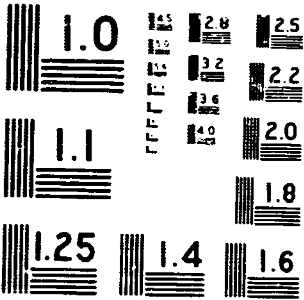
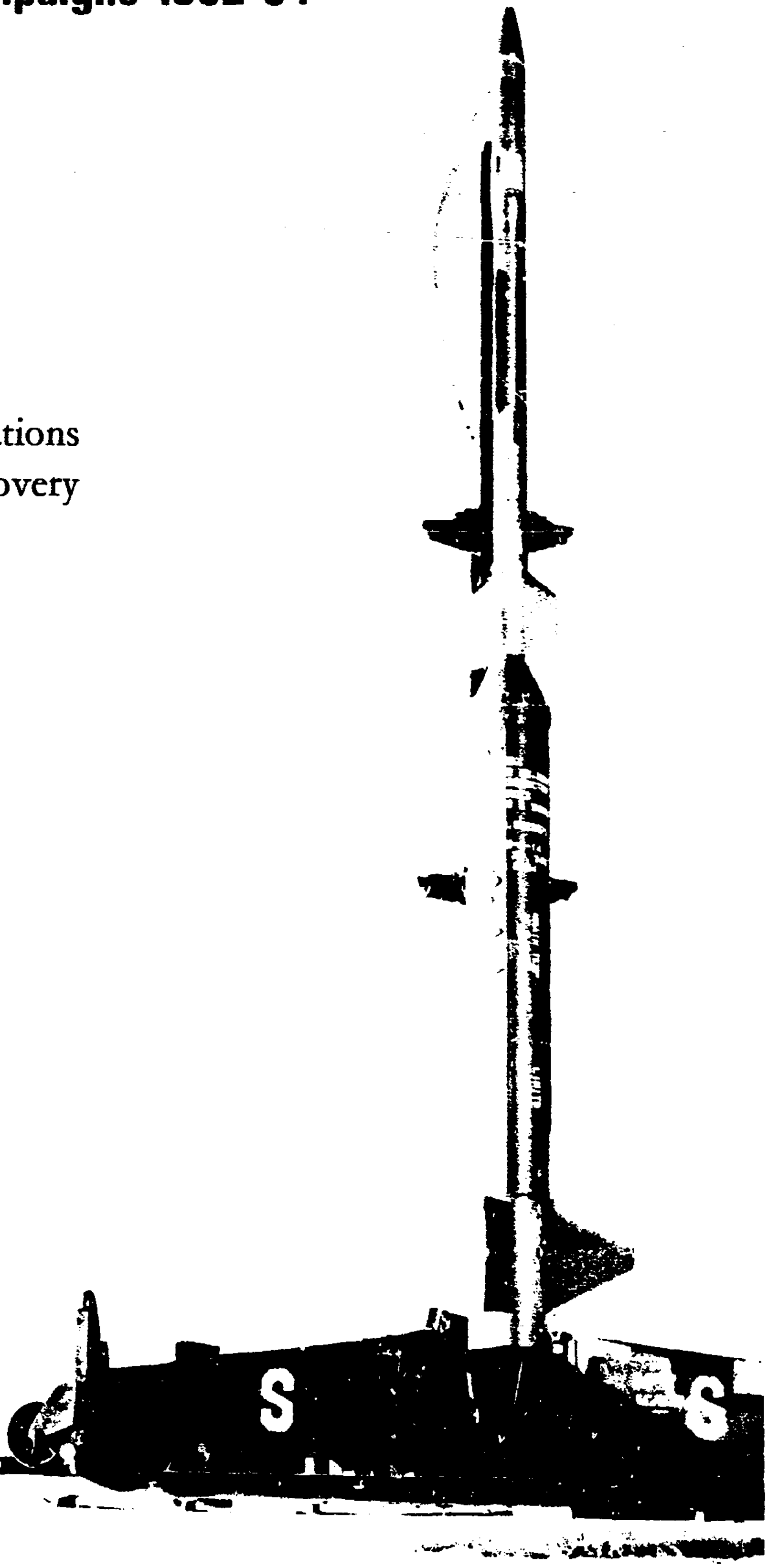


MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS – 1963

SWEDISH SPACE RESEARCH COMMITTEE
The Kronogård campaigns 1962-64



No 67-K2

Sound ranging computations for sounding rocket recovery

Johan Martin-Löf

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SOUND RANGING COMPUTATIONS FOR SOUNDING ROCKET RECOVERY

Johan Martin-Löf

ABSTRACT

An acoustic method to locate an object that enters into the earth's atmosphere at supersonic speed is presented. The direction to the object is computed from shock wave arrival times observed with the aid of infra-sonic microphones. A numerical method for the computations is described. This method gives a least squares estimate of the direction of the normal to an idealized plane shock wave. It has been successfully used in a computer program to reduce data collected during the sounding rocket experiments in northern Sweden in 1964.

FOREWORD

The Kronogård reports

During the summer of 1962, 1963 and 1964 a series of sounding rocket experiments were performed at Kronogård in northern Sweden under a cooperative agreement between the US National Aeronautics and Space Administration (NASA) and the Swedish Space Research Committee. The main experimenter on the Swedish side was the Institute of Meteorology, University of Stockholm and on the US side groups from USAF Cambridge Research Laboratories (AFCRL) and NASA Goddard Space Flight Center.

The Swedish Space Research Committee set up a technical group to take care of the technical and operational parts of the experiments. While the scientific results from the experiments have been and will be published by the experimenters, this group is preparing a special series of reports covering its activities during the campaigns.

The group is since the 1st of July 1965 a division of TUAB, Teleutredningar AB under the name of Space Technology Group.

CONTENTS

		page
1.	Introduction	4
2.	The mathematical problem	4
	2.1. Non-degenerate case	7
	2.2. Degenerate case	13
	2.3. Numerical difficulties	15
3.	Effect of the idealizations	16
4.	Practical experience	17
5.	Conclusions	18
6.	References	18
	Figures	19

1. INTRODUCTION

In many sounding rocket experiments it is necessary to recover the instrumentation after the flight for examination. Usually the payload is separated from the rocket and is slowed down by a parachute which gives a moderate descent rate. The impact can be located by means of radar or, if radar is not available, by some active homing system in the payload. In case of separation or power failure in the rocket, these systems might give no data, however, and a back-up system is desirable.

Sound ranging is very well suited for this purpose. During reentry a sounding rocket creates one or more shock waves which propagate through the atmosphere. They can easily be detected by means of low-frequency microphone systems located in surveyed positions on the ground. From the observed arrival times, it is possible to estimate the direction to the origin of the shock waves from each microphone system and thus by extrapolation to get a fix on the impact point. The minimum number of microphones in each system is three, but it is preferable to have more in order to get redundancy and an estimate of errors in the result.

This report deals with the numerical problem of computing the direction of incidence of a plane sound wave from observed arrival times to a microphone system. It will be shown that it is a linear least squares problem with a quadratic constraint. The problem is often ill-conditioned especially for low elevation angles of the wave normal. Additional difficulties arise because the problem degenerates when the microphones lie in a common plane. A method will be presented which gives a unique solution in all practical cases, also where previous methods (1, 2) have failed. In the presentation will be used a geometric interpretation to visualize the numerical problem.

2. THE MATHEMATICAL PROBLEM

The aim is to obtain an estimate of the direction of the normal and the time of arrival of a plane sound wave from observations of the arrival to a system of microphones.

The speed of sound is assumed constant in the lowest part of atmosphere over the microphones. Wind influence on the speed of sound is neglected. These idealizations are tolerable as will be discussed below in paragraph 3.

The number of microphones is m. Their positions are given relative to an arbitrary cartesian coordinate system by the column vectors:

$$\underline{r}_i = \begin{bmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{bmatrix} \qquad i = 1, \ldots m$$

In order to obtain a solution it is necessary that $m \geq 3$.

$\underline{n}$ is the unit wave normal pointing in the direction from which the wave is coming.
c is the speed of sound.
$t_1, \ldots t_m$ are the observed arrival times at the microphones.

In practice it is possible to determine the microphone position coordinates with an accuracy that is one order of magnitude better than the accuracy in the determination of the arrival times. This is due to the fact that the sound recordings are made with the aid of low-frequency microphones which are disturbed by the general noise in the atmosphere. We will thus here neglect the errors in the microphone positions $\underline{r}_i$.

The arrival time at the origin of the coordinate system can be calculated from each observation by

$$t_i' = t_i + (\underline{r}_i^T \underline{n})/c \qquad i = 1, \ldots m$$

The observations are given equal weight and the average of these quantities, t_o, is taken as estimate of the time of arrival at the origin.

Introducing the average values

$$\bar{t} = \frac{1}{m}\sum_{i=1}^{m} t_i \qquad\qquad \bar{\underline{r}} = \frac{1}{m}\sum_{i=i}^{m} \underline{r}_i$$

we get

$$t_o = \frac{1}{m}\sum_{i=1}^{m} t_i' = \bar{t} + (\bar{\underline{r}}^T \underline{n})/c$$

We choose our estimate of $\underline{n}$ so that the mean square deviation from t_o is minimized.

Thus we seek the minimum of the error function

$$\tau(\underline{n}) = \sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} (t_i' - t_o)^2 =$$

$$= \sum_{i=1}^{m} \left\{(t_1 - \bar{t}) + (\underline{r}_i^T - \bar{\underline{r}}^T)\underline{n}/c\right\}^2$$

with the subsidiary condition $|\underline{n}| = 1$

introduce new coordinates

$$u_i = t_i - \bar{t}$$

$$i = 1, \dots m$$

$$\underline{s}_i = (\underline{r}_i - \bar{\underline{r}})/c$$

and the matrix

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ \cdot & & \\ \cdot & & \\ s_{m1} & s_{m2} & s_{m3} \end{bmatrix}$$

Our problem is equivalent to obtaining the least squares solution of the system of linear equations

$$S\underline{n} = -\underline{u}$$

with the quadratic constraint

$$|\underline{n}|^2 = \underline{n}^T \underline{n} = 1$$

We have two cases depending on the rank of the matrix S.
If the vectors

$$\underline{s}_1 \dots \underline{s}_m$$

span the whole 3-dimensional space the rank is 3, otherwise it is 2 and we have a degenerate problem.
As the coordinates $\underline{s}_i$ have the property

$$\sum_{i=1}^{m} \underline{s}_i = 0$$

the vectors span the whole space only if $m \geq 4$ and if the microphones are not located in a common plane. For numerical reasons it is also necessary that the microphones are not even very close to a common plane.

2.1. Non-degenerate case

We can write the error function

$$F(\underline{n}) = |\underline{u} + S\underline{n}|^2 = (\underline{u} + S\underline{n})^T (\underline{u} + S\underline{n})$$

The symmetric matrix S^TS is positive definite and can thus be inverted. The function F can then be written:

$$F(\underline{n}) = (\underline{n} + (S^TS)^{-1}S^T\underline{u})^T S^TS(\underline{n} + (S^TS)^{-1}S^T\underline{u}) +$$

$$+ \underline{u}^T (I - S(S^TS)^{-1}S^T)\underline{u}$$

The second term is constant.
As $S^T S$ is positive definite the set of surfaces

$$F(\underline{n}) = \text{const.}$$

is a set of ellipsoids with their common center in

$$\underline{C} = -(S^T S)^{-1} S^T \underline{u}$$

Our problem is equivalent to finding the tangent point between the unit sphere

$$\underline{n}^T \underline{n} = 1$$

and the smallest ellipsoid in the set that touches this sphere. In this point the normal to the ellipsoid is also normal to the sphere. The normal to an ellipsoid is parallel to the gradient

$$\text{grad } F = 2S^T(\underline{u} + S\underline{n})$$

Thus tangent point is the solution of the non-linear equation

$$\underline{n} = \pm \frac{\text{grad } F}{|\text{grad } F|} = \pm \frac{S^T\underline{u} + S^T S\underline{n}}{|S^T\underline{u} + S^T S\underline{n}|}$$

The plus sign is used if the center $\underline{C}$ of the ellipsoids is outside the unit sphere and the minus sign if it is inside. In the special case that it is exactly on the surface of the sphere we have the immediate solution

$$\underline{n} = \underline{C}$$

which is the well known solution to the problem without the constraint.

Generally it is necessary to make an iterative solution to the problem and we will here first discuss the previous methods that have been used and why they often fail in recovery applications.

Our microphone system is for practical reasons placed on the ground so that the vertical separation between microphones is much smaller than the horisontal. This leads to poor accuracy in elevation and consequently the error ellipsoids are long and cigarshaped with their long axis roughly along the vertical. For high elevations the equi-error contours on the unit sphere are essentially circular with the minimum point in the middle. For lower elevations the equi-error contours get more and more oval and the minimum is located in a "valley" that is very sharp in the azimuth direction and shallow in the elevation direction.

In previous reports (1, 2) dealing with this problem the following iterative method has been used to find the minimum. Starting from an approximate point on the unit sphere the minimum point in the tangent plane is located. This latter point is no longer on the unit sphere, but normalizing its position vector gives a new and better approximation of the minimum on the sphere. The method has been successfully used by the author of this report in connection with the rocket grenade experiment, where the elevation is around 80°.

When trying to use this method for recovery computations, however, it was frequently found that no convergence could be obtained due to the fact that the minimum is in a narrow valley.

Therefore a quite different method has been devised. Let us study the locus of all points on the error ellipsoids from which the normal passes through the origin. Introducing a parameter k, the points on this locus are the solutions to the equation

$$\text{grad } F(\underline{n}) + k\underline{n} = 0$$

or

$$(S^T\underline{u} + S^T S\underline{n}) + k\underline{n} = 0$$

or

$$\underline{n}(k) = -(S^T S + kI)^{-1} S^T \underline{u}$$

The function $\underline{n}(k)$ is a continuous function of k and traces all points on the locus as k passes from $+\infty$ to $-\infty$. Let us study its general shape (see fig. 1). For k = 0 we have

$\underline{n}(0) = \underline{C}$

That is the center of the ellipsoids. As k increases towards $+\infty$ the trajectory approaches the origin. As k decreases towards $-\lambda_3$ where λ_3 is the smallest of the (positive) eigenvalues of S^TS the trajectory goes towards infinity along the direction of the correspon-ding eigenvector. As k passes $-\lambda_3$ the trajectory comes back from infinity along the opposite direction. As k further passes $-\lambda_2$ and $-\lambda_1$ where λ_2 and λ_1 are the next largest and the largest eigenvalue of S^TS, the trajectory shows a similar behaviour with respect to the corresponding eigenvectors. Finally, as k approaches $-\infty$ the tra-jectory approaches the origin again.

This trajectory can have 2, 4 or 6 intersections with the unit sphere and thus there exists a corresponding number of points from which a unit normal can be drawn from an ellipsoid to the origin. In half of these points the error function F is smaller, in the others it is greater than in a neighbourhood on the unit sphere.

The k-values corresponding to these points are the solutions of the equation

$$|\underline{n}(k)|^2 = 1$$

which is of order 6.

To simplify the equation let us rotate the coordinate system with an orthogonal matrix Q that diagonalizes S^TS.

$$Q^TS^TSQ = D \qquad Q^T = Q^{-1}$$

The columns of Q are the eigenvectors of S^TS. The diagonal elements of D are the corresponding eigenvalues λ_1, λ_2 and λ_3. Suppose they are ordered: $\lambda_1 > \lambda_2 > \lambda_3$. Multiplying the expressions for $\underline{n}(k)$ from the left with Q^T we get:

$$Q^T\underline{n}(k) = -Q^T(S^TS + kI)^{-1}QQ^TS^T\underline{u} =$$

$$= \begin{bmatrix} -1/(k+\lambda_1) & 0 & 0 \\ 0 & -1/(k+\lambda_2) & 0 \\ 0 & 0 & -1/(k+\lambda_3) \end{bmatrix} \underline{v} = \begin{bmatrix} -v_1/(k+\lambda_1) \\ -v_2/(k+\lambda_2) \\ -v_3/(k+\lambda_3) \end{bmatrix}$$

and

$$Q^T \underline{C} = \begin{bmatrix} -v_1/\lambda_1 \\ -v_2/\lambda_2 \\ -v_3/\lambda_3 \end{bmatrix}$$

where we have introduced the vector

$$\underline{v} = Q^T S^T \underline{u}$$

Now the subsidiary condition

$|\underline{n}|^2 = 1$ is equivalent to

$|Q^T \underline{n}| = 1$ as det $Q = 1$

Put

$$g(k) = |Q^T \underline{n}|^2 - 1 = \sum_{i=1}^{3} \frac{v_i^2}{(k+\lambda_1)^2} - 1$$

Thus our equation is transformed into $g(k) = 0$.

The general behaviour of $g(k)$ is sketched in fig. 2 which shows a case where the equation has 4 solutions. There will always be two solutions, one to the right of $-\lambda_3$ and one to the left of $-\lambda_1$. There may be 2 or 4 more depending on the position of the middle branches of the curve. To find the solutions the method of Newton-Raphson can be employed. Thus we need the derivative

$$g'(k) = -\sum_{i=1}^{3} \frac{2v_i}{(k+\lambda_i)^3}$$

Starting from a first approximation $k^{(0)}$ a sequence $k^{(1)}$, $k^{(2)}$ ... is formed where

$$k^{(j+1)} = k^{(j)} - \frac{g(k^{(j)})}{g'(k^{(j)})}$$

This sequence will converge towards the solutions and it can be broken off when the desired accuracy is reached.

A suitable choice of starting points $k^{(0)}$ is necessary. It should have the property

$$g(k^{(0)}) > 0$$

to avoid that the sequence jumps between the branches. A good choice is

$$k^{(0)} = -\lambda_i \pm |v_i| \qquad i = 1, 2, 3$$

From these points the sequence will monotonously converge to the corresponding solutions, if any. When no solution exists on the branch in question there will be a jump to another branch, and thus in the results we will find the same solution repeated several times.

The normal vector $\underline{n}$ is then easily obtained by rotating back to the original coordinate system.

$$\underline{n} = Q \begin{bmatrix} \frac{-v_1}{k+\lambda_1} \\ \frac{-v_2}{k+\lambda_2} \\ \frac{-v_3}{k+\lambda_3} \end{bmatrix}$$

From this we find the azimuth and the elevation of the wave normal. The arrival time at the origin is

$$t_0 = \bar{t} + \bar{\underline{r}}^T \underline{n}/ c$$

and the errors in the microphone arrival times

$$e_i = (t_i - \bar{t}) + (\underline{r}_i^T - \bar{\underline{r}}^T)\underline{n}/ c$$

The value of

$$F(\underline{n}) = \sum_i e_i^2$$

gives a measure of the goodness of fit in the plane wave approximation.

2.2. Degenerate case

In case that the microphones are all in a common plane the rank of the matrix S is 2. This is always the fact when the number of microphones is 3. Even if there are more microphones they may come so close to a common plane that numerical difficulties with the previously described method arise. In this case the microphone configuration should also be treated as planar. (We will here not treat the problem when the microphones are on a common straight line).

In the planar case the matrix S^TS is positive semidefinite. One of the eigenvalues is zero and the set of surfaces

$$F(\underline{n}) = |\underline{u} + S\underline{n}|^2 = \text{const.}$$

degenerates to a set of elliptical cylinders with their common axis perpendicular to the plane of the microphones. The matrix S^TS does not have a regular inverse and thus there no longer exists a unique centervector

$$\underline{C} = -(S^TS)^{-1}S^T\underline{u}$$

This is obvious as the matrix S^TS maps all vectors parallel to the cylinder axis on the zero vector and so there can be no inverse with respect to such vectors. We can however define a generalized inverse

$$(S^TS)^{-1'}$$

that behaves like an inverse with respect to vectors in the two-dimensional subspace which is orthogonal to the cylinder axis and that maps vectors parallel to the cylinder axis on the zero vector. This inverse has the following representation in a coordinate system that has been rotated as before

$$Q^T(S^TS)^{-1}Q = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The vector

$$\underline{C} = -(S^TS)^{-1}S^T\underline{u}$$

is now well defined and represents the point on the cylinder axis which is closest to the origin.

We must now distinguish between two different solutions depending on whether the cylinder axis is completely outside the unit sphere or not.

As before the coordinate system is rotated so that the coordinate axes are parallel to the principal axes of the cylinders.

In this rotated system the representation of $\underline{C}$ is

$$Q^T\underline{C} = \begin{bmatrix} -v_1/\lambda_1 \\ -v_2/\lambda_2 \\ 0 \end{bmatrix}$$

In the case that

$$|\underline{C}| = |Q^T\underline{C}| \geq 1$$

a 2-dimensional analogy to the previous method can be used. Thus, the solutions lie on the trajectory.

$$\underline{n}(k) = -(S^TS + kI)^{-1}S^T\underline{u}$$

As before the k-values corresponding to the possible solutions can be found from

$$g(k) = |Q^T\underline{n}|^2 - 1 = \sum_{i=1}^{2} \frac{v_i^2}{(k+\lambda_i)^2} - 1 = 0$$

which can have 2 or 4 solutions. These can be found exactly as before. They all lie in the plane of the microphones.

If on the other hand

$$|Q^T \underline{C}| < 1$$

the cylinder axis passes through the sphere and the normal is chosen on the unit sphere so that its projection on the microphone plane is $\underline{C}$. Its representation in the original system is then

$$\underline{n} = Q \begin{bmatrix} -v_1/\lambda_1 \\ -v_2/\lambda_2 \\ \pm\sqrt{1 - v_1^2/\lambda_1^2 - v_2^2/\lambda_2^2} \end{bmatrix}$$

There are two solutions, symmetric with respect to the plane of the microphones, as we can not tell from mathematical reasons from which side the wave arrives and so the sign before the square root is chosen so that $\underline{n}$ gets a positive elevation for physical reasons.

2.3. Numerical difficulties

A microphone system is normally placed on the ground for practical reasons. Thus the microphones are fairly close to a plane. The matrix $S^T S$ has one eigenvalue λ_3 which is 3-5 orders of magnitude smaller than the others. This means that the ellipsoids previously mentioned are very oblong and that the solution for the elevation of the wave normal is very sensitive to errors in the arrival times, at least for low elevations.

It has been shown that there may be up to six solutions to the equation

$g(k_i) = 0 \qquad i = 1, \ldots p \qquad$ where $p = 2, 4$ or 6.

To each of these is a corresponding wave normal

$$\underline{n}_i = \underline{n}(k_i)$$

and also a value of the error function

$F_i = F(\underline{n}_i) = F(\underline{n}(k_i))$

The value of F_i has been found to increase with i and thus the smallest value of the error function is assumed fo

$k = k_1$

that is the solution on the positive side of the asymptot $-\lambda_3$ for g(k) (see fig. 2). If there is no solution on the other side of this asymptot, $\underline{n}(k_1)$ is the solution to our problem. In the case that the next solution k_2 is immediately on the negative side of the same asymptot, an odd case can occur. Normally, $\underline{n}(k_1)$ has a positive and $\underline{n}(k_2)$ a negative elevation. In some cases these elevations are reversed due to the errors in the arrival times and for physical reasons we then have to accept the solution that gives the next smallest value, F_2, of the error function.

The solutions associated with the other asymptots have been found to have a much larger value of the error function and do not come into consideration. Thus, when using the analysis described above it is sufficient to calculate only the best one or two solutions and choose the one which is physically reasonable.

3. EFFECT OF THE IDEALIZATIONS

In the model three fundamental idealizations have been made. The first one is the assumption that the sound wave is planar. This is a very good approximation provided the distance to the sound source is large compared with the dimensions of the microphone system. Furthermore it will introduce systematic errors only in the estimation of the elevation angle of the wave normal as long as the microphone system is essentially horizontal.

The second idealization is that the speed of sound has a known constant value in the atmosphere over the microphone system. This is usually a good approximation as the speed of sound seldom varies more than 1% over the lowest few hundred meters of the atmosphere. The speed of sound at ground level can be accurately determined by means of a thermometer. Again the introduced errors only affect the elevation angle of the wave normal.

The third idealization is that the surface wind is negligible compared to the speed of sound. The wind speed is normally a few per cent of the speed of sound. The influence is the same as from an error in the speed of sound and furthermore it decreases with the cosine of the elevation angle.

Thus the idealizations in the first approximation have no influence on the azimuth of the wave normal which is the essential quantity for recovery purposes. It is evident that the determination of the elevation angle is fairly uncertain. Especially for low elevations the error can be considerable.

4. PRACTICAL EXPERIENCE

An ALGOL-program performing the calculations described above has been constructed and used on data collected during the experiments at the Kronogård range in Northern Sweden during the summer of 1964.

Three microphone systems were operated. One was located near the launching site and was equipped with 7 capacitor microphones. Two systems were located at down-range stations and equipped with 4 hot-wire microphones each.

All systems were operated on all 8 firings and a large number of shock waves were recorded both from vehicles during reentry and from exploding grenades in the rocket grenade experiment. All data have been successfully processed by the ALGOL-program.

The distance between the microphone systems was of the order of 20 - 30 km and it was generally possible to locate impacting objects to within a kilometer at least when the impact point was located so that the angles in the fix were not too unfavourable. This area is so small that a final helicopter search is feasible.

5. CONCLUSIONS

It has been demonstrated that it is possible to determine the direction to an impacting rocket by means of sound ranging with an accuracy that is adequate for recovery purposes. The problem is numerically cumbersome but it is possible to solve it by means of a computer. As data transmission over telephone lines is nowadays easy, it is possible to operate a computer also from a remote rocket range and thus obtain results so quickly that they can be used in search operations.

6. REFERENCES

(1) E Alan Dean,
At least squares solution for the directional cosines of a plane wave arriving at a sound ranging array.
Quarterly progress report No 2, Contract NAS 5-556,
Schellenger Research Laboratories,
Texas Western College, El Paso, Texas, 1961

(2) Emmit E Fisher, Robert P Lee, Henry Rachele,
Meteorological effects on an acoustic wave within a sound ranging array,
Technical report, MM - 435, Missile Meteorology Division
White Sands Missile Range, New Mexico, May 1962.

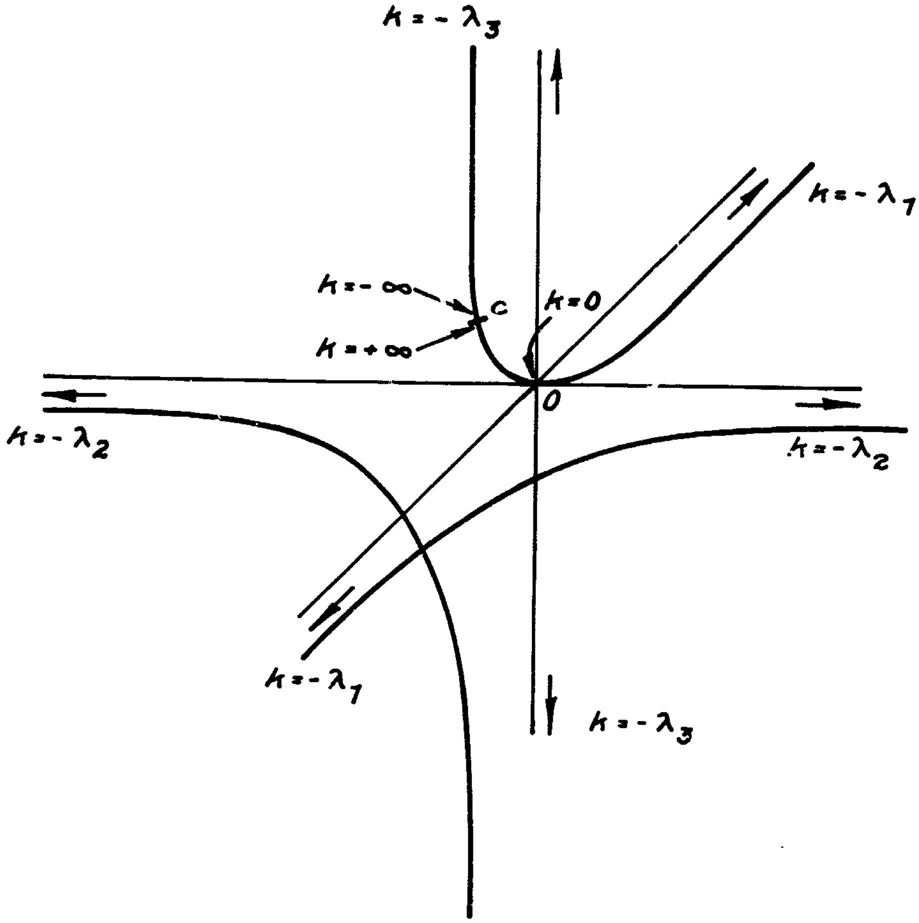

Fig. 1
The function $\underline{n}(k)$

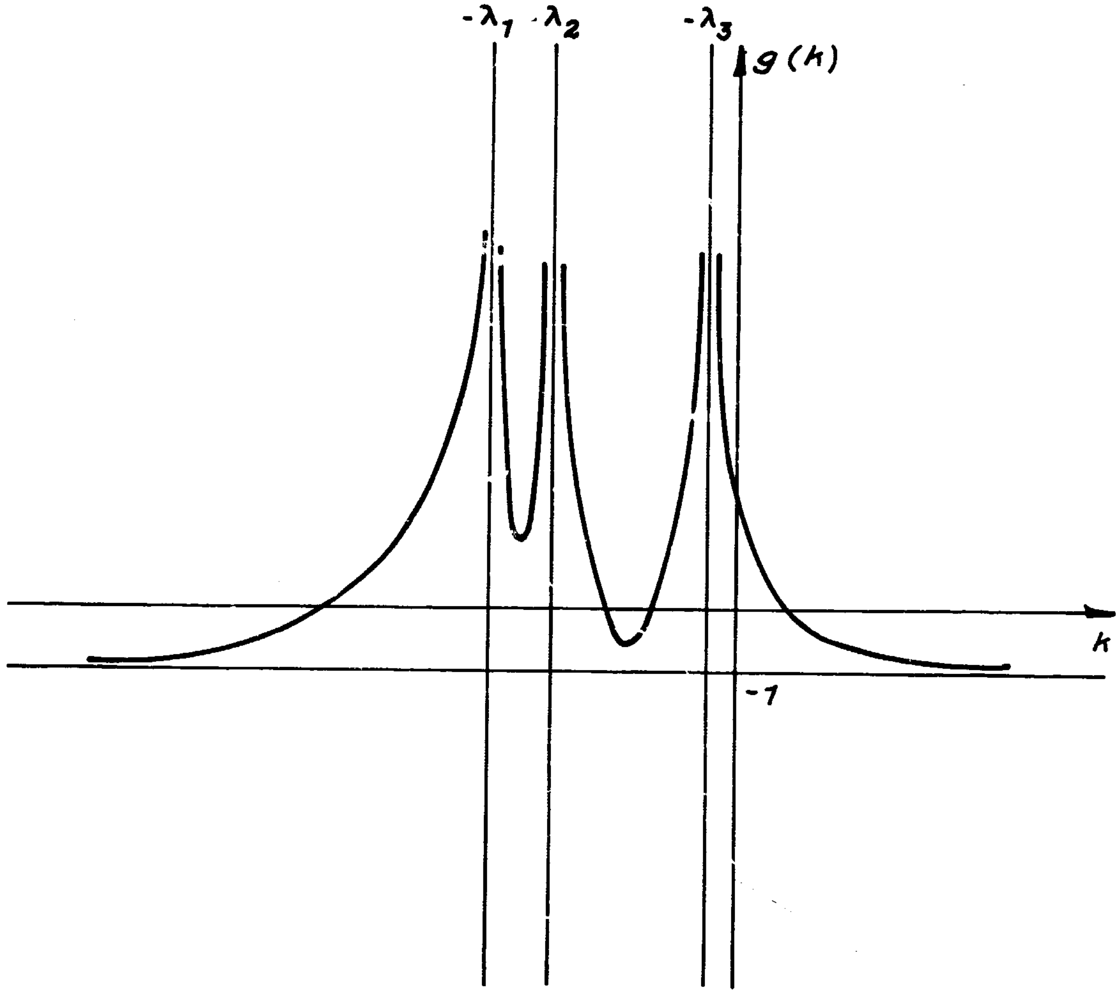

Fig. 2

The function g(k)

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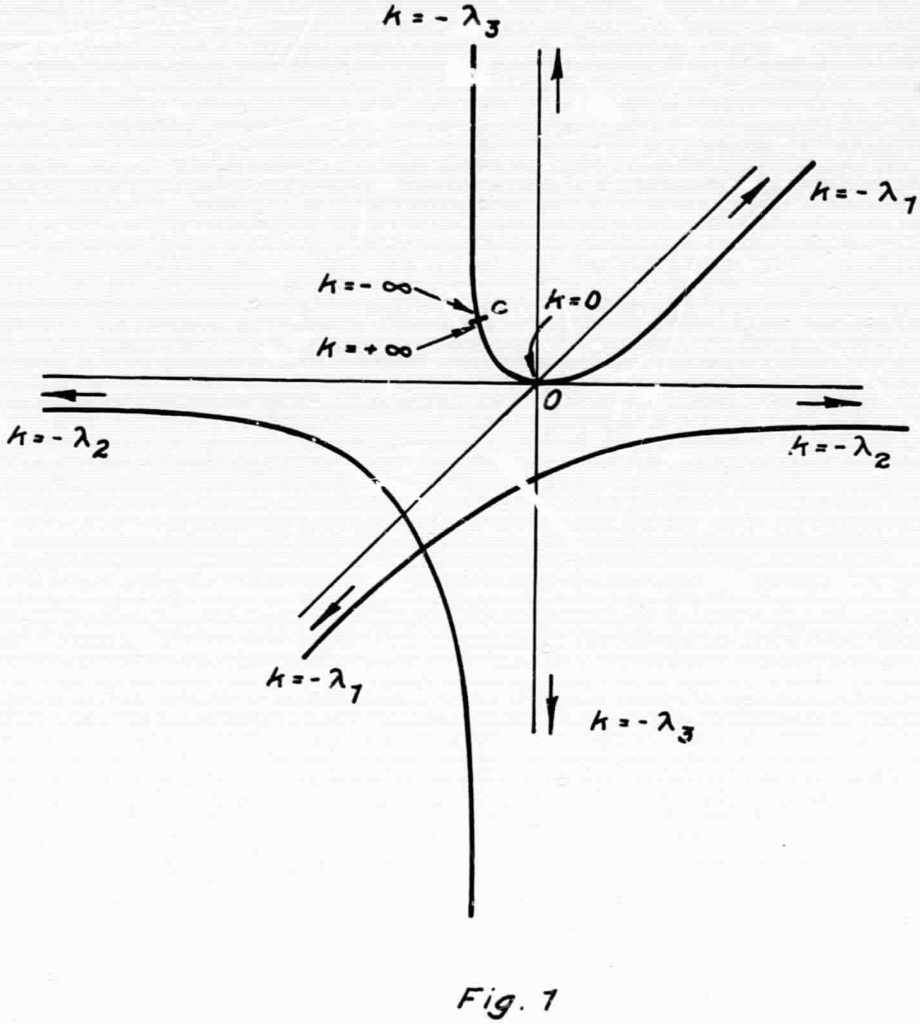

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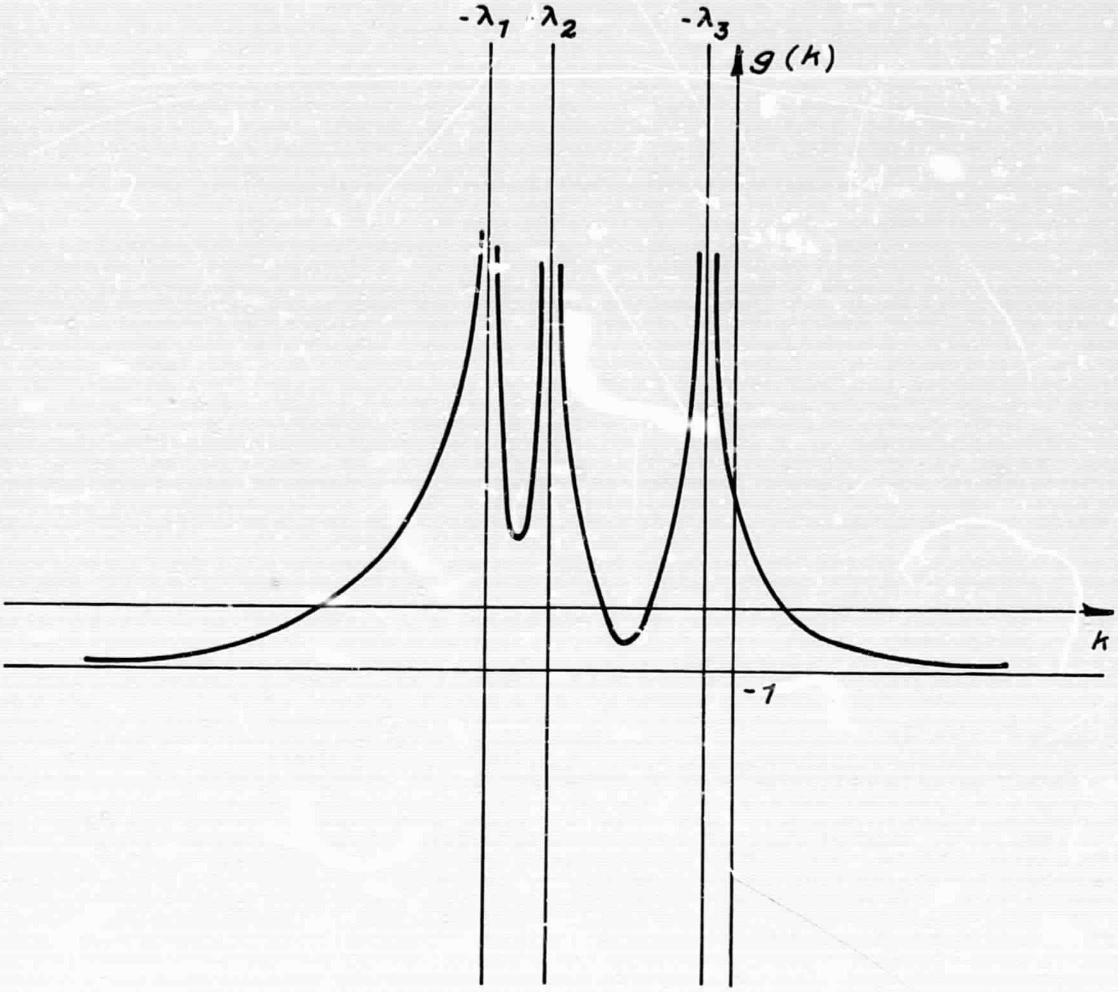

Fig. 2

The function $g(k)$