) sufficient time is not available for this type of flow to occur, and as a result energy is radiated into the far field[16].

For the rigid ring case considered here, in which the initial pressure is independent of the axial coordinate, there are no adjacent compressed and rarified regions along the axis of the shell and therefore a nonradiating mode cannot be activated. One might then assume, that the use of the approximate acoustic field equations would yield results with similar characteristics to those obtained from the exact solution (see Figs. 3 and 4).

Computer storage and time limitations limited the investigation of the higher order terms of the series expansion. The results obtained by including only the first two terms of the expansion are presented in Figure 6 for a ring spacing 2L = 0.3321. It is seen that the first term of the expansion (rigid ring term) exhibits a damped response, while the second term is periodic, with a frequency equal to that of the breathing mode of the ring. An investigation of the mode shape associated with the second term (Fig. 7) reveals that it oscillates about its equilibrium position in a manner that induces adjacent compressed and rarified regions in the fluid. In addition, the fluid wavelength is much greater than the shell wavelength. Thus, it is not unexpected to find that the steady-state component of the second term is periodic and radiates no energy into the far field.

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APPENDIX A

EIGENFUNCTIONS AND FREQUENCY EQUATION FOR RIGID RING-STIFFENED SHELL IN VACUO

The Donnell equation governing free vibration in vacuo can be written:

$$w_{,xxxx} + 4K^4 w + 4K^4 \left(\frac{c}{c_s}\right)^2 w_{,\tau\tau} = 0 \qquad \text{(A-1)}$$

where

$$w_{,x}(0,\tau) = w_{,xxx}(0,\tau) = w_{,x}(L,\tau) = w(L,\tau) = 0 \qquad \text{(A-2)}$$

The free vibration solution to Eq. (A-1) is

$$w(x,\tau) = X_j(x) e^{i\omega_j \tau} \qquad \text{(A-3)}$$

where

$$X_j^{IV} - \lambda_j^4 X_j = 0 \qquad \text{(A-4)}$$

and

$$\lambda_j^4 = [4K^4 \left(\frac{c}{c_s}\right)^2 \omega_j^2 - 4K^4] \qquad \text{(A-5)}$$

The solution of Eq. (A-4) which satisfies the boundary conditions (A-2) yields the eigenfunctions

$$X_j(x) = \cos\lambda_j x - \frac{\cos\lambda_j L}{\cosh\lambda_j L}\cosh\lambda_j x \qquad (j=1,2,3,\cdots\infty) \tag{A-6}$$

and the following transcendental equation from which the eigenvalues are obtained

$$\tan\lambda_j L + \tanh\lambda_j L = 0 \tag{A-7}$$

In a routine manner one can show that the eigenfunctions are orthogonal in the interval 0-L, i.e.,

$$\int_o^L X_i(x)X_j(x)\,dx = \begin{cases} \lambda_i L[(\frac{\cos\lambda_i L}{\cosh\lambda_i L})^2+1] & , \quad i = j \\ 0, \ i \neq j \end{cases} \tag{A-8}$$

APPENDIX B

STATIC SOLUTION OF RING-SUPPORTED SHELL

The differential equation of the shell can be written

$$w^{IV}(x) + 4K^4 w(x) = 4K^4 \left(\frac{a}{h}\right)\frac{p_o}{E_s} \qquad \text{(B-1)}$$

where the boundary conditions are

$$w_x(0) = w_{,xxx}(0) = w_{,x}(L) = 0$$

$$w(L) = \beta w_{,xxx}(L) \qquad \text{(B-2)}$$

The general solution to equation (B-1) satisfying boundary conditions (B-2) is

$$w(x) = \left(\frac{a}{h}\right)\left(\frac{p_o}{E_s}\right)\left[\frac{(\cos KL \sinh KL - \sin KL \cosh KL)\sin Kx \sinh Kx}{4K^3\beta(\sinh^2 KL + \sin^2 KL) + \sin KL \cos KL + \sinh KL \cosh KL}\right.$$

$$\left. - \frac{(\sin KL \cosh KL + \cos KL \sinh KL)\cos Kx \cosh Kx}{4K^3\beta(\sinh^2 KL + \sin^2 KL) + \sin KL \cos KL + \sinh KL \cosh KL} + 1\right]$$

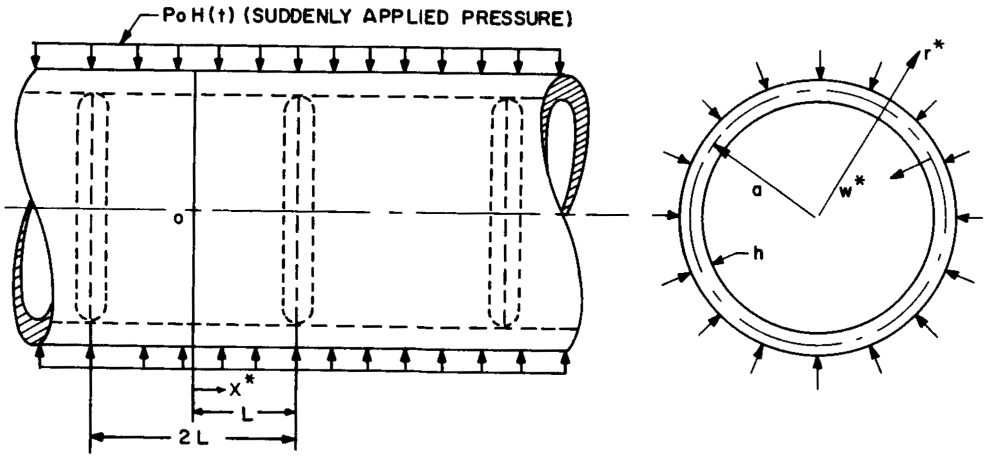

FIG. 1 SHELL GEOMETRY

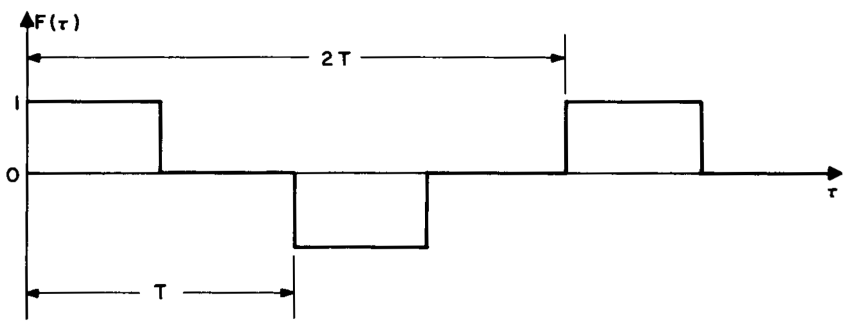

FIG. 2 FORCING FUNCTION

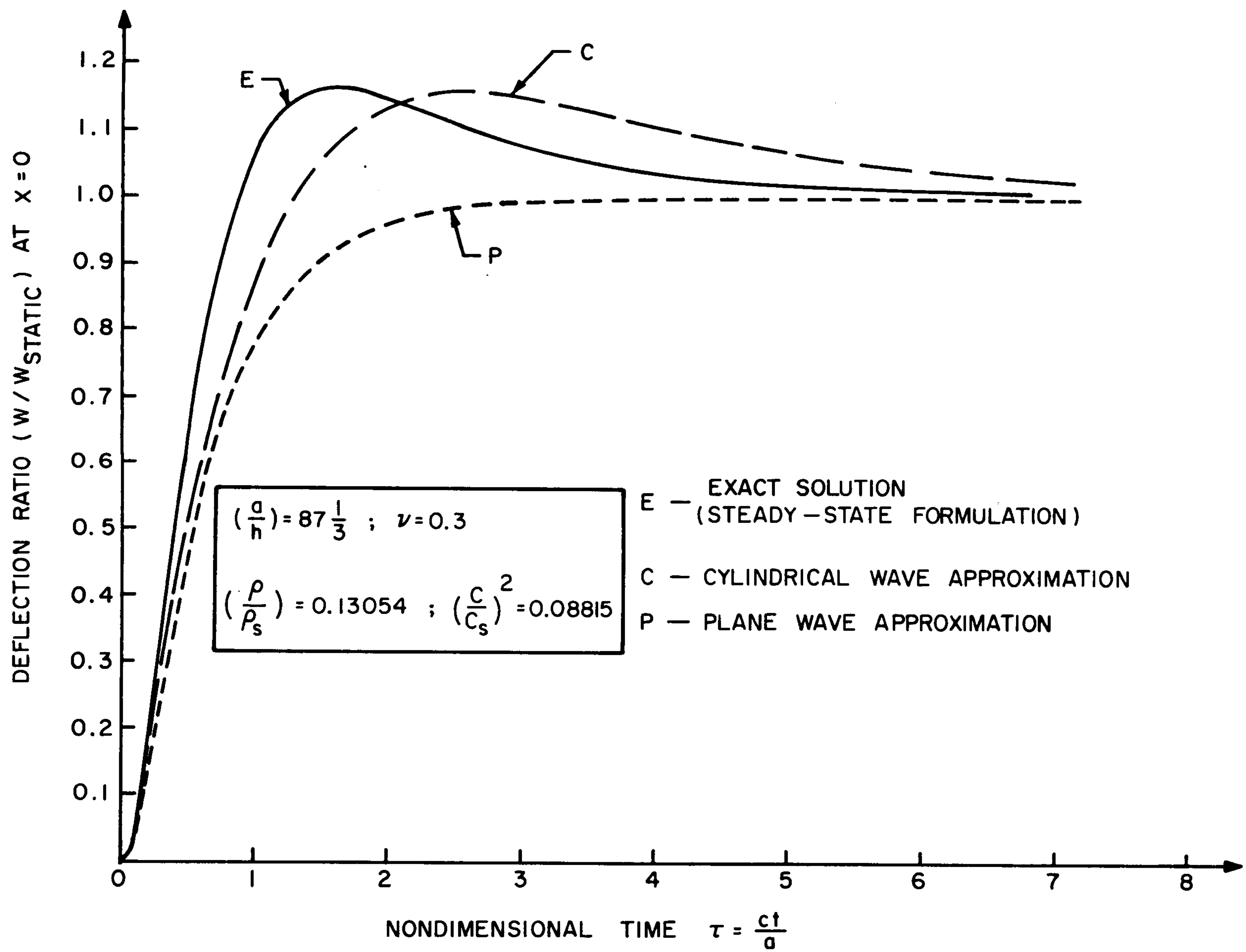

FIG. 3 TIME HISTORY OF DISPLACEMENT.
$2L = 0.3321$; $\beta = 0$ (RIGID RING)

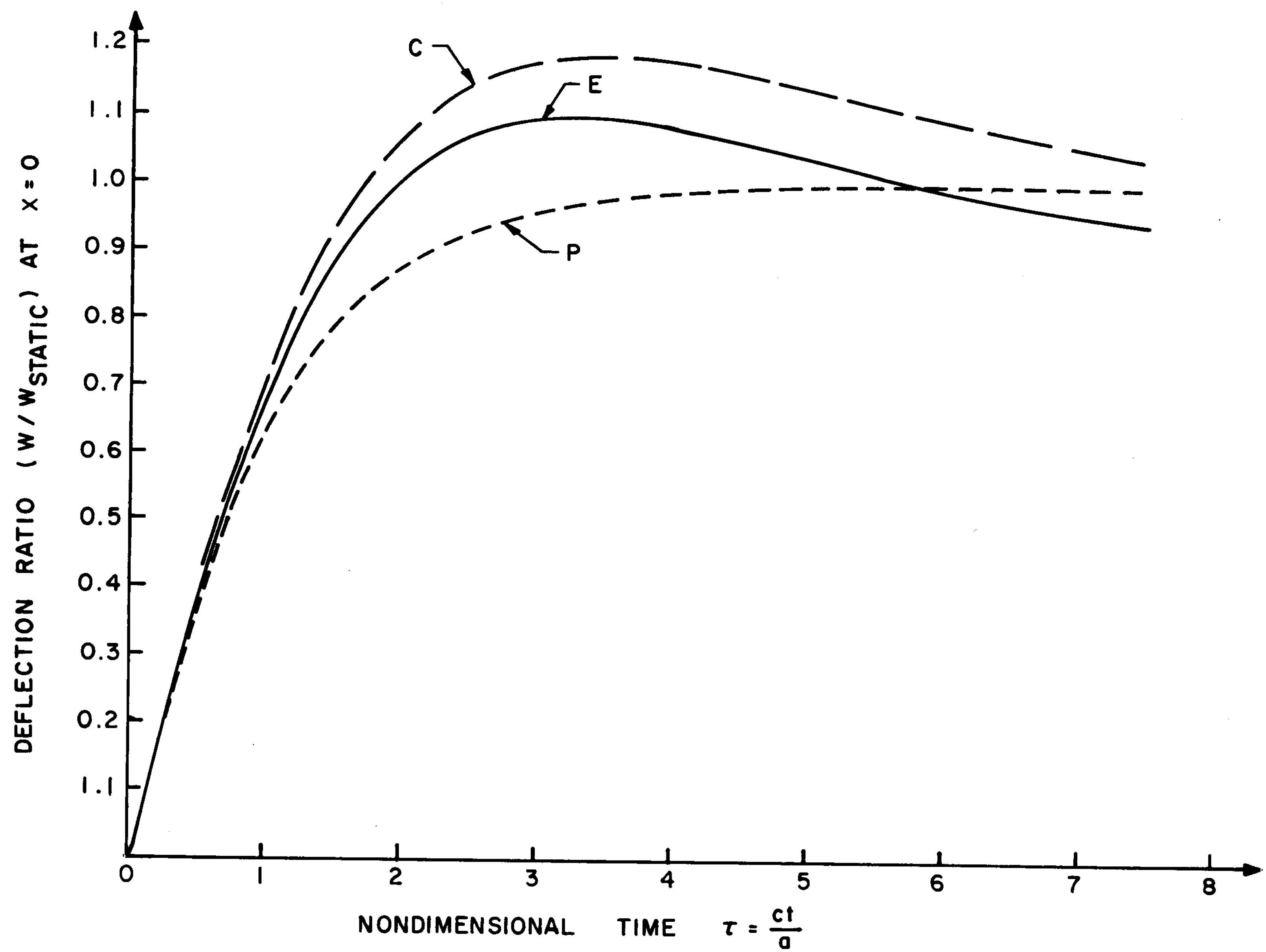

FIG. 4 TIME HISTORY OF DISPLACEMENT

$2L = 6.0$; $\beta = 0$ (RIGID RING)

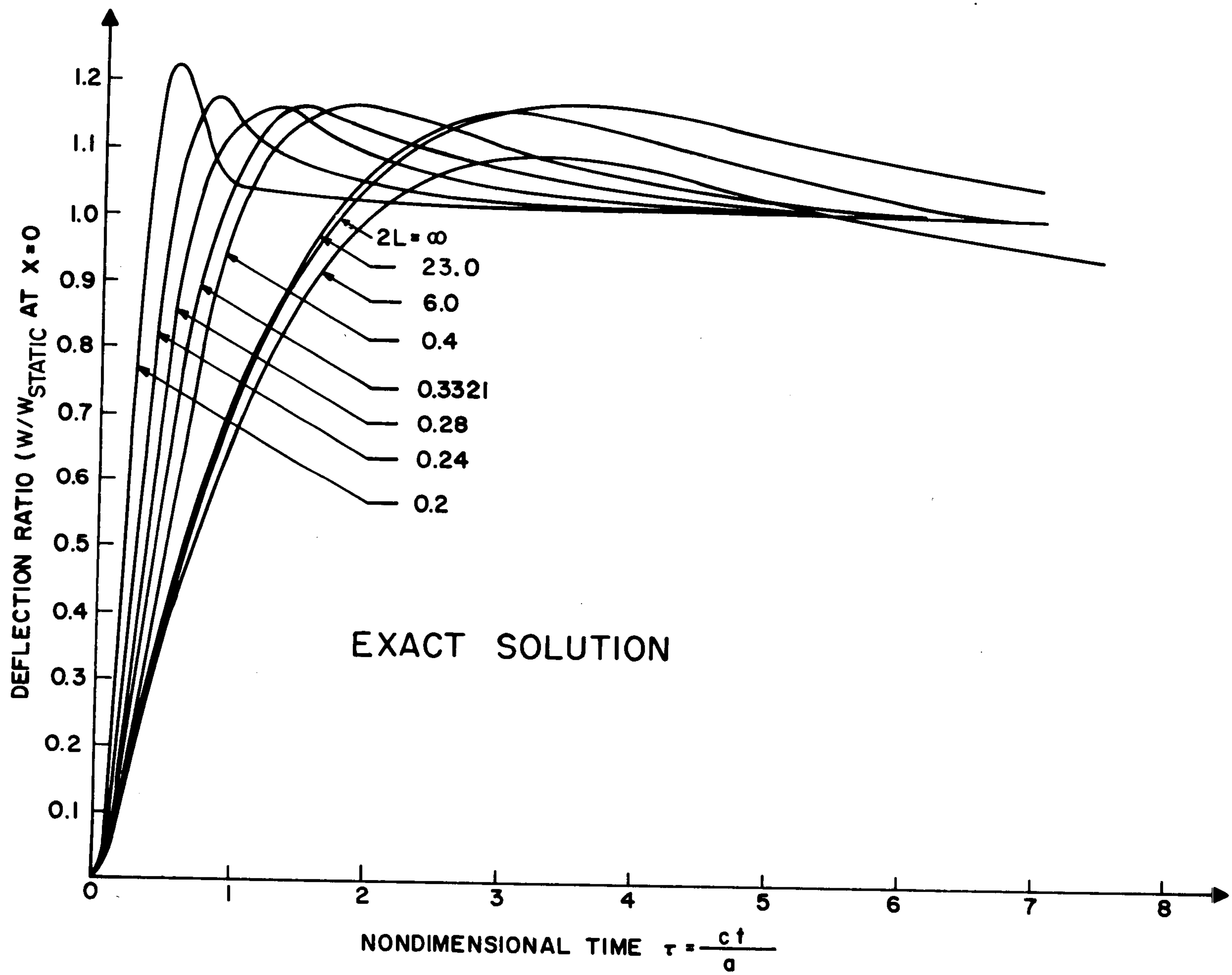

FIG. 5 TIME HISTORY OF DISPLACEMENT

$\beta=0$ (RIGID RING)

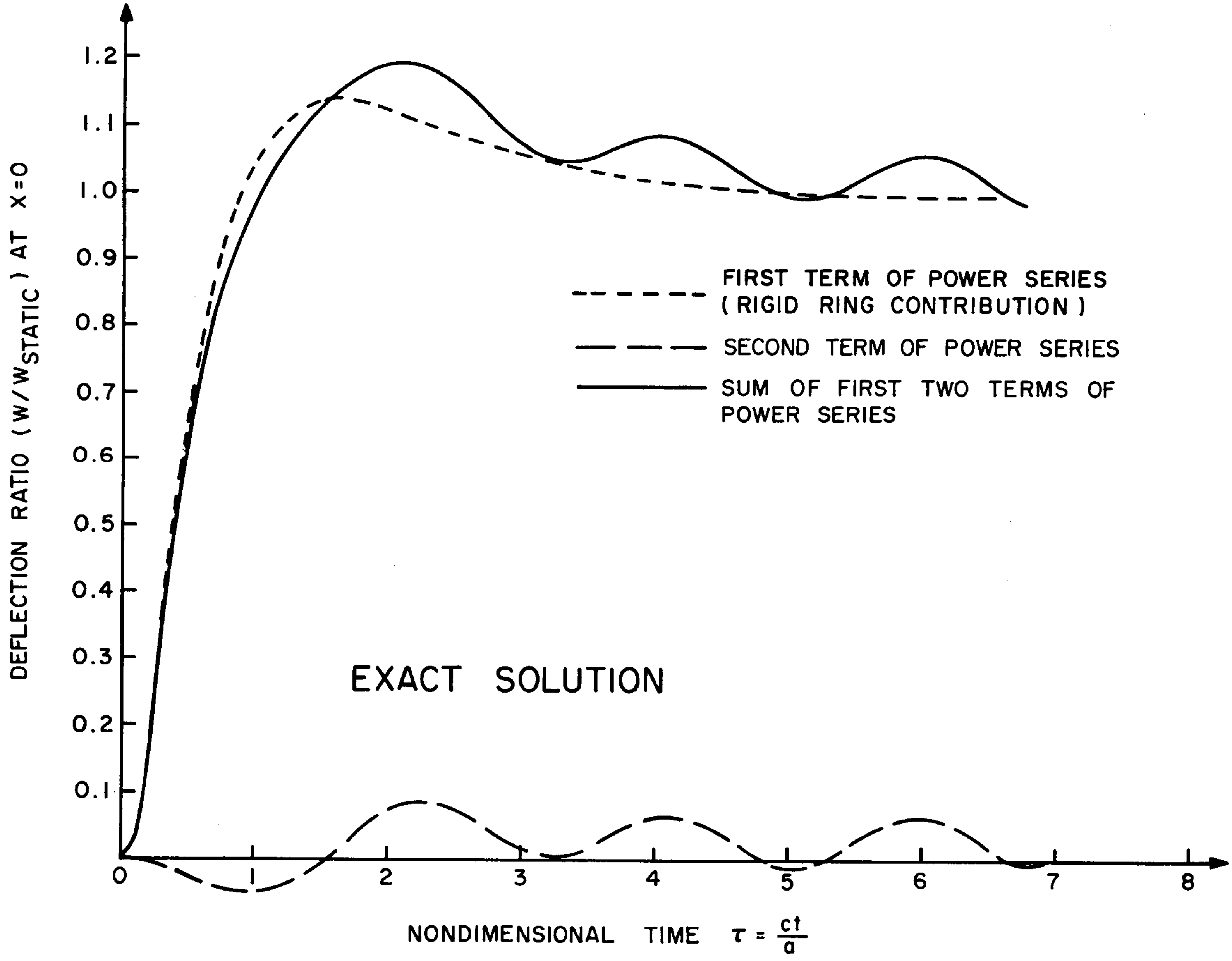

FIG. 6 TIME HISTORY OF DISPLACEMENT

$2L = 0.3321$; $\beta = 1.75 \times 10^{-5}$ (FLEXIBLE RING)

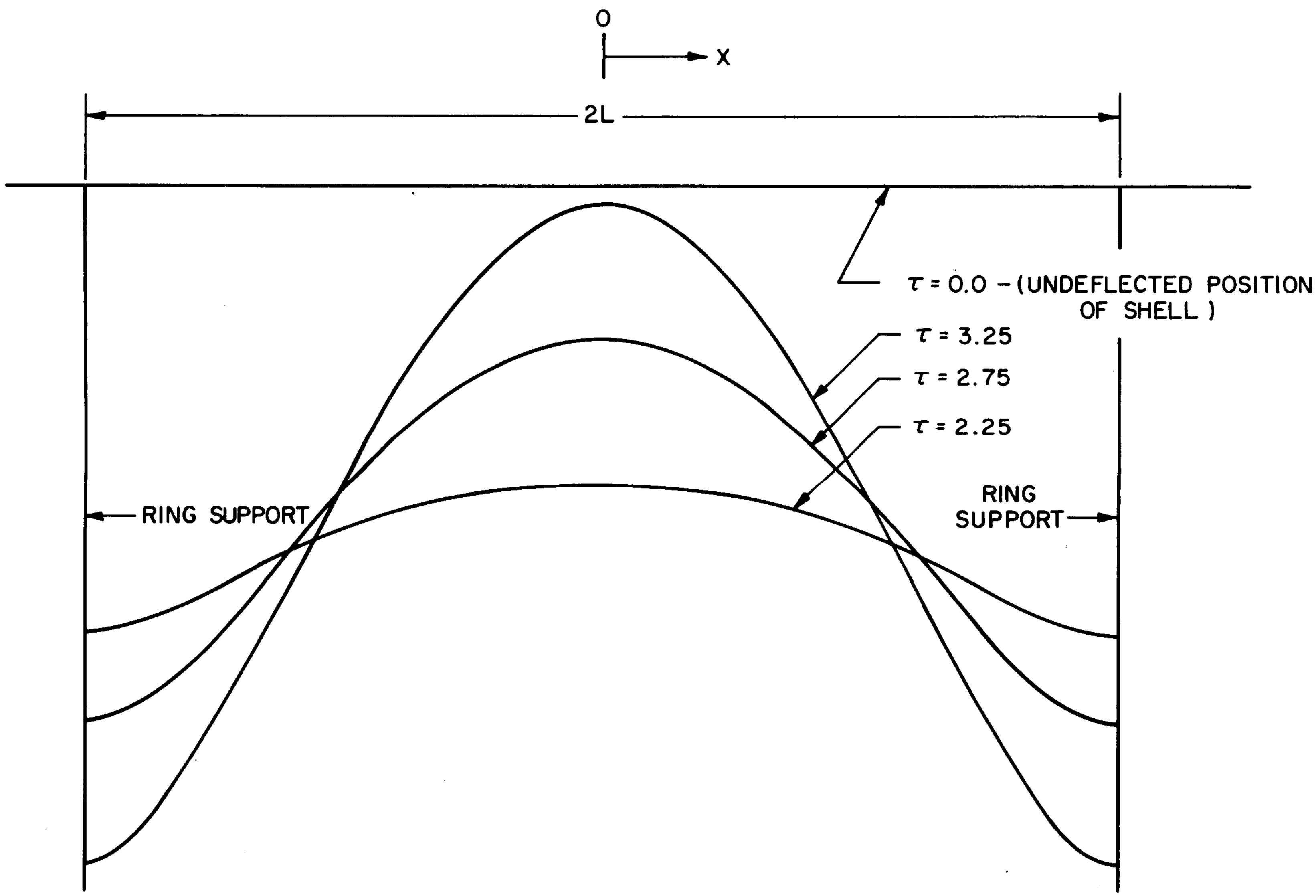

FIG.7. MODE SHAPE OF SECOND TERM OF POWER SERIES $2L = 0.3321$

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13. ABSTRACT

This work is concerned with the transient dynamic response of a periodically ring-reinforced, infinitely long, circular cylindrical shell to a uniform pressure suddenly applied through the surrounding acoustic medium. The incident particle velocity is zero and the rings are assumed to be slightly flexible.

A classical theory of the Donnell type is used to analyze the shell while the fluid is described by the linear acoustic field equation.

The solution is obtained by assuming a power series expansion in the ring stiffness parameter and utilizing a technique which reduces the transient dynamic problem to an equivalent steady-state formulation.

Numerical results are presented for a steel shell immersed in salt water for different ring spacings.

For the case of rigid rings, a cylindrical and plane wave approximation was also used to represent the fluid field. It is shown that the cylindrical wave approximation yields reasonably accurate reasonably accurate results.

Flexible ring results, although limited, indicate that undamped or nonradiating components of the shell vibration are activated.

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Shell-fluid interaction						
Transient response of a cylindrical shell						
Reinforced cylindrical shell						

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