HEAT TRANSFER IN A PLANE INCOMPRESSIBLE LAMINAR JET

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ABSTRACT

Using the velocity components derived by Bickley from a physical formulation due to Schlichting, the thermal distribution in a plane incompressible laminar jet is determined by reducing the energy equation to an ordinary differential equation by a similarity transformation. The reduced equation is solved in terms of associated Legendre functions for all Prandtl numbers, and solutions are derived for particular Prandtl numbers by elementary methods. Some numerical results are presented.

NOMENCLATURE

$\bar{x}$ and $\bar{y}$	rectangular Cartesian coordinates [l]
$\bar{u}(\bar{x},\bar{y})$ and $\bar{v}(\bar{x},\bar{y})$	corresponding velocity components [l/t]
$\bar{t}(\bar{x},\bar{y})$	temperature [°]
$T(\bar{x},\bar{y}) = \bar{t}(\bar{x},\bar{y}) - \bar{t}(\infty,\bar{y})$	excess temperature [°]
p	pressure $[m/lt^2]$
ρ	constant density $[m/l^3]$
k	thermal conductivity $[ml/°t^3]$
c	specific heat $[l^2/°t^2]$
μ	coefficient of absolute viscosity [m/lt]
M	momentum flux per unit length $[m/t^2]$
$\alpha^3 = \rho M/\mu^2$	the reciprocal of a characteristic length $[l^{-1}]$
$x = \alpha^3 \bar{x}$ and $y = \alpha^3 \bar{y}$	dimensionless coordinates [0]
$u = \rho\bar{u}/(\alpha^3\mu)$ and $v = \rho\bar{v}/(\alpha^3\mu)$	corresponding dimensionless velocities [0]
$\theta = c\rho^2 T/(\alpha^6\mu^2)$	dimensionless excess temperature [0]
$\sigma = \mu c/k$	Prandtl number [0]
$\beta = (48)^{-1/3} = 0.2751606040$	a numerical constant [0]
$\xi = y/x^{2/3}$	similarity parameter [0]
$\psi(x,y)$	dimensionless stream function [0]
$\eta = \tanh(\beta\xi)$	transformed similarity parameter [0]
$F(\xi)$ and $G(\xi)$	dimensionless auxiliary functions [0]
$P_\nu^\mu(\eta)$ and $Q_\nu^\mu(\eta)$	associated Lengendre function of order μ and degree ν of the first and second kind respectively [0]

INTRODUCTION

The theory of a plane incompressible laminar jet, i.e., the pressurized slow flow of a liquid through a linear slit in a wall into the same liquid, can be treated by boundary layer theory provided that one idealizes the problem in the obvious way. The hydrodynamic aspects of the problem were first formulated by Schlichting [1] who showed that the analytical treatment could be reduced by a similarity transformation to the solution of an ordinary differential equation with prescribed two point boundary conditions. Subsequently, a closed form solution to the differential system was obtained by Bickley [2]. Excellent reviews [3] of the analysis are readily available.

Yih [4] has incorrectly treated the associated heat transfer problem by requiring the viscous dissipation to vanish identically. This requirement was used to derive a similarity transformation in a precise analogy with Schlichting's treatment of the hydrodynamic problem where the momentum flux across any plane normal to the jet is constant. Subsequently, Yih was able to obtain a closed form solution to the truncated energy equation using his improperly obtained transformation. Needless to say, the whole procedure used by Yih is highly suspect, for there is a decided difference between neglecting the dissipation term in the energy equation after one has derived the correct similarity transformation and deriving an incorrect similarity transformation from the artificial requirement that the energy flux per unit depth of the slit be constant for each normal plane (i.e., the dissipation vanishes identically).

In view of the continuing interest in the plane jet, it appears worthwhile to present an exact treatment for the thermodynamic situation within the framework of boundary layer theory. It is interesting to note that the exact treatment is completely different from the incorrect theory due to Yih.

I. FORMULATION OF THE PROBLEM

A mathematical description of the pertinent phenomena will be aided by reference to Fig. 1, a schematic diagram of a plane laminar jet.

The differential equations that describe the flow are the continuity equation, the momentum equation with the boundary layer approximation and the energy equation in its boundary layer form; these three equations are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1.1}$$

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}\right)=-\frac{dp}{d\bar{x}}+\mu\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \tag{1.2}$$

and

$$\rho c\left(\bar{u}\frac{\partial T}{\partial\bar{x}}+\bar{v}\frac{\partial T}{\partial\bar{y}}\right)=k\left(\frac{\partial^2 T}{\partial\bar{y}^2}\right)+\mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2 \tag{1.3}$$

The boundary conditions are

$$\bar{u}(\bar{x},\pm\infty)=0,\ \bar{v}(\bar{x},o)=0,\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_{\bar{y}=0}=0,\ T\ (\infty,\bar{y})=0\ \text{and}\left(\frac{\partial T}{\partial\bar{y}}\right)_{\bar{y}=0}=0 \tag{1.4}$$

In addition there are two other important considerations: the pressure field is taken to be constant and the flux of momentum per unit depth of the jet across any plane normal to the $\bar{x}$ - axis is a constant. Analytical formulations of these considerations yield

$$\frac{dp}{d\bar{x}}=0 \qquad \text{and}\quad 2\int_0^\infty \rho\,\bar{u}^2\,d\bar{y}=\text{const.}=M \tag{1.5}$$

One immediately notes that $\alpha^3=\rho M/\mu^2$ is a reciprocal length, and the fundamental equations may then be put into dimensionless forms:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1.6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2} \tag{1.7}$$

and

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{1}{\sigma}\frac{\partial^2\theta}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2 \tag{1.8}$$

The transformed boundary conditions are

$$u(x,\pm\infty)=0,\ \ v(x,0)=0,\left(\frac{\partial u}{\partial y}\right)_{y=0}=0,\ \theta(\infty,y)=0\ \text{and}\left(\frac{\partial\theta}{\partial y}\right)_{y=0}=0 \tag{1.9}$$

II. SOLUTION OF THE HYDRODYNAMIC PROBLEM

As previously remarked Schliching has shown that the introduction of a stream function, $\Psi=x^{1/3}F(\xi)$, where $\xi=y/x^{2/3}$, reduces the solution of Eqs. (1.6) and (1.7) to the integration of an ordinary differential equation, and Bickley has integrated the resulting equation into the following closed forms:

$$u=6\beta^2\,[\text{sech}^2(\beta\xi)]\,/x^{1/3} \tag{2.1}$$

and $$v = 2\beta\,[2\beta\xi\,\mathrm{sech}^2(\beta\xi) - \tanh(\beta\xi)]/x^{2/3} \tag{2.2}$$

The dimensionless streamlines, $\psi(x,y)$ = const., are shown in Fig. 2(a) for the case in which α^{-3} is equal to the unit of length in terms of which the other physical quantities are measured, i.e., it is the dimensionless momentum, $\rho M/\mu^2$, per unit depth of the line along which the unidimensional force is applied. Fig. 2(b) shows the dimensionless velocity components for the same case; the u-profiles are also shown.

III. SOLUTION OF THE THERMODYNAMIC PROBLEM

If Eqs. (2.1) and (2.2) are inserted into Eq. (1.8), the energy equation becomes

$$\frac{6\beta^2\mathrm{sech}^2(\beta\xi)}{x^{1/3}}\frac{\partial\theta}{\partial y} + \frac{2\beta[2\beta\xi\,\mathrm{sech}^2(\beta\xi) - \tanh(\beta\xi)]}{x^{2/3}}\frac{\partial\theta}{\partial y} = \frac{1}{\sigma}\frac{\partial^2\theta}{\partial y^2} + \frac{144\beta^6\mathrm{sech}^4(\beta\xi)\tanh^2(\beta\xi)}{x^2} \tag{3.1}$$

Some algebraic manipulation shows that the similarity transformation, $\theta = \beta^4 x^{-2/3}G(\xi)$, reduces the partial differential equation (3.1) to the ordinary differential equation:

$$-2[\tanh(\beta\xi)]\,G' - 4[\mathrm{sech}^2(\beta\xi)]G = G''/\sigma + 144\,\mathrm{sech}^4(\beta\xi)\tanh^2(\beta\xi) \tag{3.2}$$

where the primes denote differentiation with respect to $\beta\xi$. The boundary condition due to the symmetry of the thermal field about the axis of the jet becomes

$$G'(0) = 0 \tag{3.3}$$

The form of Eq. (3.2) may be simplified further by putting $\eta = \tanh(\beta\xi)$; the simplified equation is

$$(1-\eta^2)G'' - 2(1-\sigma)\,\eta G' + 4\sigma G = -\ 144\,\sigma\,\eta^2(1-\eta^2) \tag{3.4}$$

where the primes now denote differentiation with respect to η. Eq. (3.4) can be easily transformed into the standard form for associated Legendre functions, and the general solution for the homogeneous part is

$$G(\eta) = A(\eta^2-1)^{-\sigma/2}P_\nu^{-\sigma}(\eta) + B\,(\eta^2-1)^{-\sigma/2}Q_\nu^{-\sigma}(\eta) \tag{3.5}$$

where $\nu = \left[-1 \pm (4\sigma^2 + 20\sigma + 1)^{\frac{1}{2}}\right]/2$ and where P_ν^μ and Q_ν^μ are the associated Legendre functions of order μ and degree ν of the first and second kinds respectively [5]. The complete solution may be obtained in terms of quadratures by the method of variation of parameters, and one notes that the Wronskian has a particularly simple form $e/(n^2-1)$ [6].

Whenever the Prandtl number is an integer or whenever ν is an integer, the associated Legendre functions become derivatives of Legendre functions or simple polynomials in η [7]. However, the specific properties and the numerical values of the associated Legendre functions are generally unknown; consequently, it seems appropriate to give as illustrative examples some solutions of Eq. (3.4) for particular σ which can be obtained by elementary methods.

The complete solution of Eq. (3.4) may be written in the following form:

$$G(\eta) = A\,G_1(\eta) + B\,G_2(\eta) + G_p(\eta) \tag{3.6}$$

where $G_1(\eta)$ and $G_2(\eta)$ are general solutions of the homogeneous equation and where $G_p(\eta)$ is a particular integral. A particular integral in polynomial form may be found by the method of undetermined constants; this integral is

$$G_p(\eta) = q + r\,\eta^2 + s\eta^4 \tag{3.7}$$

where

$$q = \frac{36\,(3\sigma - 2)}{(3\sigma - 5)\,(4\sigma - 3)}$$

$$r = \frac{-72\sigma\,(3\sigma - 2)}{(3\sigma - 5)\,(4\sigma - 3)}$$

and

$$s = \frac{36\sigma}{(3\sigma - 5)}$$

Obviously the Prandtl numbers, $\sigma = 3/4$ and $\sigma = 5/3$, are not included, and these values require the more elaborate treatment which was previously outlined. If the parametric dependence of the solution on the Prandtl number is indicated by $G(\eta,\sigma)$ and if it is assumed that

$$G(\eta,\sigma) = \sum_{i} C_{2i}\,\eta^{2i} \tag{3.8}$$

where the even powers of η are required by the symmetry of the thermal field, one can derive a recurrence relation between the coefficients by putting Eq. (3.8) into the homogeneous part of Eq. (3.4). This procedure yields

$$\Big[(2i + 2)\,(2i + 1)\Big]\quad C_{2i+2} = \Big[2i(2i + 1 - 2\sigma) - 4\sigma\Big]\,C_{2i} \tag{3.9}$$

If the series in Eq. (3.8) is to terminate, then $2i(2i + 1 - 2\sigma) - 4\sigma = 0$ or

$$\sigma = \frac{i(2i + 1)}{2(i + 1)} \tag{3.10}$$

Unfortunately, the cases i = 1 and i = 2 are the exceptionally restricted ones, $\sigma = 3/4$ and $\sigma = 5/3$, so that the first simply complete solution occurs for i = 3 or $\sigma = 21/8$. This case will be treated as a particularly simple and illustrative example. The complete solution for $\sigma = 21/8$ is

$$G(\eta, 21/8) = A\left[1 - (21/4)\eta^2 + (105/16)\eta^4 - \left(\frac{161}{64}\right)\eta^6\right] + \left[3780\eta^4 - 5922\eta^2 + 1128\right]/115 \tag{3.11}$$

The constant A is determined by the strength of the heat source at the origin or, equivalently, by assigning the value of the temperature at a point. Figs. 3,4 and 5 show the thermal distributions for A = 0, 10, and 100 respectively. The case A = 0 yields the thermal field due to viscous dissipation alone. The case A = 10 is one in which the effects of viscous dissipation and convective transfer are comparable. The case A = 100 is the usual case in which viscous dissipation may be neglected.

Of course, there are many other cases which have simple solutions in different forms; an interesting example of these is the case $\sigma = 1/2$. When $\sigma = 1/2$, the differential equation becomes

$$(1 - \eta^2)\, G''(\eta) - \eta\, G'(\eta) + 2\, G(\eta) = 0 \tag{3.12}$$

which can be transformed to

$$G''(\phi) + 2\, G(\phi) = 0 \tag{3.13}$$

by the substitution $\eta = \sin \phi$ [8]. The pertinently complete solution is

$$G(\eta, 1/2) = A \cos (2^{1/2} \sin^{-1} \eta) - 36\,(\eta^4 - \eta^2 + 1)/7 \tag{3.14}$$

CONCLUSION

It is shown that the partial differential equation for the heat transfer in a plane incompressible laminar jet can be reduced by a similarity transformation to an ordinary differential equation that can be solved in terms of associated Legendre functions. The particular instance when the Prandtl number of the fluid is in the form, $\sigma = [i(2i + 1)]/[2/i + 1)]$, is treated by elementary methods; the case, $\sigma = 1/2$, is also treated by another elementary method.

ACKNOWLEDGMENT

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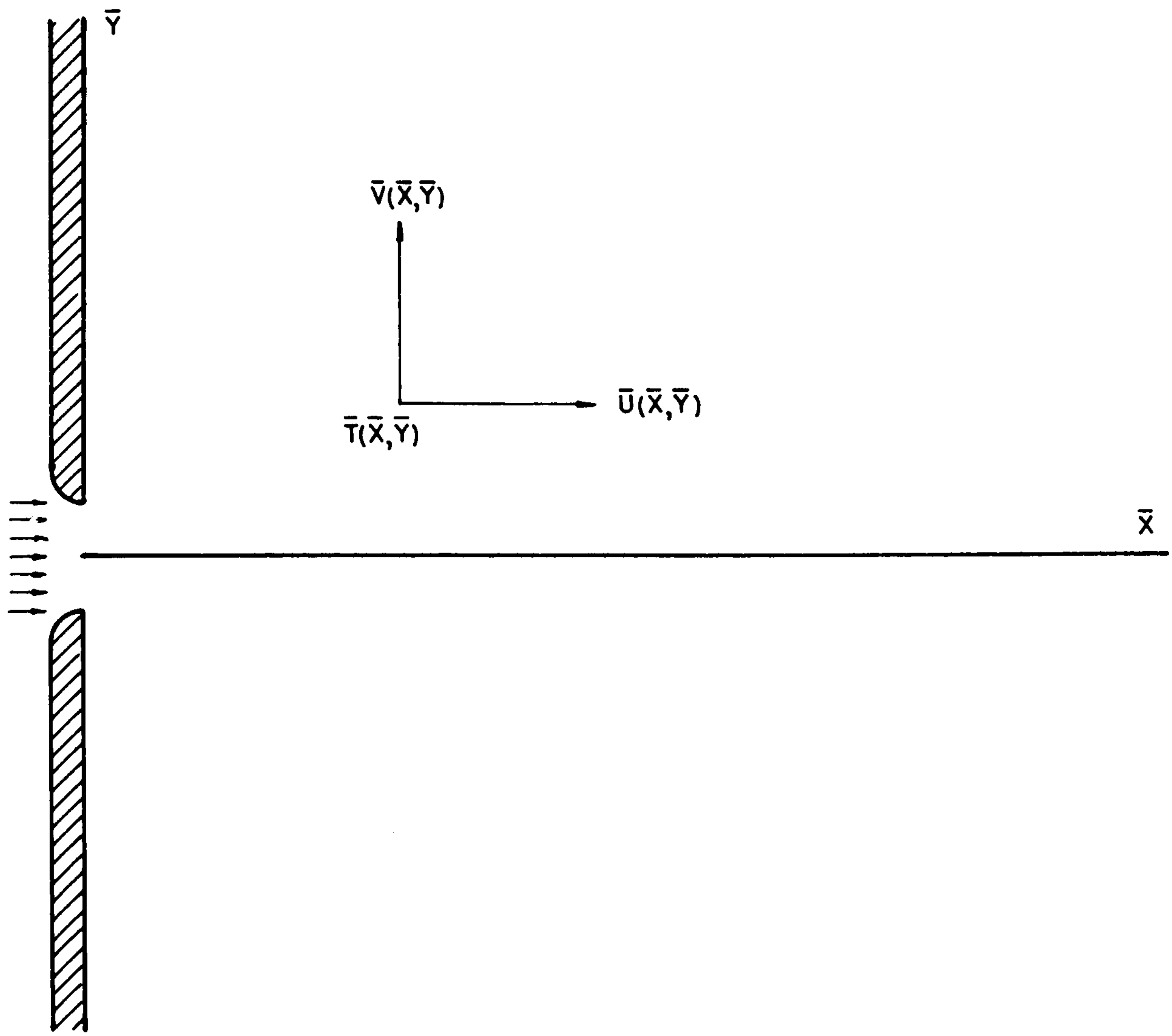

Fig. 1, A Schematic Diagram of a Plane Laminar Jet.

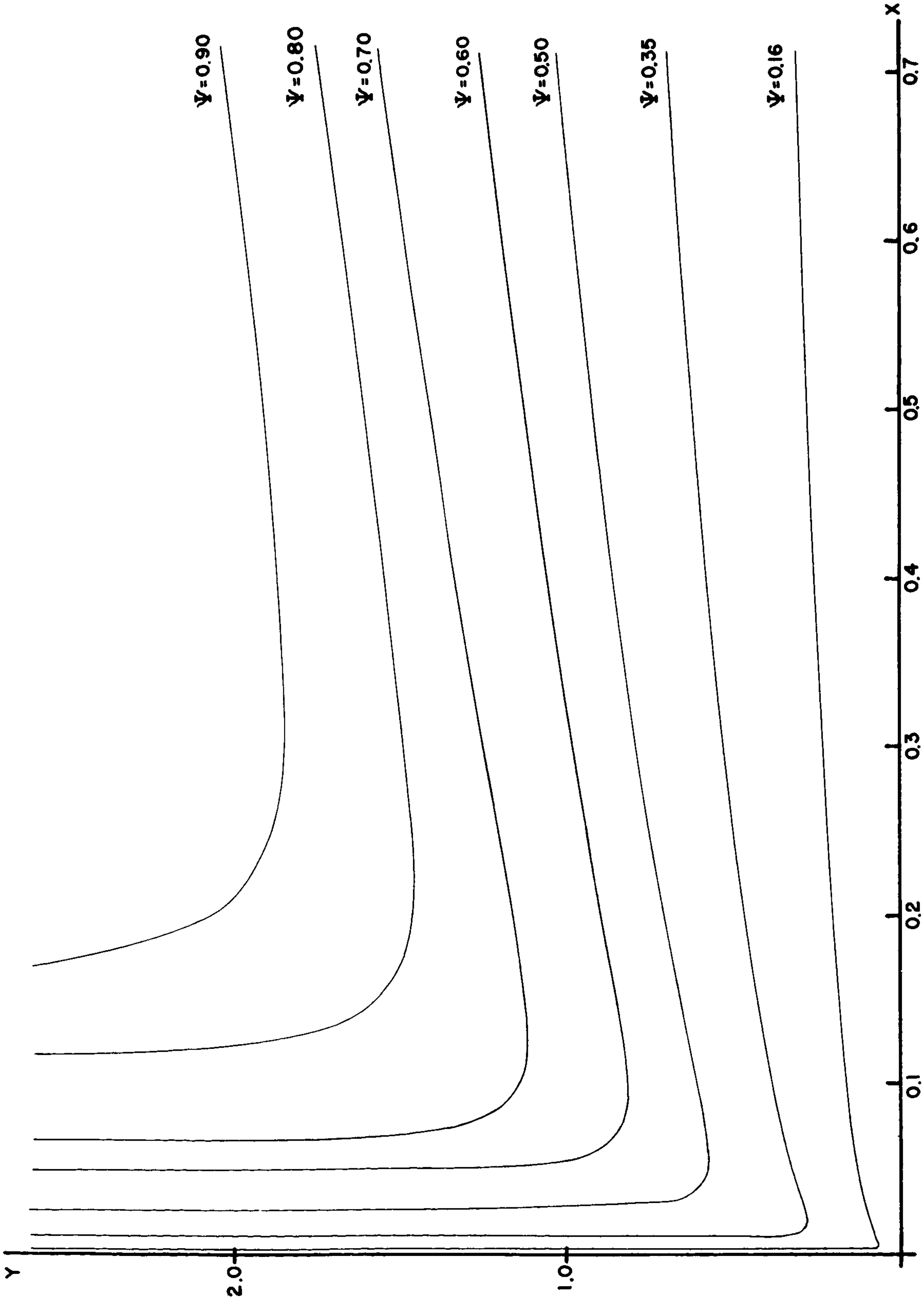

FIG. 2(a)

Fig. 2(a), Dimensionless Streamlines for a Plane Incompressible Laminar Jet for α^{-3} Equal to the Unit Length.

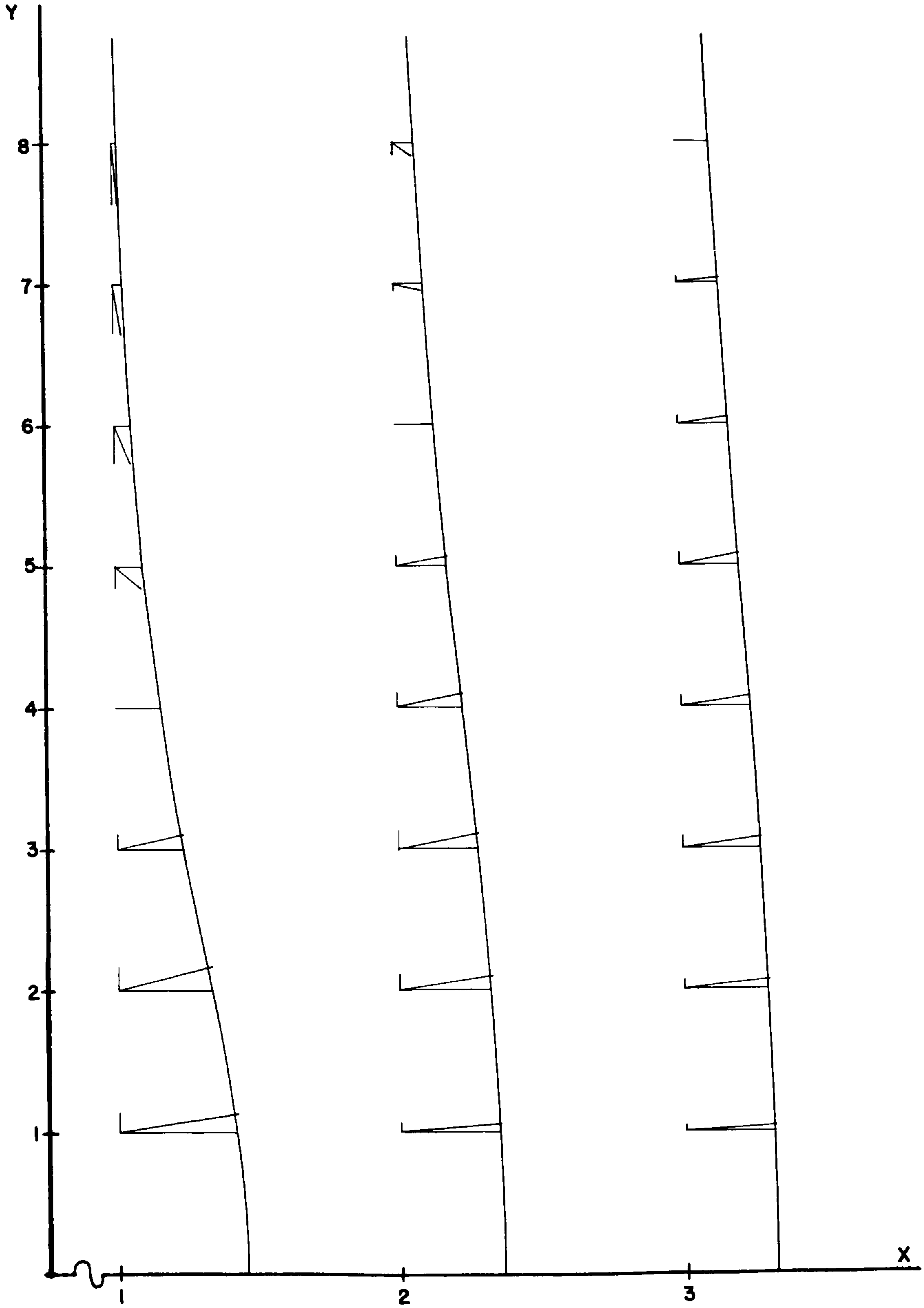

Fig. 2(b), Selected Dimensionless Velocity Components and Corresponding Horizontal Dimensionless Velocity Profiles for a Plane Incompressible Laminar Jet for a^{-3} Equal to the Unit Length.

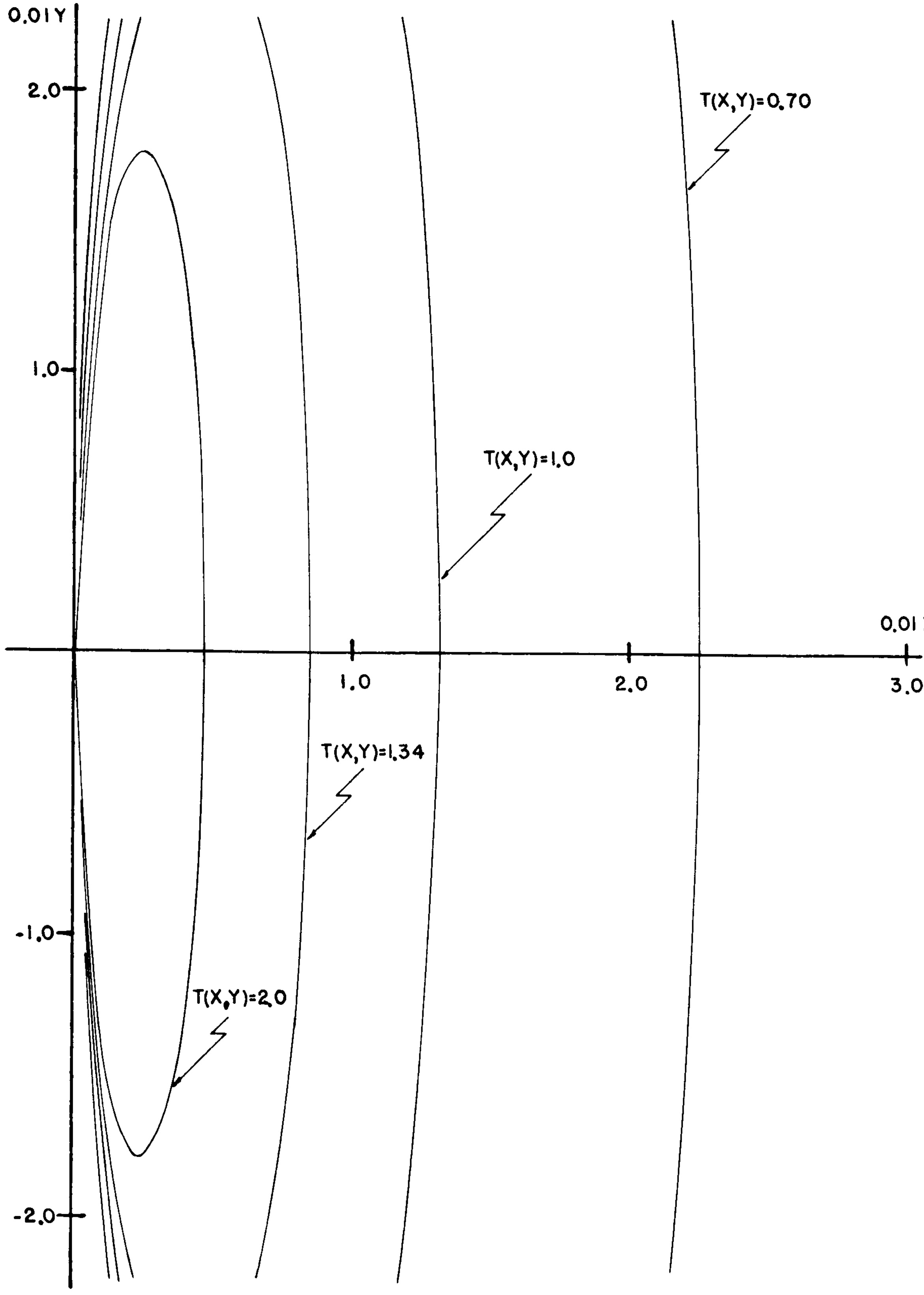

Fig. 3, The Dimensionless Thermal Field due to Viscous Dissipation in a Plane Incompressible Laminar Jet for α^{-3} Equal to the Unit Length. The Prandtl number is 21/8.

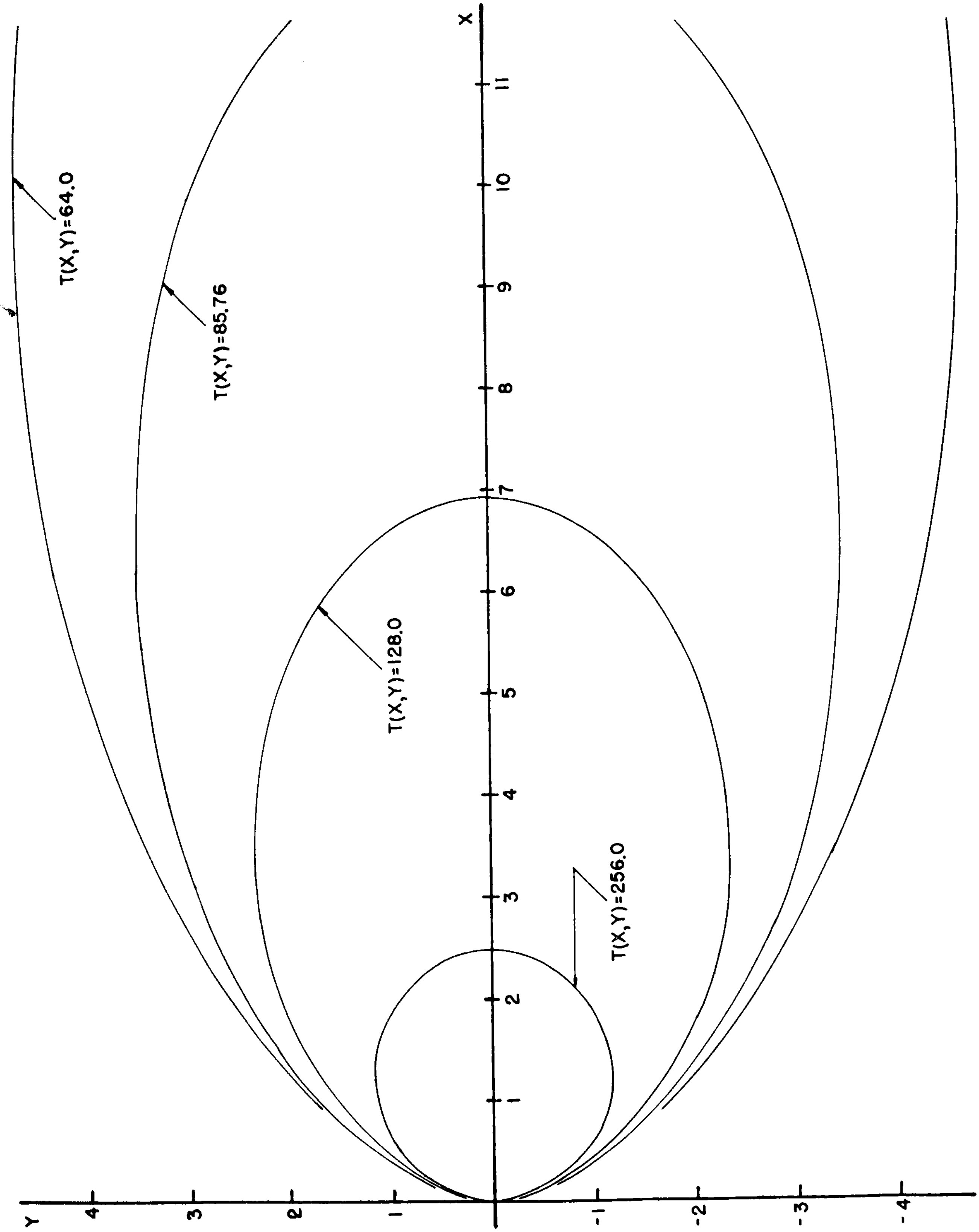

FIG. 4

Fig. 4, The Dimensionless Thermal Field for Comparable Effects due to Viscous Dissipation and Convective Processes in a Plane Incompressible Laminar Jet for α^{-3} Equal to the Unit Length. The Prandtl number is 21/8.

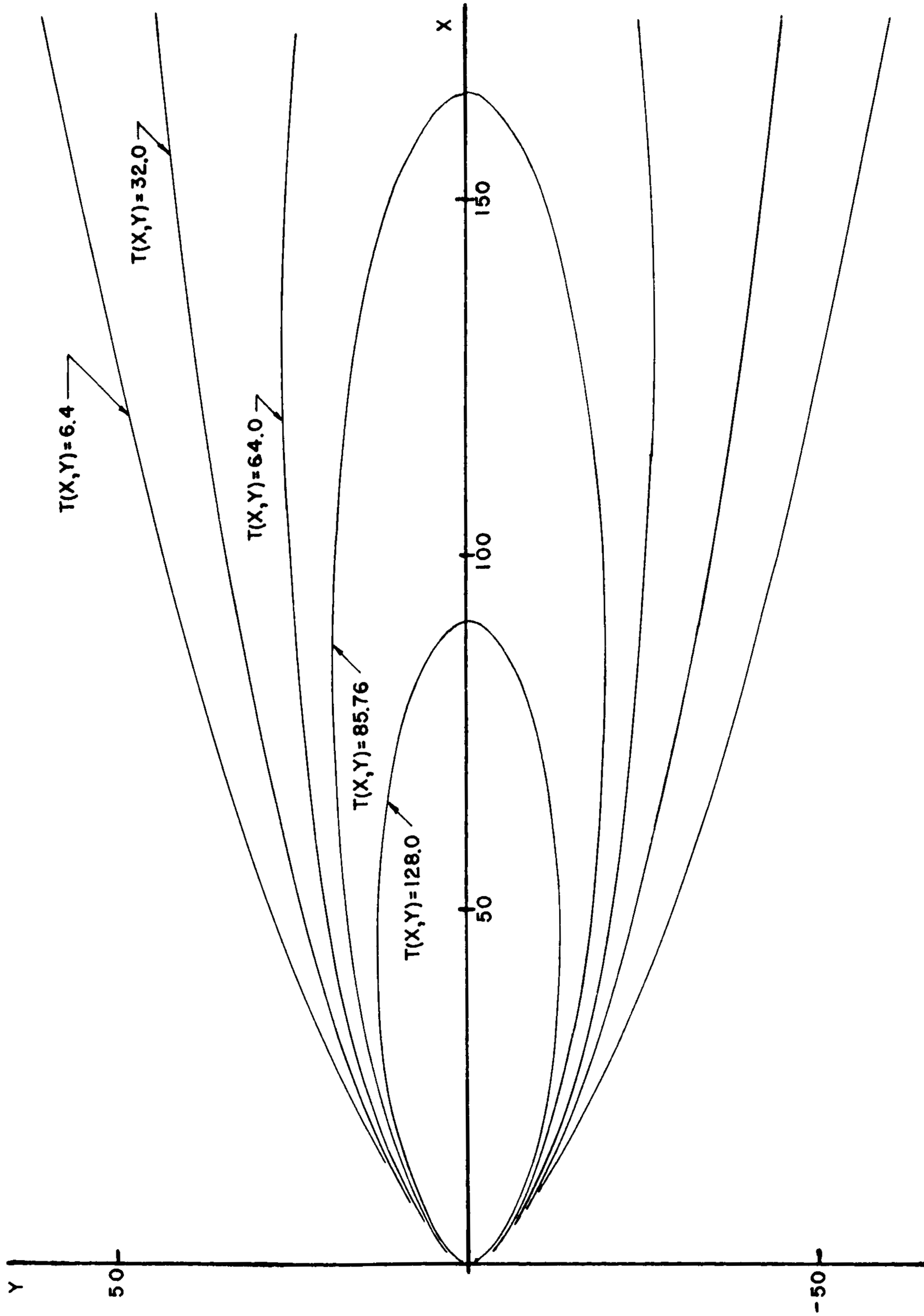

FIG. 5

Fig. 5, The Dimensionless Thermal Field Neglecting Viscous Dissipation in a Plane Incompressible Laminar Jet for a^{-3} Equal to the Unit Length. The Prandtl number is 21/8.