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## SENSITIVITY REDUCTION USING OPTIMALLY DERIVED CONTROLLERS

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SENSITIVITY REDUCTION USING OPTIMALLY  
DERIVED CONTROLLERS

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Abstract

Feedback structures which reduce the parameter sensitivity of a linear system are derived from the solution of the classical linear regulator problem. Linear, time varying systems with several inputs and outputs are treated, and simplifications in the design are noted for the time invariant case.

Throughout the discussion, problems of implementation are considered as constraints on the design of the system. Specifically, unbounded elements in the controller are now allowable as the solution to the sensitivity problem. It is shown that for the structure, herein called Nth order feedback, the sensitivity of the system may be reduced to an arbitrarily small value.

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## 1. INTRODUCTION

In the design of automatic control systems, the sensitivity of the control system to the variation of its parameters is an important consideration. In fact, one of the primary reasons for the introduction of feedback into a system is the ability of feedback to diminish the influence of parameter variations on the properties of the system.

Before design techniques could be developed, analysis of the sensitivity problem had to be made. The basic concepts in this area were first formulated by Bode [4]. His definition of system sensitivity to parameter variations is very useful for single input, single output, time invariant systems. However, his definition of system sensitivity was not easily generalized to systems with several inputs or outputs. To satisfy the need for a generalization, Cruz and Perkins [5,6,7,8] defined a new sensitivity matrix which compared the output errors of two system structures. In their case the two structures were the open loop system and the closed loop system with each structure restricted to realize the nominal transfer characteristics of the system in the absence of parameter variations. Perkins and Cruz [12] also showed that for single input, time invariant systems the conditions for feedback to reduce the sensitivity of the system to parameter variations imply that the feedback must be an optimal control law. The control is optimal in the sense that a performance index, quadratic in the state variables and the input to the system, is minimized for all initial states.

If the converse of the above implication were true, a design technique could be based on solving the optimal control problem which is known as the linear regulator problem. Recently, Anderson [1,2] has shown that optimally derived controllers reduce the sensitivity of the system to parameter variations for a particular weighting of the output errors. However, this weighting of the output errors is not selected beforehand, but is a consequence of the optimization. Since the above publications indicate a relationship between sensitivity reduction and optimal control, this present work is devoted to establishing a closer tie between the two notions.

Controllers derived from the solution of the linear regulator problem are presented as a design procedure. It is shown that these controllers reduce the sensitivity of the system to parameter variations. As far as possible, ease of implementation is carried through the design as an implicit constraint. Chapter 2 is devoted to the mathematical description of the system considered and to the derivation of the feedback structures. Chapter 3 gives the main results of this presentation. The comparative sensitivities of the structures derived in the second chapter are given explicitly. For the restriction to time invariant systems, simplifications are indicated. In Chapter 4, large variations in parameters are considered in a slightly modified form and the controller is shown to reduce the sensitivity of the system. Throughout this work, examples are presented purely as illustrations of the techniques of design for sensitivity reduction, because the

introduction of other design considerations obscures the results emphasized in this work. Also, since the feedback structure of the system realizes the nominal characteristics of the open loop system in the absence of parameter variations, other design requirements can be satisfied in the same manner as they would be for the open loop system.



## 2. DERIVATION OF THE CONTROLLERS

### 2.1 Description of the Plant

The plant to be considered is shown in block diagram form in Figure 1. It is described by the following set of linear differential equations:

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t), \quad \underline{x}(t_0) = 0 \quad (2.1)$$

$$\underline{y}(t) = C(t)\underline{x}(t), \quad (2.2)$$

where

$$(\dot{\cdot}) \equiv d/dt .$$

The  $rx1$  dimensional vector  $\underline{u}$  is called the input to the plant. The  $px1$  dimensional vector  $\underline{y}$  is the output of the plant, and  $mx1$  dimensional vector  $\underline{x}$  is the state of the plant. The time varying matrices  $A$ ,  $B$ , and  $C$  are of appropriate dimensions.

Equations (2.1) and (2.2) define a linear differential operator  $\mathcal{P}$ . It is assumed that the plant is completely controllable and completely observable [10]. A test for complete controllability is that  $W(t_0, t)$  is positive definite for some finite  $t > t_0$ , where

$$W(t_0, t) = \int_{t_0}^t \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau) d\tau \quad (2.3)$$

$\Phi(t_0, t)$  is the state transition matrix, and superscript  $T$  denotes transpose. Similarly, by the duality theorem of Kalman, an analogous test can be made for complete observability.

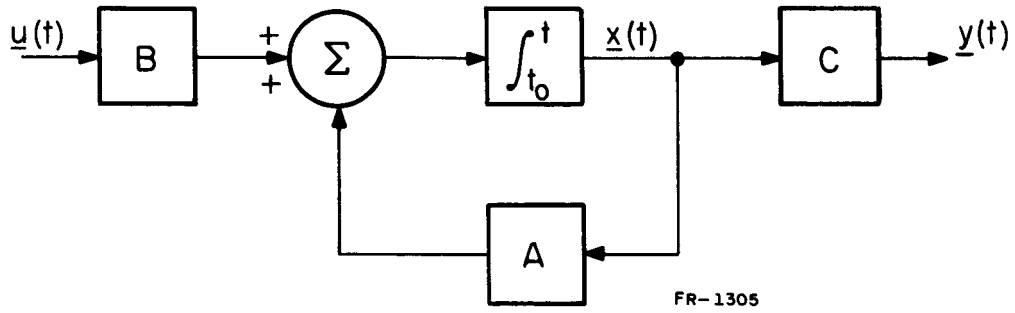


Figure 1. The nominal plant.

The operator  $P$  is called the nominal plant. Since variations occur in the elements of the matrices A, B, and C; a different operator  $P'$  is defined by the plant equations (2.1) and (2.2). The operator  $P'$  is called the varied plant. The open loop system is called  $P_0$ , where

$$P_0 = P'. \quad (2.4)$$

For each input  $\underline{u}$  and system configuration the following definitions are used:

- 1) A superscript ' denotes a varied quantity such as the varied output  $\underline{y}'_i$  and the varied state  $\underline{x}'_i$ .
- 2) The subscript i denotes the system configuration such as  $P_0$ . Other system configurations to be defined later are designated  $P_1, P_2$ , etc.
- 3) The error signal  $\underline{e}_i$  in the output of  $P_i$  is defined by

$$\underline{e}_i = \underline{y}'_i - \underline{y}_i \quad (2.5)$$

- 4) The error signal  $\underline{z}_i$  in the state of  $P_i$  is defined by

$$\underline{z}_i = \underline{x}'_i - \underline{x}_i \quad (2.6)$$

- 5) The sensitivity of a system  $P_i$  is measured by

$$\Delta_i = \int_{t_0}^{\infty} \underline{e}_i^T W \underline{e}_i dt, \quad W \geq 0, \quad (2.7)$$

for each  $P_i$  and  $\underline{u}$  such that  $\Delta_i$  exists.

Since feedback systems introduce a dependence of the plant input on the plant output, feedback systems can be designed which are less sensitive than the equivalent open loop system. However, Porter [13] shows that not all variations can be compensated for by varying the input. Therefore, it is assumed that the varied plant is output equivalent to the nominal plant.

Definition:  $P'$  is output equivalent to  $P$  if and only if for each  $\underline{u} \in U$  (input space) there exists  $\underline{u}' \in U$  such that

$$P' \underline{u}' = P \underline{u} . \quad (2.8)$$

Output equivalence, it should be noted, is similar to the complete compatibility of adaptive controllers [3] in that either notion guarantees that the system performance can be maintained. The difference is that no structure is specified for the generation of the varied input in the former case.

With this restriction on the variations, the open loop output error can be written

$$\underline{e}_0 = P' \underline{u} - P \underline{u} = P' (\underline{u} - \underline{u}') = P' \delta \underline{u}_0 . \quad (2.9)$$

If the variations are small enough so that the approximation

$$\underline{e}_0 \approx P \cdot \delta \underline{u} \quad (2.10)$$

can be made,

$$\underline{e}_0 = C \underline{z}_0 \quad (2.11a)$$

$$\dot{\underline{z}}_0 = A\underline{z}_0 + B\delta\underline{u}_0, \quad \underline{z}_0(t_0) = \underline{0} \quad (2.11b)$$

and

$$\underline{e}(t_0) = \underline{0}. \quad (2.11c)$$

Hence, the output error can be considered as a consequence of an error in the input to the nominal plant.

Finally, it is assumed that the nominal plant is uniformly asymptotically stable. This assumption is not restrictive. In fact, if the nominal plant is unstable and the open loop system is tested for sensitivity, the error signal becomes unbounded even for small variations in the plant parameters. This unbounded error signal occurs because unstable modes are excited whenever the plant varies from its nominal value. This result is not new since it has always been considered "bad practice" to cancel instabilities with an open loop input. For time invariant systems this procedure would amount to cancellation of poles in the right half of the complex frequency plane with zeros in the right half plane.

## 2.2 The Linear Regulator Problem

The linear regulator problem is well known [9,11] and is presented here for completeness because it is the basis for the design techniques presented later.

The plant is the same as given by Equations (2.1) and (2.2) except that  $\underline{x}(t_0) \neq \underline{0}$ . The problem is to select a control of the form

$$\underline{u}_* = k(\underline{x}, t), \quad k \in C^1 \quad (2.12)$$

which minimizes

$$J = \lim_{t_1 \rightarrow \infty} \int_{t_0}^{t_1} \underline{y}^T W \underline{y} + \underline{u}_*^T R \underline{u}_* dt \quad (2.13)$$

where  $W \geq 0$ ,  $R > 0$  for all  $t \geq t_0$ . The solution is given by

$$\underline{u}_* = -F \underline{x}, \quad (2.14)$$

where

$$-\dot{P} + PBR^{-1}B^T P - A^T P - PA = C^T W C, \quad (2.15a)$$

$$P(t_1) = 0, \quad P \geq 0, \quad (2.15b)$$

$$\lim_{t_1 \rightarrow \infty} P = \bar{P}, \quad (2.15c)$$

and

$$F = R^{-1}B^T \bar{P}. \quad (2.15d)$$

Since the plant is completely controllable,  $\bar{P}$  exists for all  $t \geq t_0$  (proposition (6.6), [9]), and the system

$$\dot{\underline{x}} = (A - BF)\underline{x}, \quad \underline{x}(t_0) \neq \underline{0} \quad (2.16)$$

is asymptotically stable. Complete observability insures that the control can be generated from measurements of the output  $\underline{y}$ . That is, with  $\emptyset$  the operator defined by the state transition matrix, the equation

$$F \emptyset B = \mathcal{H} C \emptyset B. \quad (2.17)$$

can be solved for the operator  $\mathcal{H}$ .

Example 1: Given the plant described by

$$\ddot{y} + \dot{y} + y = u,$$

the state equations are

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = [1 \quad 0] \underline{x} .$$

Let

$$J = \int_0^{\infty} 3y^2 + u^2 dt$$

so that

$$F = [1 \quad .732] .$$

Thus the operator  $\mathcal{H}$  has the transform

$$H(s) = (1 + .732s) .$$

### 2.3 Closed Loop Controllers

The operator  $\mathcal{H}$  is now used to form a set of feedback structures.

The first member of this set is constructed in Figure 2, where the prefilter  $\mathcal{G}$  realizes the transfer properties of the nominal plant.

By a simple calculation,

$$\mathcal{G} = I + F \emptyset B \quad (2.18)$$

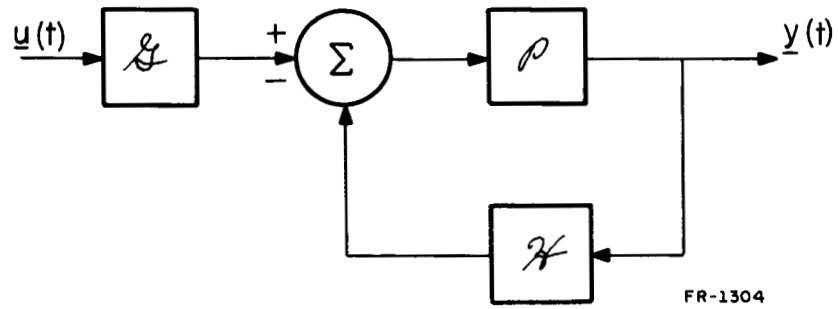


Figure 2. The first order feedback structure.



The operator  $\mathcal{G}$  is described in state form by

$$\underline{v} = \mathcal{G} \underline{w} \quad (2.19a)$$

where

$$\dot{\underline{s}} = \underline{A}\underline{s} + \underline{B}\underline{w}, \quad \underline{s}(0) = \underline{0}, \quad (2.19b)$$

$$\underline{v} = \underline{F}\underline{s} + \underline{w}. \quad (2.19c)$$

The structure in Figure 2 is called the first order feedback system and is represented by  $\mathcal{P}_1$ . The additional members of the set of feedback structures are formed by a simple algorithm. The Nth member of this set is called the Nth order feedback system and is denoted  $\mathcal{P}_n$ . The algorithm is depicted in Figure 3 and is

- 1) With the system  $\mathcal{P}_{i-1}$  given, connect the feedback loop  $\mathcal{H}$ .
- 2) Insert the prefilter  $\mathcal{G}$ , and
- 3) Define the resultant operator as  $\mathcal{P}_i$ .

Example 2: From Example 1,

$$F = [ 1 \quad .732 ]$$

so  $\mathcal{G}$  has transform

$$G(s) = \frac{s^2 + 1.732s + 2}{s^2 + s + 1}.$$

The second order feedback system is shown in Figure 4.

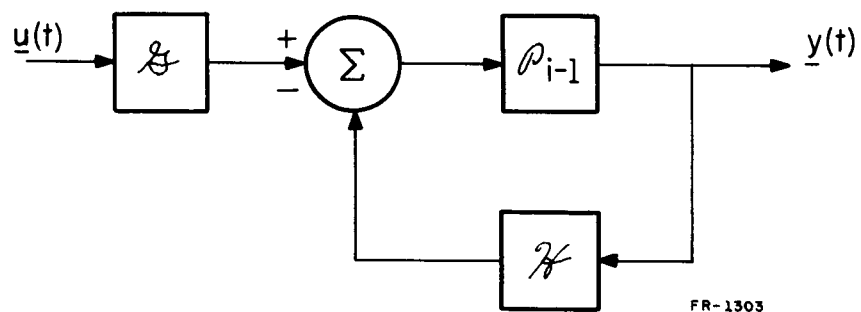
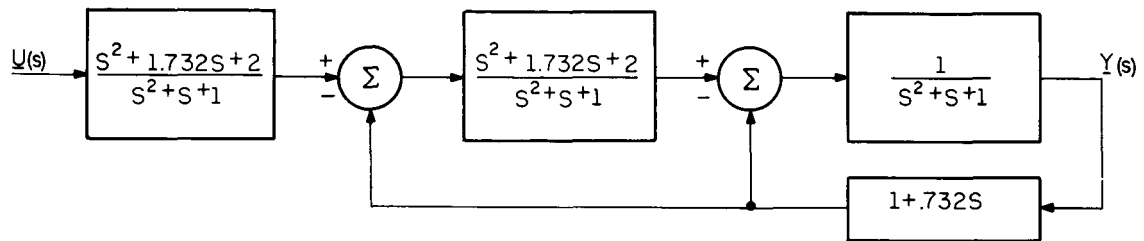


Figure 3. The formation of  $\mathcal{P}_i$  from  $\mathcal{P}_{i-1}$ .



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Figure 4. The second order feedback structure of Example 2.

### 3. SENSITIVITY ANALYSIS

#### 3.1 Introduction

The elements of the set of structures

$$\{ P_0, P_1, P_2, \dots \} \quad (3.1)$$

defined in Section 2.3 and Equation (2.4) satisfy

$$P_i = P \quad \text{for } i = 0, 1, 2, \dots \quad (3.2)$$

when  $P' = P$ . However, if the plant deviates from its nominal value  $P$ ,  $P'_i$  defines a different operator for each  $i = 0, 1, 2, \dots$ . It is this difference which is important and which is exploited by the design to reduce the sensitivity of the system to variations in the plant.

As previously defined in Equation (2.7), the sensitivity of the system is measured by  $\Delta$ . With  $\Delta$  as the measure of system performance, the notion of comparative sensitivity is used for the selection of the controller parameters. In general, the requirement for an adequate design is

$$\Delta \leq \alpha \Delta_0 \quad \text{for all } P' \quad (3.3)$$

with  $0 < \alpha < 1$ , where  $\alpha$  is a constant selected to satisfy the tolerances placed on the system output. In the next section the actual design of a system which satisfies Equation (3.3) is presented with the basis for the design on the solution of the linear regulator problem.

### 3.2 Time Varying Plants

Up to this point in the discussion, the selection of the matrices  $W$  and  $R$  has not been specified beyond that given in Equation (2.6). Now,  $W$  is chosen to correspond to the weighting of the output errors used in the measure of system performance, Equation (2.7). The choice is a natural one, but unfortunately, only justifiable by the results it produces. The selection of the matrix  $R$  is deferred until analysis of the design is given.

Lemma 1: For small variations in the plant parameters,

$$\Delta_0 + U_0 - \Delta_1 - U_1 = \Delta_0, \quad (3.4)$$

where

$$U_0 = \int_{t_0}^{\infty} \delta u_0^T R \delta u_0 dt, \quad (3.5)$$

and

$$U_1 = \int_{t_0}^{\infty} (\delta \underline{u}_0 - F \underline{z}_1)^T R (\delta \underline{u}_0 - F \underline{z}_1) dt. \quad (3.6)$$

Proof: The calculation of  $\Delta_0$  and  $\Delta_1$  follows from Equations (2.7), (2.11) and (2.15).

For  $\Delta_0$ ,

$$\begin{aligned} \underline{e}_0^T W \underline{e}_0 &= - \underline{z}_0^T P \underline{z}_0 + \underline{z}_0^T F^T R F \underline{z}_0 \\ &\quad - \underline{z}_0^T A^T P \underline{z}_0 - \underline{z}_0^T P A \underline{z}_0 \end{aligned} \quad (3.7)$$

or

$$\begin{aligned}
 \underline{e}_0^T \underline{W}_{\underline{e}_0} &= - \underline{z}_0^T \dot{\underline{P}} \underline{z}_0 - \dot{\underline{z}}_0^T \underline{P} \underline{z}_0 - \underline{z}_0^T \underline{P} \dot{\underline{z}}_0 \\
 &+ \underline{z}_0^T \underline{F}^T \underline{R} \underline{F} \underline{z}_0 + \underline{z}_0^T \underline{F}^T \underline{R} \delta \underline{u}_0 + \delta \underline{u}_0^T \underline{R} \underline{F} \underline{z}_0 \\
 &+ \delta \underline{u}_0^T \underline{R} \underline{S} \underline{u}_0 - \delta \underline{u}_0^T \underline{R} \delta \underline{u}_0
 \end{aligned} \tag{3.8}$$

Integration of Equation (3.8) for all time yields

$$\begin{aligned}
 \int_{t_0}^{\infty} \underline{e}_0^T \underline{W}_{\underline{e}_0} dt &= \int_{t_0}^{\infty} (\delta \underline{u}_0 + \underline{F} \underline{z}_0)^T \underline{R} (\delta \underline{u}_0 + \underline{F} \underline{z}_0) \\
 &- \delta \underline{u}_0^T \underline{R} \delta \underline{u}_0 dt - \underline{z}_0^T \underline{P} \underline{z}_0 \Big|_{t_0}^{\infty} .
 \end{aligned} \tag{3.9}$$

The last term vanishes since  $\bar{P}$  exists and

$$\lim_{t \rightarrow \infty} \underline{z}_0(t) = \underline{0} \tag{3.10}$$

from the existence of  $\Delta_0$ .

For  $\Delta_1$ ,

$$\underline{e}_1^T \underline{W}_{\underline{e}_1} = - \underline{z}_1^T \dot{\underline{P}} \underline{z}_1 - \underline{z}_1^T \underline{A}^T \underline{P} \underline{z}_1 - \underline{z}_1^T \underline{P} \underline{A} \underline{z}_1 + \underline{z}_1^T \underline{F}^T \underline{R} \underline{F} \underline{z}_1 \tag{3.11}$$

or

$$\begin{aligned}
 \underline{e}_1^T \underline{W}_{\underline{e}_1} &= - \underline{z}_1^T \dot{\underline{P}} \underline{z}_1 - \dot{\underline{z}}_1^T \underline{P} \underline{z}_1 - \underline{z}_1^T \underline{P} \dot{\underline{z}}_1 - \underline{z}_1^T \underline{F}^T \underline{R} \underline{F} \underline{z}_1 \\
 &+ \delta \underline{u}_0^T \underline{R} \underline{F} \underline{z}_1 + \underline{z}_1^T \underline{F}^T \underline{R} \delta \underline{u}_0 - \delta \underline{u}_0^T \underline{R} \delta \underline{u}_0 \\
 &+ \delta \underline{u}_0^T \underline{R} \delta \underline{u}_0 .
 \end{aligned} \tag{3.12}$$

Integration of Equation (3.12) for all time yields

$$\int_{t_0}^{\infty} \underline{e}_1^T \underline{W} \underline{e}_1 dt = \int_{t_0}^{\infty} \delta \underline{u}_0^T R \delta \underline{u}_0 - (\delta \underline{u}_0 - F \underline{z}_1)^T R (\delta \underline{u}_0 - F \underline{z}_1) dt - \underline{z}_1^T P \underline{z}_1 \Big|_{t_0}^{\infty} \quad (3.13)$$

Again the last term vanishes since

$$\lim_{t \rightarrow \infty} \underline{z}_1(t) = \underline{0} . \quad (3.14)$$

Subtraction of (3.13) from (3.9) and rearrangement of the terms yields (3.4).

Theorem 1: For small variations in the parameters of the plant,

$$\lim_{n \rightarrow \infty} \Delta_n = 0 . \quad (3.15)$$

Proof: Lemma 1 generalizes by induction and the repetition of the proof for Lemma 1 with  $\delta \underline{u}_i$  in place of  $\delta \underline{u}_{i-1}$ . Then,

$$\Delta_1 + U_1 - \Delta_2 - U_2 = \Delta_1 \quad (3.16)$$

$$\Delta_2 + U_2 - \Delta_3 - U_3 = \Delta_2 \quad (3.17)$$

and in general

$$\Delta_{i-1} + U_{i-1} - \Delta_i - U_i = \Delta_{i-1} \quad (3.18)$$

Summation of  $\Delta_i$  from  $i = 0$  to  $i = n$  gives the relationship

$$\sum_{i=0}^n \Delta_i = \Delta_0 + U_0 - \Delta_n - U_n . \quad (3.19)$$

The sequence of partial sums defined by Equation (3.19) is monotone increasing since  $\Delta_i$  is positive for all  $i$ . This sequence is also bounded

from above by  $\Delta_0 + U_0$ . Hence, the sequence converges and (3.15) follows. Theorem 1 means that the error can be reduced to an arbitrarily small value at the cost of increasing the number of sections in the feedback structure. Then the rate of decrease for each step and the uniformity of the rate of decrease become important.

From Equation (3.18), the rate of decrease is a function of the weighting matrix  $R$ , albeit a very complicated function. However, for the general time varying plant the assertion can be made that a sufficiently smaller norm of  $R$  increases the rate of convergence. This statement follows from Theorem 2.

Theorem 2: With the replacement of  $R$  by  $\lambda R$ ,  $\lambda > 0$ , in the design of the controller as given in Sections 2.2 and 2.3,

$$\lim_{\lambda \rightarrow 0} \Delta_1(\lambda) = 0 . \quad (3.20)$$

Proof: From Equation (3.13),

$$\Delta_1(\lambda) = \lambda \int_{t_0}^{\infty} \delta \underline{u}_0^T R \delta \underline{u}_0 = \lambda U_0 . \quad (3.21)$$

Since  $U_0$  exists and is independent of  $\lambda$ , (3.20) is proved. The interpretation of Theorem 2 is that  $R$  should be selected as small as possible in order to achieve the greatest decrease in the sensitivity per order of feedback structure. However, smaller  $R$  matrices generally yield feedback operators with larger parameter values. Hence, the choice of  $R$  is restricted by the value of feedback which can be implemented.



On an intuitive basis, Theorem 1 and Theorem 2 are similar results. The property of feedback which allows the possibility of sensitivity reduction derives from the manner in which the input to the plant is formed. For the nominal plant, the difference between the feedback signal and the output of the prefilter is the nominal input. When variations in the plant occur, the input to the plant will change to compensate for these variations if the system is designed properly. Hence, the change in the input relies on the output actually deviating from its nominal value. Since this deviation is to be kept small, the controller must be able to detect small variations and to produce large changes at the input.

To circumvent the need for large gains in the system, the higher order feedback structures are introduced in this work. Theorem 1 verifies that the desired reduction in sensitivity is possible. The power to reduce the sensitivity of the plant results from the manner in which the input to the plant is formed by the system. Instead of forming the input as the difference of two large signals, the higher order feedback systems perform the repeated differences of many manageable signals. Then, if variations occur, the change in the input is the result of the sum of many small signals. In this way, the undesirable use of very large gains in the controller is eliminated. The latter method is not without limitations, however. Inspection of the high order feedback structures reveals that the feedback loop must have the power to supply the feedback signal to many summation

points. Hence, the magnitude restriction is replaced by a power requirement.

It should not be assumed that the algorithm given for forming the higher order feedback structures is necessarily the actual method of construction. Many other realizations are possible but since any particular choice of construction relies on factors in the design not considered here, this question is left for future study.

### 3.3 Time Invariant Plants

Since time invariant systems can be analyzed as finite dimensional linear operators in the frequency domain, improvements in the design are possible. Application of Parseval's theorem to Equation (3.9) and Equation (3.13) yields

$$\Delta_0 - \Delta_1 = \int_{-\infty}^{\infty} \delta \underline{U}_0^*(j\omega) \{ K^*(j\omega) R K(j\omega) + K^{-1*}(j\omega) R K^{-1}(j\omega) - 2R \} \delta \underline{U}_0(j\omega) d\omega \quad (3.22)$$

where

$$K(j\omega) = I + F \Phi_p(j\omega) B, \quad (3.23)$$

$$\Phi_p(j\omega) = [j\omega I - A]^{-1}, \quad (3.24)$$

and superscript \* denotes conjugate transpose. Now frequency domain conditions on the R matrix can be formulated. Since each increase in the order of the feedback structure necessitates a greater amount of implementation, the choice of R should guarantee that each feedback

structure is better than the one before. Better in the sense that

$$\Delta_j \geq \Delta_{j+1} \quad \text{for all } \rho' \quad (3.25)$$

for  $j = 1, 2, 3, \dots$

Lemma 2:  $\Delta_j \geq \Delta_{j+1}$  for all  $\delta U_j$  if and only if

$$K^* R K + K^{-1*} R K^{-1} - 2R \geq 0 \quad \text{for all real } \omega . \quad (3.26)$$

Proof: See [7].

Equation (3.26) can be stated in a more convenient form. Since  $R > 0$ ,  $R$  has a unique square root defined by

$$R = Q^2 \quad (3.27)$$

and

$$Q > 0 . \quad (3.28)$$

Premultiplication and postmultiplication of Equation (3.26) by  $Q^{-1}$  yields the equivalent necessary and sufficient condition of Lemma 2

$$K'^* K' + K'^{-1*} K'^{-1} - 2I \geq 0 \quad \text{for all real } \omega , \quad (3.29)$$

where

$$K' = QKQ^{-1} . \quad (3.30)$$

Theorem 3: If  $K'$  is a normal operator for all real  $\omega$ , Equation (3.29) is satisfied.

Proof:  $K'$  normal implies, by definition,

$$K'^* K' = K' K'^* = D > 0 . \quad (3.31)$$

Then, Equation (3.29) becomes

$$D^{-1} \{D-I\}^2 \quad (3.32)$$

which is positive semidefinite since  $D^{-1}$  is positive definite,  $(D-I)^2$  is positive semidefinite, and  $D^{-1}$ ,  $(D-I)^2$  commute.

For single input systems  $K'$  is always normal since  $K'$  is a scalar. The next example shows that systems exist for which  $R$  can be chosen to satisfy Theorem 3.

Example 3: Given

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Choose

$$R = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

then

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$F = \begin{bmatrix} \sqrt{2} - 1 & 2 - \sqrt{5} \\ 1 - \sqrt{2} & 2\sqrt{5} - 4 \end{bmatrix}.$$

Finally

$$K' = \frac{\begin{bmatrix} s^2 + (2 + \sqrt{2})s + 2\sqrt{2} & 0 \\ 0 & s^2 + (1 + \sqrt{5})s + \sqrt{5} \end{bmatrix}}{s^2 + 3s + 2}$$

which is a normal operator for  $s = j\omega$ ,  $\omega$  real.

With this example as a basis the design for the performance criterion

$$\Delta \leq \frac{1}{2} \Delta_0 \quad \text{for } \omega^2 \leq 6$$

is produced. For the values given above, it can be verified from the sensitivity criterion of Cruz and Perkins that the  $n^{\text{th}}$  order feedback systems meet the requirement for the ranges shown in Table 1. Hence, in order to satisfy the criterion for system performance,  $n$  must be chosen greater than or equal to eight.

The order of the feedback structure may be decreased by increasing the gains of the feedback elements. For this case, choose a new weighting of the input

$$R' = \frac{1}{2} R$$

then

$$Q' = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$F' = \begin{bmatrix} \sqrt{3} - 1 & 2 - \sqrt{6} \\ 1 - \sqrt{3} & 2\sqrt{6} - 4 \end{bmatrix}$$

The performance criterion is met for the ranges of  $\omega$ ,  $\omega$  real, shown in Table 2. Thus, for this choice of weighting of the inputs, the order of the feedback system can be reduced to four.

Table 1

Order of feedback system n	Range of $\omega$ for which $\Delta \leq \frac{1}{2} \Delta_0$
1	none
2	none
3	none
4	$\omega^2 \leq 1.29$
5	$\omega^2 \leq 2.70$
6	$\omega^2 \leq 4.12$
7	$\omega^2 \leq 5.70$
8	$\omega^2 \leq 6.93$
9	$\omega^2 \leq 8.50$
10	$\omega^2 \leq 9.90$

Table 2

Order of feedback system n	Range of $\omega$ for which $\Delta \leq \frac{1}{2} \Delta_0$
1	none
2	$\omega^2 \leq .83$
3	$\omega^2 \leq 3.69$
4	$\omega^2 \leq 6.60$
5	$\omega^2 \leq 9.40$
6	$\omega^2 \leq 12.4$
7	$\omega^2 \leq 15.4$
8	$\omega^2 \leq 18.2$
9	$\omega^2 \leq 21$
10	$\omega^2 \leq 23.8$



## 4. LARGE VARIATIONS IN TIME INVARIANT SYSTEMS

In the preceding section, the variations in the parameters were assumed to be small so that the approximation

$$P' \delta \underline{u} \approx P \delta \underline{u} \quad (4.1)$$

could be made for all  $\delta \underline{u}$ . For the discussion of large variations a different approach is taken in this section. The type of variation is still assumed to leave the varied plant output equivalent to the nominal plant, which by definition implies the existence of the input  $\underline{u}'$ , where  $\underline{u}'$  is the input to the varied plant which yields the nominal output. Therefore, the input to the open loop system is in error by

$$\delta \underline{u}_0 = \underline{u}' - \underline{u} \quad (4.2)$$

Similarly, the inputs to the feedback structures are in error by  $\delta u_1, \delta u_2$ , etc. It is not surprising that these errors are related.

The relation between  $\delta \underline{u}_0$  and  $\delta \underline{u}_1$  is now derived. Since

$$P'(\delta \underline{u}_0 + \underline{u}) = \underline{y} \quad (4.3)$$

and

$$P'(\underline{y}(u + \delta \underline{u}_1) - \mathcal{H} \underline{y}) = \underline{y}, \quad (4.4)$$

$$\delta \underline{u}_0 + \underline{u} = \underline{y}(u + \delta \underline{u}_1) - \mathcal{H} \underline{y} \quad (4.5)$$

But

$$\underline{y} = I + F \emptyset B \quad (2.8)$$

$$\mathcal{H} C \emptyset B = F \emptyset B \quad (2.7)$$

and

$$\underline{y} = C \emptyset B \underline{u} \text{ (nominal system)} \quad (4.6)$$

so that

$$\delta \underline{u}_0 = \mathcal{Y} \delta \underline{u}_1 \quad (4.7)$$

For time invariant systems,  $\mathcal{Y}$  has transform

$$G(s) = I + F \Phi_p(s) B = K(s) \quad (4.8)$$

and Equation (4.7) has transform

$$\delta \underline{U}_0(s) = K(s) \delta \underline{U}_1(s) \quad (4.9)$$

Now  $F$  is optimal for regulator problem which implies

$$K^* R K > R \text{ for all real } \omega . \quad (4.10)$$

Hence,

$$\delta \underline{U}_0^* R \delta \underline{U}_0 > \delta \underline{U}_1^* R \delta \underline{U}_1 \quad (4.11)$$

for all real  $\omega$ . Equation (4.11) shows that the input  $\underline{U}$ , which is fixed, is closer to the correct control  $\underline{U}'$  for the first order feedback system than for the open loop system. Similarly, it can be shown for higher order feedback structures that  $\underline{U}$  approaches the correct input; that is,

$$\delta \underline{U}_{i-1}^* R \delta \underline{U}_{i-1} > \delta \underline{U}_i^* R \delta \underline{U}_i \quad (4.12)$$

for all real  $\omega$  and  $i \geq 1$ . In fact, this error goes to zero for very large order feedback structures,

$$\lim_{n \rightarrow \infty} \delta \underline{U}_n^* R \delta \underline{U}_n = 0 \text{ for all real } \omega . \quad (4.13)$$

To prove Equation (4.13), define the monotonic sequence

$$R > K^{-1*} R K^{-1} > K^{-2*} R K^{-2} > \dots > K^{-n*} R K^{-n} \quad (4.14)$$

which is bounded from below by the zero matrix. Hence,

$$\lim_{n \rightarrow \infty} K^{-n*} R K^{-n} \text{ exists for all } \omega, \quad (4.15)$$

and

$$\lim_{n \rightarrow \infty} K^{-n} \text{ exists for all } \omega. \quad (4.16)$$

For any  $\delta U_0$  with finite norm,

$$\lim_{n \rightarrow \infty} K^{-n} \delta U_0 = \lim_{n \rightarrow \infty} \delta U_n = \underline{a} \quad (4.17)$$

for some  $\underline{a}$ , and

$$\begin{aligned} K \underline{a} &= K \lim_{n \rightarrow \infty} K^{-n} \delta U_0 = \lim_{n \rightarrow \infty} K^{-(n-1)} \delta U_0 \\ &= \lim_{n' \rightarrow \infty} K^{-n'} \delta U_0 = \underline{a}. \end{aligned} \quad (4.18)$$

But  $K = I + F \Phi_p B$ , so that

$$F \Phi_p B \underline{a} = \underline{0} \quad (4.19)$$

Now, the pair  $[A, B]$  is by assumption completely controllable and the pair  $[F, A]$  is by optimality completely observable. Hence, by [14]

Lemma 6

$$F \Phi_p B \text{ is nonsingular} \quad (4.20)$$

so that

$$\underline{a} = \underline{0} \quad (4.21)$$

and Equation (4.12) follows.

Thus, even if the variations are large the higher order feedback structures are capable of sensitivity reduction if the varied system remains stable.

## 5. CONCLUSIONS

### 5.1 General

It has been shown that optimally derived controllers yield systems which can be made arbitrarily insensitive to parameter variations. Theorem 1 shows that the sensitivity of a linear plant approaches zero with very high order feedback structures. Theorem 2 states that the same goal is achieved with an increase in the gains associated with the feedback structure. However, in any practical situation the gains must of course remain bounded. For any specified problem, therefore, there exists a trade-off between the number of stages which must be implemented and the gains which must be met in the implementation of any one stage. Hence, it is suggested that the relationships given here be used as guidelines, tempered by other design considerations such as realizability as a passive network, weight limitations, power requirements, and size restrictions.

The realizability of the feedback operator is the primary limitation of this method of design. Since the exact realization of the feedback operator requires infinite bandwidth, only an approximation of the feedback operator can be implemented. This limitation is partially reduced by the fact that automatic control systems are generally low pass systems. Thus the input may be restricted so that the input frequencies are bounded. Then the feedback operator can be approximated over this band of frequencies by the insertion of poles which lie

sufficiently to the left of the imaginary axis. In this process, great care must be taken to insure that high frequency instabilities are not introduced.

## 5.2 Problems for Future Study

The most severe restriction or assumption made in this development is the condition that the varied plant is output equivalent to the nominal plant. A sufficient condition for output equivalence is the existence of an inverse operator for the varied plant. However, the existence of an inverse operator is not a necessary condition for output equivalence. Consider a single input, time invariant,  $m^{\text{th}}$  order differential plant whose outputs are the states. Expression of the nominal plant and the varied plant in canonical form yields

$$A' = A + \underline{b}\underline{g}^T \text{ for some } \underline{g} \quad (5.1)$$

Thus, the varied plant is output equivalent to the nominal plant and  $u'$  is given by

$$u' = u - \underline{g}^T \underline{x} \quad (5.2)$$

where  $\underline{x}(0) = \underline{0}$ ,  $\dot{\underline{x}} = A\underline{x} + \underline{b}u$  for each  $u$ . The formulation of necessary and sufficient conditions for one system to be output equivalent to another system merits some attention.

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<p>Feedback structures which reduce the parameter sensitivity of linear system are derived from the solution of the classical linear regulator problem. Linear, time varying systems with several inputs and outputs are treated, and simplifications in the design are noted for the time invariant case.</p> <p>Throughout the discussion, problems of implementation are considered as constraints on the design of the system. Specifically, unbounded elements in the controller are now allowable as the solution to the sensitivity problem. It is shown that for the structure, herein called Nth order feedback, the sensitivity of the system may be reduced to an arbitrarily small value.</p>			

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